Sums of 2m-th powers of rational functions in one variable over real closed fields

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Abstract

We give a characterization of polynomials $f \in R[X]$, R real closed, which are representable as

$$\sum_{i=1}^{N} \left(\frac{g_i}{h}\right)^{2m} \text{ with } g_i, h \in R[X]$$

by introducing a suitable invariant $\rho(f) \in R$. Moreover, we give estimates for N and $\deg h$ in terms of m, $\deg f$ and $\rho(f)$. In the special case $m = 2$, we prove $N \leq 6$, independently of $\deg f$ and $\rho(f)$.

1 Introduction and results

In this paper we will deal with the following questions. Let $f \in R[X]$ be a polynomial in one variable over the real closed field R and let $m \geq 2$ be a natural number, fixed throughout this paper. Then the questions are:

(1.1) *For which f does there exist a representation*

$$f = \sum_{i=1}^{N} \left(\frac{g_i}{h}\right)^{2m} \textit{ with } g_i, h \in R[X]\,?$$

(1.2) *What can be said about the length N of such a representation?*

(1.3) *What can be said about the degree of h ?*

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Concerning (1.1), there is a very simple and satisfactory answer in case R is the field $\mathbb{R}$ of real numbers (or any real closed subfield of $\mathbb{R}$):

(1.4) *$f \in \mathbb{R}[X]$ has a representation (1.1) if and only if*

(a) $f \geq 0$ (i.e. $f(a) \geq 0$ for all $a \in \mathbb{R}$),

(b) $2m | \deg f$, and

(c) $2m$ divides the multiplicity of $X - a$ in f for every $a \in \mathbb{R}$.

This characterization is an easy consequence of Becker's *Valuation Criterion* (see [Be], Theorem (1.9)):

(1.5) *Let K be a formally real field. Then $f \in K$ is a sum of $2m$-th powers of elements of K if and only if f is a sum of squares in K and $2m$ divides $w(f)$ in Γ for every valuation $w : K^{\times} \longrightarrow \Gamma$ where Γ is an ordered abelian group and the residue field $\overline{K}_w$ of w is formally real.*

The reason why (1.5) is equivalent to (1.4) for $K = \mathbb{R}(X)$ is that the restriction of w in (1.5) to the field of coefficients $\mathbb{R}$ must be trivial (since $\overline{K}_w$ is formally real). This no longer holds if the field of coefficients of $R(X)$ is non-archimedean real closed. In such a case we find infinitely large positive elements $\omega \in R$ (i.e. $n < \omega$ for all $n \in \mathbb{N}$). Then the polynomial

$$f = X^{2m} + \omega X^2 + 1, \tag{1.6}$$

although satisfying (a),(b),(c) of (1.4), does not admit a representation (1.1) (cf. [P_2], Proof of Theorem 2, or the beginning of §2). Thus, the conditions (a)-(c) are no longer sufficient for f to have a representation (1.1). Clearly, they are still necessary as one easily sees from (1.5). We will now explain what has to be added to (1.4) in the non-archimedean case.

Let $f \in R[X]$ belong to $\Sigma R(X)^{2m}$, the set of sums of $2m$-th powers in $R(X)$. Now f can be decomposed into a product $f_1 \cdots f_r$ of polynomials $f_i \in R[X]$ which all belong again to $\Sigma R(X)^{2m}$ and are *indecomposable*, in the sense that f_i is not a product of non-constant polynomials from $R[X]$, all belonging to $\Sigma R(X)^{2m}$. Clearly every polynomial $f \in \Sigma R(X)^{2m}$ admits such a decomposition which, however, need not be uniquely determined. If f is monic, the f_i's may of course also be taken to be monic. It suffices to determine the monic indecomposable polynomials $f \in \Sigma R(X)^{2m}$.

In case of $R = \mathbb{R}$, it follows immediately from (1.4) that a monic polynomial f is an indecomposable element in $\Sigma R(X)^{2m}$ if and only if

(1.7) *(i) $f = (X - a)^{2m}$ for some $a \in R$, or*

(ii) $f > 0$ (i.e. $f(b) > 0$ for all $b \in R$) and $\deg f = 2m$.

The main result of §2 will be

Theorem 1.8 *Let R be a real closed field and $f \in R[X]$ be monic. Then f is an indecomposable element of $\Sigma R(X)^{2m}$ (in the above sense) if and only if (i) $f = (X-a)^{2m}$ for some $a \in R$, or (ii) $f > 0$, $\deg f = 2m$ and $\rho(f)$ is finite in R.*

The invariant $\rho(f)$ is defined for strictly positive (i.e. $f > 0$) monic polynomials $f \in R[X]$ of degree $2m$ as follows:

Write $f = q_1 \cdots q_m$ where each q_i is an irreducible monic polynomial from $R[X]$ of degree two, say

$$q_i = (X - a_i)^2 + b_i^2, \quad b_i > 0$$

for $1 \le i \le m$. We then let

$$\rho'(q_i, q_j) = \frac{(a_i - a_j)^2 + (b_i - b_j)^2}{(\min\{b_i, b_j\})^2}, \quad \text{and} \tag{1.9}$$

$$\rho(f) = \max\{\rho'(q_i, q_j) \mid 1 \le i < j \le m\}. \tag{1.10}$$

This invariant is a positive element of R. It is called *finite in* R, if $\rho(f) \le n$ for some $n \in \mathbb{N}$.

One easily checks that $\rho(f)$ is not finite in R for the polynomial f in (1.6). On the other hand, if R is archimedean, every $\rho(f)$ is finite. This is why the condition "$\rho(f)$ is finite in R" does not show up in (1.4).

The proof of Theorem 1.8 uses Becker's Criterion (1.5). For this reason it is not constructive and gives no answer to the questions (1.2) and (1.3). Nevertheless, Theorem 1.8 is helpful in getting such answers in §3 and §4.

Concerning (1.2), let us call the least N suitable for (1.1) and an arbitrary $f \in \Sigma R(X)^{2m}$ the $2m$-th *Pythagoras Number* $P_{2m}(R(X))$ of $R(X)$. By a result of Becker [Be], Theorem (2.9)), this number is known to be finite for all $m \in \mathbb{N}$ and all real closed fields R. In §3 we will prove (cf. Theorem 3.4) in a constructive way that

$$P_{2m}(R(X)) \le \binom{2m+3}{3},$$

which for small m is a slight improvement of Becker's bound. In the special case $m = 2$, Becker gave the estimate $P_4(R(X)) \le 36$, and Schmid ([Sch$_1$], Satz 3.1) gave $P_4(R(X)) \le 24$. By a geometric argument tailored for the case $m = 2$, we are able to prove in §4 (cf.(4.1)) that

$$P_4(R(X)) \le 6.$$

On the other hand, we also show that $P_{2m}(R(X)) \ge 3$ (for any $m \ge 2$), by an elementary argument at the end of §3.

Concerning (1.3), it is known from [P_2], Theorem 2 that there is no bound for $\deg h$ depending only on $\deg f$. In [B-P], Theorem B, it is shown for the case $R = \mathbb{R}$ that there is a bound on $\deg h$ depending on $\deg f$, the "size" of the coefficients of f, and the "distance" of the non-real zeros of f from the x-axis. The proof in [B-P] depends on the Compactness Theorem from Model Theory and is thus non-constructive. By the constructive proof of Theorem 3.4 in §3 we find an effective bound for $\deg h$ in terms of $\deg f$ and $\rho(f)$ for any strictly positive polynomial $f \in \Sigma R(X)^{2m}$. This estimate re-proves in an effective way the result of [B-P], since from the data there it is possible to estimate $\rho(f)$. In case $m = 2$, the special argument in §4 can be used to give the bound

$$\deg h \leq 3\left(\frac{\deg f}{4}\right)^2 \max\{\rho, 1\},$$

for any strictly positive polynomial $f \in \Sigma R(X)^4$ (cf. the proof of Theorem 4.12).

2 Characterization of sums of 2m-th powers

The main purpose of this section is to prove Theorem 1.8. In order to do so, we need to fix first some further notations and prove some lemmata.

From now on we use the following *notations*:

R	a real closed field,
$\leq$	its unique ordering,
$\mathcal{O}$	the convex hull of $\mathbb{Z}$ in R (which is a valuation ring),
$\mathcal{O}^\times$	the units of $\mathcal{O}$,
M	the maximal ideal of $\mathcal{O}$,
v	the valuation corresponding to $\mathcal{O}$,
$v(R)$	the value group of v (which is a divisible ordered abelian group),
$\overline{R}$	the residue field of $\mathcal{O}$,
$\overline{}$	the residue map from $\mathcal{O}$ to $\overline{R}$,
$\overline{f}$	the polynomial of $\overline{R}[X]$ obtained from $f \in \mathcal{O}[X]$ by taking residues of its coefficients.

We also note some basic facts, frequently used in the sequel:

(2.1) *(a)* $a \leq b \Rightarrow \overline{a} \leq \overline{b}$ *for all* $a, b \in \mathcal{O}$.

(b) $v(\sum_{i=1}^{m} a_i^2) = \min\{v(a_i^2) \mid 1 \leq i \leq n\}$ *for all* $n \in \mathbb{N}$ *and* $a_1, \ldots, a_n \in R$.

Note that (2.1) also holds for every valuation v of $R(X)$ if its residue field is formally real.

Concerning sums of $2m$-th powers from $R(X)$ we note the following obvious consequence of Becker's Criterion (1.5):

(2.2) Let $f \in R[X]$ belong to $\Sigma R(X)^{2m}$. Then

(a) $f \geq 0$,

(b) $2m \mid \deg f$,

(c) $2m$ divides the multiplicity of $x - a$ in f for every $a \in R$.

Moreover we have

Lemma 2.3 *Let $g \in \mathcal{O}[Y]$. If $g \in \Sigma R(Y)^{2m}$, then $\overline{g} \in \Sigma \overline{R}(Y)^{2m}$.*

Proof: Extend the valuation v from R to $R(Y)$ by setting (cf. [Be], Ch.6, §10, Prop. 2):

$$v'(a_n Y^n + \ldots + a_0) := \min\{v(a_j) \mid 0 \leq j \leq n\}.$$

Then the residue field of v' is $\overline{R}(\overline{Y})$ with $\overline{Y}$ transcendental over $\overline{R}$. Thus we write again Y for $\overline{Y}$. Since $g \in \mathcal{O}[Y]$, we have $v'(g) \geq 0$. Thus if $g = \Sigma r_j^{2m}$ with $r_j \in R(Y)$, we get $v'(r_j) \geq 0$ from the note after (2.1). Thus

$$\overline{g} = \Sigma \overline{r_j}^{2m} \in \Sigma \overline{R(Y)}^{2m}.$$

This proves $\overline{g} \in \Sigma \overline{R}(Y)^{2m}$. □

From (2.3) it is now easy to prove that the polynomial (1.6), say for $m = 2$, cannot belong to $\Sigma R(X)^4$. If it would belong to $\Sigma R(X)^4$, we would find

$$g(Y) = Y^4 + Y^2 + \frac{1}{\omega^2} \in \Sigma R(Y)^4$$

by taking $X = \sqrt{\omega} Y$. Now Lemma 2.3 implies

$$\overline{g}(Y) = Y^2(Y^2 + 1) \in \Sigma \overline{R}(Y)^4,$$

contradicting (2.2)(c).

We now turn to the proof of Theorem 1.8. As a first step we will show

Lemma 2.4 *Let $f \in R[X]$ be monic, strictly positive and of degree $2m$. If $\rho(f)$ is finite in R, we have $f \in \Sigma R(X)^{2m}$. (Clearly, such f is indecomposable.)*

Before we prove Lemma 2.4 let us recall from §1 that $f = q_1 \cdots q_m$ with

$$q_i = (X - a_i)^2 + b_i^2, \ b_i > 0.$$

With these notations and the definitions (1.8) and (1.9), we see that the finiteness of $\rho(f)$ in R implies that

$$\rho'(q_i, q_j) \in \mathcal{O} \quad \text{for all } 1 \leq i < j \leq m. \tag{2.5}$$

This property and the next Proposition will enable us to prove Lemma 2.4.

Proposition 2.6 *Let $q_1 = (X-a_1)^2+b_1^2$ and $q_2 = (X-a_2)^2+b_2^2$ be polynomials with $a_1, a_2, b_1, b_2 \in R$ and $b_1, b_2 > 0$. Then the following are equivalent:*

(1) $\rho' = \rho'(q_1, q_2) \in \mathcal{O}$;

(2) there exists $M \in \mathbb{N}$ such that for all $x \in R$,

$$M^{-1} \le \frac{q_1(x)}{q_2(x)} \le M;$$

(3) q_1/q_2 is a unit in the real holomorphy ring $H(R(X))$ of $R(X)$;

(4) $b_1/b_2 \in \mathcal{O}^\times$ and $(a_1-a_2)/b_1 \in \mathcal{O}$.

Proof:

(1) $\Rightarrow$ (2): Let $z_i = a_i + b_i\sqrt{-1}$. Then $q_i(x) = \|z_i - x\|^2$. Now (2) follows from

$$\frac{\|z_1 - x\|}{\|z_2 - x\|} \le \frac{\|z_2 - x\| + \|z_1 - z_2\|}{\|z_2 - x\|} \le 1 + \frac{\|z_1 - z_2\|}{b_2} \le 1 + \sqrt{\rho'}$$

and the symmetry of (1).

(2) $\Rightarrow$ (3): Recall that the real holomorphy ring $H(R(X))$ is defined to be the intersection of all valuation rings $\mathcal{O}_1$ of $R(X)$ which have a formally real residue field (cf. [Be]). Every such valuation ring $\mathcal{O}_1$ is a convex subring of $R(X)$ with respect to some ordering $\le$ of $R(X)$ (see e.g. [P_1]). From (2) we then get $M^{-1} \le q_1/q_2 \le M$ by a simple application of the Artin-Lang Theorem. This implies that $q_1/q_2 \in H(R(X))^\times$.

(3) $\Rightarrow$ (4): Let $\varphi : R(X) \to R \cup \{\infty\}$ be the place extending the substitution $X \mapsto x \in R$. Let furthermore $\mathcal{O}_1 = \varphi^{-1}(\mathcal{O})$. Then $\mathcal{O}_1$ is a valuation ring of $R(X)$ with residue field $\overline{R}$. Hence by (3), q_1/q_2 is a unit in $\mathcal{O}_1$. Applying the place φ, we get $q_1(x)/q_2(x) \in \mathcal{O}^\times$. Choosing $x = a_2$, we have therefore

$$\frac{(a_2-a_1)^2 + b_1^2}{b_2^2} \in \mathcal{O}.$$

By (2.1)(b), this implies $(a_1-a_2)/b_2$, $b_1/b_2 \in \mathcal{O}$. Similarly, we get $b_2/b_1 \in \mathcal{O}$. Hence $b_1/b_2 \in \mathcal{O}^\times$.

(4) $\Rightarrow$ (1): From (4) we get at once that

$$\frac{(a_1-a_2)^2}{b_1^2} \in \mathcal{O}, \quad \text{and} \quad \left(\frac{b_2}{b_1} - 1\right)^2 = \frac{(b_1-b_2)^2}{b_1^2} \in \mathcal{O}.$$

This together with $b_1/b_2 \in \mathcal{O}^\times$ gives (1). □

Proof (of Lemma 2.4): From (2.5) and Proposition 2.6, we get

$$\frac{q_2}{q_1}, \dots, \frac{q_m}{q_1} \in H(R(X))^\times .$$

Therefore, for every valuation w of $R(X)$ whose residue field is formally real,

$$w(f) = w(q_1 \cdots q_m) = w(q_1^m) = m\, w(q_1).$$

Since $2|w(q_1)$ by (2.1)(b), we find $2m|w(f)$. Thus, by Becker's Criterion (1.5), f belongs to $\Sigma\, R(X)^{2m}$.

The proof of Theorem 1.8 will therefore be complete once we establish the following lemma:

Lemma 2.7 *Let $f \in R[X]$ be a monic and strictly positive element of $\Sigma\, R(X)^{2m}$. If f is indecomposable, then $\deg f = 2m$ and $\rho(f)$ is finite in R.*

Proof: Assuming that $f \in R[X] \cap \Sigma\, R(X)^{2m}$ is monic and strictly positive, we will prove that there exists a monic factor f_1 of f of degree $2m$ such that $\rho(f_1) \in \mathcal{O}$. By Lemma 2.4, this implies that $f_1 \in \Sigma\, R(X)^{2m}$. Since f is assumed to be indecomposable, this gives $f = f_1$. Therefore we have $\deg f = 2m$ and $\rho(f) \in \mathcal{O}$

Let us write $f = q_1 \cdots q_n$ with $n = rm$ and

$$q_j = (X - a_j)^2 + b_j^2, \quad a_j, b_j \in R, \; b_j > 0.$$

Since both properties, $\rho(f) \in \mathcal{O}$ and $f \in \Sigma\, R(X)^{2m}$, are invariant under linear substitutions, we may freely use them. Thus w.l.o.g. we may first assume that

$$a_1 = 0. \tag{2.8}$$

Secondly, we will make the a_j's and b_j's integral as follows. Let $\delta \in R, \delta > 0$ be such that

$$v(\delta) = \min\{v(a_j), v(b_j) |\; 1 \le j \le n\}.$$

We then define $Y := X/\delta$ and

$$g(Y) := \delta^{-2n} f(X) = \prod_{j=1}^{n} [(Y - a_j')^2 + b_j'^2]$$

where we set $a_j' := a_j/\delta$ and $b_j' := b_j/\delta$. The polynomial $g(Y)$ now has coefficients in $\mathcal{O}$. Thus Lemma 2.3 and $f \in \Sigma\, R(X)^{2m}$ imply

$$\overline{g}(Y) \in \Sigma\, \overline{R}(Y)^{2m} . \tag{2.9}$$

Now we distinguish two cases:

Case 1: $\overline{b'_j} \neq 0$ for some $j \leq n$ (say $j = n$).
If the polynomial $\overline{g}$ has a zero $\overline{a_e}$ in R, by (2.2)(c) its multiplicity is divisible by $2m$. Thus the number of factors q_e with $\overline{b'_e} \neq 0$ is divisible by m. Hence we can find $m-1$ additional factors, say $q_{n-m+1}, \ldots, q_{n-1}$, with $b'_e \neq 0$. If we then set

$$f_1 = q_{n-m+1} \cdots q_n ,$$

we find $\rho(f_1) \in \mathcal{O}$, since $\overline{b'_e} \neq 0$ implies $b'_e \in \mathcal{O}^\times$ and thus for all $n-m+1 \leq j < e \leq n$:

$$\frac{(a_j - a_e)^2 + (b_j - b_e)^2}{\min\{b_j, b_e\}^2} = \frac{(a'_j - a'_e)^2 + (b'_j - b'_e)^2}{\min\{b'_j, b'_e\}^2} \in \mathcal{O}. \tag{2.10}$$

Case 2: $\overline{b'_j} = 0$ for all $j \leq n$.
In this case, necessarily, $\overline{a'_j} \neq 0$ for some j (say $j = n$). After a suitable renumbering, we choose $n_1 \in \mathbb{N}$ such that $\overline{a'_j} = 0$ for all $j \leq n_1$ and $\overline{a'_j} \neq 0$ for all $n_1 < j \leq n$. Thus (2.8), Lemma 2.3 and (2.2)(c) imply

$$1 \leq n_1 < n \text{ and } m | n_1 . \tag{2.11}$$

Next we choose $\delta_1 \in R$, $\delta_1 > 0$, such that

$$v(\delta_1) = \min\{v(a'_j), v(b'_j) | \ 1 \leq j \leq n_1\} , \tag{2.12}$$

and define $Y_1 := Y/\delta_1$ and

$$g_1(Y_1) := \delta_1^{-2n_1} g(Y) = \prod_{j=1}^{n_1} [(Y_1 - a''_j)^2 + b''^2_j] \cdot \prod_{j=n_1+1}^{n} [(\delta_1 Y_1 - a'_j)^2 + b'^2_j] , \tag{2.13}$$

where we set $a''_j = a'_j/\delta_1$ and $b''_j = b'_j/\delta_1$. The polynomial $g_1(Y_1)$ has coefficients in $\mathcal{O}$. Thus by Lemma 2.3 and $g(Y) \in \Sigma R(Y)^{2m}$ we obtain $\overline{g}_1(Y_1) \in \Sigma \overline{R}(Y_1)^{2m}$. Taking residue classes in g_1 and observing that

$$(\overline{\delta_1} Y_1 - \overline{a'_j})^2 + \overline{b'_j}^2 = \overline{a'_j}^2 \in \overline{R}^{2m} ,$$

we finally get

$$\overline{h_1}(Y_1) = \prod_{j=1}^{n_1} [(Y_1 - \overline{a''_j})^2 + \overline{b''_j}^2] \in \Sigma \overline{R}(Y_1)^{2m}. \tag{2.14}$$

Now (2.14) puts us into a situation similar to (2.9). The difference, however, is that $\deg \overline{h_1}$ is less than $\deg \overline{h}$ (where we set $\overline{h} = \overline{g}$). Thus we can repeat the argument following (2.9). There are again two cases:

Case 1: $\overline{b''_j} \neq 0$ for some $j \leq n_1$. In this case we finish similarly as we did before, since (2.10) also holds for a', b' replaced by a'', b''.

Case 2: $\overline{b_j''} = 0$ for all $j \le n_1$. In this case there exists $\overline{a_j''} \ne 0$ for some $j \le n_1$ (say $j = n_1$). Thus, if after a suitable renumbering, we define $n_2 \in \mathbb{N}$ by $\overline{a_j''} = 0$ for all $j \le n_2$ and $\overline{a_j''} \ne 0$ for all $n_2 < j \le n_1$, we find as in (2.11):

$$1 \le n_2 < n_1 \text{ and } m|n_2.$$

We then define $\delta_2 \in R$ as in (2.12), replacing n_1 there by n_2, and define $g_2(Y_2)$ as in (2.13) with the obvious changes. We finally end up with a polynomial $\overline{h_2}(Y_2) \in \Sigma\,\overline{R}(Y_2)^{2m}$ similar to (2.14). Now $\deg \overline{h_2} < \deg \overline{h_1}$, and we may repeat the whole procedure.

After repeating this procedure finitely many times, we have to end up with "Case 1". In fact, we can only be forced into Case 2 if all the b's under consideration have values bigger (i.e. are essentially smaller) than the corresponding a's. But in this case the number of a's under consideration is reduced. Since we have assumed $a_1 = 0$ in (2.8), a repeated application of Case 2 will finally give some $n_t < \cdots < n_2 < n_1 < n$ such that $a_j = 0$ for all $j \le n_t$ and $m|n_t$. Defining now δ_t as in (2.12) and $\overline{h}_t$ as in (2.14), we must be in Case 1 for $\overline{h_t}$. □

3 Effective bounds

With the notations from §§1-2, let $f \in R[X]$ be monic and assume that f admits a representation

$$(3.1) \qquad f = \sum_{i=1}^{N} \left(\frac{g_i}{h}\right)^{2m} \text{ for some } g_i, h \in R[X].$$

In this section we will find effective bounds for N and $\deg h$.

Assuming that f belongs to $\Sigma\,R(X)^{2m}$, we know from Theorem 1.8 that f admits a factorization

$$f = f_1 \cdots f_r$$

where each f_i is monic, has degree $2m$ and is either equal to some $(X-a)^{2m}$ or is strictly positive with its invariant $\rho(f_i)$ finite in R. Since the factors $(X-a)^{2m}$ do not contribute anything to N or $\deg h$ in a representation (3.1), we may assume that each f_i is of the second type. Let us then take

$$(3.2) \qquad \rho := \max\{\rho(f_i) \mid 1 \le i \le r\}.$$

The bound N obtained below will depend only on m, while the bound on $\deg h$ will depend on m, $\deg f$, ρ and some positive $\varepsilon \in \mathbb{Q}$, to be introduced now. This ε is determined by m.

By a result of Hilbert (in his work on Waring's Problem: cf. [El], or [R$_2$: (5.14)]), the form

$$(X_1^2 + X_2^2 + X_3^2 + X_4^2)^m$$

is interior to the cone $Q_{4,2m}(\mathbb{R})$ of sums of $2m$-th powers of linear forms from $\mathbb{R}[X_1,...,X_4]$. Hence there exists some positive $\varepsilon = \varepsilon(m) \in \mathbb{Q}$ such that

(3.3) $[X_1^2+X_2^2+(1+\varepsilon)(X_3^2+X_4^2)]\cdot[(1+\varepsilon)(X_1^2+X_2^2)+X_3^2+X_4^2]^{m-1} \in Q_{4,2m}(\mathbb{R})$.

(It can be shown, for instance, that $\varepsilon(m)$ can be taken to be any positive rational number $\leq \sqrt{m+1}$. However, the actual value of ε is going to be immaterial. Therefore, we will not digress here to get an explicit ε.) Since the $\mathbb{R}$-vector space of forms of degree $2m$ from $\mathbb{R}[X_1,...,X_4]$ has dimension $\binom{2m+3}{3}$, it follows from Caratheodory's Theorem (see Proposition 2.3 in [R$_1$]) that every element of $Q_{4,2m}(\mathbb{R})$ is actually a sum of $\binom{2m+3}{3}$ $2m$-th powers of linear forms from $\mathbb{R}[X_1,...,X_4]$. Thus the fact (3.3) can be expressed in a first order formula over $\mathbb{R}$. By Tarski's Transfer Principle we therefore obtain that for every real closed field R the form in (3.3) is a sum of $\binom{2m+3}{3}$ $2m$-th powers of linear forms from $R[X_1,...,X_4]$ (for the same $\varepsilon \in \mathbb{Q}$).

Now we can state the main theorem of this section.

Theorem 3.4 *Let R be real closed, $f \in R[X]$ be monic, strictly positive and an element of $\Sigma R(X)^{2m}$, of degree $2mr$. Let ρ and ε be defined as above. Then f admits a representation (3.1) with $N \leq \binom{2m+3}{3}$ and*

$$\deg h \leq \left(2m(2m-1)T_0 + \frac{m-1}{m}\right)\deg f\,,$$

where T_0 is the smallest integer satisfying

$$m^2(m-1)[(1+\sqrt{\rho})^r - 1]^2 < 4T_0\log(1+\varepsilon).$$

For the proof of this theorem we shall need several lemmata.

Lemma 3.5 *Let $k_1,...,k_m \in R[X]$ be strictly positive and assume that there exists a $\delta > 1$ in R such that $\delta^{m-1} < 1+\varepsilon$ and $k_i(x)/k_j(x) < \delta$ for every $x \in R$ and $i,j \in \{1,...,m\}$. Then $f = \prod_{i=1}^m k_i$ has a representation (3.1) with $N \leq \binom{2m+3}{3}$ and $\deg h \leq (m-1)\deg k_1$.*

Proof: [1] If we set

$$k = (k_1)^{-(m-1)}\cdot\prod_{i=1}^m k_i \in R(X)\,,$$

the assumption of the lemma gives us

$$k < (1+\varepsilon)k_1 \quad\text{and}\quad k_1 < (1+\varepsilon)k$$

[1] In an earlier version of this lemma we had $N \leq \binom{4m-1}{2m}$ and $\deg h = 0$. We are grateful to J. Schmid for pointing out this improvement on N. (It is more important here to get a smaller N than to have $\deg h = 0$, since denominators will be introduced later in any case.)

on R, and therefore

$$[(1+\varepsilon)k_1 - k]k_1^{2(m-1)} \geq 0 \text{ and } [(1+\varepsilon)k - k_1]k_1^{2(m-1)} \geq 0$$

on R. Since every positive semidefinite polynomial in $R[X]$ is a sum of two squares of polynomials, there exist rational functions $p_1, p_2, q_1, q_2 \in R(X)$ having common denominator k_1^{m-1} such that

$$(1+\varepsilon)k_1 - k = q_1^2 + q_2^2, \tag{3.6}$$

$$(1+\varepsilon)k - k_1 = p_1^2 + p_2^2. \tag{3.7}$$

Solving for k and k_1 from (3.6) and (3.7), we get

$$\varepsilon(2+\varepsilon)k = (1+\varepsilon)(p_1^2+p_2^2) + q_1^2 + q_2^2,$$

$$\varepsilon(2+\varepsilon)k_1 = p_1^2 + p_2^2 + (1+\varepsilon)(q_1^2+q_2^2).$$

Therefore,

$$\prod_{i=1}^{m} k_i = k_1^{m-1} \cdot k = \frac{1}{\varepsilon^m(2+\varepsilon)^m}[p_1^2+p_2^2+(1+\varepsilon)(q_1^2+q_2^2)]^{m-1}[(1+\varepsilon)(p_1^2+p_2^2)+q_1^2+q_2^2].$$

Now by the analog of (3.3) in R, $\prod_{i=1}^{m} k_i$ is a sum of at most $\binom{2m+3}{3}$ $2m$-th powers of linear forms in p_1, p_2, q_1, q_2. Thus the conclusion of the lemma follows. □

Let us now return for a moment to Theorem 3.4 and see what is missing in order to deduce this theorem from Lemma 3.5.

We have already assumed that f factors into $f_1 \cdots f_r$ with $f_i \in R[X]$ monic, strictly positive of degree $2m$ and $\rho(f_i)$ finite in R. Writing each f_i as

$$f_i = q_{i1} \cdots q_{im} \tag{3.8}$$

with $q_{i\nu} \in R[X]$ monic and irreducible of degree two, we see from the proof of Proposition 2.6, (1)⇒(2), that for all $\nu, \mu \in \{1, ..., m\}$

$$\frac{q_{i\nu}(x)}{q_{i\mu}(x)} < 1 + \sqrt{\rho} \text{ for all } x \in R \tag{3.9}$$

where ρ is defined in (3.2). Taking now $k_\nu = \prod_{i=1}^{r} q_{i\nu}$ for $1 \leq \nu \leq m$, we find from (3.9) that for all ν, $\mu \in \{1, ..., m\}$ and $x \in R$:

$$\frac{k_\nu(x)}{k_\mu(x)} < (1+\sqrt{\rho})^r. \tag{3.10}$$

In case $(1+\sqrt{\rho})^{r(m-1)} < 1 + \varepsilon$, we could apply Lemma 3.5 to

$$f = \prod_{\nu=1}^{m} k_\nu \tag{3.11}$$

to derive the desired result. But, of course, $(1+\sqrt{\rho})^{r(m-1)} < 1+\varepsilon$ need not hold in general! What remains is to find a "refined" factorization (3.11) of f where the quotients k_ν/k_μ are "small enough" to allow the application of Lemma 3.5. This will be accomplished in the two lemmas below.

Lemma 3.12 *Suppose that r and s are positive elements in R satisfying the inequality $M^{-1} < r/s < M$ for some $M > 1$, and for $\lambda \in R$, define $A_\lambda(r,s) = (1-\lambda)r + \lambda s$. If λ, α and β are positive elements of R so that $\lambda \pm \alpha$ and $\lambda \pm \beta$ all lie in the interval $[0,1]$, then*

$$\left| \frac{A_{\lambda+\alpha}(r,s)A_{\lambda-\alpha}(r,s)}{A_{\lambda+\beta}(r,s)A_{\lambda-\beta}(r,s)} - 1 \right| \le |\beta^2 - \alpha^2|\,(M-1)^2.$$

Proof: Observe that $0 \le \lambda \pm \alpha \le 1$ implies that $0 \le \lambda \le 1$. Furthermore, the conditions on r and s are symmetric, and $A_{1-\lambda}(s,r) = A_\lambda(r,s)$, so we may assume without loss of generality that $\lambda \in [0,1/2]$. A straightforward calculation gives

$$A_{\lambda+t}(r,s)A_{\lambda-t}(r,s) = (A_\lambda(r,s)+t(s-r))(A_\lambda(r,s)-t(s-r)) = A_\lambda^2(r,s) - t^2(s-r)^2.$$
(3.13)

Thus we obtain the identity:

$$(3.14) \qquad \left| \frac{A_{\lambda+\alpha}(r,s)A_{\lambda-\alpha}(r,s)}{A_{\lambda+\beta}(r,s)A_{\lambda-\beta}(r,s)} - 1 \right| = |\beta^2 - \alpha^2| \frac{(s-r)^2}{A_{\lambda+\beta}(r,s)A_{\lambda-\beta}(r,s)}.$$

For fixed r, s and λ, the right hand side of (3.14) is maximized when the denominator is minimized. By (3.13), $A_{\lambda+t}(r,s)A_{\lambda-t}(r,s)$ is a decreasing function for $t > 0$; hence the denominator is minimized by making β as large as possible. Since $0 \le \lambda \pm \beta \le 1$ (and $\lambda \in [0,1/2]$), this value occurs when $\beta = \lambda$. Let $v = s/r$ (so $1/M < v < M$) and consider

$$\begin{aligned} \frac{(s-r)^2}{A_{\lambda+\beta}(r,s)A_{\lambda-\beta}(r,s)} &\le \frac{(s-r)^2}{A_{2\lambda}(r,s)A_0(r,s)} = \frac{(s-r)^2}{((1-2\lambda)r + 2\lambda s)r} \\ &= \frac{(v-1)^2}{(1-2\lambda)+2\lambda v} := G(\lambda, v). \end{aligned}$$

Since $2\lambda \in [0,1]$, we have $(1-2\lambda)+2\lambda v \ge \min(1,v)$. If $1 \le v \le M$, then $G(\lambda,v) \le (v-1)^2 \le (M-1)^2$. If, instead, $1/M < v < 1$, then

$$G(\lambda,v) \le v^{-1}(v-1)^2 \le M(1-M^{-1})^2 \le (M-1)^2.$$

Thus, (3.13) and (3.14) combine to give the conclusion in the Lemma. □

Lemma 3.15 *For any given m let $g, g' \in R[X]$ be strictly positive and $1 < M \in \mathbb{Q}$ be such that $M^{-1} < g(x)/g'(x) < M$ for all $x \in R$. To every $\delta > 1$ in $\mathbb{Q}$, there exist strictly positive polynomials $p, p_1, \ldots, p_m \in R[X]$ such that $p^m \cdot g = p_1 \cdots p_m \cdot g'$ and $p_i(x)/p_j(x) < \delta$ for all $x \in R$ and all $i,j \in \{1,\ldots,m\}$.*

Proof: For rational λ with $0 \leq \lambda \leq 1$, let

$$A_\lambda := (1-\lambda)g + \lambda g' \in R[X] \quad \text{and} \quad B_\mu = A_{\mu/2mT} \quad \text{for } 0 \leq \mu \leq 2mT\,,$$

where T is a large integer to be chosen later. Clearly these polynomials are again strictly positive, with $B_0 = A_0 = g$ and $B_{2mT} = A_1 = g'$. Now let

$$p := \prod_{\mu=1}^{2mT} B_\mu \quad \text{and} \quad C_\mu := B_{\mu-1} B_\mu^{m-1} \;\; (1 \leq \mu \leq 2mT).$$

Then

$$g \cdot p^m = B_0 \left(\prod_{\mu=1}^{2mT} B_\mu \right)^m = \left(\prod_{\mu=1}^{2mT} B_{\mu-1} B_\mu^{m-1} \right) B_{2mT} = \left(\prod_{\mu=1}^{2mT} C_\mu \right) g'.$$

For each integer i with $1 \leq i \leq m$, we group together the factors C_μ with $\mu \equiv i$ or $1-i \pmod{2m}$ to form the subproducts

$$p_i = \prod_{\nu=0}^{T-1} C_{2\nu m+i} C_{2(\nu+1)m+1-i}\,,$$

so that we have $g \cdot p^m = (\prod_{i=1}^m p_i)\, g'$. It remains to check that the quotients p_i/p_j have the desired size. First we note that

$$\text{(3.16)} \qquad \frac{p_i}{p_j} = \prod_{\nu=0}^{T-1} \frac{B_{2\nu m+i-1} B_{2(\nu+1)m-i}}{B_{2\nu m+j-1} B_{2(\nu+1)m-j}} \left(\frac{B_{2\nu m+i} B_{2(\nu+1)m+1-i}}{B_{2\nu m+j} B_{2(\nu+1)m+1-j}} \right)^{m-1}.$$

Now the two different quotients in the product (3.16) can be estimated by using Lemma 3.12 for suitable choices of λ, α, and β. In fact, for the first quotient we may take

$$\lambda = \frac{(2\nu+1)m - \frac{1}{2}}{2mT}, \quad \alpha = \frac{m-i+\frac{1}{2}}{2mT}, \quad \beta = \frac{m-j+\frac{1}{2}}{2mT},$$

and for the second one

$$\lambda = \frac{(2\nu+1)m + \frac{1}{2}}{2mT}, \quad \alpha \text{ and } \beta \text{ as above.}$$

For $i, j \in \{1, \ldots, m\}$ we then have $|\alpha|, |\beta| \leq m/2mT = 1/2T$. Now (3.12) (applied with $r = g(x)$ and $s = g'(x)$) shows that each quotient of (3.16) is bounded by $1 + (M-1)^2/4T^2$. Thus we get

$$\frac{p_i}{p_j} \leq \left(1 + \frac{(M-1)^2}{4T^2} \right)^{mT} \leq \left(e^{(M-1)^2/4T^2} \right)^{mT} = e^{m(M-1)^2/4T}.$$

Choosing now the natural number T so large that

$$4T\log\delta > m(M-1)^2\,, \tag{3.17}$$

we have achieved the desired estimate $p_i/p_j < \delta$. □

Now we are ready to give the

Proof of Theorem 3.4: In (3.10) and (3.11) we already obtained a factorization of f into $k_1\cdots k_m$ with all k_i's monic and strictly positive such that for all $i,j\in\{1,...,m\}$ and all $x\in R$:

$$k_i(x)/k_j(x) < (1+\sqrt{\rho})^r \le M \in \mathbb{Q}. \tag{3.18}$$

We now apply Lemma 3.15 to $g' = k_1$ and $g = k_i$ for each $1\le i\le m$ with $\delta > 1$ in $\mathbb{Q}$ chosen such that

$$\delta^{m(m-1)} < 1+\varepsilon\,. \tag{3.19}$$

Thus, for every $i\in\{1,...,m\}$, we obtain strictly positive polynomials p_i and $p_{i1},...,p_{im}\in R[X]$ such that

$$p_i^m k_i = p_{i1}\cdots p_{im}k_1 \quad \text{and} \quad p_{i\nu}(x)/p_{i\mu}(x) < \delta \tag{3.20}$$

for all $x\in R$ and $\nu,\mu\in\{1,...,m\}$. Therefore,

$$(p_1\cdots p_m)^m f = (p_{11}\cdots p_{1m})\cdots(p_{m1}\cdots p_{mm})k_1^m\,,$$

and after setting

$$p' = p_1\cdots p_m,\quad p'_\nu = p_{1\nu}\cdots p_{m\nu}\,(p'k_1)\quad (1\le\nu\le m)\,, \tag{3.21}$$

we have

$$(p')^{2m}f = p'_1\cdots p'_m \;\text{ with }\; p'_\nu(x)/p'_\mu(x) < \delta^m$$

for all $x\in R$ and all $\nu,\mu\in\{1,...,m\}$. Thus the choice of δ with (3.19) allows us to conclude from Lemma 3.5 that $(p')^{2m}f$ admits a representation as in (3.1) with $N\le\binom{2m+3}{3}$, and with denominator (say) h' satisfying

$$\deg h' \le (m-1)\deg p'_1. \tag{3.22}$$

To compute $\deg p'_1$, note that from (3.20) and (3.21):

$$\begin{aligned}
\deg p'_1 &= \sum_i \deg p_{i1} + \deg p' + \deg k_1 && (3.23)\\
&= \sum_i \deg p_i + \deg p' + \deg k_1 && (3.24)\\
&= 2\deg p' + \deg k_1\,. && (3.25)
\end{aligned}$$

Here we use the fact that, for strictly positive polynomial $a, b \in R[X]$, the boundedness of a/b and b/a on R implies $\deg a = \deg b$. From this fact, we also get (in view of (3.10)):

$$(3.26) \qquad \deg k_1 = 2r \quad \text{where} \quad \deg f = 2rm.$$

It remains to estimate the degrees of the p_i's. Note that for each fixed i, p_i corresponds to the p in Lemma 3.15, whose degree actually depended on the choice of the natural number T as in (3.17). In view of (3.17), (3.18) and (3.19), we find that the smallest integer T_0 satisfying

$$m^2(m-1)[(1+\sqrt{\rho})^r - 1]^2 < 4T_0 \log(1+\varepsilon)$$

will be sufficient. With the use of this T_0, the proof of Lemma 3.15 gives

$$\deg p_i = 2mT_0 \deg k_1 = 4rmT_0 \quad \text{(for all } i\text{)},$$

and hence

$$(3.27) \qquad \deg p' = \sum_{i=1}^{m} \deg p_i = 4rm^2T_0.$$

Therefore, using the denominator $h := h'p'$, the final estimate on $\deg h$ is obtained from (3.22), (3.25) and (3.27) as

$$\begin{aligned} \deg h &\leq (m-1)\deg p_1' + \deg p' \\ &= (m-1)[2\deg p' + 2r] + \deg p' \\ &= (2m-1)\deg p' + 2r(m-1) \\ &= (2m-1)4rm^2T_0 + 2r(m-1). \end{aligned}$$

Taking out the factor $\deg f = 2rm$, we obtain the estimate on $\deg h$ in Theorem 3.4.

As a consequence of Theorem 3.4, we see that

$$P_{2m}(R(X)) \leq \binom{2m+3}{3}.$$

This bound slightly improves that of Becker in [Be] for small numbers m. It is, however, still very crude as the next section will show. In fact, for $m = 2$ we obtain only $P_4(R(X)) \leq 35$, while in the next section we shall improve this bound to $P_4(R(X)) \leq 6$.

Concerning lower bounds for $P_{2m}(R(X))$, we have the following

Proposition 3.28 *For any $m \geq 2$, we have $P_{2m}(R(X)) \geq 3$.*

Proof: Taking $f = (X^2+1)^m$ it follows that $\rho(f) = 0$. Hence by Theorem 1.8, f is a sum of $2m$-th powers in $R(X)$. (In fact, if we specialize the form $(X_1^2+X_2^2+X_3^2+X_4^2)^m \in Q_{4,2m}(\mathbb{R})$ introduced at the beginning of §3 by setting $(X_1, X_2, X_3, X_4) = (X, 1, 0, 0)$, we see that f is even a sum of $2m$-th powers of linear forms.) Let us assume that

$$(3.29) \qquad (X^2+1)^m = \frac{g_1^{2m}+g_2^{2m}}{h^{2m}} \quad \text{with } g_1, g_2, h \in R[X].$$

Clearly, we can choose the g_i 's to be relatively prime. Multiplying by the denominator h^{2m} in (3.29) and factorizing over $R(i)$ with $i = \sqrt{-1}$ as usual, we find

$$(3.30) \qquad l_1^{2m} \cdots l_s^{2m}(X+i)^m(X-i)^m = \prod_{j=0}^{m-1} (g_1 - \zeta^{2j+1}g_2)(g_1 - \zeta^{-(2j+1)}g_2)$$

where ζ denotes a primitive $4m$-th root of unity, and the l_i 's are linear forms over $R(i)$. Clearly, the $2m$ factors on the right hand side of (3.30) are pairwise relatively prime. Therefore,

$$(3.31) \qquad (X+i)^m \mid (g_1 - \zeta^{2k+1}g_2) \text{ for some } 0 \le k \le m-1.$$

By conjugation, this implies

$$(3.32) \qquad (X-i)^m \mid (g_1 - \zeta^{-(2k+1)}g_2).$$

Since $m \ge 2$, there exists $0 \le k' \le m-1$ different from k. Now the factor $g_1 - \zeta^{2k'+1}g_2$ is neither divisible by $(X+i)^m$ nor by $(X-i)^m$. Thus we get

$$(3.33) \qquad g_1 - \zeta^{2k'+1}g_2 = c\, l_{\nu_1}^{2m} \cdots l_{\nu_r}^{2m}$$

for a suitable choice of indices and a nonzero constant c. This is, however, impossible since the right hand side of (3.33) has degree $\equiv 0 \pmod{2m}$, while it follows from (3.31) and (3.32) that $g_1 - \zeta^{2k+1}g_2$ and hence also $g_1 - \zeta^{2k'+1}g_2$ have degree $\equiv m \pmod{2m}$. (Note that g_1 and g_2 have coefficients from R, so the leading coefficients cannot cancel.) □

4 The case of 4-th powers

The main result of this section will be

$$(4.1) \qquad P_4(R(X)) \le 6$$

for any real closed field R. This follows immediately from the basic identity

$$(4.2) \qquad (U^2+V^2)^2 = \frac{1}{2}\left(U + \frac{1}{\sqrt{3}}V\right)^4 + \frac{1}{2}\left(U - \frac{1}{\sqrt{3}}V\right)^4 + \frac{8}{9}V^4$$

and the result of Theorem 4.12 below which states that to every strictly positive $f \in R[X] \cap \Sigma R(X)^4$ there exist $h, g_1, g_2 > 0$ in $R[X]$ such that

$$h^2 f = g_1^2 + g_2^2. \tag{4.3}$$

In fact, the rational functions g_1/h and g_2/h, being positive semidefinite, are in $R(X)^2 + R(X)^2$. Thus (4.2) implies (4.1).

The main idea for proving (4.3) is contained in Lemma 4.7 below. Actually, this idea had also inspired Lemma 3.15 and its proof. The proof of Lemma 4.7 will depend on the geometry of the euclidean plane over $\mathbb{R}$. Thus (4.3) will first only follow for $R = \mathbb{R}$. By the characterization of $\Sigma R(X)^4$ in Theorem 1.8 and the existence of bounds (see Lemma 4.7), however, we will be able to transfer the statement (4.3) to any real closed field R.

Let us now consider the case $R = \mathbb{R}$. Since $f \in \mathbb{R}[X] \cap \Sigma \mathbb{R}(X)^4$, all its real zeros have order divisible by 4 (see (1.4)). Thus, in order to obtain (4.3), we may as well assume that f is monic and strictly positive. Then f has a representation

$$f = g_1^2 + g_2^2 \tag{4.4}$$

with $g_i \in \mathbb{R}[X]$. For every $x \in \mathbb{R}$, let us consider the complex number $z(x) = g_1(x) + ig_2(x)$ where i denotes as usual $\sqrt{-1}$. Let us define the argument $\arg z$ of any complex number z to lie between $-\pi$ and π. Now the assertion that g_1 and g_2 in (4.4) are strictly positive is equivalent to

$$0 < \arg(g_1(x) + ig_2(x)) < \frac{\pi}{2} \quad \text{for all } x \in \mathbb{R}. \tag{4.5}$$

In case we know that $f = g_1'^2 + g_2'^2$ with $g_i' \in \mathbb{R}[X]$ and

$$|\arg(g_1'(x) + ig_2'(x))| < \frac{\pi}{4} \quad \text{for all } x \in \mathbb{R}, \tag{4.6}$$

we could simply multiply $g_1'(x) + ig_2'(x)$ by the 8-th root of unity $\zeta = e^{2\pi i/8} = (1+i)/\sqrt{2}$. We would then obtain (4.5) with g_1 and g_2 defined by

$$g_1(x) + ig_2(x) = \zeta \cdot (g_1'(x) + ig_2'(x))$$

for all $x \in \mathbb{R}$. This would yield a representation (4.4) with g_1, g_2 strictly positive, since

$$f(x) = \zeta(g_1'(x) + ig_2'(x))\zeta^{-1}(g_1'(x) - ig_2'(x)) = g_1(x)^2 + g_2(x)^2.$$

Unfortunately, (4.5) or (4.6) cannot be achieved in general for a strictly positive $f \in \mathbb{R}[X] \cap \Sigma \mathbb{R}(X)^4$. In the next lemma, however, we will show how to obtain (4.6) (and hence (4.5)) for f replaced by a suitable product $f \cdot h^2$ with $h \in \mathbb{R}[X]$ strictly positive.

Lemma 4.7 *Let* $f = q_1 \cdots q_{2r}$ *with* $q_\nu = (X - a_\nu)^2 + b_\nu^2$, $a_\nu, b_\nu \in \mathbb{R}$, $b_\nu > 0$ *and* $1 \le \nu \le 2r$. *Then there exist strictly positive polynomials* $g_1, g_2, h \in \mathbb{R}[X]$ *such that* $f \cdot h^2 = g_1^2 + g_2^2$, $\deg g_1 = \deg g_2$ *and* $\deg h \le (T_0 - \frac{1}{2}) \deg f$ *where* T_0 *is the smallest* $T \in \mathbb{N}$ *such that* $4\pi T^2 \ge \rho(5rT + \pi)$ *and* ρ *is the maximum of* $\rho'(q_{2j-1}, q_{2j})$ *for* $1 \le j \le r$.

Proof: Let us take $f_j = q_{2j-1}q_{2j}$ for $1 \le j \le r$. It will suffice to prove the existence of polynomials h_j, g_{1j}, $g_{2j} \in \mathbb{R}[X]$ such that

(4.8) (a) $|\arg(g_{1j} + i g_{2j})| \le \pi/5r$ *on* $\mathbb{R}$.

(b) h_j *is strictly positive and* $\deg h_j \le 2(2T_0 - 1)$ *where* T_0 *is the smallest* $T \in \mathbb{N}$ *such that* $4\pi T^2 \ge \rho(5rT + \pi)$.

(c) $h_j^2 \cdot f_j = g_{1j}^2 + g_{2j}^2$ *for* $1 \le j \le r$.

In fact, once we have (4.8), we can take $h = h_1 \cdots h_r$ (with $\deg h \le 2r(2T_0 - 1) = (T_0 - \frac{1}{2}) \deg f$) to get

$$fh^2 = \prod_{j=1}^{r} (f_j h_j^2) = \prod_{j=1}^{r} (g_{1j}^2 + g_{2j}^2) = g_1'^2 + g_2'^2, \quad \text{where}$$

$$g_1' + i g_2' := (g_{11} + i g_{21}) \cdots (g_{1r} + i g_{2r})$$

has argument $\le \pi/5$ on $\mathbb{R}$ in absolute value. Taking $g_1^* + i g_2^* = \zeta(g_1' + i g_2')$ as before, we find $fh^2 = g_1^{*2} + g_2^{*2}$ with $\arg(g_1^* + i g_2^*) \in [\pi/20, 9\pi/20]$. At this point, we can then further multiply $g_1^* + i g_2^*$ by some "very small" root of unity, to get $g_1 + i g_2$ satisfying (4.5), and also satisfying the *additional* condition $\deg g_1 = \deg g_2$. Thus it only remains for us to prove (4.8).

In order to simplify the notations, let us omit the index j in (4.8) and simply consider the case $f = q_1 q_2$. We shall now show how to find $h, g_1, g_2 \in \mathbb{R}[X]$ satisfying (4.8). Let us denote the complex zeros of q_1 and q_2 with positive imaginary parts by

$$z_1 = a_1 + i b_1 \text{ and } z_2 = a_2 + i b_2 .$$

Then any point on the line from z_1 to z_2 is given by

$$z(\lambda) = (1 - \lambda) z_1 + \lambda z_2$$

for some $\lambda \in \mathbb{R}$ such that $0 \le \lambda \le 1$. For every $0 \le \mu \le 2T$ we define the linear polynomials

$$A_\mu = X - z\left(\frac{\mu}{2T}\right) \text{ and } B_\mu = X - \overline{z\left(\frac{\mu}{2T}\right)} = \overline{A_\mu}$$

where T is a large integer to be chosen later, and $\overline{z}$ denotes the complex number conjugate of z (similarly for polynomials). With this notation we clearly have $f = A_0 B_0 A_{2T} B_{2T}$.

We next define the strictly positive polynomial

$$h = \prod_{\mu=1}^{2T-1} A_\mu B_\mu \quad \text{with} \quad \deg h = 2(2T-1).$$

If we then define for $0 \le \nu \le T-1$,

$$C_\nu = A_{2\nu} B_{2\nu+1}^2 A_{2\nu+2} \text{ and } C = \prod_{\nu=0}^{T-1} C_\nu$$

and write $C = g_1 + i g_2$ with $g_1, g_2 \in \mathbb{R}[X]$, it is easy to check that

$$h^2 f = C\overline{C} = g_1^2 + g_2^2.$$

Now it remains to estimate the argument of $C(x)$ for $x \in \mathbb{R}$.

For a fixed ν, let us write $\alpha + i\beta$ for $z\left(\frac{2\nu+1}{2T}\right)$ with $\alpha, \beta \in \mathbb{R}$. Then

$$C_\nu(x) = \left(x - (\alpha + i\beta) - \frac{z_1 - z_2}{2T}\right)(x - (\alpha - i\beta))^2 \left(x - (\alpha + i\beta) + \frac{z_1 - z_2}{2T}\right).$$

Hence we get

$$C_\nu(x + \alpha) = (x^2 + \beta^2)^2 - \left(\frac{z_1 - z_2}{2T}\right)^2 (x + i\beta)^2.$$

The real part of this expression is estimated by

$$\operatorname{Re} C_\nu(x+\alpha) \ge (x^2 + \beta^2)\left[x^2 + \beta^2 - \frac{|z_1 - z_2|^2}{4T^2}\right] \ge (x^2 + \beta^2)\left[\beta^2 - \frac{|z_1 - z_2|^2}{4T^2}\right].$$

The right hand side above is positive, if we choose T such that

$$4T^2 > \rho \ge \frac{|z_1 - z_2|^2}{\beta^2}. \tag{4.9}$$

The imaginary part of $C_\nu(x+\alpha)$ is estimated by

$$|\operatorname{Im} C_\nu(x+\alpha)| \le (x^2 + \beta^2)\frac{|z_1 - z_2|^2}{4T^2}. \tag{4.10}$$

Thus, the argument of $C_\nu(x+\alpha)$ is estimated by

$$|\arg C_\nu(x+\alpha)| \le \frac{|\operatorname{Im} C_\nu(x+\alpha)|}{\operatorname{Re} C_\nu(x+\alpha)} \le \frac{1}{\frac{4T^2\beta^2}{|z_1 - z_2|^2} - 1} \le \frac{1}{\frac{4T^2}{\rho} - 1}$$

for all $x \in \mathbb{R}$. Therefore, we have

$$|\arg C| \le \sum_{\nu=0}^{T-1} |\arg C_\nu| \le \frac{T}{\frac{4T^2}{\rho} - 1} \quad \text{on } \mathbb{R}.$$

It now remains to choose T_0 to be the smallest integer T satisfying

$$(4.11) \qquad T \le \frac{\pi}{5r}\left(\frac{4T^2}{\rho} - 1\right), \quad \textit{i.e.}\ 4\pi T^2 \ge \rho(5rT + \pi).$$

This proves (4.8) and hence the lemma. □

We can now prove the main theorem of this section:

Theorem 4.12 *Let R be a real closed field and $f \in R[X]$ monic and strictly positive. Then $f \in \Sigma R(X)^4$ if and only if there exist strictly positive polynomials $g_1, g_2, h \in R[X]$ such that $h^2 f = g_1^2 + g_2^2$ and $\deg g_1 = \deg g_2$.*

Proof: As we already saw at the beginning of this section, the existence of strictly positive polynomials g_1, g_2, $h \in R[X]$ with $h^2 f = g_1^2 + g_2^2$ implies $f \in \Sigma R(X)^4$.

Conversely let us assume that $f \in \Sigma R(X)^4$. Then by Theorem 1.8 there is a decomposition $f = f_1 \cdots f_r$ where each f_j is a monic polynomial from $\Sigma R(X)^4$ such that $\rho(f_j)$ is finite in R. Let $\rho \in \mathbb{Q}$ be an upper bound for $\rho(f_1)$, ..., $\rho(f_r)$. Now Lemma 4.7 tells us that for the case $R = \mathbb{R}$ there exist strictly positive polynomials g_1, g_2, h such that

$$(4.13) \qquad h^2 f = g_1^2 + g_2^2, \ \deg g_1 = \deg g_2 \text{ and } \deg h \le \left(T_0 - \frac{1}{2}\right) \deg f,$$

where $T_0 \in \mathbb{N}$ is the smallest integer satisfying (4.11).

For any fixed upper bound d of $\deg f$, (4.13) can be expressed by some first order formula $\Phi_{d,\rho}$ depending only on the bounds d and ρ. By Tarski's Transfer Principle, $\Phi_{d,\rho}$ also holds in any real closed field R. Thus, we find that (4.13) holds for our given $f \in R[X]$. □

5 Epilog

Usually it takes one year for one author to write a paper. For four authors it may, therefore, take at least four years to finish a joint paper. In the case of this paper it took even more years. No wonder that meanwhile other papers based on the results of this one have been written or even published. We take this occasion to mention three of them.

In [Sch$_2$] Schmid has positively answered Schülting's question whether a totally positive unit in the real holomorphy ring $H(K)$ of a formally real field K is a sum of squares of totally positive units of $H(K)$. The method used by Schmid was (according to him) inspired by the "partition" method used in this paper. In [Sch$_3$] Schmid treats the special case where $P_2(K) = 2$, e.g. $K = R(X)$ with R real closed. He extends our result (4.1) on the 4-th Pythagoras Number as follows: If K is formally real, $\sqrt{3} \in K$ and $P_2(K) = 2$, then $P_4(K) \le 6$.

Schmid's result applies in particular to algebraic function fields in one variable over a real closed field R. This case is not covered by our methods. In [P_3] the third author of this paper relates Schülting's question for the *relative* real holomorphy ring $H(R(X)/R)$ of $R(X)$ to the sums of 4-th powers in $R(X)$ and thus makes the results of this paper applicable to questions about totally positive units in $H(R(X)/R)$.

References

[Be] E. BECKER: The real holomorphy ring and sums of $2m$-th powers. In: Géométrie Algébrique Réelle et Formes Quadratiques. *LNM 959*, pp. 139-181, Springer-Verlag, 1982.

[Bo] N. BOURBAKI: "Éléments de Mathématique", Algèbre Commutative, Hermann, Paris 1972.

[B-P] M. BRADLEY–A. PRESTEL: Representation of a real polynomial $f(X)$ as a sum of 2m-th power of rational functions. In: J. Martinez (ed.), Ordered Algebraic Structures, 197-207, Kluwer 1989.

[El] W. J. ELLISON: Waring's Problem, Amer. Math. Monthly 78(1971), 10-36.

[P_1] A. PRESTEL: Lectures on formally real fields, *LNM 1093*, Springer 1984.

[P_2] A. PRESTEL: Model theory of fields: An application to positive semidefinite polynomials. Mem. Soc. Math. France, 2 serié, 16(1984), 53-66.

[P_3] A. PRESTEL: On a variation of Hilbert's 17-th Problem. Proceedings: Algebra and Number Theory. De Gruyter Lecture Notes (to appear).

[R_1] B. REZNICK: Sums of even powers of real linear forms. Memoirs AMS, Nr. 463(1992).

[R_2] B. REZNICK: Uniform denominators in Hilbert's Seventeenth Problem. Preprint 1993.

[Sch_1] J. SCHMID: Eine Bemerkung zu den höheren Pythagoraszahlen reeller Körper. Manuscripta Math. 61(1988), 195-202.

[Sch_2] J. SCHMID: On totally positive units of real holomorphy rings. Israel J. of Math. **85**(1994, 339-350.

[Sch_3] J. SCHMID: Sums of fourth powers of real algebraic functions. Preprint.