Discrete Mathematics 60 (1986) 219–242
North-Holland

LATTICE POINT SIMPLICES

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Received 23 August 1983
Revised 6 December 1984

We consider simplices in $\mathbb{R}^m$ with lattice point vertices, no other boundary lattice points and n interior lattice points, with an emphasis on the barycentric coordinates of the interior points. We completely classify such triangles under unimodular equivalence and enumerate. For example, in a lattice point triangle with exactly one interior point, that point must be the centroid.

We discuss the literature for fundamental tetrahedra and prove that there are seven possible barycentric coordinates for a one-point tetrahedron. Following suggestions of P. Erdös, we prove that, for fixed m and n, there are only finitely many possible sets of barycentric coordinates for the interior points. We also discuss a generalization of Beatty's problem in combinatorial number theory which has arisen several times in recent years.

1. Introduction

In [16] I proved, in passing, the following result about plane lattice point triangles. Suppose $T = T(\underline{v}_0, \underline{v}_1, \underline{v}_2)$ is the triangle with vertices $\underline{v}_0$, $\underline{v}_1$, $\underline{v}_2$ and suppose $T \cap \mathbb{Z}^2 = \{\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{w}\}$ where $\underline{w}$ is strictly interior to T. Then $\underline{w} = \frac{1}{3}(\underline{v}_0 + \underline{v}_1 + \underline{v}_2)$ is the centroid of T. This paper contains generalizations of this result to n interior lattice points in a triangle, and to higher dimensional simplices.

I should note at the outset two features which distinguish this paper from the rest of the literature in this subject. First, we shall be exclusively concerned with lattice points simplices (*not* polytopes) which have no lattice points on their boundary. This specialization allows us to define barycentric coordinates unambiguously and simplifies some technical considerations. Second, we are not directly interested in the volume of the simplices. This distinction is somewhat deceptive, as the volume determines the denominator of the barycentric coordinates. Despite these eccentricities, this paper inevitably poaches on the work of others. Some methods are so natural that their use is unavoidable, and no novelty is claimed for them. I have endeavored to credit non-trivial poaching.

As one of this paper's referees has noted, many of the two dimensional problems have been rediscovered several times. For example, consider the characterization of fundamental tetrahedra (tetrahedra T with lattice point

* Supported in part by the National Science Foundation and the Alfred P. Sloan Foundation

0012-365X/86/$3.50

vertices and no other lattice points on the boundary or interior). Reeve (in 1957) and White (in 1964) independently studied this problem, and the solution is the compositum of their work. Further, Howe (in 1977) solved this problem independently and Scarf has used it in his study of integer programming. One referee also reports that it was solved also in 1982 by Betke and Gritzmann. In this context, I cannot guarantee the full allocation of due credit. A good source of information about the mainstream of the subject is Hammer's book [7]. The rest of this introduction serves as a guide to the body of the paper.

First, let me give two different proofs of the result in the first paragraph. By Pick's Theorem (see Section 3), T has area $1\frac{1}{2}$ and $T(\underline{v}_i, \underline{v}_j, \underline{w}) = \frac{1}{2}$ for $1 \leqslant i < j \leqslant 3$. The centroid is the unique point which triangulates T into three equal parts. For a more synthetic proof (which works even if T is not in $\mathbb{R}^2$), let $\underline{w} = \lambda_0 \underline{v}_0 + \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2$ be the expression of the interior point in barycentric coordinates, so $\lambda_i > 0$ and $\sum \lambda_i = 1$. Assuming $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2$, there are two possibilities: $\lambda_0 \geqslant \frac{1}{2}$ or $\frac{1}{2} > \lambda_i$. In the first case, $2\underline{w} - \underline{v}_0 = (2\lambda_0 - 1)\underline{v}_0 + 2\lambda_1 \underline{v}_1 + 2\lambda_2 \underline{v}_2$ is another lattice point interior to T and $\underline{w} \neq 2\underline{w} - \underline{v}_0$. In the second, $\underline{v}_0 + \underline{v}_1 + \underline{v}_2 - 2\underline{w} = \sum (1 - 2\lambda_i)\underline{v}_i$ is also a lattice point interior to T, hence $\underline{v}_0 + \underline{v}_1 + \underline{v}_2 - 2\underline{w} = \underline{w}$. These two approaches exemplify the differences in our study of triangles and higher-dimensional simplices: we can look at the figure itself, or look at the barycentric coordinates of $\mathbb{Z}^m$ with respect to the vertices of the simplex.

In Section 2, we introduce some notations. Let T be a closed non-degenerate simplex with vertices $\underline{v}_0, \ldots, \underline{v}_m \in \mathbb{Z}^m$ and suppose $T \cap \mathbb{Z}^m = \{\underline{v}_0, \ldots, \underline{v}_m, \underline{w}_1, \ldots, \underline{w}_n\}$, where the $\underline{w}_i$'s are strictly interior to T. Then T is called an n-point m-simplex; $S(m, n)$ is the set of all n-point m-simplices. The configuration of T, $M_T = [\lambda_{ij}]$ is the $n \times (m+1)$ matrix of barycentric coordinates: $\underline{w}_i = \sum \lambda_{ij} \underline{v}_j$. Two matrices have the same configuration, $T \sim T'$, if $M_T = M_{T'}$ after a relabeling of points. If h is a volume-preserving affine map with integer coefficients then h gives a bijection of $\mathbb{Z}^m$ to itself and preserves inclusion. Indeed, if $T \in S(m, n)$ then so is $h(T)$ and $T \sim h(T)$. However, equivalence under unimodular h is generally a stronger condition than having the same configuration in higher dimensions.

In Section 3, we classify n-point triangles up to unimodular equivalence. For example, every 2-point triangle is unimodularly equivalent to the triangle with vertices $(0,0)$, $(1,0)$, $(2,5)$ and must have configuration (1.1), up to a permutation of rows and columns.

$$\begin{bmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \tag{1.1}$$

For any n, the barycentric coordinates are rational with denominator $2n + 1$. The number of classes of configurations for fixed n is readily computable as the number of orbits of $\{e \pmod{2n+1}) : (e, 2n+1) = (e-1, 2n+1) = 1\}$ under the action of the group of order six generated by $f_1(x) = 1 - x$ and $f_2(x) = x^{-1}$. The key to our analysis is Pick's Theorem, which fixes the area of an n-point triangle as $\frac{1}{2}(2n+1)$. Using Pick's Theorem we can prove that any two fundamental (0-point) triangles are unimodularly equivalent to each other. From this we prove that T and T' have the same configuration iff $T'=h(T)$ for some unimodular h.

Pick's Theorem does not generalize simply to higher dimensions—see [15], [12]. For example, Reeve constructs a family of fundamental tetrahedra: T_m has vertices $(0,0,0),(1,0,0),(0,1,0),(1,1,m)$. It is easy to see that T_m has volume $\frac{1}{6}m$; no two are unimodularly equivalent to each other.

In Section 4 we develop machinery to generalize the second method of proof of the centroid theorem. Let $A_m=\{(\lambda_0,\ldots,\lambda_m): 0<\lambda_i<1, \sum \lambda_i \in \mathbb{Z}\}$ and define addition componentwise mod 1; A_m forms a $\mathbb{Z}$-module. For a submodule $H\subseteq A_m$, let $G(H)=\{\underline{\lambda}\in H: \sum \lambda_i=1\}$. Suppose $\underline{w}=\sum \lambda_j \underline{v}_j$ and $\underline{\lambda}'\in G(\langle\underline{\lambda}\rangle)$, where $\langle\underline{\lambda}\rangle$ is the module generated by $\underline{\lambda}$. Then $\lambda_j'=k\lambda_j-t_j$ for integers k and t_j, so $\underline{w}'=\sum \lambda_j'\underline{v}_j=k\underline{w}-\sum t_j\underline{v}_j$ is a lattice point. Since $\sum \lambda_j'=1$ and $\lambda_j'\geq 0$, $\underline{w}'$ is expressed in barycentric coordinates, so w 'generates' $\underline{w}'$. We can look at submodules generated by more vectors. Indeed, suppose $T\in S(m,n)$ and $M_T=[\lambda_{ij}]$ then, necessarily, $G(\langle\underline{\lambda}_1,\ldots,\underline{\lambda}_n\rangle)=\{\underline{\lambda}_1,\ldots,\underline{\lambda}_n\}$. The converse is an open question; that is, if $G(\langle\underline{\lambda}_1,\ldots,\underline{\lambda}_n\rangle)=\{\underline{\lambda}_1,\ldots,\underline{\lambda}_n\}$ for $\underline{\lambda}_i\in A_m$, does there exist $T\in S(m,n)$ with $M_T=[\lambda_{ij}]$? We give a construction, under apparently restrictive hypotheses, which allows us to answer "yes" for $m=2$ and 3 and to show that $S(m,n)\neq\emptyset$ for $m\geq 2$, $n\geq 0$.

In Section 5, we return to simplices, per se. Using the machinery of the previous section, we show that every n-point tetrahedron is unimodularly equivalent to $T=T((0,0,0),(1,0,0),(0,1,0),(a,b,c))$ with $(a,c)=(b,c)=(a+b-1,c)=1$. By combining the work of Reeve and White, we show that the fundamental tetrahedra correspond to $a=1$ above. As noted above, this problem has several solutions; we also discuss the Howe–Scarf work. (Unimodular equivalence appears to be a far more restrictive condition in higher dimensions than it is in the plane.) We show that there are, up to permutation of components, exactly seven configurations for 1-point tetrahedra. The proof is tedious and requires a large number of cases. Finally, we show that if $T\in S(m,n)$ then there is a universal upper bound on the denominators in the configuration $M_T=[\lambda_{ij}]$. The proof is relatively short, self-contained and due to Erdös. This theorem is also a consequence of a stronger theorem of Hensley [9, Theorem 3.4]. Hensley constructs a universal upper bound on the volume of m-simplices with n interior lattice points (and any number of boundary lattice points.)

We conclude with a number of open problems, suggestions of other directions of exploration and a list of acknowledgments.

2. Notations and preliminaries

In this section we fix notation and collect some simple but useful results. A point $\underline{x}=(x_1,\ldots,x_m)\in\mathbb{R}^m$ is a *lattice point* if $x\in\mathbb{Z}^m$. Given $\underline{v}_j\in\mathbb{Z}^m$, $0\leq j\leq m$, let $V = \{\underset{\sim}{v}_0, \ldots, \underset{\sim}{v}_m\}$ and let $T = T(V) = T(\underset{\sim}{v}_0, \ldots, \underset{\sim}{v}_m)$ denote the closed convex hull of the $\underset{\sim}{v}_j$'s. *Throughout, $T(V)$ will be assumed non-degenerate*; that is, the vectors $\underset{\sim}{v}_j - \underset{\sim}{v}_0$, $1 \leq j \leq m$, form a linearly independent set in $\mathbb{R}^m$.

For fixed $\underset{\sim}{v}_0, \ldots, \underset{\sim}{v}_m$ in $\mathbb{Z}^m$ as above, any $\underset{\sim}{w} \in \mathbb{R}^m$ can be written uniquely in barycentric coordinates: $\underset{\sim}{w} = \sum \lambda_j \underset{\sim}{v}_j$, $\sum \lambda_j = 1$; we write $\underset{\sim}{\lambda} = \mathrm{BC}(\underset{\sim}{w})$. As usual, $\underset{\sim}{w}$ is in the *interior* of T iff $\lambda_j > 0$ for all j and $\underset{\sim}{w}$ is on the *boundary* of T iff $\lambda_j \geq 0$ for all j and $\lambda_k = 0$ for some k. We say that $\underset{\sim}{w}$ is *in* $T(V)$ if $\underset{\sim}{w}$ is in the interior or on the boundary of T, but $\underset{\sim}{w} \neq \underset{\sim}{v}_j$. If $\underset{\sim}{w} \in \mathbb{Z}^m$ then the equations for $\underset{\sim}{\lambda}$ form an $(m+1) \times (m+1)$ linear system with non-zero determinant $\Delta \in \mathbb{Z}$: $|\Delta| = m!\,\mathrm{Vol}\,(T(V))$. Thus, if $\underset{\sim}{w} \in \mathbb{Z}^m$, then $\lambda_j \in \mathbb{Z}/\Delta$. We shall write a vector $\underset{\sim}{\lambda} \in \mathbb{Q}^m$ as $\underset{\sim}{\lambda} = (a_0, \ldots, a_m)/D$ and say that $\underset{\sim}{\lambda}$ has *denominator* D if $\lambda_j = a_j/D$ and D is the least common denominator of the $\underset{\sim}{\lambda}_j$'s.

A simplex $T = T(\underset{\sim}{v}_0, \ldots, \underset{\sim}{v}_m)$, $m \geq 2$, is called an *n-point m-simplex* ($T \in S(m, n)$) if $\underset{\sim}{v}_j \in \mathbb{Z}^m$ and there are exactly n lattice points in T, none on the boundary. (If $T \in S(m, 0)$ then T is a *fundamental* m-simplex.) Given $T \in S(m, n)$ and interior points $\underset{\sim}{w}_i = \sum \lambda_{ij} \underset{\sim}{v}_j$, where $\underset{\sim}{\lambda}_i = \mathrm{BC}(\underset{\sim}{w}_i)$, form the $n \times (m+1)$ matrix $M_T = [\lambda_{ij}]$. We call M_T the *configuration* of T. Two simplices $T, T' \in S(m, n)$ have the *same configuration*, written $T \sim T'$, if $M_T = M_{T'}$ after a possible permutation of rows and columns (that is, a relabeling of points). By default, two fundamental m-simplices have the same configuration. A natural question is this: do there exist n-point m-simplices for all $n \geq 0$, $m \geq 2$. The answer is "yes", but we defer the construction to Section 4.

By a *unimodular* map, we mean an affine function $h: \mathbb{R}^m \to \mathbb{R}^m$, $h(\underset{\sim}{x}) = M\underset{\sim}{x} + \underset{\sim}{b}$, where M and $\underset{\sim}{b}$ have integer components and $\det M = \pm 1$. Observe that a volume-preserving affine map is unimodular iff $h(0, \ldots, 0)$ and $h(\underset{\sim}{e}_i)$, $1 \leq i \leq m$ are lattice points, where $\underset{\sim}{e}_i$ denotes the ith unit vector. A unimodular map h gives a bijection of $\mathbb{Z}^m$ to itself and preserves barycentric coordinates: if $\underset{\sim}{w} = \sum \lambda_j \underset{\sim}{v}_j$ and $\sum \lambda_j = 1$ then $h(\underset{\sim}{w}) = \sum \lambda_j h(\underset{\sim}{v}_j)$. Thus, if $T = T(V) \in S(m, n)$ and h is unimodular then $h(T) := T(h(V)) \in S(m, n)$ and $T \sim T'$. Suppose $T' = h(T)$ for a unimodular h, we write $T \simeq T'$. One caveat is needed: if $T = T(V)$, $T' = T(V')$ and $T \simeq T'$, it does *not* follow that the affine map h defined by $\underset{\sim}{v}'_j = h(\underset{\sim}{v}_j)$ is unimodular: even though h is volume-preserving, it need not have integer coefficients. However, for *some* permutation π of $\{1, \ldots, m\}$, the map h_π defined by $h_\pi(\underset{\sim}{v}_j) = \underset{\sim}{v}'_{\pi(j)}$ *is* unimodular. We prove below (Theorem 3.2) that $T \sim T'$ iff $T \simeq T'$ for $T, T' \in S(2, n)$. The examples of Reeve from the introduction show that this needn't hold for $m \geq 3$.

3. Lattice point triangles

In this section we determine all n point triangles up to unimodular equivalence. First we need two elementary formulas for the area of a triangle. One is that

$T((0,0),(a,b),(c,d))$ has area $\frac{1}{2}|ad-bc|$, the other is a special case of Pick's theorem (see [3] for a proof): if T is a triangle with lattice point vertices, e other lattice points on its edges and i lattice points in its interior, then T has area $i+\frac{1}{2}(e+1)$. These lead immediately to our first result.

Proposition 3.1. *If $T \in S(2, n)$ then the entries of M_T are in $(2n+1)^{-1}\mathbb{Z}$.*

Proof. By Pick's Theorem, T has area $\frac{1}{2}(2n+1)$ and by translation (a unimodular map) we may assume that $T=T((0,0),(a,b),(c,d))$. Let $\underline{w}=(x,y)$ be an interior point and compute its barycentric coordinates. As noted in the last section, the equations $\lambda_0+\lambda_1+\lambda_2=1$, $\lambda_1 a+\lambda_2 c=x$, $\lambda_1 b+\lambda_2 d=y$ give a 3×3 system with determinant $ad-bc=\pm(2n+1)$. The conclusion follows by Cramer's Rule. □

For $n=1$, $\underline{w}=\sum \lambda_j \underline{v}_j$ implies $\lambda_j \in \frac{1}{3}\mathbb{Z}$: since $\lambda_j>0$ and $\sum \lambda_j=1$ we must have $\lambda_0=\lambda_1=\lambda_2=\frac{1}{3}$, giving yet another proof of the centroid theorem.

We start by analyzing $S(2,0)$. The following lemma has already been mentioned, and leads to a complete classification of $S(2,n)$.

Lemma 3.2. *If T and T' are in $S(2,0)$ then $T \simeq T'$. Indeed, the map h may be chosen to permute the vertices in any prescribed order.*

Proof. Let $T_0=T((0,0),(1,0),(0,1))$ and, by a (unimodular) tranlation assume that $T=T((0,0),(x_1,y_1),(x_2,y_2))$. By Pick's Theorem (as in Proposition 3.1), $|x_1y_2-x_2y_1|=1$, so h defined by $h(x,y)=(x_1x+x_2y, y_1x+y_2y)$ is unimodular and $T=h(T_0)$. Similarly, $T'=h'(T_0)$ for a unimodular h'. The six unimodular functions $g_i(x,y)=(x,y)$, (y,x), $(1-x,y)$, $(x,1-y)$, $(1-x-y,x)$, $(y,1-x-y)$ permute the vertices of T_0 in all ways, so $T'=h'(g_i(h^{-1}(T)))$. □

Theorem 3.3. *If T and T' are in $S(2,n)$ then $T \sim T'$ if and only if $T \simeq T'$.*

Proof. We have already shown that $T \simeq T'$ implies $T \sim T'$. To prove the converse, suppose $T \sim T'$ and relabel the vertices so that $\underline{w}_i=\sum \lambda_{ij}\underline{v}_j$ and $\underline{w}'_i=\sum \lambda_{ij}\underline{v}'_j$. Since T and T' have the same area, the affine map h defined by $h(\underline{v}_j)=\underline{v}'_j$ is volume-preserving; we must show that it is unimodular. Let L be the edge $\overline{\underline{v}_0\underline{v}_1}$ and suppose $\lambda_{k2} \leq \lambda_{i2}$ for $1 \leq i \leq n$. Consider $\bar{T}=T(\underline{v}_0,\underline{v}_1,\underline{w}_k)$; we claim $\bar{T} \in S(2,0)$. Otherwise, a lattice point in T could be written $\mu_0\underline{v}_0+\mu_1\underline{v}_1+\mu_2(\lambda_{k0}\underline{v}_0+\lambda_{k1}\underline{v}_1+\lambda_{k2}\underline{v}_2)$ and we must have $\lambda_{k2} \leq \mu_2\lambda_{k2}$, a contradiction. Similarly, $\bar{T}'=T(\underline{v}'_0,\underline{v}'_1,\underline{w}'_k) \in S(2,0)$. By Lemma 3.2, the affine map $\bar{h}$ defined by $\bar{h}(\underline{v}_0)=\underline{v}'_0$, $\bar{h}(\underline{v}_1)=\underline{v}'_1$ and $\bar{h}(\underline{w}_k)=\underline{w}'_k$ is unimodular. By linearity, $\bar{h}(\underline{v}_2)=\underline{v}'_2$ and so $h \equiv \bar{h}$ is unimodular. □

Theorem 3.4. *If $T \in S(2, n)$, then $T \simeq T_e := T((0, 0), (1, 0), (e, 2n+1))$ for some $0 < e < 2n+1$ with $(e, 2n+1) = (e-1, 2n+1) = 1$. Further, $T_e \simeq T_{e'}$ iff $e' \equiv f(e) \pmod{2n+1}$, where f is in the group G generated by $1-x$ and x^{-1}. In other words, $e' \equiv e$, $1-e$, e^{-1}, $1-e^{-1}$, $(1-e)^{-1}$ or $e(e-1)^{-1} \pmod{2n+1}$.*

Proof. Take $T \in S(2, n)$ and translate so that $T = T((0, 0), (x_1, y_1), (x_2, y_2))$, where $x_1 y_2 - x_2 y_1 = 2n+1$. Since there are no lattice points on the edges of T, x_1 and y_1 are relatively prime; choose integers a and b so that $ax_1 + by_1 = 1$. Now let $h(x, y) = (ax + by, -y_1 x + x_1 y)$. By construction, h is unimodular and $T \simeq h(T) = T_e$ for $e = ax_2 + by_2$. Since $h_r(x, y) = (x + ry, y)$ is unimodular and $h_r(T_e) = T_{e + r(2n+1)}$, we may take $0 \le e < 2n+1$. Since there are no lattice points on the edges of T_e, $(e, 2n+1) = (e-1, 2n+1) = 1$. Conversely, T_e has area $\frac{1}{2}(2n+1)$, and if $(e, 2n+1) = (e-1, 2n+1) = 1$ then there are no edge points, so $T_e \in S(2, n)$.

Now suppose $T_e \simeq T_{e'}$ or $T_{e'} = h(T_e)$. There are six ways for the vertices to be mapped. As both T_e and $T_{e'}$ have area $\frac{1}{2}(2n+1)$ it is enough to show that $h(0, 0)$, $h(1, 0)$ and $h(0, 1)$ are lattice points. Fortunately the first two are vertices of $T_{e'}$! Let $\{\underline{v}_0, \underline{v}_1, \underline{v}_2\} = \{(0, 0), (1, 0), (e', 2n+1)\}$ and suppose $h(0, 0) = \underline{v}_0$, $h(1, 0) = \underline{v}_1$, $h(e, 2n+1) = \underline{v}_2$. It is easily checked that h affine implies $h(0, 1) = ((2n+e)\underline{v}_0 - e\underline{v}_1 + \underline{v}_2)/(2n+1)$. Thus the second component of $h(0, 1)$ is always an integer. An enumeration of the permutations gives the six possibilities in the conclusion. Alternatively, any permutation of $\{0, 1, 2\}$ is generated by (01) and (12), thus h will be a composition of x^{-1} and $1-x$, using the same formula for $h(0, 1)$. □

Let $f(m, n)$ denote the number of distinct configurations in $S(m, n)$; we are now in a position to compute $f(2, n)$. In view of Theorem 3.4, $f(2, n)$ is the number of orbits of C_{2n+1} under the group G:

$$C_{2n+1} = \{e \pmod{2n+1} : (e, 2n+1) = (e-1, 2n+1) = 1\}. \tag{3.5}$$

We must therefore compute $|C_{2n+1}|$ and determine the degenerate orbits. A major tool is the Chinese Remainder Theorem (see also Section 4).

Lemma 3.6. *Suppose $2n+1 = p_1^{a_1} \cdots p_r^{a_r}$, $p_j < p_{j+1}$, $a_j > 0$, then*

$$|C_{2n+1}| = (2n+1) \prod_{j=1}^{r} \left(1 - \frac{2}{p_j}\right) := \phi_2(2n+1). \tag{3.7}$$

Proof. By the Chinese Remainder Theorem, $e \in C_{2n+1}$ iff $e \not\equiv 0$ or $1 \pmod{p_j}$, or $e \not\equiv tp_j$ or $tp_j + 1 \pmod{p_j^{a_j}}$ for $0 \le t < p_j^{a_j - 1}$. This leaves $(p_j - 2)p_j^{a_j - 1}$ residues and (3.7) follows immediately. The notation ϕ_2 can be found in several textbooks, e.g., [13, p. 37]. □

An orbit of $e \in C_{2n+1}$ under G has six elements unless e is fixed by a non-trivial

$f_i \in G$. That is, e is in a degenerate orbit provided $e^2 \equiv 1$, $2e \equiv 1$, $e \equiv 2$ or $e^2 - e + 1 \equiv 0 \pmod{2n+1}$. If $e^2 \equiv 1 \pmod{\prod p_j^{a_j}}$, then $e^2 \equiv 1 \pmod{p_j^{a_j}}$, so $p_j^{a_j} \mid (e-1)(e+1)$. Since $2n+1$ is odd, $p_j \geq 3$ and $e \equiv \pm 1 \pmod{p_j^{a_j}}$. But $e \in C_{2n+1}$ so $e \equiv -1 \pmod{p_j^{a_j}}$ and $e \equiv 2n \pmod{2n+1}$ by the Chinese Remainder Theorem. It is easily checked that $\{2, n+1, 2n\}$ forms a degenerate orbit for all $n > 1$. The case $e^2 - e + 1 \equiv 0$ requires a lemma.

Lemma 3.8. *Suppose* $2n+1 = 3^t p_1^{a_1} \cdots p_r^{a_r}$, $t \geq 0$, $a_j \geq 1$, $3 < p_1 < \cdots < p_r$, *then* $e^2 - e + 1 \equiv 0$ *has no solutions* $\bmod (2n+1)$ *unless* $t \leq 1$ *and* $p_j \equiv 1 \pmod 3$ *for all* j, *in which case it has* 2^r *solutions. All solutions are in* C_{2n+1}.

Proof. If e is a solution then $e^2 - e + 1 \equiv 0 \pmod{3^t}$ (if $t \geq 1$) and $e^2 - e + 1 \equiv 0 \pmod{p_j^{a_j}}$. It is easily checked that $e^2 - e + 1 \equiv 0 \pmod 3$ has one solution and $e^2 - e + 1 \equiv 0 \pmod 9$ has no solutions. Now suppose $e^2 - e + 1 \equiv 0 \pmod{p_j^{a_j}}$, $p_j \geq 5$; since $e \equiv -1$ is not a solution, this is equivalent to $e^3 \equiv -1 \pmod{p_j^{a_j}}$, $e \not\equiv -1$. Letting $f \equiv -e$, we have $f^3 \equiv 1$, $f \not\equiv 1$. Let r be a primitive root $\pmod{p_j^{a_j}}$, then r has order $\phi(p_j^{a_j}) = (p_j - 1)p_j^{a_j - 1}$ and $f = r^t$ for some t. If $p_j \equiv 2 \pmod 3$, then $(3, \phi(p_j^{a_j})) = 1$ so that $f^3 \equiv 1$ implies $f \equiv 1$ and there are no solutions to $e^2 - e + 1 \equiv 0$. If $p_j \equiv 1 \pmod 3$ then $f^3 \equiv 1$ implies $f \equiv 1, a, a^2$ where $a = r^u$, $u = \frac{1}{3}\phi(p_j^{a_j})$, and there are two solutions to $e^2 - e + 1 \equiv 0$. Finally, $e \equiv 0$ or $1 \pmod p$ implies $e^2 - e + 1 \equiv 1 \pmod p$, hence all solutions are in C_{2n+1}. □

Theorem 3.9. *Let* $f(2, n)$ *denote the number of distinct configurations in* $S(2, n)$ *and write* $2n+1 = 3^t p_1^{a_1} \cdots p_r^{a_r}$ *as in Lemma* 3.8. *Then*

$$f(2, n) = \begin{cases} \frac{1}{6}(\phi_2(2n+1) + 3) & \text{if } t \geq 2 \text{ or some } p_j \equiv 2 \pmod 3 \\ \frac{1}{6}(\phi_2(2n+1) + 3 + 2^{r+1}) & \text{otherwise} \end{cases} \tag{3.10}$$

Proof. The case $n = 1$ is special and it is readily checked that $f(1) = 1$. Suppose $n \geq 2$; if $e^2 - e + 1 \equiv 0 \pmod{2n+1}$ then $e \not\equiv 2, n+1, 2n$ and $\{e, 1-e\}$ forms an orbit under G. In the first case of (3.10) there are no orbits of this kind and all elements of C_{2n+1} except $\{2, n+1, 2n\}$ fall into orbits of size 6. In the second case, there are 2^{r-1} orbits $\{e, 1-e\}$ and so $f(2, n) = \frac{1}{6}(\phi_2(2n+1) - 3 - 2^r) + 1 + 2^{r-1} = \frac{1}{6}(\phi_2(2n+1) + 3 + 2^{r+1})$. □

It is clear that $\lim_{n\to\infty} f(2, n) = \infty$, but the growth is not uniform. If $6s+1$ is prime then $f(2, 3s) = s + 1$; a standard lower estimate, à la Hardy–Wright, is $\phi_2(2n+1) \geq cn/(\log\log n)^2$. Here is a short table—Table 1—of $f(2, n)$.

Table 1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$f(n)$	1	1	2	1	2	3	1	3	4	2	4	3	2	5	6	2	3	7	3	7	8	2	8	7	3

Table 2

n	orbit(s)	triangle	configurations ($x(2n+1)$)
1	{2}	(0, 0), (1, 0), (2, 3)	|1 1 1|
2	{2, 3, 4}	(0, 0), (1, 0), (2, 5)	|1 3 1|
			|2 1 2|
3	{2, 4, 6}	(0, 0), (1, 0), (2, 7)	|1 5 1|
			|2 3 2|
			|3 1 3|
	{3, 5}	(0, 0), (1, 0), (3, 7)	|2 4 1|
			|4 1 2|
			|1 2 4|
4	{2, 5, 8}	(0, 0), (1, 0), (2, 9)	|1 7 1|
			|2 5 2|
			|3 3 3|
			|4 1 4|
5	{2, 6, 10}	(0, 0), (1, 0), (2, 11)	|1 9 1|
			|2 7 2|
			|3 5 3|
			|4 3 4|
			|5 1 5|
	{3, 4, 5, 7, 8, 9}	(0, 0), (1, 0), (3, 11)	|2 8 1|
			|4 5 2|
			|6 2 3|
			|1 4 6|
			|3 1 7|
6	{2, 7, 12}	(0, 0), (1, 0), (2, 13)	|1 11 1|
			|2 9 2|
			|3 7 3|
			|4 5 4|
			|5 3 5|
			|6 1 6|
	{3, 5, 6, 8, 9, 11}	(0, 0), (1, 0), (3, 13)	|2 10 1|
			|4 7 2|
			|6 4 3|
			|8 1 4|
			|1 5 7|
			|3 2 8|
	{4, 10}	(0, 0), (1, 0), (4, 13)	|3 9 1|
			|6 5 2|
			|9 1 3|
			|2 6 5|
			|5 2 6|
			|1 3 9|
7	{2, 8, 14}	(0, 0), (1, 0), (2, 15)	|1 13 1|
			|2 11 2|
			|3 9 3|
			|4 7 4|
			|5 5 5|
			|6 3 6|
			|7 1 7|

Table 2 (continued)

n	orbit(s)	triangle	configurations $(x(2n+1))$		
8	{2, 9, 16}	(0, 0), (1, 0), (2, 17)	1	15	1
			2	13	2
			3	11	3
			4	9	4
			5	7	5
			6	5	6
			7	3	7
			8	1	8
	{3, 6, 8, 10, 12, 15}	(0, 0), (1, 0), (3, 17)	2	14	1
			4	11	2
			6	8	3
			8	5	4
			10	2	5
			1	7	9
			3	4	10
			5	1	11
	{4, 5, 7, 11, 13, 14}	(0, 0), (1, 0), (4, 17)	3	13	1
			6	9	2
			9	5	3
			12	1	4
			1	10	6
			4	6	7
			7	2	8
			2	3	12

We amplify Table 1 in Table 2 for $1 \leq n \leq 8$, giving the orbits of C_{2n+1}, the possible configurations and representative triangles. For ease of reading, the matrices M_T are multiplied by $2n+1$. Thus the only configuration for $T \in S(2, 2)$ was given in (1.1). Geometrically, this means that in a two point triangle, one interior point is the midpoint of a segment containing a vertex and the other interior point. For three point triangles there are two configurations: either the interior points are on a line with one vertex or they form a triangle with cyclically symmetric barycentric coordinates.

We conclude this section with a more detailed analysis of the configurations in $S(2, n)$. Quite a few patterns are apparent in Table 2. We discuss a few which have ramifications in $S(m, n)$, $m \geq 3$.

Corollary 3.11. *If* $T \in S(2, n)$ *then every column of* M_T *contains the entry* $(2n+1)^{-1}$.

Proof. This is a corollary of Theorem 3.3. Using the notation from that proof, note that the triangles T and $\bar{T}$ share the base L and have areas $\frac{1}{2}$ and $\frac{1}{2}(2n+1)$. Hence their altitudes have ratio $1:(2n+1)$. Since $\underline{w}_k = \lambda_{k2}\underline{v}_2 + (1-\lambda_{k2})\underline{v}$ for $\underline{v}$ on L, $\lambda_{k2} = 1/(2n+1)$ by similar triangles. The choice of the third column is purely arbitrary. Alternatively, using Theorem 3.4, $T \simeq T_e$ and the barycentric coordinates of $(1, 1)$ in T_e are $(e-1, 2n+1-e, 1)/(2n+1)$. Again, the choice of the third column is arbitrary: in the proof of Theorem 3.4, any vertex can be translated to the origin. □

Corollary 3.12. *If $T \in S(2, n)$ and $\underline{\lambda} = (a_0, a_1, a_2)/D$ is a row in M_T, then $(a_0, D) = (a_1, D) = (a_2, D) = 1$ and each column of M_T contains distinct entries. further, there exists an interior point $\underline{w}$ which generates all interior points.*

In fact, $\underline{w} = (1, 1)$ generates all interior points in T_e, $0 < e < 2n+1$. The proofs of this, and the other assertions of Corollary 3.12 are not difficult, and since they follow immediately from Corollary 4.7, we omit them.

4. Lattice point generation

Before we discuss $S(m, n)$ for $m \geq 3$, we want to expand on the idea of lattice points generating other lattice points. For $m \geq 2$ let $A_m = \{\underline{\lambda} = (\lambda_0, \ldots, \lambda_m): 0 \leq \lambda_j < 1, \sum \lambda_j \in \mathbb{Z}\}$ and define addition in A_m componentwise mod 1: for integers k, k' and $\underline{\lambda}, \underline{\lambda}' \in A_m$,

$$k\underline{\lambda} \oplus k'\underline{\lambda}' = (\{k\lambda_0 + k'\lambda_0'\}, \ldots, \{k\lambda_m + k'\lambda_m'\}),$$

where $\{x\} = x - [x]$ is the fractional part of x. To avoid confusion with ordinary addition, we write $k\underline{\lambda} \oplus k'\underline{\lambda}'$ as $\{k\underline{\lambda} + k'\underline{\lambda}'\}$. Under these definitions, A_m forms a $\mathbb{Z}$-module. Of course λ_m is determined by the other components and A_m is isomorphic to the m-torus, but it is easier to work in 'homogeneous' coordinates.

Given $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\} \subseteq A_m$ let $\langle \underline{\lambda}_1, \ldots, \underline{\lambda}_n \rangle$ be the submodule of A_m generated by the $\underline{\lambda}_i$'s over $\mathbb{Z}$. For any $\mathbb{Z}$-submodule $H \subseteq A_m$, $G(H) = \{\underline{\lambda} \in H: \sum \lambda_j = 1\}$ is the *good part* of H. We call $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$ a *good* set if $\lambda_{ij} > 0$ and $G(\langle \underline{\lambda}_1, \ldots, \underline{\lambda}_n \rangle) = \{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$. If $\underline{\lambda} = (a_0, \ldots, a_m)/D$ and $(a_j, D) = 1$ for all j, then $G(\langle \underline{\lambda} \rangle)$ is itself a good set because the components of $\{k\underline{\lambda}\}$ are not zero unless $\{k\underline{\lambda}\} \equiv 0$. For convenience, we will write $|G(\langle \lambda \rangle)| = g(\underline{\lambda})$.

Why are we doing this? Suppose $\underline{w}_1, \ldots, \underline{w}_r$ are lattice points in $T(V)$ with $\underline{w}_i = \sum \lambda_{ij}\underline{v}_j$ and suppose $\underline{\lambda} \in G(\langle \underline{\lambda}_1, \ldots, \underline{\lambda}_r \rangle)$. Then $\underline{\lambda} = \{\sum k_i\underline{\lambda}_i\}$ for some integers k_i; that is, $\lambda_j = \sum k_i\lambda_{ij} - t_j$ for some integers t_j. Hence $\underline{w} = \sum \lambda_j\underline{v}_j = \sum k_i\underline{w}_i - \sum t_j\underline{v}_j$ is a lattice point; since $\sum \lambda_j = 1$, $\underline{\lambda} = \mathrm{BC}(w)$ and since $\lambda_j \geq 0$, $\underline{w}$ is in $T(V)$. Thus, $\underline{w}_1, \ldots, \underline{w}_r$ *generate* $\underline{w}$. With this notation, if $T \in S(m, n)$ then $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$ is necessarily a good set. A natural, and open, question is the converse:

If $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$ is a good set in T_m, does there exist $T \in S(m, n)$ with $M_T = [\lambda_{ij}]$?

We can answer affirmatively if $m = 2$ or 3 (see Theorem 4.3) and we have found no counterexample.

We need one more definition: given $T \in S(m, n)$ with $M_T = [\lambda_{ij}]$, the *rank* of T is the rank of $\langle \underset{\sim}{\lambda}_1, \ldots, \underset{\sim}{\lambda}_n \rangle$ as a $\mathbb{Z}$-submodule of T_m. We shall prove (Theorem 4.7) that for $m = 2$ or 3, rank$(T) = 1$, but there exists $T \in S(7, 2)$ with rank 2.

For any given $T \in S(m, n)$ or set $\{\underset{\sim}{\lambda}_1, \ldots, \underset{\sim}{\lambda}_r\}$, we will be dealing with rational vectors with common denominator D, say. For fixed D, $A_m \cap (D^{-1}\mathbb{Z})^{m+1}$ is isomorphic to $\{(a_0, \ldots, a_m): a_i \in \mathbb{Z}/D\mathbb{Z}, \sum a_i \equiv 0 \pmod D\}$. Unfortunately it is desirable to maintain flexibility in denominators. For example $H_1 = \langle(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\rangle \neq H_2 = \langle(0.85, 0.65, 0.45, 0.05)\rangle$, but $G(H_1) = G(H_2) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ $(= \{5(0.85, 0.65, 0.45, 0.05)\}.)$

We now give two lemmas on rank 1 submodules of A_m. First we need a version of the Chinese Remainder Theorem which is usually left as an exercise. (See e.g., [13, p. 33, #14c].)

Chinese Remainder Theorem

The congruences $x \equiv a_i \pmod{m_i}$ *have a simultaneous (unique) solution* $\bmod[m_1, \ldots, m_n]$ *iff* $(m_i, m_j) \mid (a_i - a_j)$ *for each* (i, j), $i < j$. *Here,* $[m_1, \ldots, m_n]$ *denotes the least common multiple and* (m_i, m_j) *the greatest common divisor. In particular, if the* m_i*'s are pairwise relatively prime, there is no restriction on the* a_i*'s.*

Lemma 4.1. *If* $\underset{\sim}{\lambda} \in A_m$ *has denominator* D, *then* $\{k\underset{\sim}{\lambda}\} = \{k'\underset{\sim}{\lambda}\}$ *iff* $k \equiv k' \pmod D$.

Proof. Write $\underset{\sim}{\lambda} = (a_0, \ldots, a_m)/D$; then $\{k\underset{\sim}{\lambda}\} = \{k'\underset{\sim}{\lambda}\}$ iff $ka_j \equiv k'a_j \pmod D$ for $0 \leq j \leq m$. Let $D = \prod p_i^{e_i}$, since D is the denominator, for each i, $p_i \nmid a_j$ for some j, hence $k \equiv k' \pmod{p_i^{e_i}}$ for all i. By the Chinese Remainder Theorem this is equivalent to $k \equiv k' \pmod D$. □

Lemma 4.2. *Suppose* $H = \langle \underset{\sim}{\lambda}_0, \underset{\sim}{\lambda}_1 \rangle$, *where* $\underset{\sim}{\lambda}_k = (a_{k0}, \ldots, a_{km})/D_k$. *Then* H *has rank* 1 *if and only if* $a_{0i}a_{1j} - a_{0j}a_{1i} \equiv 0 \pmod{(D_0, D_1)}$ *for all* i *and* j.

Proof. First suppose H has rank 1, then there exists $\underset{\sim}{\lambda} = (b_0, \ldots, b_m)/D$ and integers r_0, r_1 so that $\underset{\sim}{\lambda}_0 = \{r_0\underset{\sim}{\lambda}\}$ and $\underset{\sim}{\lambda}_1 = \{r_1\underset{\sim}{\lambda}\}$. That is, there exist integers t_{kj} so that $a_{kj}/D_k = r_kb_j/D + t_{kj}$. Thus

$$0 = \frac{r_0b_i}{D}\frac{r_1b_j}{D} - \frac{r_0b_j}{D}\frac{r_1b_i}{D} = \left(\frac{a_{0i}}{D_0} - t_{0i}\right)\left(\frac{a_{1j}}{D_1} - t_{1j}\right) - \left(\frac{a_{0j}}{D_0} - t_{0j}\right)\left(\frac{a_{1i}}{D_1} - t_{1i}\right). \quad (4.3)$$

Let $(D_0, D_1) = \bar{D}$, $D_0 = \bar{D}E_0$ and $D_1 = \bar{D}E_1$ and multiply (4.3) by $\bar{D}E_0E_1$. It follows that $(a_{0i}a_{1j} - a_{0j}a_{1i})/\bar{D}$ is an integer.

Conversely, suppose $a_{0i}a_{1j} \equiv a_{0j}a_{1i} \pmod{\bar{D}}$ for all (i, j). Let $\bar{D} = \prod p_k^{e_k}$; since $\bar{D} \mid D_0$, for each prime p_k there exists a_{0l} with $p_k \nmid a_{0l}$. We have $a_{0i}a_{1j} \equiv a_{0j}a_{1i}$ $\pmod{p_k^{e_k}}$, or, since a_{0i} is invertible, $a_{1j} \equiv (a_{0i}^{-1}a_{1i})a_{0j} \pmod{p_k^{e_k}}$. That is,

$a_{1j} \equiv s_k a_{0j} \pmod{p_k^{e_k}}$ for each k and $(s_k, p_k) = 1$. By the Chinese Remainder Theorem there exists $s_0 \equiv s_k \pmod{p_k^{e_k}}$ with $a_{1j} \equiv s_0 a_{0j} \pmod{\bar{D}}$ and $(s_0, \bar{D}) = 1$. Finally we may choose $s \equiv s_0 \pmod{\bar{D}}$ with $(s, \bar{D}E_0) = 1$, again by the Chinese Remainder Theorem (we need only consider primes in E_1 which are not in $\bar{D}$). Consider the simultaneous congruences

$$\begin{cases} b_j \equiv s a_{0j} \pmod{\bar{D}E_0} \\ b_j \equiv a_{1j} \pmod{\bar{D}E_1}. \end{cases} \tag{4.4}$$

Since $\bar{D} \mid s a_{0j} - a_{1j}$, there exist $b_j \pmod{\bar{D}E_0E_1}$ satisfying (4.4) for each j. Now take $r_0 \equiv s^{-1}E_1$ and $r_1 \equiv E_0 \pmod{\bar{D}E_0E_1}$ and let $D = \bar{D}E_0E_1$. Then $\{r_0 b_j/D\} = \{s^{-1}b_j/\bar{D}E_0\} = a_{0j}/\bar{D}E_0 = a_{0j}/D_0$ and $\{r_1 b_j/D\} = a_{1j}/D_0$, hence $\underline{\lambda}_k = \{r_k \underline{\lambda}\}$. Further, $(r_0, r_1) = 1$, so $\underline{\lambda} \in \langle \underline{\lambda}_1, \underline{\lambda}_2 \rangle$, thus $H = \langle \underline{\lambda} \rangle$ has rank 1. □

We now give our most general construction for building $T \in S(m, n)$. Two corollaries show that the hypothesis is not as restrictive as might first be thought.

Theorem 4.5. *Suppose $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$ is a good set, $\langle \underline{\lambda}_1, \ldots, \underline{\lambda}_n \rangle$ has rank 1 with generator $(a_0, \ldots, a_m)/D$ and $(a_j, D) = 1$ for at least one j. Then there exists $T \in S(m, n)$ with $M_T = [\lambda_{ij}]$.*

Proof. Without loss of generality assume that $(a_m, D) = 1$. If $(s, D) = 1$, then the sets $\{\{k\underline{\lambda}\}\}$ and $\{\{ks\underline{\lambda}\}\}$, $0 \leq k \leq D - 1$ are identical, hence we may take $\{s\underline{\lambda}\}$ as a generator. Choosing $s \equiv a_m^{-1} \pmod{D}$, we may therefore assume that $a_m = 1$.

Let $T = T(V)$ be the simplex with vertices $\underline{v}_0 = (0, \ldots, 0)$, $\underline{v}_j = \underline{e}_j$ for $1 \leq j \leq m - 1$ and $\underline{v}_m = (-a_1, \ldots, -a_{m-1}, D)$. Since $\operatorname{Vol} T = D/m!$, T is a non-degenerate simplex. Let $\underline{w} = (w_1, \ldots, w_{m-1}, k)$ be a lattice point; it is easy to compute $\underline{\mu} = \mathrm{BC}(\underline{w})$: $\mu_m = k/D$, $\mu_j = w_j + ka_j/D$ and $\mu_0 = 1 - \sum_{j=1}^m \mu_j$. If $\underline{w}$ is in T then $0 \leq k \leq D$ and $k = 0$ or D clearly imply that $\underline{w} = \underline{v}_j$ for some j. Otherwise, $0 < k < D$ and $0 \leq \mu_j < 1$ implies $w_j = -[ka_j/D]$ for $1 \leq j \leq m - 1$; that is, k determines $(w_1, \ldots, w_{m-1})$, and further, $\mu_j = \{ka_j/D\}$ for $1 \leq j \leq m$. Since $D \mid \sum_{j=0}^m ka_j$, $\{\mu_0\} = \{ka_0/D\}$. Therefore $\underline{w}$, as described, is a lattice point in T iff $\sum \{ka_j/D\} = 1$; that is, iff $\{k\underline{\lambda}\} \in G(\langle \underline{\lambda} \rangle)$. In other words, $M_T = [\lambda_{ij}]$. □

Corollary 4.6. $S(m, n) \neq \emptyset$ *for* $m \geq 2$, $n \geq 0$.

Proof. Let $\underline{\lambda} = (mn + 1 - m, 1, \ldots. 1)/(mn + 1)$. It is easy to see that $G(\langle \underline{\lambda} \rangle) = \{\{k\underline{\lambda}\} : 1 \leq k \leq n\}$. Since $(a_m, D) = 1$, by the last theorem, there exists $T \in S(m, n)$. Indeed, the vertices of T are $\underline{0}, \underline{e}_1, \ldots, \underline{e}_{m-1}$ and $(-1, -1, \ldots, -1, mn + 1)$ and the n interior points are $\{(0, \ldots, 0, k) : 1 \leq k \leq n\}$. □

Theorem 4.7. *For $m = 2$ or 3, every good set is realized as the configuration of*

$T \in S(m, n)$. In particular, $\operatorname{rank}(T) = 1$, *and if* $\underline{\lambda} = (a_0, \ldots, a_m)/D$ *generates* $\langle \underline{\lambda}_1, \ldots, \underline{\lambda}_n \rangle$, *then* $(a_j, D) = 1$ *for all* j.

Proof. Suppose $\{\underline{\lambda}_1, \ldots, \underline{\lambda}_n\}$ is a good set in A_m for $m = 2$ or 3 and let $H = \langle \underline{\lambda}_1, \ldots, \underline{\lambda}_n \rangle$. We first show that $\underline{\lambda} \in H$, $\underline{\lambda} \neq \underline{0}$ implies $\lambda_j > 0$ for all j. (A priori, all we know is that $\lambda_j > 0$ for $\underline{\lambda} \in G(H)$.) Using this, we prove that $\operatorname{rank}(H) = 1$ and, if $\underline{\lambda} = (a_0, \ldots, a_m)/D \in H$, then $(a_j, D) = 1$ for all j. In particular, if $H = \langle \underline{\lambda} \rangle$, then the corollary follows by Theorem 4.5.

First suppose $\underline{\lambda} = (0, \lambda_1, \lambda_2) \in H \subseteq A_2$, $\underline{\lambda} \neq \underline{0}$; $\sum \lambda_j \in \mathbb{Z}$ implies $\underline{\lambda} \in G(H)$, a contradiction. Now suppose $\underline{\lambda} = (0, \lambda_1, \lambda_2, \lambda_3) \in H \subseteq A_3$. Then $\{-\underline{\lambda}\} = (0, \{-\lambda_1\}, \{-\lambda_2\}, \{-\lambda_3\}) \in H$ as well. Since $\{x\} + \{-x\} \leqslant 1$, $\sum \lambda_j + \sum \{-\lambda_j\} = \sum (\{\lambda_j\} + \{-\lambda_j\}) \leqslant 3$. Thus $\underline{\lambda} \in G(H)$ or $\{-\underline{\lambda}\} \in G(H)$, another contradiction. The same proofs obviously work for $\lambda_j = 0$, $j \geqslant 1$.

If $\operatorname{rank}(H) \geqslant 2$ then $\operatorname{rank}(\langle \underline{\mu}_0, \underline{\mu}_1 \rangle) = 2$ for some $\underline{\mu}_0, \underline{\mu}_1 \in H$. Write $\underline{\mu}_0 = (a_{00}, \ldots, a_{0m})/D_0$ and $\underline{\mu}_1 = (a_{10}, \ldots, a_{1m})/D_1$, $D_0 = \bar{D}E_0$ and $D_1 = \bar{D}E_1$ with $(E_0, E_1) = 1$. By Lemma 4.2, $a_{0i}a_{1j} \not\equiv a_{0j}a_{1i} \pmod{\bar{D}}$ for some (i, j). But $\underline{\lambda} = \{a_{1j}E_0\underline{\mu}_0 - a_{0j}E_1\underline{\mu}_1\} \in H$ and $\lambda_l = \{(a_{0l}a_{1j} - a_{0j}a_{1l})/\bar{D}\}$ hence $\lambda_i \neq 0$, $\lambda_j = 0$. This is a contradiction by the last paragraph, hence $\operatorname{rank}(H) = 1$.

Finally, suppose $\underline{\lambda} = (a_0, \ldots, a_m)/D \in H$ and $(a_j, D) = d > 1$. Writing $D = dd'$, $\underline{\lambda}' = \{d'\underline{\lambda}\} = (\{d'a_0/D\}, \ldots, \{d'a_m/D\})$, so $\lambda_j' = 0$. Since $d' < D$, $\underline{\lambda}' \neq \underline{0}$ and we have another contradiction. Taking $\underline{\lambda}$ as the generator of H_1, we are done. □

Several comments are in order. First, Corollary 3.12 follows immediately. Second, there is no hope of generalizing this proof to $m \geqslant 4$, since the intermediate steps are false. Consider $\underline{\lambda} = (1, 2, 3, 4, 5)/15$. It is not hard to see that $G(\langle \underline{\lambda} \rangle) = \{\underline{\lambda}\}$ and $(3, 15) \neq 1$. Further, $\underline{\lambda}' = \{5\underline{\lambda}\} = \frac{1}{3}(1, 2, 0, 1, 2) \in \langle \underline{\lambda} \rangle$ but $\lambda_2' = 0$. On the other hand, $(1, 15) = 1$ so there exists $T \in S(4, 1)$ with $M_T = [\underline{\lambda}]$. A more complicated example is $\underline{\lambda} = (5, 7, 11, 303, 1984)/2310$. Notice that $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and each a_i is divisible by a different prime. Suppose that, say, $\{k\lambda_0\} = 0$, then $\{5k/2310\} = 0$ so $k = 2 \cdot 3 \cdot 7 \cdot 11 \cdot t$ and $\{2 \cdot 3 \cdot 7 \cdot 11 \cdot \underline{\lambda}\} = \frac{1}{5}(0, 2, 1, 3, 4)$. Since $\{\frac{2}{5}t\} + \{\frac{3}{5}t\} = \{\frac{1}{5}t\} + \{\frac{4}{5}t\} = 1$, $\{2 \cdot 3 \cdot 7 \cdot 11 \cdot t\underline{\lambda}\} \notin G(\langle \underline{\lambda} \rangle)$. The same thing happens if $\{k\lambda_j\} = 0$ for $1 \leqslant j \leqslant 4$. (Indeed $\underline{\lambda}$ was found by applying the Chinese Remainder Theorem algorithm.) It is not hard to establish by computer that $g(\underline{\lambda}) = 105$. Since $(a_j, 2310) > 1$ for $0 \leqslant j \leqslant 4$, we cannot apply Theorem 4.5, and we have been unable to determine whether or not there exists $T \in S(4, 105)$ with $M_T = [\{k_i\lambda_j\}]$. One way we might be able to construct T would be to find $\underline{\mu} = (b_0, \ldots, b_4)/D$ with $(b_j, D) = 1$, $\{s\underline{\mu}\} = \underline{\lambda}$ and $\{k\underline{\mu}\} \in G(\langle \underline{\mu} \rangle)$ if $k = sk_i$. Such a phenomenon occurs for $\underline{\mu} = (0.85, 0.65, 0.45, 0.05)$, $\underline{\lambda} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, as noted above.

We conclude with one 'sporadic' construction of $T \in S(7, 2)$ with $\operatorname{rank}(T) = 2$. Let $\underline{\lambda}_1 = (1, 1, 1, 1, 3, 3, 3, 3)/16$ and $\underline{\lambda}_2 = (3, 3, 3, 3, 1, 1, 1, 1)/16$. Then $\langle \underline{\lambda}_1, \underline{\lambda}_2 \rangle$ has rank 2 by Lemma 4.2 since $16 \nmid 3^2 - 1^2$. It is not hard to show that

$G\langle \underline{\lambda}_1, \underline{\lambda}_2 \rangle = \{\underline{\lambda}_1, \underline{\lambda}_2\}$ so it is a good set. Let $T = T(V)$, where the $\underline{v}_i$'s are given in (4.8).

$$\begin{aligned} \underline{v}_0 &= (1, 1, 0, 0, 3, 3, 2, 2), & \underline{v}_4 &= (3, 3, 2, 2, 1, 0, 0, 1),\\ \underline{v}_1 &= (0, 1, 1, 0, 3, 2, 3, 2), & \underline{v}_5 &= (3, 2, 2, 3, 0, 1, 1, 0),\\ \underline{v}_2 &= (1, 0, 0, 1, 2, 3, 3, 3), & \underline{v}_6 &= (2, 3, 3, 3, 0, 0, 1, 1),\\ \underline{v}_3 &= (0, 0, 1, 1, 2, 2, 2, 3), & \underline{v}_7 &= (2, 2, 3, 2, 1, 1, 0, 0). \end{aligned} \tag{4.8}$$

We must first show that T is non-degenerate: this is tantamount to showing that $\det[v_{ij}] \neq 0$. In fact, a routine application of row reduction shows that $|\det T| = 96$. Notice also that for $i \neq j$, $\underline{v}_i \not\equiv \underline{v}_j \pmod 2$, hence there are no lattice points on the edges $\overline{\underline{v}_i\underline{v}_j}$. Now suppose $\underline{w} = (w_0, \ldots, w_7)$ is a lattice point in T and write $\underline{w} = \sum \lambda_j \underline{v}_j$. Then $w_0 - w_1 = \lambda_2 + \lambda_5 - \lambda_1 - \lambda_6$, so $w_0 - w_1 \in \{-1, 0, 1\}$. If $w_0 - w_1 = 1$, then $\lambda_2 + \lambda_5 = 1$ so $\underline{w}$ is on the edge $\overline{\underline{v}_2\underline{v}_5}$, an impossibility; if $w_0 - w_1 = -1$, then $\lambda_1 + \lambda_6 = 1$, which leads to another impossibility, therefore $w_0 = w_1$. Similarly, $w_1 = w_2$, $w_2 = w_3$, $w_4 = w_5$, $w_5 = w_6$ and $w_6 = w_7$. Thus $\underline{w} = (a, a, a, a, b, b, b, b)$; since $11 \leq \sum_j v_{ij} \leq 13$, $11 \leq 4a + 4b \leq 13$ so either $a = 2$ and $b = 1$ or $a = 1$ and $b = 2$. These points have barycentric coordinates $\underline{\lambda}_1$ and $\underline{\lambda}_2$ respectively. Thus $T \in S(7, 2)$ and $\operatorname{rank}(T) = 2$.

5. Lattice point simplices

In this section we return to simplices. We start with a classification theorem similar to Theorem 3.4 for $T \in S(3, n)$. It is a non-trivial problem to characterize $S(3, 0)$ up to unimodular equivalence, let alone $S(3, n)$.

Lemma 5.1. *Suppose $y = (y_0, \ldots, y_m) \in \mathbb{Z}^{m+1}$ and $(y_i, y_j) = d$. Then there is a unimodular map which only affects the* ith *and* jth *components and sends* $(\ldots, y_i, \ldots, y_j, \ldots)$ *to* $(\ldots, d, \ldots, 0, \ldots)$.

Proof. Without loss of generality, take $(i, j) = (0, 1)$ and choose integers a and b so that $ay_0 + by_1 = d$. Then the map h defined by $h(x_0, \ldots, x_m) = (ax_0 + bx_1, -y_1d^{-1}x_0 + y_0d^{-1}x_1, x_2, \ldots, x_m)$ has the desired properties. □

Theorem 5.2. *If $T \in S(3, n)$, then $T \simeq T_{a,b,c} := T((0, 0, 0), (1, 0, 0), (0, 1, 0), (a, b, c))$, where $0 < a \leq b < c$ and $(a, c) = (b, c) = (a + b - 1, c) = 1$, or $a = b = 0$, $c = 1$. Conversely, such a $T_{a,b,c} \in S(3, n)$ for $n = g((a + b - 1, c - a, c - b, 1)/c) \leq \frac{1}{2}(c - 1)$.*

Proof. Given $T \in S(3, n)$, translate so that $\underline{v}_0 = (0, 0, 0)$. If $\underline{v}_1 = (v_{10}, v_{11}, v_{12})$ then the iterated $\gcd(v_{10}, (v_{11}, v_{12})) = 1$ since there are no points on the edge $\overline{\underline{v}_0\underline{v}_1}$. Applying Lemma 4.1 twice, we have $T \simeq T((0, 0, 0), (1, 0, 0), \underline{v}_2', \underline{v}_3')$.

Applying the lemma once more to the last two coordinates of $\underline{v}_2'$, we have $T \simeq T((0, 0, 0), (1, 0, 0), (s, t, 0), \underline{v}_3'')$.

Since the face $T((0, 0, 0), (1, 0, 0), (s, t, 0))$ may be considered in $S(2, 0)$ we must have $t = 1$ by Pick's theorem and we may take $s = 0$ by applying Lemma 4.1 with $(i, j) = (1, 0)$. Thus $T \simeq T_{a,b,c}$ for some (a, b, c). By applying the unimodular maps which send (x, y, z) to $(x + jz, y + kz, z)$ we can put a and b in $[0, c)$, and by permuting if necessary, we may assume $a \leq b$.

If $c = 1$ then $a = b = 0$ and $T_{0,0,1} \in S(3, 0)$; henceforth assume $c \geq 2$. As in the proof of Theorem 4.5, we compute the barycentric coordinates of a lattice point (i, j, k) in $T_{a,b,c}$

$$\mathrm{BC}(i, j, k) = \underline{\lambda}_k = \left(\frac{(a + b - 1)k}{c} + 1 - (i + j), i - \frac{ak}{c}, j - \frac{bk}{c}, \frac{k}{c}\right). \tag{5.3}$$

As before, for each k, $0 < k < c$, there is a unique choice of (i, j) making $0 \leq \lambda_{k1}, \lambda_{k2} < 1$ and (i, j, k) is in T if $\{k\underline{\lambda}\} \in G(\langle\underline{\lambda}\rangle)$ for $\underline{\lambda} = (a + b - 1, c - a, c - b, 1)/c$, where 1 has been added to λ_1 and λ_2 to ensure $\underline{\lambda} \in A_3$. By Theorem 4.7, $T \in S(3, n)$ implies $(c - a, c) = (c - b, c) = (a + b - 1, c) = 1$.

Conversely, if $(a, c) = (b, c) = (a + b - 1, c) = 1$ and $\underline{\lambda} = (a + b - 1, c - a, c - b, 1)/c$ then $G(\langle\underline{\lambda}\rangle)$ is a good set (as remarked in the last section) so that $T_{a,b,c} \in S(3, n)$ for some n. Indeed $\{k\underline{\lambda}\} \in G(\langle\underline{\lambda}\rangle)$ implies $\sum \{k\lambda_i\} = 1$, so $\sum \{(c - k)\lambda_i\} = 3$ and $\{(c - k)\underline{\lambda}\} \notin G(\langle\underline{\lambda}\rangle)$. Since $\langle\underline{\lambda}\rangle = \{\{k\underline{\lambda}\} : 0 \leq k \leq c - 1\}$, $n = g(\underline{\lambda}) \leq (c - 1)/2$. □

The first part of Theorem 5.2 was proved by Reeve for $S(3, 0)$. He also observed that $a = 1$ or $b = 1$ imply $T_{a,b,c} \in S(3, 0)$ but that this condition is not necessary: $T_{2,5,7} \in S(3, 0)$. White studied $S(3, 0)$ in his work on admissible lattices [19] and nearly characterized them geometrically. It is not clear how to apply White's criteria, which involve 'consecutive' planes of lattice points, to $T_{a,b,c}$. Nonetheless, White's Theorem 2 (see below) can be used to complete the characterization of $S(3, 0)$. Lemma 5.4 can also be derived from a recent paper of Noordzij [14], see the last section for more discussion. For another proof of White's Theorem 2, see Hossain [10], but read the review first.

Scarf [17] has studied a class of convex polyhedra in $\mathbb{R}^n$ which arises in the study of integer programming in n variables. His analysis is based on a remarkable unpublished theorem of Roger Howe about fundamental lattice point octahedra in $\mathbb{R}^3$. Such a polyhedron is unimodularly equivalent to the figure with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, a, b)$, $(1, c, d)$, $(1, a + c, b + d)$, where $(a + c, b + d) = 1$ and $ad - bc = 1$. In the course of proving this theorem, Theorem 5.5 naturally arises as a lemma.

White's Theorem 2. *Let* a_0, a_1, a_2, D *be integers satisfying* $D \not\equiv 0$ *and* $a_i + a_j \not\equiv 0 \pmod D$ *for* $i \not\equiv j$. *Then there exists an integer* k_0, $1 \leq k_0 \leq D - 1$ *for which* $0 \leq \sum \{k_0 a_i / D\} < 1$.

Lemma 5.4. *If* $\underline{\lambda} = (a_0, a_1, a_2, a_3)/D \in A_3$ *and* $g(\underline{\lambda}) = 0$, *then* $(a_j, D) = 1$ *and the* a_i*'s decompose into two pairs which sum to* D *(or* $\underline{\lambda} = \underline{0}$*).*

Proof. As in the proof of Theorem 4.7, if $(a_j, D) = d > 1$, then $\{d'\underline{\lambda}\}$ or $\{-d'\underline{\lambda}\}$ is in $G(\langle\underline{\lambda}\rangle)$, so $(a_j, D) = 1$. As $G(\langle\underline{\lambda}\rangle) = 0$, $\sum_{i=0}^{3} \{ka_i/D\} \geqslant 2$ for all k, $1 \leqslant k \leqslant D-1$. If the a_i's do not pair off then the hypothesis of White's Theorem 2 is satisfied, so that $\{k_0 a_3/D\} > 1$, a contradiction. □

Theorem 5.5 (Reeve–White–Howe–Scarf). *A tetrahedron* T *is fundamental iff* $T \simeq T_{a,b,c}$ *with* $0 \leqslant a \leqslant b < c$ *and* $c = 1$ *or* $a = 1$ *or* $a + b = c$ *with* $(a, c) = 1$.

Proof. Returning to the notation of Theorem 5.2, $T \in S(3, 0)$ if $g(\underline{\lambda}) = 0$ for $\underline{\lambda} = (a + b - 1, c - a, c - b, 1)/c$. If $c \geqslant 2$, then $\underline{\lambda} = 0$, so by Lemma 4.4, $a = 1$ or $b = 1$ or $a + b = c$. □

We strengthen this theorem somewhat below. It is too much to expect that $T \sim T'$ implies $T \simeq T'$. In fact, unimodular equivalence is hard to come by. As an application of Theorem 5.6, only two of the 24 affine maps which permute the vertices of $T_{1,b,c}$ are unimodular: the identity and the map which permutes $(0, 0, 0)$ with $(0, 1, 0)$ and $(1, 0, 0)$ with $(1, b, c)$.

Theorem 5.6. *Suppose* $T_{a,b,c} \simeq T_{d,e,f}$. *Then* $|c| = |f|$ *and* $(d, e) = f_i(a, b) \pmod{|c|}$ *for some* f_i, $1 \leqslant i \leqslant 24$, *where* f_i *is in the group generated by* $f_1(x, y) = (1 - (x + y), y)$, $f_2(x, y) = (x, 1 - (x + y))$ *and* $f_3(x, y) = (x(x + y - 1)^{-1}, y(x + y - 1)^{-1})$. *In particular, if* $T_{1,b,c} \simeq T_{d,e,c}$, *then* (d, e) *or* (e, d) *is in the set*

$$\{(1, b), (1, c - b), (1, b'), (1, c - b'), (b, c - b), (b', c - b')\},$$

where $bb' \equiv 1 \pmod c$.

Proof. If $T \simeq T'$ then they have the same volume, so $\frac{1}{6}|c| = \frac{1}{6}|f|$. Now suppose $T_{a,b,c}$ and $T_{d,e,c}$ are given: we argue as in the proof of Theorem 3.4. Let h be the affine map defined by $h(0, 0, 0) = \underline{w}_0$, $h(1, 0, 0) = \underline{w}_1$, $h(0, 1, 0) = \underline{w}_2$ and $h(a, b, c) = \underline{w}_3$, where

$$\{\underline{w}_0, \underline{w}_1, \underline{w}_2, \underline{w}_3\} = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (d, e, c)\},$$

then $T_{a,b,c} \simeq T_{d,e,c}$ iff one of the 24 possible maps has integral coordinates (note that h preserves volume). As before, it suffices to show that

$$h(0, 0, 1) = \frac{1}{c}[(a + b + c - 1)\underline{w}_0 - a\underline{w}_1 - b\underline{w}_2 + \underline{w}_3]$$

is a lattice point. Each permutation gives rise to a different condition on $(d, e) \pmod c$. For example, if $\underline{w}_0 = (1, 0, 0)$, $\underline{w}_1 = (0, 1, 0)$, $\underline{w}_2 = (d, e, c)$ and $\underline{w}_3 = (0, 0, 0)$, then h is unimodular iff $a + b - 1 - bd \equiv -a - be \equiv 0 \pmod c$, or

$(d, e) \equiv (b^{-1}(a+b-1), -b^{-1}a) \pmod c$. Since the symmetric group on $\{0, 1, 2, 3\}$ is generated by (01), (02) and (03), the functional dependence is a composition of f_1, f_2 and f_3. In the above example, $(0123) = (01)(02)(03)$ and $(b^{-1}(a+b-1), -b^{-1}a) = f_3(f_2(f_1(a, b)))$. Finally, suppose that $a = 1$, then (d, e) is one of the twelve displayed possibilities. Since the orbit of T_{1bc} has twelve members, one non-trivial permutation fixes $(1, b)$; as noted above, this is $(02)(13)$. □

We see that Lemma 3.2 fails dramatically in higher dimensions. One consequence of this theorem is that Theorem 5.5 can be strengthened somewhat. For example, Reeve's fundamental tetrahedron $T_{2,5,7} \simeq T_{1,2,7}$.

Corollary 5.7. *A tetrahedron T is fundamental iff* $T \simeq T_{0,0,1}$ *or* $T \simeq T_{1,b,c}$ *with* $1 \leq b < c$ *and* $(b, c) = 1$.

Finally we can generalize Theorem 5.2 partially for $m \geq 4$. We omit the proof and state it for $m = 4$ only. The necessary conditions on $\{b, c, d, e, f, g\}$ seem hard to find.

Corollary 5.8. *If* $T \in S(4, n)$, *then* $T \simeq T((0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (1, b, c, 0), (d, e, f, g))$.

Now we turn to $S(3, 1)$: we need a theorem whose tedious proof we defer to the end of the section.

1 **Theorem 5.9.** *Suppose* $\underset{\sim}{\lambda} \in A_3$ *and* $\{\underset{\sim}{\lambda}\}$ *is good; then, up to a permutation of*
6 *coordinates,* $\underset{\sim}{\lambda}$ *is one of the following seven vectors:*

$$\begin{array}{l} (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}), \quad (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}), \quad (\tfrac{3}{7}, \tfrac{2}{7}, \tfrac{1}{7}, \tfrac{1}{7}), \quad (\tfrac{5}{11}, \tfrac{3}{11}, \tfrac{2}{11}, \tfrac{1}{11}), \\ (\tfrac{5}{13}, \tfrac{4}{13}, \tfrac{3}{13}, \tfrac{1}{13}), \quad (\tfrac{7}{17}, \tfrac{5}{17}, \tfrac{3}{17}, \tfrac{2}{17}), \quad (\tfrac{7}{19}, \tfrac{5}{19}, \tfrac{4}{19}, \tfrac{3}{19}). \end{array} \tag{5.10}$$

Corollary 5.11. *If* $T \in S(3, 1)$, *then* M_T *is listed in* (5.10); $f(3, 1) = 7$.

Proof. Given $T \in S(3, 1)$ and $M_T = \underset{\sim}{\lambda}$, $\{\underset{\sim}{\lambda}\}$ is a good set and so $\underset{\sim}{\lambda}$ is in (5.10). By Theorem 4.7, every good set in A_3 is achieved. □

As might be expected, there exist $T \sim T'$, $T \not\simeq T'$ in $S(3, 1)$. Indeed, recall that $G(\langle(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})\rangle) = G(\langle(0.85, 0.65, 0.45, 0.05)\rangle)$, and using the construction of Theorem 4.5, we have $T_{-1,-1,4} \sim T_{-13,-9,20}$ with common configuration $(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$ but $T_{-1,-1,4} \not\simeq T_{-13,-9,20}$, since their volumes differ.

Finally, we turn to a non-constructive existence theorem which bounds the denominators in M_T for $T \in S(m, n)$. The proof is essentially due to Erdös [4] who has graciously permitted us to reproduce it here. Although this result is a consequence of a theorem of Hensley [9], we have chosen nevertheless to include it.

Theorem 5.12. *Suppose $\underset{\sim}{\lambda} = (a_0, \ldots, a_m)/D$ is given with $\sum_{j=0}^{m} \lambda_j = 1$. There is a computable function $D(m, n)$ so that $D > D(m, n)$ implies $g(\underset{\sim}{\lambda}) > n$.*

Corollary 5.13. *The function $f(m, n)$ is finite for each $m \geq 2$, $n \geq 0$.*

Proof of Corollary. Suppose $T \in S(m, n)$ and $\underset{\sim}{\lambda}_i = (a_0, \ldots, a_m)/D$ is a row in M_T, then $\underset{\sim}{\lambda} \in G(\langle \underset{\sim}{\lambda}_i \rangle)$ implies $\underset{\sim}{\lambda}$ is also a row in M_T, hence $D \leq D(m, n)$. As there are finitely many partitions of $D \leq D(m, n)$, there are finitely many possible rows and so $f(m, n) < \infty$. Note that this proof works whether or not $\operatorname{rank}(T) = 1$. □

To prove Theorem 5.12 we need some notation and two easy lemmas. Let $\|x\| = \min(\{x\}, \{-x\})$ denote the distance from x to the nearest integer. The first lemma collects some simple inequalities satisfied by $\| \ \|$. The second may be found in Hardy and Wright [8; Chapter 11] and can be proved by a standard pigeonhole principle argument. We omit the proofs.

Lemma 5.14.
(i) $\|x + y\| \leq \|x\| + \|y\|$,
(ii) $\|x - y\| \leq |\{x\} - \{y\}|$,
(iii) $\|nx\| \leq n\,\|x\|$,
(iv) $\|y\| \leq \{x\}$ *implies* $\{x + y\} \leq \{x\} + \|y\|$.

Lemma 5.15. (Dirichlet). *Given $\frac{1}{2} > \varepsilon > 0$ and $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_m > 0$, $\sum \lambda_j = 1$, there exists t, $1 \leq t < (1 + \varepsilon^{-1})^{m+1}$ such that $\|t\lambda_j\| < \varepsilon$ for $0 \leq j \leq m$.*

Proof of Theorem 5.12. Write $\underset{\sim}{\lambda} = (a_0, \ldots, a_m)/D$ and assume without loss of generality that $\lambda_0 \geq \cdots \geq \lambda_m$. We also assume that $g(\underset{\sim}{\lambda}) \leq n$ and will get the bound on D by showing that, for D sufficiently large, there exists t so that $1 + nt < D$ and $\{(1 + kt)\underset{\sim}{\lambda}\} \in G(\langle \underset{\sim}{\lambda} \rangle)$ for $0 \leq k \leq n$.

Since $\lambda_0 \geq \cdots \geq \lambda_m$, $\lambda_0 \geq (m + 1)^{-1}$ and $\lambda_1 \geq (1 - \lambda_0)/m$. Suppose $\lambda_0 \geq n/(n + 1)$, then $\{s\lambda_0\} = s\lambda_0 - (s - 1)$ for $1 \leq s \leq n + 1$, hence $\sum \{s\lambda_j\} \leq \{s\lambda_0\} + \sum_{j=1}^{m} \{s\lambda_j\} \leq s\lambda_0 - (s - 1) + \sum_{j=1}^{m} s\lambda_j = 1$ for $1 \leq s \leq n + 1$, violating the hypothesis. Therefore, $\lambda_0 < n/(n + 1)$ and so $\lambda_1 \geq (m(n + 1))^{-1}$. We now proceed by induction, supposing $\lambda_j \geq (N(j, m, n))^{-1}$ and proving $\lambda_{j+1} \geq (N(j + 1, m, n))^{-1}$ for a suitably defined integer $N(j, m, n)$. Indeed take $N(1, m, n) = m(n + 1)$ and define

$$N(j + 1, m, n) = 2(m - j)(1 + n(1 + 2nN(j, m, n)^{m+1})).$$

Suppose $\lambda_j \geq (N(j, m, n))^{-1}$ and consider two possibilities for D: $D \leq N(j + 1, m, n)$ or $D > N(j + 1, m, n)$. In the first case, $\lambda_{j+1} = a_{j+1}/D \geq 1/D$ so $\lambda_{j+1} \geq$

$(N(j+1, m, n))^{-1}$. We must work harder in the second case. By Lemma 5.15 there exists t, $1 \leq t \leq (1+2n/\lambda_j)^{m+1} \leq (1+2nN(j, m, n))^{m+1} < D/n$, such that $\|t\lambda_i\| < \lambda_j/2n$ for $0 \leq i \leq m$. We consider separately $0 \leq i \leq j$ and $j+1 \leq i \leq m$. First,

$$\{\lambda_i\} = \lambda_i \geq \lambda_j > n\lambda_j/2n > n\ \|t\lambda_i\| > \|nt\lambda_i\|,$$

by Lemma 5.14, hence for $0 \leq k \leq n$,

$$\sum_{i=0}^{j} \{(1+kt)\lambda_i\} \leq \sum_{i=0}^{j} \lambda_i + \|kt\lambda_i\| \leq \sum_{i=0}^{j} \lambda_i + \tfrac{1}{2}\lambda_j \leq \sum_{i=0}^{j} \tfrac{3}{2}\lambda_i < 1\tfrac{1}{2}.$$

But $|G(\langle\lambda\rangle)| \leq n$ and $1+nt < D$, so $\sum_{i=j+1}^{m} \{(1+kt)\lambda_i\} > \frac{1}{2}$ for at least one k, $0 \leq k \leq n$. Hence

$$\begin{aligned}\tfrac{1}{2} < \sum_{i=j+1}^{m} \{(1+kt)\lambda_i\} &\leq \sum_{i=j+1}^{m} (1+kt)\lambda_i \leq \sum_{i=j+1}^{m} (1+kt)\lambda_{j+1} \\ &\leq (m-j)(1+n(1+2nN(j, m, n))^{m+1})\lambda_{j+1},\end{aligned}$$

or

$$\lambda_{j+1} \geq (N(j+1, m, n))^{-1}.$$

By repeating the argument we see that $\lambda_j \geq \lambda_m \geq (N(m, n, n))^{-1}$. By one final application of Lemma 5.15, there exists t, $1 \leq t \leq (2+nN(m, m, n))^{m+1}$ such that $\|t\lambda_j\| \leq (nN(m, m, n)+1)^{-1} < \lambda_m/n$ for $0 \leq j \leq m$. Thus, for $0 \leq k \leq n$,

$$\sum_{j=0}^{m} \{(1+kt)\lambda_j\} \leq \sum_{j=0}^{m} \lambda_j + \|kt\lambda_j\| < \sum_{j=0}^{m} \lambda_j + \lambda_m \leq 2.$$

That is, $\{(1+kt)\underset{\sim}{\lambda}_j\} \in G(\langle\underset{\sim}{\lambda}\rangle)$. Since $g(\underset{\sim}{\lambda}) \leq n$, $1+nt \geq D$. Putting together these arguments, if $D \geq D(m, n) = 2 + n(2+nN(m, m, n))^{m+1}$, then $g(\underset{\sim}{\lambda}) \geq n+1$. □

As might be expected, the bound $D(m, n)$ given in the previous proof is terrible; for example, $D(3, 1) \simeq 8.8 \times 10^{86}$. In fact, by Theorem 5.9, $\sum \lambda_j = 1$, $\lambda_j > 0$, $g(\underset{\sim}{\lambda}) = 1$ imply $D \leq 19$. Neither theorem covers $(0.85, 0.65, 0.45, 0.05)$ $(D = 20)$. However, by Hensley's Theorem 3.4 [9], there is an upper bound of the volume of any m-simplex with n interior points, disregarding possible boundary lattice points. The bounds are somewhat better, but still huge. By refining the induction at early stages, the upper bound for denominators in the configuration of a one-point tetrahedron is reduced to $74\,088 = 42^3$. In general, however, $D(m, n)$ and Hensley's upper bound have asymptotic log log log's. The recent construction of Zaks, Perles and Wills [20] gives a lower bound which compares in this way with Hensley's (better) upper bound in the same way. The example in [20] has many lattice points on the boundary, so perhaps $D(m, n)$ can be significantly improved.

Proof of Theorem 5.9. Suppose $\{\underset{\sim}{\lambda}\} \in A_3$ is good; $\underset{\sim}{\lambda} = (a_0, a_1, a_2, a_3)/D$ with

$(a_j, D) = 1$. Assume without loss of generality that $a_0 \geq a_1 \geq a_2 \geq a_3$. By hypothesis, $\sum \{k\lambda_j\} \geq 2$ for $2 \leq k \leq D-1$; and as $\sum \{k\lambda_j\} + \{(D-k)\lambda_j\} = 4$ it follows that $\sum \{k\lambda_j\} = 2$ for $2 \leq k \leq D-2$. Hence $\sum [k\lambda_j] = \sum (k\lambda_j - \{k\lambda_j\}) = k-2$ for $2 \leq k \leq D-2$, or $\sum [2\lambda_j] = 0$ and $\sum ([(k+1)\lambda_j] - [k\lambda_j]) = 1$ for $2 \leq k \leq D-3$. Thus $\frac{1}{2} > \lambda_0$ and for $2 \leq k \leq D-3$, the equation

$$(k+1)\lambda_j > m > k\lambda_j \tag{5.16}$$

has a solution for exactly one j and some integer m. (Since $(a_j, D) = 1$, $k\lambda_j$ is not an integer.) Equation (5.16) is equivalent to $k+1 > m\lambda_j^{-1} > k$. Hence the four sequences $\{[\lambda_j^{-1}], [2\lambda_j^{-1}], \ldots\}$, $0 \leq j \leq 3$, partition $\{2, 3, \ldots, D-3\}$. In particular, $\lambda_j = \lambda_{j+1}$ implies $[\lambda_j^{-1}] \geq D-3$ or $D \geq (D-3)a_j$; that is, $a_j = 1$. We now let $r_j = \lambda_j^{-1}$ and remove the understanding that $r_j = D/a_j$. We shall prove below that the four sequences $\{[mr_j]\}$, $m \geq 1$, $r_j > 2$, can partition $\{2, 3, \ldots, D-3\}$ only if $D \leq 19$.

Suppose $2 < r_0 \leq r_1 \leq r_2 \leq r_3$ and $D \geq 10$ and $\{[mr_j]\}$ partition $\{2, \ldots, D-3\}$. Then clearly $[r_0] = 2$, so $[2r_0] = 4$ or 5, hence $[r_1] = 3$. Since $[2r_1] = 6$ or 7, $[r_2] = 4$ or 5, depending on what $[2r_0]$ is. This analysis quickly becomes unwieldy and it seems easier to first consider the possible beginnings for $\{[mr_0]\}$ and $\{[mr_1]\}$ with $[mr_j] \leq 15$. In Table 3, constructed with the aid of the Farey sequence, we list the 13 choices for $\{[mr_0]\}$ and 7 choices for $\{[mr_1]\}$. This gives 91 cases! For example, J represents $\{[mr_0]\}$, $1 \leq m \leq 6$ when $\frac{13}{5} \leq r_0 < \frac{8}{3}$. In Table 4 we present $\{A, \ldots, M\}$ **vs** $\{N, \ldots, T\}$. An entry Xn means $n = [m_0 r_0] = [m_1 r_1]$ so the case is excluded. An entry such as $\{4, 7, 11, 14\}$ for K **vs** N means that no integers between 2 and 15 are duplicated and that $\{[mr_2]\} \cup \{[mr_3]\}$ must partition $\{4, 7, 11, 14\}$.

We are now down to 21 cases. However, the smallest appearing number k in an entry must be $[r_3]$ and so $[2r_3] = 2k$ or $2k+1$ must also appear. This rules out AT, for example, since 5 appears in $\{5, 9, 13\}$ but 10 and 11 do not. We are now down to seven cases and five different remainders: 5, 8, 11, 14; 4, 8, 11, 14; 4, 6, 9, 12, 15; 4, 7, 9, 12, 15; and 4, 6, 9, 12, 14. In the first, we must have $[r_2] = 5$ and $[2r_2] = 11$, so $[r_3] = 8$, but 14 is uncovered. In the second, $[r_2] = 4$ so $[2r_2] = 8$ and $[r_3] = 11$, but $[3r_3] = 12$ or 13 is already taken. In the third and fourth, $[r_2] = 4$ and $[2r_2] = 9$ so $[3r_2] = 13$ or 14, which are also already taken. Finally, there is no

Table 3

A: 2, 4, 6, 8, 10, 12, 14	H: 2, 4, 7, 9, 12, 14	N: 3, 6, 9, 12, 15
B: 2, 4, 6, 8, 10, 12, 15	I: 2, 5, 7, 10, 12, 15	O: 3, 6, 9, 12
C: 2, 4, 6, 8, 10, 13, 15	J: 2, 5, 7, 10, 13, 15	P: 3, 6, 9, 13
D: 2, 4, 6, 8, 11, 13, 15	K: 2, 5, 8, 10, 13	Q: 3, 6, 10, 13
E: 2, 4, 6, 9, 11, 13, 15	L: 2, 5, 8, 11, 13	R: 3, 7, 10, 14
F: 2, 4, 6, 9, 11, 13	M: 2, 5, 8, 11, 14	S: 3, 7, 11, 14
G: 2, 4, 7, 9, 11, 14		T: 3, 7, 11, 15

Table 4

	N	O	P	Q	R	S	T
A	$X6$	$X6$	$X6$	$X6$	$X10$	$X14$	5, 9, 13
B	$X6$	$X6$	$X6$	$X6$	$X10$	5, 9, 13	$X15$
C	$X6$	$X6$	$X6$	$X6$	$X10$	5, 9, 12	$X15$
D	$X6$	$X6$	$X6$	$X6$	5, 9, 12	$X11$	$X11$
E	$X6$	$X6$	$X6$	$X6$	5, 8, 12	$X11$	$X11$
F	$X6$	$X6$	$X6$	$X6$	5, 8, 12, 15	$X11$	$X11$
G	$X9$	$X9$	$X9$	5, 8, 12, 15	$X7$	$X7$	$X7$
H	$X9$	$X9$	$X9$	5, 8, 11, 14	$X7$	$X7$	$X7$
I	$X12$	$X12$	4, 8, 11, 14	$X10$	$X7$	$X7$	$X7$
J	$X15$	4, 8, 11, 14	$X13$	$X10$	$X7$	$X7$	$X7$
K	4, 7, 11, 14	4, 7, 11, 14, 15	$X13$	$X10$	$X10$	4, 6, 9, 12, 15	4, 6, 9, 12, 14
L	4, 7, 10, 14	4, 7, 10, 14, 15	$X13$	$X13$	4, 6, 9, 12, 15	$X11$	$X11$
M	4, 7, 10, 13	4, 7, 10, 13, 15	4, 7, 10, 12, 15	4, 7, 9, 12, 15	$X14$	$X11$	$X11$

contradiction in KT: 4, 6, 9, 12, 14 and we must have $[r_2]=4$, $[2r_2]=9$, $[3r_2]=14$, $[r_3]=6$, $[2r_3]=12$. Recalling what K and T stand for, $\{[mr_j]\}$, up to 15, are $\{2,5,8,10,13\}$, $\{3,7,11,15\}$, $\{4,9,14\}$, $\{6,12\}$. Thus $[6r_0]=15$ or 16; since $[4r_1]=15$, $[6r_0]=16$, but then no $[mr_j]$ can equal 17. Therefore $D-3\leqslant 16$ or $D\leqslant 19$.

We are not done yet! However, we need only check partitions of $D\leqslant 19$ into relatively prime parts with $\frac{1}{2}D>a_0>\frac{1}{3}D>a_1>\frac{1}{4}D$ if $D\geqslant 10$. For each partition we only have to check that $\sum\{ka/D\}=2$ for $2\leqslant k\leqslant\frac{1}{2}D$. This is easily done by hand and we omit the details, which lead to (5.10). □

We should point out that Beatty's Problem—see [1], and [5], for a survey—says that $\{[nx]\}$ and $\{[ny]\}$ partition the positive integers if and only if $x>0$, $y>0$, $x^{-1}+y^{-1}=1$ and both are irrational. Uspensky [18] proved that $\{[nx_1]\},\ldots,\{[nx_k]\}$ can partition $\mathbb{Z}^+$ only if $k\leqslant 2$, see Graham [6] for a short proof.

6. Open questions and related matters

The principal question raised by this paper has already been stated: If $\{\underline{\lambda}_1,\ldots,\underline{\lambda}_n\}\in A_m$ is a good set, does there exist $T\in S(m,n)$ with $M_T=[\lambda_{ij}]$? It would be desirable to characterize submodules $H\subseteq A_m$ with a specified good part and to find conditions on $\underline{\lambda}_1,\ldots,\underline{\lambda}_n$ so that $\underline{\lambda}\in G(\langle\underline{\lambda}_1,\ldots,\underline{\lambda}_n\rangle)$ implies $\lambda_{ij}>0$. Let $r(m)=\sup\{\text{rank}(T)\colon T\in S(m,n)\}$; we know that $r(2)=r(3)=1$ and $r(7)\geqslant 2$. How does $r(m)$ behave? For that matter, is there a good algorithm for determining rank(H)? Another numerical question involves $f(m,n)$, the number of distinct configurations in $S(m,n)$. All that is known is $f(2,n)$ from Theorem 3.9, $f(3,1)=7$ from Theorem 5.9 and $f(m,n)<\infty$ from Theorem 5.12.

One further generalization we have not discussed is relaxing the condition that interior points be off the boundary of $\dot{T}$. Theorem 3.1 applies with minor alterations; if there are k edge points and $n-k$ interior points, then $\lambda_{ij} \in \mathbb{Z}/(2n+1-k)$ by Pick's Theorem. Theorem 5.12 also applies since any boundary point $\underline{w}$ can be thought of as interior to an m'-dimensional face. Thus if $\mathrm{BC}(\underline{w}) = (a_1, \ldots, a_{m'}, 0, \ldots, 0)/D$ and $D > D(m', n)$, then $\underline{w}$ generates n points on that face. As remarked after the proof of Theorem 5.12, there is some literature on vol T; see, e.g. [9] and [20].

We now turn to a family of related questions. It is perhaps best to start with a question posed in 1979 by Kimberling [11].

$$\text{For which } a, b, c, d \text{ does } [ka] + [kb] = [kc] + [kd], \text{ for all } k? \tag{6.1}$$

Before the combined solution [2] of many authors could be printed, Noordzij solved it as well [14], crediting Heath-Brown with the problem. Actually, Pomerance had told Heath-Brown about [11].

Theorem 6.2 (Noordzij, [14]). *If* (6.1) *holds, then at least one of* $a+b$, $a-c$ *or* $a-d$ *is an integer.*

Since the condition in (6.1) implies $a+b=c+d$, it is equivalent to:

$$\text{For which } a, b, c, d \text{ does } \{ka\} + \{kb\} = \{kc\} + \{kd\}, \text{ for all } k? \tag{6.3}$$

Thus, Theorem 6.2 is equivalent to Theorem 6.4.

Theorem 6.4. *If* (6.3) *holds, then* $\{\{a\}, \{b\}\} = \{\{c\}, \{d\}\}$, *or* $\{a\} + \{b\} = \{c\} + \{d\} = 1$.

We now show that Theorem 6.4 implies Lemma 4.4. Suppose $\sum \{ka_j/D\} = 2$ for $1 \leqslant k \leqslant D-1$, then, as before, $(a_j, D) = 1$ so that $\{ka_j/D\} = 0$ if $k \equiv 0 \pmod D$. If $k \not\equiv 0 \pmod D$, then $\{k(D-a_j)/D\} = 1 - \{ka_j/D\}$, hence (6.3) holds for $(a, b, c, d) = (a_0, a_1, D-a_2, D-a_3)/D$. By Theorem 6.4, $a_0 + a_j \equiv 0 \pmod D$ for some j. The proofs in [19], [2] and [14] are neither direct nor short. We outline an alternative proof which seems to be simpler than either. (It is, however, not as strong as Theorem 6.2.)

By considering $t(a_0, a_1, a_2, a_3)/D$, where $t \equiv a_0^{-1} \pmod D$ we may assume that $a_0 = 1$. Suppose $\sum \{ka_j/D\} = \sum \{2ka_j/D\} = 2$, then exactly two of the $\{ka_j/D\}$'s are $\geqslant \frac{1}{2}$. For $1 \leqslant k < \frac{1}{2}D$, let $h_D^+(a)$ (resp. $h_D^-(a)$) denote the number of $\{ka/D\}$'s which are greater than (resp. less than) $\frac{1}{2}$ and let $\Delta_D(A) = h_D^+(a) - h_D^-(a)$. Then $\sum \Delta_D(a_j) = 0$, and since $\Delta_D(1) = -[\frac{1}{2}(D-1)]$, $\Delta_D(a_j)$ must be large for some j $1 \leqslant j \leqslant 3$. Since $\Delta_D(a) = -\Delta_D(D-a)$, we assume $a < \frac{1}{2}D$ and partition the multiples ka/D into 'runs' with constant integer part. Each full run contributes -1, 0 or 1 to $\sum \Delta_D(a)$, so $|\Delta_D(a)|$ is bounded by the number of runs plus the size of a partial run. By a delicate casing out, it can be proved that $|\Delta_D(a)| < \frac{1}{3} |\Delta_D(1)|$ for $a \not\equiv 1$, $D-1$ unless $D = 6s+1$, $r = 3$, $2s$, $4s+1$, $6s-2$ or $D = 6s+2$, $r = 3$,

$2s+1$, $4s+1$, $6s-1$, where there is equality. Ultimately, this implies that some $a_j = D-1$.

More generally, one can ask for conditions on $\underset{\sim}{\lambda} = (a_0, \ldots, a_m)/D$ which imply that $g(\underset{\sim}{\lambda}) = 0$. When $m > 3$, this means that $2 \leq \sum \{ka_j/D\} \leq m-1$, so we cannot easily generalize the proof of Theorem 5.9. There seems to be a wider class of examples; it is not necessary that the a_j's form groups which sum to $0 \pmod D$. For example, $g((2, 3, 4, 5, 12)/13) = 0$. Indeed, if $p > \frac{1}{2}(m^2 - m - 2)$ is a prime, then $g(\underset{\sim}{\lambda}) = 0$ for

$$\underset{\sim}{\lambda} = (2, 3, 4, \ldots, m-1, p - (m^2 - m - 4)/2, p-1)/p.$$

The key to the proof of this is the following lemma, whose proof we omit.

Lemma 6.5. *Suppose $a, a+b, \ldots, a+lb$ is an arithmetic progression with $l \geq 3$ and $\sum \{a+kb\} < 1$. Then $\{a\}, \{a+b\}, \ldots, \{a+lb\}$ is also an arithmetic progression.*

(This lemma is false for $l = 2$ as 1/13, 8/13, 15/13 illustrates.) Examples such as these, when combined with the construction of Theorem 4.5, indicate that the classification of higher fundamental simplices will not be easy.

Finally, we can jettison the geometric rationale and ask the following question. For fixed m, which sequences of integers $\{r_k\}$ can be written $r_k = \sum_{j=0}^{m} \{k\lambda_j\}$ for some $(\lambda_0, \ldots, \lambda_m)$? Two immediate necessary conditions are $0 \leq r_k \leq m$ and $r_k + r_{k'} \leq r_{k+k'}$. These are not sufficient as $r_1 = 1$, $r_k = 2$, $2 \leq k \leq 17$ is not achievable for $m = 3$. (The question seems to be open whether the sequences are finite or infinite.) An argument similar to, but easier than, that proving Theorem 5.9 shows that $r_1 = 2$, $r_2 = 1$, $r_k = 2$, $2 \leq k \leq s$, $m = 3$, is achievable only for $s \leq 16$, with $\underset{\sim}{\lambda} = (0.68, 0.63, 0.58, 0.11)$ as one solution. Alternatively, let $s(m)$ be the largest s so that $r_1 = 1$, $r_k \geq 2$, $2 \leq k \leq s$ is achievable for a given m. How does $s(m)$ grow? Heuristically, the $\{k\lambda_j\}$'s may be thought of as linearly independent random variables with a uniform distribution on $[0, 1]$. In this case, the 'probability' that $\sum_{j=0}^{m} \{k\lambda_j\} = 1$ is $1/m!$. (For $\underset{\sim}{\lambda} = (5, 7, 11, 303, 1984)/2310$, $g(\underset{\sim}{\lambda}) = 105$ and $2310/24 \simeq 96$.)

Acknowledgments

I gratefully acknowledge the advice of Professor Paul Erdös for showing me how Theorem 5.12 'ought' to be proved, and then proving it. I would also like to thank the Geometric Potpourri Seminar for their suggestions, especially Ralph Alexander and Dick Bishop, R. Balasubramanian for the estimate of $\phi_2(m)$ and Robin Sahner for computer assistance. I also thank Doug Hensley for calling my attention to [9] and [20]. I finally thank the referees for their helpful suggestions.

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