NORTH-HOLLAND

Some Constructions of Spherical 5-Designs

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ABSTRACT

Spherical designs were introduced by Delsarte, Goethals, and Seidel in 1977. A *spherical t-design in* $\mathbf{R}^n$ is a finite set $X \subset S^{n-1}$ with the property that for every polynomial p with degree $\leqslant t$, the average value of p on X equals the average value of p on S^{n-1}. This paper contains some existence and nonexistence results, mainly for spherical 5-designs in $\mathbf{R}^3$. Delsarte, Goethals, and Seidel proved that if X is a spherical 5-design in $\mathbf{R}^3$, then $|X| \geqslant 12$ and if $|X| = 12$, then X consists of the vertices of a regular icosahedron. We show that such designs exist with cardinality 16, 18, 20, 22, 24, and every integer $\geqslant 26$. If X is a spherical 5-design in $\mathbf{R}^n$, then $|X| \geqslant n(n+1)$; if $|X| = n(n+1)$, then X has been called *tight*. Tight spherical 5-designs in $\mathbf{R}^n$ are known to exist only for $n = 2, 3, 7, 23$ and possibly $n = u^2 - 2$ for odd $u \geqslant 7$. Any tight spherical 5-design in $\mathbf{R}^n$ must consist of $n(n+1)/2$ antipodal pairs of points. We show that for $n \geqslant 3$, there are no spherical 5-designs in $\mathbf{R}^n$ consisting of $n(n+1)/2 + 1$ antipodal pairs of points.

1. INTRODUCTION AND OVERVIEW

The natural way to place m points equally on a circle is to use the vertices of a regular m-gon. Unfortunately, there is no natural way to place m points equally on the n-sphere $S^{n-1} \subset \mathbf{R}^n$ for most (m, n) with $n \geqslant 3$. Many

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interesting geometric investigations have arisen from attempts to accommodate this situation. One of the most beautiful approaches is that of the spherical design, developed by Delsarte, Goethals, and Seidel in a powerful series of papers in the late 1970s and early 1980s [D1, G1, G2, S1]. Other general references include the surveys of Bannai (e.g. [B3]) and Seidel [S2], parts of the book [C1] by Conway and Sloane, and papers by Lyubich and Vaserstein [L2] and the author [R1].

A *spherical t-design in* $\mathbf{R}^n$ is a finite set $X = \{\xi_k\} \subset S^{n-1}$ with the property that for all polynomials $p(x_1, \dots, x_n)$ of degree $\leqslant t$, the average value of p on X is equal to the average value of p on S^{n-1}. Write $\xi_k = (\xi_{k1}, \dots, \xi_{kn})$ and let μ denote the normalized rotation-invariant Lebesgue measure satisfying $\mu(S^{n-1}) = 1$. Then this assertion becomes

$$\frac{1}{|X|} \sum_{k=1}^{|X|} p(\xi_{k1}, \dots, \xi_{kn}) = \int \cdots \int_{S^{n-1}} p(u)\, d\mu. \tag{1.1}$$

The union of two spherical t-designs is also a spherical t-design. The rotational invariance of μ implies that if X is a spherical t-design, then so is $\rho(X)$ for any rotation ρ of S^{n-1}.

The properties of spherical designs described in the next two paragraphs were all proved by Delsarte, Goethals, and Seidel in their seminal paper [D1]. For other detailed references see e.g. [R1, p. 122]. If $X \subset \mathbf{R}^n$ is a spherical t-design, then

$$|X| \geqslant \binom{n+s-1}{n-1} + \binom{n+s-2}{n-1} \qquad \text{if } t = 2s, \tag{1.2a}$$

$$|X| \geqslant 2\binom{n+s-1}{n-1} \qquad \text{if } t = 2s+1. \tag{1.2b}$$

A spherical t-design X of minimal cardinality (with respect to (1.2)) is called *tight*. A tight spherical $(2s+1)$-design X must be *antipodal*; that is, $\xi \in X$ implies $-\xi \in X$. (An antipodal spherical $2s$-design is automatically a spherical $(2s+1)$-design and hence cannot be tight.) There is a small corpus of known tight spherical $(2s+1)$-designs in $\mathbf{R}^n$, which exhausts the possibilities, except for $(2s+1, n) = (5, u^2-2)$ for odd integers $u \geqslant 7$ and $(2s+1, n) = (7, 3v^2-4)$ for integers $v \geqslant 4$, in which the questions of existence are open.

For $X = \{\xi_k\} \subset S^{n-1}$, let $A(X) = \{\xi_j \cdot \xi_k : j \neq k\}$. If $X = \{\xi_k\}$ is a tight spherical $(2s+1)$-design, then $|A(X)| = s+1$, in fact, it consists of -1

(from the antipodal pairs) and the s roots of the associated Gegenbauer polynomial. For $2s + 1 = 5$, these roots are $\pm\sqrt{1/(n+2)}$; for $2s + 1 = 7$, they are $0, \pm\sqrt{3/(n+4)}$. This property is related to the original context of spherical designs, spherical codes. If $Y = \{\pm \eta_k\} \subset S^{n-1}$ is antipodal and $|A(Y) \setminus \{-1\}| \leqslant s$, then $|Y| \leqslant 2\binom{n+s-1}{n-1}$. If $|Y| = 2\binom{n+s-1}{n-1}$, then Y is a tight spherical $(2s+1)$-design. Thus the tight spherical $(2s+1)$-designs have maximal cardinality (with respect to a limited number of distinct angles) and minimal cardinality (with respect to being a spherical design).

It is not too hard to show that the vertices of a regular n-gon in the plane form a spherical t-design for $n \geqslant t + 1$, which is tight for $n = t + 1$. (A proof follows Lemma 2.1.) In this sense, tight spherical designs generalize regular polygons in the plane. Hong [H4] proved in 1982 that if X is a spherical t-design in $\mathbf{R}^2$ and $|X| = n \leqslant 2t + 1$, then X must consist of the vertices of a regular n-gon. Hong's Theorem suggests that there might not be too many spherical t-designs which are "snug," if not actually tight. (We do not wish to make "snug" precise; the proposed snug spherical t-designs in this paper have one more antipodal pair than a tight spherical t-design.)

In the other direction, Seymour and Zaslavsky proved in 1984 [S3] that for each fixed (n, t), there exists $M = M'(n, t)$ (in the later notation of Bajnok [B2]) so that there exist spherical t-designs in $\mathbf{R}^n$ of *every* cardinality $\geqslant M$. Since regular n-gons are spherical t-designs for $n \geqslant t + 1$, $M'(2, t) = t + 1$.

Many known spherical designs are sets of points which are familiar from their other combinatorial or geometric properties. The *strength* of $X \subset S^{n-1}$ is the largest t so that X is a spherical t-design. (By (1.1), every such X has strength $\geqslant 0$; by symmetry, an antipodal X has odd strength $\geqslant 1$). The strengths of all regular polytopes in $\mathbf{R}^n$ are analyzed in [G2]. Another approach is to use unions of the orbits of points under finite subgroups of the orthogonal group. In this paper, we construct designs in $\mathbf{R}^3$ from the union of regular polygons placed at varying "latitudes" on the sphere. This sort of construction has previously been used by Bajnok [B1, B2] and Hardin and Sloane [H2].

We shall use an equivalent criterion for spherical designs discovered by Goethals and Seidel [G2], but unstressed there. (See [R1, p. 114] for a proof.) A set $X = \{\xi_k\} \subset S^{n-1}$ is a spherical t-design if and only if the following identities hold when $0 \leqslant 2s, 2\bar{s} + 1 \leqslant t$:

$$\frac{1}{|X|}\sum_{k=1}^{|X|}(\xi_{k1}x_1 + \cdots + \xi_{kn}x_n)^{2s} = \left(\prod_{j=0}^{s-1}\frac{1+2j}{n+2j}\right)(x_1^2 + \cdots + x_n^2)^s, \tag{1.3a}$$

$$\frac{1}{|X|}\sum_{k=1}^{|X|}(\xi_{k1}x_1+\cdots+\xi_{kn}x_n)^{2\bar{s}+1}=0. \tag{1.3b}$$

In fact, it suffices to verify that the identities in (1.3) hold for $\{2s, 2\bar{s}+1\} = \{t-1, t\}$. (These imply the others upon repeated application of the Laplacian to both sides.) If $X = \{\pm\xi_k\}$ is antipodal, then (1.3b) is automatic and (1.3a) need only be checked for $\{\xi_k\}$. It follows from (1.3) that the polynomials $\Sigma_k(\xi_k \cdot)^j$ (cf. (3.2)) depend only on $|X|$ and $\Sigma_j x_j^2$ for a spherical t-design X in $\mathbf{R}^n$. Thus, if $X = Y \cup Z$ is a spherical t-design in $\mathbf{R}^n$ and $Y \subset V$, where V is a k-dimensional subspace of $\mathbf{R}^n$ and if ρ is any rotation of V, then $\rho(Y) \cup Z$ is also a spherical t-design in $\mathbf{R}^n$. This observation extends to affine subspaces as well, and is implicit in the constructions of Section 2.

Formulas of the shape (1.3a), without the assumption $|\xi_k| = 1$ or the constant on right-hand side, are familiar objects in number theory and functional analysis. They were essential to Hilbert's solution of Waring's problem, and are equivalent to the isometric embedding of l_2^n into $l_{2s}^{|X|}$. These equivalences were discovered independently by Lyubich and Vaserstein [L2] and the author [R1], and the reader is directed to these references for more details.

In this paper, we concentrate our attention of spherical 5-designs in $\mathbf{R}^3$, although the methods undoubtedly generalize to $(2s+1)$-designs for $s \geqslant 3$. In section 2, we construct spherical 5-designs in $\mathbf{R}^3$ from building blocks of regular m-gons ($m \geqslant 6$) which are either equatorial in S^2 or come in pairs at symmetric latitudes. We may also use an antipodal pair of regular pentagons or the north and south poles together. After introducing some technical machinery (Theorems 2.9, 2.12), we use these sets to construct spherical 5-designs in $\mathbf{R}^3$ with every even cardinality $\geqslant 18$ and every cardinality $\geqslant 26$ (Corollary 2.19). (Compare with the value $M'(3,5) = 72$ from [B2].) We also show that an explicit 16-point spherical 4-design from [H2] is actually a 5-design. (Since the first version of this paper was circulated, Hardin and Sloane [H3] have constructed spherical 5-designs in $\mathbf{R}^3$ with 23 and 25 points.)

The methods in Section 2 generalize in a straightforward way to spherical designs of greater strength and a greater number of dimensions. We do not develop these ideas here, except to conjecture that there exist spherical $(2s+1)$-designs in $\mathbf{R}^3$ with every cardinality $\geqslant (2s+1)^2+1$ (Conjecture 2.20).

Very recently, Korevaar and Meyers [K1] have proved that there exist spherical t-designs in $\mathbf{R}^3$ with $\mathscr{O}(t^3)$ points and conjecture that $\mathscr{O}(t^2)$ should be possible. Hardin and Sloane [H3] present a conjectured sequence of spherical t-designs in $\mathbf{R}^3$ with cardinality $(t^2/2)(1+o(1))$ and have verified it numerically for $t \leqslant 21$. By comparison, the lower bounds in (1.2) reduce to $(t^2/4)(1 + o(1))$. Conjecture 2.20 would imply that there exist spherical t-designs in $\mathbf{R}^3$ with *every* cardinality $\geqslant t^2(1 + o(1))$.

Another mathematical tool is needed for the negative results—the catalecticant. In Section 3, we recall just enough algebraic machinery from [R1] to make the definitions comprehensible. If $p(x_1, \dots, x_n)$ has degree $2s$, then one can define the associated *Hankel form* $\mathscr{H}_p$, which is a quadratic form in $\binom{n+s-1}{n-1}$ variables. The determinant of the matrix of $\mathscr{H}_p$ is called its *catalecticant* and denoted $C(p)$. If p is a sum of r $2s$-th powers of linear forms, then $\mathscr{H}_p$ is a positive semidefinite quadratic form whose rank is at most r. In particular, if p is a sum of fewer than $\binom{n+s-1}{n-1}$ $2s$-th powers of linear forms in n variables, then $C(p) = 0$. In Section 4, we first reprise the derivation of the icosahedron as the unique tight spherical 5-design in $\mathbf{R}^3$ from [R1], and then combine arguments from that paper with some determinantal identities to show that there is no spherical 5-design in $\mathbf{R}^n$ for $n \geqslant 3$ consisting of $\binom{n+1}{2} + 1$ pairs of antipodal points (Theorems 4.6, 4.20). The final twist in the last proof is the observation that $n(n^2 - 4)$ is not a square for $n \geqslant 3$.

2. THERE ARE MANY SPHERICAL 5-DESIGNS IN $\mathbf{R}^3$

In order to establish the assertion made in the heading of this section, we first need a lemma, the nontrivial portion of which can be found in [R1, (8.29), Thm. 9.5]. The earliest citation we have found for these identities is [F1], written in 1957. The implicit generalization in this lemma to sums with arbitrary exponents can also be found in [D2].

LEMMA 2.1. *Suppose $m > 2s, 2\bar{s} + 1$. Then for any real θ,*

$$\sum_{k=0}^{m-1} \left(\cos\left(\theta + k\frac{2\pi}{m} \right) x_1 + \sin\left(\theta + k\frac{2\pi}{m} \right) x_2 \right)^{2s} = \frac{m}{2^{2s}} \binom{2s}{s} \left(x_1^2 + x_2^2 \right)^s, \tag{2.2a}$$

$$\sum_{k=0}^{m-1} \left(\cos\left(\theta + k\frac{2\pi}{m} \right) x_1 + \sin\left(\theta + k\frac{2\pi}{m} \right) x_2 \right)^{2\bar{s}+1} = 0. \tag{2.2b}$$

Proof. Let $\zeta = \exp(2\pi i/m)$ and observe that

$$\cos\left(\theta + k\frac{2\pi}{m}\right)x_1 + \sin\left(\theta + k\frac{2\pi}{m}\right)x_2$$

$$= \frac{x_1 - ix_2}{2}e^{i\theta}\zeta^k + \frac{x_1 + ix_2}{2}e^{-i\theta}\zeta^{-k}.$$

Then by reversing the order of summation, and using the fact that $\sum_{k=0}^{m-1}\zeta^{rk}$ vanishes (unless m divides r, in which case it equals m), we find that, more generally,

$$\sum_{k=0}^{m-1}\left(\cos\left(\theta + k\frac{2\pi}{m}\right)x_1 + \sin\left(\theta + k\frac{2\pi}{m}\right)x_2\right)^N$$

$$= \sum_{j=0}^{N}\binom{N}{j}\left(\frac{x_1 - ix_2}{2}e^{i\theta}\right)^j\left(\frac{x_1 + ix_2}{2}e^{-i\theta}\right)^{N-j}\left(\sum_{k=0}^{m-1}\zeta^{(j-(N-j))k}\right)$$

$$= m\sum_{m|2j-N}\binom{N}{j}\left(\frac{x_1 - ix_2}{2}e^{i\theta}\right)^j\left(\frac{x_1 + ix_2}{2}e^{-i\theta}\right)^{N-j}. \tag{2.3}$$

The last sum in (2.3) is taken over all multiples of m in

$$\{-N, -(N-2), \dots, (N-2), N\}.$$

If $N = 2s < m$, the only such multiple of m is 0, and occurs when $j = s$, giving the sum asserted in (2.2a). If $N = 2\bar{s} + 1 < m$, then no such j exists, and the sum is vacuous. ■

Since $2^{-2s}\binom{2s}{s} = \frac{1}{2}\frac{3}{4}\cdots\frac{2s-1}{2s}$, Lemma 2.1 and (1.3) give another proof that a regular m-gon is a spherical t-design in the plane if $m > t$.

Suppose now that $|c| < 1$ and $s = \sqrt{1-c^2}$ and suppose $Y = \{\eta_k\} = \{(\cos\theta_k, \sin\theta_k)\} \subset S^1$. We can suspend Y at the parallel of latitude $z = c$: let $(sY, c) = \{(s\cos\theta_k, s\sin\theta_k, c)\}$. In case $c = 0$, $(Y, 0)$ will be called *equatorial*; it is the standard embedding of Y into S^2. We use the notation $\{m, \theta\}$ to denote the regular m-gon for which $\theta_k = \theta + 2\pi k/m$ (writing $\{m\}$ when θ is unimportant).

The next two lemmas are computational, and show how pairs of $\{m\}$'s suspended at $\pm c$ can be used as building blocks for making spherical 5-designs. These combine with (1.3) to generalize in higher dimensions, but we shall not pursue this idea here. Note that none of the sums depends on θ.

LEMMA 2.4. *If $m \geqslant 6$, then*

$$\sum_{k=0}^{m-1}\left(s\cos\left(\theta+\frac{2\pi k}{m}\right)x+s\sin\left(\theta+\frac{2\pi k}{m}\right)y+cz\right)^4$$

$$=\frac{3m}{8}s^4(x^2+y^2)^2+\binom{4}{2}\frac{m}{2}c^2s^2(x^2+y^2)z^2+mc^4z^4, \quad \text{(a)}$$

$$\sum_{k=0}^{m-1}\left(s\cos\left(\theta+\frac{2\pi k}{m}\right)x+s\sin\left(\theta+\frac{2\pi k}{m}\right)y+cz\right)^5$$

$$=\binom{5}{1}\frac{3m}{8}cs^4(x^2+y^2)^2z+\binom{5}{3}\frac{m}{2}c^3s^2(x^2+y^2)z^3+mc^5z^5. \quad \text{(b)}$$

Proof. Expand each sum in powers of z. Since $m \geqslant 6$, Lemma 2.1 is operative: the sums of the odd powers of $\cos(\theta + 2\pi k/m)x + \sin(\theta + 2\pi k/m)y$ all vanish, and the sums of the 0-th, 2-nd and 4-th powers are m, $m\frac{1}{2}(x^2 + y^2)$, and $m\frac{1}{2}\,\frac{3}{4}(x^2 + y^2)^2$, respectively. ■

LEMMA 2.5.

$$\sum_{k=0}^{4}\left(s\cos\left(\theta+\frac{2\pi k}{5}\right)x+s\sin\left(\theta+\frac{2\pi k}{5}\right)y+cz\right)^4$$

$$=\tfrac{15}{8}s^4(x^2+y^2)^2+15c^2s^2(x^2+y^2)z^2+5c^4z^4. \quad \text{(a)}$$

$$\sum_{k=0}^{4}\left(s\cos\left(\theta+\frac{2\pi k}{5}\right)x+s\sin\left(\theta+\frac{2\pi k}{5}y\right)+cz\right)^5$$

$$+\sum_{k=0}^{4}\left(-s\cos\left(\theta+\frac{2\pi k}{5}\right)x-s\sin\left(\theta+\frac{2\pi k}{5}y\right)-cz\right)^5=0. \quad \text{(b)}$$

Proof. Part (a) follows in the same way as the proof of the last lemma, and is consistent with Lemma 2.4(a) on taking $m = 5$. Although Lemma 2.1 does not apply to the coefficient of s^5 in (b), the summands cancel pairwise. ∎

We now introduce an auxiliary case of designs, whose generalizations are apparent. A finite set $Z = \{\zeta_k\} \subset S^2$ is a *circular* 5-*design* if there exist A, B and C such that

$$\sum_{k=1}^{|Z|} (\zeta_{k1}x + \zeta_{k2}y + \zeta_{k3}z)^4$$

$$= A(x^2 + y^2)^2 + B(x^2 + y^2)z^2 + Cz^4, \qquad (2.6a)$$

$$\sum_{k=1}^{|Z|} (\zeta_{k1}x + \zeta_{k2}y + \zeta_{k3}z)^5 = 0. \qquad (2.6b)$$

By Lemma 2.4, $(s\{m, \theta_1\}, c) \cup (s\{m, \theta_2\}, -c)$ is a circular 5-design for $m \geq 6$ and any θ_1, θ_2. By Lemma 2.5, $(s\{5, \theta_1\}, c) \cup (s\{5, -\theta_1\}, -c)$ is a circular 5-design. While keeping in mind that the pairs of $\{5\}$'s (only) must be rotated to be antipodal, we shall write these pairs of $\{m\}$'s for $m \geq 5$ as $(s\{m\}, \pm c)$ for short. The sets $(\{m\}, \pm 0)$ consist of two equatorial $\{m\}$'s rotated so that the points are distinct.

Two other types of circular 5-designs are $\{(0,0,\pm 1)\}$ and $(\{m, \theta\}, 0)$ for $m \geq 6$. (Any spherical 5-design in $\mathbf{R}^2$, when embedded in $\mathbf{R}^3$ on the first two coordinates, is a circular 5-design in $\mathbf{R}^3$.) These of course do not exhaust the arsenal of potential circular 5-designs.

LEMMA 2.7. *If Z is a circular* 5-*design, then* $|Z| = \frac{8}{3}A + \frac{2}{3}B + C$.

Proof. Since $Z \subset S^2$, a comparison with (2.6a) shows that

$$|Z| = \sum_{k=1}^{|Z|} (\zeta_{k1}^2 + \zeta_{k2}^2 + \zeta_{k3}^2)^2$$

$$= \sum_{k=1}^{|Z|} \zeta_{k1}^4 + \zeta_{k2}^4 + \zeta_{k3}^4 + 2\zeta_{k1}^2\zeta_{k2}^2 + 2\zeta_{k1}^2\zeta_{k3}^2 + 2\zeta_{k2}^2\zeta_{k3}^2$$

$$= A + A + C + 2\frac{2A}{6} + 2\frac{B}{6} + 2\frac{B}{6}.$$

(A more insightful proof, in the spirit of [R1], would take the "inner product" of both sides of (2.6a) with $(x^2 + y^2 + z^2)^2$.) ■

Suppose Z is a circular 5-design satisfying (2.6). Define the *discrepancy* of Z to be

$$\Delta(Z) = \left(B - \tfrac{2}{5}|Z|, C - \tfrac{1}{5}|Z|\right). \tag{2.8}$$

Note also that if $\{Z_k\}$ is a finite set of circular 5-designs, then $\bigcup_k Z_k$ is also a circular 5-design and $\Delta(\bigcup_k Z_k) = \sum_k \Delta(Z_k)$.

THEOREM 2.9. *A circular 5-design Z is a spherical 5-design if and only if $\Delta(Z) = (0, 0)$.*

Proof. By (1.3), Z is a spherical 5-design if and only if

$$\frac{1}{|Z|}\sum_{k=1}^{|Z|}\left(\zeta_{k1}x + \zeta_{k2}y + \zeta_{k3}z\right)^4 = \tfrac{1}{3}\tfrac{3}{5}\left(x^2 + y^2 + z^2\right)^2, \tag{2.10a}$$

$$\sum_{k=1}^{|Z|}\left(\zeta_{k1}x + \zeta_{k2}y + \zeta_{k3}z\right)^5 = 0. \tag{2.10b}$$

Suppose Z is a circular 5-design. Then (2.10b) is automatic and (2.10a) holds if and only if $(A, B, C) = (|Z|/5, 2|Z|/5, |Z|/5)$. This implies $\Delta(Z) = (0, 0)$ by (2.8); conversely, if $\Delta(Z) = 0$, then $(B, C) = (2|Z|/5, |Z|/5)$, and by Lemma 2.7, $A = \frac{3}{8}|Z| - \frac{1}{4}B - \frac{3}{8}C = \frac{3}{8}|Z| - \frac{1}{10}|Z| - \frac{3}{40}|Z| = \frac{1}{5}|Z|$, as required. ■

We wish to find sets of circular 5-designs whose discrepancies cancel. A lemma is helpful.

LEMMA 2.11.

(a) *If $m \geqslant 5$, then $\Delta(s\{m\}, \pm c) = (6ms^2c^2 - \frac{4}{5}m, 2mc^4 - \frac{2}{5}m)$.*
(b) *If $m \geqslant 6$, then $\Delta(\{m\}, 0) = (-\frac{2}{5}m, -\frac{1}{5}m)$.*
(c) *If $Z = \{(0, 0, \pm 1)\}$, then $\Delta(Z) = (-\frac{4}{5}, \frac{8}{5})$.*

Proof. For (a), the circular 5-design $Z = (s\{m\}, \pm c)$ consists of two $\{m\}$'s, thus $|Z| = 2m$, and by Lemmas 2.4 or 2.5, $(A, B, C) = 2(\frac{3}{8}ms^4, 3ms^2c^2, mc^4)$ in (2.6). (As a check of Lemma 2.7, $\frac{8}{3}A + \frac{2}{3}B + C = 2ms^4 + 4mc^2s^2 + 2mc^4 = 2m(c^2+s^2)^2 = 2m$.) For (b), take one $\{m\}$ in Lemma 2.4, with $c = 0$, so $(A, B, C) = (\frac{3}{8}m, 0, 0)$. For (c), $m = 2$ and $(A, B, C) = (0, 0, 2)$ trivially. ∎

THEOREM 2.12. *In the following constructions, assume* $m_0 \geqslant 6$ *and* $m_j \geqslant 5$ *for* $j \geqslant 1$.

(a) $Z_a = \bigcup_{j=1}^{r}(s_j\{m_j\}, \pm c_j)$ *is a spherical* 5-*design if and only if*

$$\sum_{j=1}^{r} m_j c_j^2 = \frac{1}{3}\sum_{j=1}^{r} m_j, \qquad \sum_{j=1}^{r} m_j c_j^4 = \frac{1}{5}\sum_{j=1}^{r} m_j. \tag{2.13a}$$

(b) $Z_b = (\{m_0\}, 0) \cup \bigcup_{j=1}^{r}(s_j\{m_j\}, \pm c_j)$ *is a spherical* 5-*design if and only if*

$$\sum_{j=1}^{r} m_j c_j^2 = \frac{1}{6}m_0 + \frac{1}{3}\sum_{j=1}^{r} m_j, \qquad \sum_{j=1}^{r} m_j c_j^4 = \frac{1}{10}m_0 + \frac{1}{5}\sum_{j=1}^{r} m_j. \tag{2.13b}$$

(c) $Z_c = (0, 0, \pm 1) \cup \bigcup_{j=1}^{r}(s_j\{m_j\}, \pm c_j)$ *is a spherical* 5-*design if and only if*

$$\sum_{j=1}^{r} m_j c_j^2 = -\frac{2}{3} + \frac{1}{3}\sum_{j=1}^{r} m_j, \qquad \sum_{j=1}^{r} m_j c_j^4 = -\frac{4}{5} + \frac{1}{5}\sum_{j=1}^{r} m_j. \tag{2.13c}$$

(d) $Z_d = (0, 0, \pm 1) \cup (\{m_0\}, 0) \cup \bigcup_{j=1}^{r}(s_j\{m_j\}, \pm c_j)$ *is a spherical* 5-*design if and only if*

$$\sum_{j=1}^{r} m_j c_j^2 = -\frac{2}{3} + \frac{1}{6}m_0 + \frac{1}{3}\sum_{j=1}^{r} m_j,$$

$$\sum_{j=1}^{r} m_j c_j^4 = -\frac{4}{5} + \frac{1}{10}m_0 + \frac{1}{5}\sum_{j=1}^{r} m_j. \tag{2.13d}$$

Proof. In each case, we compute the discrepancy by Lemma 2.11 and apply Theorem 2.9; this is done in detail only for Z_a. First,

$$\Delta(Z_a) = \sum_{j=1}^{r} \left(6m_j s_j^2 c_j^2 - \tfrac{4}{5}m_j, 2m_j c_j^4 - \tfrac{2}{5}m_j\right)$$

$$= \sum_{j=1}^{r} \left(6m_j c_j^2 (1 - c_j^2) - \tfrac{4}{5}m_j, 2m_j c_j^4 - \tfrac{2}{5}m_j\right),$$

hence $\Delta(Z_a) = (0, 0)$ if and only if

$$\sum_{j=1}^{r} 6m_j c_j^2 (1 - c_j^2) = \frac{4}{5} \sum_{j=1}^{r} m_j, \qquad \sum_{j=1}^{r} 2m_j c_j^4 = \frac{2}{5} \sum_{j=1}^{r} m_j,$$

which is easily seen to reduce to (2.13a). For the record, the other discrepancies are

$$\Delta(Z_b) = \left(-\tfrac{2}{5}m_0, -\tfrac{1}{5}m_0\right) + \sum_{j=1}^{r} \left(6m_j s_j^2 c_j^2 - \tfrac{4}{5}m_j, 2m_j c_j^4 - \tfrac{2}{5}m_j\right),$$

$$\Delta(Z_c) = \left(-\tfrac{4}{5}, \tfrac{8}{5}\right) + \sum_{j=1}^{r} \left(6m_j s_j^2 c_j^2 - \tfrac{4}{5}m_j, 2m_j c_j^4 - \tfrac{2}{5}m_j\right),$$

$$\Delta(Z_d) = \left(-\tfrac{4}{5}, \tfrac{8}{5}\right) + \left(-\tfrac{2}{5}m_0, -\tfrac{1}{5}m_0\right)$$

$$+ \sum_{j=1}^{r} \left(6m_j s_j^2 c_j^2 - \tfrac{4}{5}m_j, 2m_j c_j^4 - \tfrac{2}{5}m_j\right),$$

and (2.13b–d) follow in the same way. ■

These conditions are clearly related to quadrature formulas of strength 5 on $[-1, 1]$. As an application of Theorem 2.12, take $r = 1$, $m_1 = 5$ and $c_1^2 = \frac{1}{5}$ in (2.13c). Then it is easy to check that $5\frac{1}{5} = -\frac{2}{3} + \frac{1}{3}5$ and $5\frac{1}{25} = -\frac{4}{3} + \frac{1}{5}5$. Thus the set of 12 points consisting of the north and south poles and two antipodal regular pentagons suspended at $z = \sqrt{\frac{1}{5}}$ is a spherical 5-design. This set is the regular icosahedron.

COROLLARY 2.14. *The set* $(s_1\{m_1\}, \pm c_1) \cup (s_2\{m_2\}, \pm c_2)$ *with* $5 \leqslant m_1 \leqslant m_2$ *is a spherical* 5*-design with* $2m_1 + 2m_2$ *points if either* (a) *or* (b) *holds*:

(a) $m_1 \leqslant m_2 < 5m_1$ *and*

$$c_1^2 = \frac{1}{3} + \sqrt{\frac{4m_2}{45m_1}}, \qquad c_2^2 = \frac{1}{3} - \sqrt{\frac{4m_1}{45m_2}};$$

(b) $m_1 \leqslant m_2 \leqslant \frac{5}{4}m_1$ *and*

$$c_1^2 = \frac{1}{3} - \sqrt{\frac{4m_2}{45m_1}}, \qquad c_2^2 = \frac{1}{3} + \sqrt{\frac{4m_1}{45m_2}}.$$

Proof. By Theorem 2.12(a), we need only verify that

$$m_1c_1^2 + m_2c_2^2 = \tfrac{1}{3}(m_1 + m_2), \qquad m_1c_1^4 + m_2c_2^4 = \tfrac{1}{5}(m_1 + m_2)$$

and $0 \leqslant c_j^2 < 1$. The solutions to $m_1c_1^2 + m_2c_2^2 = \frac{1}{3}(m_1 + m_2)$ are parameterized by $(c_1^2, c_2^2) = (\frac{1}{3} + m_2\alpha, \frac{1}{3} - m_1\alpha)$. Then $m_1c_1^4 + m_2c_2^4 = \frac{1}{5}(m_1 + m_2)$ implies $45m_1m_2\alpha^2 = 4$, giving (a) and (b). The conditions on c_j^2 ultimately reduce to the bounds on m_2/m_1. ■

When $m_1 = m_2 = 5$, $c_j^2 = (5 \pm 2\sqrt{5})/15$, and the resulting figure (with a suitable rotation of the pentagons) is the regular dodecahedron. We can verify the values of c_j by a kind of reverse argument. It is known (see [G2]) that the dodecahedron is a spherical 5-design; if rested on one facial pentagon, the 20 vertices lie on two pairs of antipodal regular pentagons. Thus the dodecahedron has the shape of Z_a with $m_1 = m_2 = 5$ and (2.13a) must be satisfied.

COROLLARY 2.15. *The set* $X = (s_1\{m_1\}, \pm c_1) \cup (s_2\{m_1\}, \pm c_2) \cup (\{m_0\}, 0)$ *is a spherical* 5*-design with* $4m_1 + m_0$ *points if* $5 \leqslant m_1$, $6 \leqslant m_0 \leqslant \frac{16}{5}m_1$ *and*

$$\{c_1, c_2\} = 5v \pm \sqrt{3v - 25v^2}, \quad \text{where } v = \frac{4m_1 + m_0}{60m_1}. \qquad (2.16)$$

Proof. We apply Theorem 2.12(b). We must have

$$m_1\left(c_1^2 + c_2^2\right) = \tfrac{1}{6}m_0 + \tfrac{1}{3}(m_1 + m_1),$$

$$m_1\left(c_1^4 + c_2^4\right) = \tfrac{1}{10}m_0 + \tfrac{1}{5}(m_1 + m_1),$$

so

$$c_1^2 + c_2^2 = \frac{4m_1 + m_0}{6m_1} = 10v, \qquad c_1^4 + c_2^4 = \frac{4m_1 + m_0}{10m_1} = 6v. \quad (2.17)$$

A similar argument to the last proof shows that (2.17) implies (2.16). This construction makes sense only if $3v - 25v^2 \geqslant 0$, $0 \leqslant 5v - \sqrt{3v - 25v^2}$ and $5v + \sqrt{3v - 25v^2} \leqslant 1$. These imply that $\frac{3}{50} \leqslant v \leqslant \frac{3}{25}$, which gives the condition on m_0/m_1. ∎

Finally we note two "sporadic" constructions.

COROLLARY 2.18. *Let* $\alpha = \sqrt{\frac{2}{5}}$ *and* $\beta = \sqrt{\frac{3}{5}}$. *Then the following two sets with* 18 *points are spherical* 5*-designs*: $(\{8\}, 0) \cup (\alpha\{5\}, \pm\beta)$ *and* $(0, 0, \pm 1) \cup (\{6\}, 0) \cup (\beta\{5\}, \pm\alpha)$.

Proof. Apply Lemmas 2.12(b) and (d), respectively. In the first case, note that $5\tfrac{3}{5} = \tfrac{1}{6}8 + \tfrac{1}{3}5$ and $5\tfrac{9}{25} = \tfrac{1}{10}8 + \tfrac{1}{5}5$. In the second case, note that $5\tfrac{2}{5} = -\tfrac{2}{3} + \tfrac{6}{6} + \tfrac{1}{3}5$ and $5\tfrac{4}{25} = -\tfrac{4}{5} + \tfrac{6}{10} + \tfrac{1}{5}5$. ∎

Mimura [M1] has found that there are spherical 2-designs in $\mathbf{R}^n$, $n \geqslant 3$, with m points for $m = n + 1$ and $m \geqslant n + 3$, Bajnok has an unpublished manuscript constructing spherical 3-designs in $\mathbf{R}^3$ with m points for $m = 6$ and $m \geqslant 8$, Hardin and Sloane [H2] have conjectured all cardinalities for spherical 4-designs in $\mathbf{R}^n$ for $n \leqslant 10$. In particular, there exist spherical 4-designs in $\mathbf{R}^3$ with m points for $m = 12$, 14 and $m \geqslant 16$. The 4-designs with 14 and 16 points are given explicitly in [H2, p. 260]. Since they are already 4-designs, they will be 5-designs if and only if they satisfy (1.3b) with $2\bar{s} + 1 = 5$. A computation (which we omit) shows that their 16-point 4-design is actually a 5-design, but their 14-point 4-design is not a 5-design.

The 16-point design has a fundamentally different shape from the other designs of this paper:

$$\{\pm_1 a, \pm_2 b, \pm_3 c\} \cup \{\pm_1 b, \pm_2 c, \pm_3 a\}$$
$$\cup \{\pm_1 c, \pm_2 a, \pm_3 b\} \cup \{\pm_4 d, \pm_5 d, \pm_6 d\},$$

where $\pm_1 \pm_2 \pm_3 = 1$, $\pm_4 \pm_5 \pm_6 = -1$, a, b and c are the square roots of the zeros of the cubic $t^3 - t^2 + \frac{7}{45}t - \frac{1}{243}$, and $d = \sqrt{\frac{1}{3}}$.

COROLLARY 2.19. *There exist spherical* 5*-designs in* $\mathbf{R}^3$ *with cardinality* 12, 16, 18, 20, 22, 24 *and every integer* $m \geqslant 26$.

Proof. Designs with 12, 16 and 18 points are given by the icosahedron, the Hardin-Sloane design and Corollary 2.18. By taking $m_2 = m_1$ and $m_2 = m_1 + 1$ in Corollary 2.14, we obtain designs with $4m_1$ and $4m_1 + 2$ points for $m_1 \geqslant 5$ and hence every even cardinality $\geqslant 20$. Finally, since $m_1 + 5 \leqslant \frac{16}{5}m_1$ for $m_1 \geqslant 5$, we may take $m_0 = m_1 + j$ for $1 \leqslant j \leqslant 5$ in Corollary 2.15, obtaining designs with $5m_1 + j$ points. This gives designs of every cardinality $\geqslant 26$. ∎

As noted in the introduction, Hardin and Sloane [H3] have very recently constructed spherical 5-designs with cardinality 23 and 25 and conjecture that the list is complete: that is, there are no spherical 5-designs in $\mathbf{R}^3$ with 13, 14, 15, 17, 19 or 21 points. This conjecture is based on extensive sophisticated numerical experimentation. Theorem 4.6 below states that there are no *antipodal* 5-designs with 14 points.

It seems likely that these constructions of 5-designs in $\mathbf{R}^3$ generalize, both in strength and dimension. To wit, we suspect that if $2s + 1 \leqslant m_1 \leqslant \cdots \leqslant m_s \leqslant m_1 + 1$, then there exist $c_1, \dots, c_s \in (0, 1)$ so that the set consisting of antipodal regular $\{m_j\}$'s suspended at $\pm c_j$ is a spherical $(2s + 1)$-design in $\mathbf{R}^3$. If this is true, then there exist spherical $(2s + 1)$-designs in $\mathbf{R}^3$ with every even cardinality $\geqslant 2s(2s + 1)$. Further, we suspect that if $2s + 1 \leqslant m_1$, $2s + 2 \leqslant m_0$ and m_0/m_1 is not too large, then there exist $\gamma_1, \dots, \gamma_s$ so that the set consisting of antipodal pairs of $\{m_1\}$'s suspended at $\pm\gamma_j$, together with an equatorial $\{m_0\}$, is also a spherical $(2s + 1)$-design in $\mathbf{R}^3$. If these suspicions are true, then so is the following conjecture.

CONJECTURE 2.20. If $n \geqslant (2s + 1)^2 + 1$, then there exists a spherical $(2s + 1)$-design in $\mathbf{R}^3$ with cardinality n.

The difficulty in proving this conjecture lies in the fact that the systems of equations satisfied by c_j and γ_j for fixed $\{m_j\}$ have degree s. Quadratic systems are much easier to solve! Another direction of generalization would be to construct spherical 5-designs in $\mathbf{R}^4$ by suspending the designs constructed here. We hope to study these ideas elsewhere.

3. CATALECTICANTS

In this section, we review some seemingly irrelevant algebraic topics from [R1]. The payoff will be in the next section.

Let $H_d(K^n)$ denote the set of homogeneous polynomials (forms) in n variables with degree d and coefficients in a field K of characteristic 0. We are only interested in $K = \mathbf{R}$, but the machinery is more generally applicable. The main result cited in this section, Proposition 3.7, would be applicable to spherical designs in any formally real field, not just $\mathbf{R}$. (A notational remark: in [R1], we wrote $F_{n,d}$ for $H_d(\mathbf{R}^n)$. The change to a more standard notation also suggests a change from H_p to $\mathscr{H}_p$.)

Suppose $n \geqslant 1$ and $d \geqslant 0$. The index set for monomials in $H_d(K^n)$ is

$$\mathscr{I}(n,d) = \left\{ i = (i_1, \dots, i_n) : 0 \leqslant i_k \in \mathbf{Z}, \quad \sum_{k=1}^{n} i_k = d \right\}.$$

Write $N(n,d) = \binom{n+d-1}{n-1} = |\mathscr{I}(n,d)|$ and for $i \in \mathscr{I}(n,d)$, let $c(i) = d!/(i_1! \cdots i_n!)$ be the associated multinomial coefficient. The multinomial abbreviation u^i means $u_1^{i_1} \cdots u_n^{i_n}$, where u may be an n-tuple of constants or variables. Every $f \in H_d(K^n)$ can be written as

$$f(x_1, \dots, x_n) = \sum_{i \in \mathscr{I}(n,d)} c(i) a(f;i) x^i. \tag{3.1}$$

(We need char$(K) = 0$ to ensure that $c(i) \neq 0$ in K, and so can be factored in (3.1) from the coefficent of x^i in f.) For $\xi \in K^n$, define $(\xi\cdot)^d \in H_d(K^n)$ by

$$(\xi\cdot)^d(x) = \left(\sum_{k=1}^{n} \xi_k x_k \right)^d = \sum_{i \in \mathscr{I}(n,d)} c(i) \xi^i x^i. \tag{3.2}$$

Suppose now that $d = 2s$ is even. For $p \in H_{2s}(K^n)$, we define the Hankel form $\mathscr{H}_p \in H_2(K^{N(n,s)})$ as follows. Index $\mathscr{I}(n, s)$ in any fixed way as $\{l_1, \dots, l_{N(n,s)}\}$. Then

$$\mathscr{H}_p(t_1, \dots, t_{N(n,s)}) = \sum_{j=1}^{N(n,s)} \sum_{k=1}^{N(n,s)} a(p; l_j + l_k) t_j t_k. \tag{3.3}$$

The *catalecticant* of p is the determinant of the associated matrix and is independent of the ordering chosen for $\mathscr{I}(n, s)$:

$$C(p) = \det\left[a(p; l_j + l_k)\right]. \tag{3.4}$$

For example, suppose $n = 2$ and $m = 4$, and write $a_{(j,4-j)}$ for

$$a(p; (j, 4-j))$$

and $\binom{4}{j}$ for $c(j, 4-j)$, so (3.1) becomes

$$p(x_1, x_2) = a_{(4,0)} x_1^4 + 4a_{(3,1)} x_1^3 x_2 + 6a_{(2,2)} x_1^2 x_2^2 + 4a_{(1,3)} x_1 x_2^3 + a_{(0,4)} x_2^4.$$

Taking $t_1 = (2, 0)$, $t_2 = (1, 1)$ and $t_3 = (0, 2)$, we find that

$$\mathscr{H}_p(t_1, t_2, t_3) = a_{(4,0)} t_1^2 + a_{(3,1)}(t_1 t_2 + t_2 t_1)$$
$$+ a_{(2,2)}(t_1 t_3 + t_2^2 + t_3 t_1) + a_{(1,3)}(t_2 t_3 + t_3 t_2) + a_{(0,4)} t_3^2.$$

The pattern of the catalecticant is easier to see in the matrix

$$\mathscr{H}(p) = \begin{pmatrix} a_{(4,0)} & a_{(3,1)} & a_{(2,2)} \\ a_{(3,1)} & a_{(2,2)} & a_{(1,3)} \\ a_{(2,2)} & a_{(1,3)} & a_{(0,4)} \end{pmatrix}. \tag{3.5}$$

The diagonal pattern in (3.5) holds for all binary forms, but is more obscure for forms in three or more variables, because there is no obvious linear ordering for $\mathscr{I}(n, s)$ for $n \geqslant 3$.

If $p = (\xi \cdot)^{2s}$, then by (3.1) and (3.2), $a(p; i) = \xi^i$ for $i \in \mathscr{I}(n, 2s)$. Thus

$$\mathscr{H}_{(\xi\cdot)^{2s}}(t) = \sum_{j=1}^{N(n,s)} \sum_{k=1}^{N(n,s)} \xi^{l_j + l_k} t_j t_k = \left(\sum_{j=1}^{N(n,s)} \xi^{l_j} t_j \right)^2 \tag{3.6}$$

is a perfect square. In the following proposition, we restrict $K = \mathbf{R}$, although the only place this is used is in (c), where the actual requirement is that K be formally real.

PROPOSITION 3.7 (see [R1], pp. 6–8]). *Suppose* $K = \mathbf{R}$ *and* $\{\xi_k\} \subset \mathbf{R}^n$.

(a) *If* $p = \sum_{k=1}^r (\xi_k \cdot)^{2s}$, *then* $\mathscr{H}_p(t) = \sum_{k=1}^r (\sum_{j=1}^{N(n,s)} \xi_k^{l_j} t_j)^2$.

(b) *If* $p = \sum_{k=1}^r (\xi_k \cdot)^{2s}$ *and* $r < N(n, s)$, *then* $C(p) = 0$.

(c) *If* $p = \sum_{k=1}^r (\xi_k \cdot)^{2s}$ *and* $\mathscr{H}_p(v) = 0$, *then* $L(v; \xi_k) = 0$ *for* $1 \leqslant k \leqslant r$, *where*

$$L(v; x) = \sum_{j=1}^{N(n,s)} v_j x^{l_j} \in H_s(\mathbf{R}^n).$$

Proof. Since the map $p \to \mathscr{H}_p$ is linear, (a) follows from (3.6); (b) is then immediate. In (c), we have

$$0 = \mathscr{H}_p(v) = \sum_{k=1}^r \left(\sum_{j=1}^{N(n,s)} \xi_k^{l_j} v_j \right)^2 = \sum_{k=1}^r L^2(v; \xi_k),$$

so each summand must vanish. ■

We end this section with a historical paragraph on catalecticants. In the early 1850s, Sylvester studied complex binary forms ($H_d(\mathbf{C}^2)$) and their representations as a sum of d-th powers of linear forms. He proved [S4] that $p(x, y) \in H_{2s}(\mathbf{C}^2)$ is a sum of s or fewer $2s$-th powers of linear forms if and only if $C(p) = 0$. Many years later, he used the catalecticant to show that "most" forms in $H_{2s}(\mathbf{C}^n)$ are not a sum of $N(n, s) - 1$ $2s$-th powers of linear forms, which violated the "constant-counting" heuristic of nineteenth century mathematicians in a few cases. There are $N(3, 4) = 15$ coefficients in the general ternary quartic $p \in H_4(\mathbf{C}^3)$, and since $3 \cdot 5 = 15$, it ought to have a representation $p(x, y, z) = \sum_{k=1}^5 (a_k x + b_k y + c_k z)^4$. This is wrong.

Clebsch proved geometrically in 1861 that six 4-th powers are needed. In 1886 (at age 72!), Sylvester observed [S5] that the catalecticant of a sum of five 4-th powers vanishes. This gives a nontrivial relation among the coefficients of a sum of five 4-th powers, hence the general ternary quartic requires at least six 4-th powers. For more on the role of catalecticants in the algebra of binary forms, see [K2, K3, R1].

4. TIGHT AND SNUG ANTIPODAL SPHERICAL 5-DESIGNS IN $\mathbf{R}^n$

Suppose $X = \{\pm \xi_1, \dots, \pm \xi_r\} \subset S^2$ is an antipodal set of points in $\mathbf{R}^3$, and let $\xi_k = (a_k, b_k, c_k)$. By (1.3), X is a spherical 5-design if and only if

$$\sum_{k=1}^{r} (a_k x + b_k y + c_k z)^4 = r\frac{1}{3}\,\frac{3}{5}(x^2 + y^2 + z^2)^2$$

$$= \frac{r}{5}(x^2 + y^2 + z^2)^2. \qquad (4.1)$$

Consider $p(x, y, z) = (x^2 + y^2 + z^2)^2$. Since $c(4, 0, 0) = c(0, 4, 0) = c(0, 0, 4) = 1$ and $c(2, 2, 0) = c(2, 0, 2) = c(0, 2, 2) = 4!/(2!2!0!) = 6$, we have $a(p; (4, 0, 0)) = 1$, etc. and $a(p; (2, 2, 0)) = \frac{2}{6} = \frac{1}{3}$, etc. Index $\mathscr{I}(3, 2)$ in the following order; $(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)$. With this ordering,

$$\mathscr{H}(p) = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \qquad (4.2)$$

A tight spherical 5-design X must contain $N(3, 2) = 6$ antipodal pairs of points. We shall prove that X consists of the vertices of a regular icosahedron. The following derivation is based on the one in [R1, p. 128]; the original observation is in [D1, p. 375]. We present it here as a warm-up to the new results.

Without loss of generality, first rotate X so that $\xi_1 = (1,0,0)$. By (4.1), we have

$$x^4 + \sum_{k=2}^{6} (a_k x + b_k y + c_k z)^4 = \tfrac{6}{5}(x^2 + y^2 + z^2)^2,$$

so $q(x, y, z) = \frac{6}{5}(x^2 + y^2 + z^2)^2 - x^4$ is a sum of five 4-th powers, and

$$C(q) = \begin{vmatrix} \frac{6}{5} - 1 & \frac{2}{5} & \frac{2}{5} & 0 & 0 & 0 \\ \frac{2}{5} & \frac{6}{5} & \frac{2}{5} & 0 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{6}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} \end{vmatrix} = 0.$$

This determinant identity is easily verified; in fact, the upper 3×3 block is singular. It is also easily checked that $\mathscr{H}_q(v) = 0$ for $v = (4, -1, -1, 0, 0, 0)^t$, since v is a null eigenvalue of the matrix. By Proposition 3.7(c),

$$4a_k^2 + (-1)b_k^2 + (-1)c_k^2 + 0a_k b_k + 0a_k c_k + 0b_k c_k = 0, \qquad 2 \leqslant k \leqslant 6.$$

Since $(a_k, b_k, c_k) \in S^2$, we have $a_k^2 + b_k^2 + c_k^2 = 1$; hence $5a_k^2 = 1$, so $a_k = \pm 1/\sqrt{5}$. But $a_k = \xi_1 \cdot \xi_k$, and this dot product is unchanged by rotation. Further, the selection of ξ_1 as the vector rotated to $(1,0,0)$ was arbitrary. Thus, if $X = \{\pm \xi_k\}$ is a tight spherical 5-design, then

$$\xi_j \cdot \xi_k = \pm \frac{1}{\sqrt{5}} \quad \text{for all } j \neq k. \tag{4.3}$$

Haantjes [H1] proved in 1948 that the only set $X \subset S^2$ of six antipodal pairs satisfying (4.3) consists of the vertices of a regular icosahedron. We shall argue directly, introducing the methods to be used later.

Observe that if ξ_1 and ξ_2 are any two unit vectors in $\mathbf{R}^3$, then there is a rotation of X after which $\xi_1 = (c, s, 0)$ and $\xi_2 = (c, -s, 0)$, where $c^2 + s^2 = 1$. Let $X = \{\pm \xi_k : 1 \leqslant k \leqslant 6\}$ be a tight spherical 5-design. By replacing ξ_k by $-\xi_k$ for $2 \leqslant k \leqslant 6$ if necessary, we may assume without loss of generality that $\xi_1 \cdot \xi_k = +1/\sqrt{5}$ for $2 \leqslant k \leqslant 6$. In particular, $\xi_1 \cdot \xi_2 =$

$c^2 - s^2 = 1/\sqrt{5}$. Let $\Phi = (1 + \sqrt{5})/2$. Since $(\Phi^2 - 1)/(\Phi^2 + 1) = \Phi/(\Phi + 2) = (1 + \sqrt{5})/(5 + \sqrt{5}) = 1/\sqrt{5}$, the choice

$$(c, s) = \left(\frac{\Phi}{\sqrt{\Phi^2 + 1}}, \frac{1}{\sqrt{\Phi^2 + 1}} \right)$$

satisfies the requirements for $c^2 \pm s^2$. Now consider $\xi_k = (a_k, b_k, c_k)$ for $3 \leqslant k \leqslant 6$. By (4.3), the following system of equations must be satisfied by (a_k, b_k):

$$\xi_1 \cdot \xi_k = \frac{\Phi}{\sqrt{\Phi^2 + 1}} a_k + \frac{1}{\sqrt{\Phi^2 + 1}} b_k = \frac{1}{\sqrt{5}} = \frac{\Phi}{\Phi^2 + 1}, \tag{4.4a}$$

$$\xi_2 \cdot \xi_k = \frac{\Phi}{\sqrt{\Phi^2 + 1}} a_k - \frac{1}{\sqrt{\Phi^2 + 1}} b_k = \pm \frac{1}{\sqrt{5}} = \pm \frac{\Phi}{\Phi^2 + 1}. \tag{4.4b}$$

If "$\pm$" = "$+$" in (4.4b), then $b_k = 0$ and $a_k = 1/(\sqrt{\Phi^2 + 1})$; since $a_k^2 + b_k^2 + c_k^2 = 1$, we must have $c_k = \pm[\Phi/(\sqrt{\Phi^2 + 1})]$. If "$\pm$" = "$-$" in (4.4b), then $a_k = 0$ and $b_k = \Phi/(\sqrt{\Phi^2 + 1})$; since $a_k^2 + b_k^2 + c_k^2 = 1$, we must have $c_k = \pm[1/(\sqrt{\Phi^2 + 1})]$. We have determined the four distinct solutions to (4.4), which can only be $\xi_3, \xi_4, \xi_5, \xi_6$. Therefore,

$$\begin{aligned} X &= \{\pm \xi_1, \ldots, \pm \xi_6\} \\ &= \frac{1}{\sqrt{\Phi^2 + 1}} \{(\pm\Phi, \pm 1, 0), (0, \pm \Phi, \pm 1), (\pm 1, 0, \pm \Phi)\}. \end{aligned} \tag{4.5}$$

These are the Schönemann coordinates for the regular icosahedron [C2, p. 52]. It is easy to check that (4.1) now holds for $\{\xi_1, \ldots, \xi_6\}$, as given in (4.5).

We now prove the first new result of this section.

THEOREM 4.6. *There is no set* $X = \{\pm \xi_k : 1 \leqslant k \leqslant 7\} \subset S^2$ *which is a spherical* 5-*design.*

Proof. Suppose to the contrary. Then by (4.1), we would have for $\xi_k = (a_k, b_k, c_k)$,

$$\sum_{k=1}^{7}(a_k x + b_k y + c_k z)^4 = \tfrac{7}{5}(x^2 + y^2 + z^2)^2. \tag{4.7}$$

Choose any two ξ_k's, which might as well be ξ_1 and ξ_2, and rotate X so that $\xi_1 = (c, s, 0)$ and $\xi_2 = (c, -s, 0)$, where $c^2 + s^2 = 1$ and $\xi_1 \cdot \xi_2 = c^2 - s^2$. Define

$$\begin{aligned} q(x, y, z) &= \sum_{k=3}^{7}(a_k x + b_k y + c_k z)^4 \\ &= \tfrac{7}{5}(x^2 + y^2 + z^2)^2 - (cx + sy)^4 - (cx - sy)^4. \end{aligned} \tag{4.8}$$

Then q is a sum of five 4-th powers and so $C(q) = 0$ by Proposition 3.7(b). Since

$$(cx + sy)^4 + (cx - sy)^4 = \binom{4}{0}2c^4x^4 + \binom{4}{2}2c^2s^2x^2y^2 + \binom{4}{4}2s^4y^4,$$

we have

$$C(q) = 0 = \begin{vmatrix} \frac{7}{5} - 2c^4 & \frac{7}{15} - 2c^2s^2 & \frac{7}{15} & 0 & 0 & 0 \\ \frac{7}{15} - 2c^2s^2 & \frac{7}{5} - 2s^4 & \frac{7}{15} & 0 & 0 & 0 \\ \frac{7}{15} & \frac{7}{15} & \frac{7}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{15} - 2c^2s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{15} \end{vmatrix}$$

$$= \tfrac{196}{675}(7 - (12c^4 - 6c^2s^2 + 12s^4))(\tfrac{7}{15} - 2c^2s^2)(\tfrac{7}{15})^2, \tag{4.9}$$

where the factors follow the order of the determinants of the blocks. By (4.9),

$$7 = 12c^4 - 6s^2s^2 + 12s^4 = 12(c^2 + s^2)^2 - 30c^2s^2 \quad \text{or} \quad c^2s^2 = \tfrac{7}{30}. \tag{4.10}$$

Since $(c^2 + s^2)^2 = 1$, (4.10) reduces to $c^2s^2 = \frac{1}{6}$ or $\frac{7}{30}$. Therefore,

$$(\xi_1 \cdot \xi_2)^2 = (c^2 - s^2)^2 = (c^2 + s^2)^2 - 4c^2s^2$$
$$\in \{1 - \tfrac{4}{6}, 1 - \tfrac{28}{30}\} = \{\tfrac{1}{3}, \tfrac{1}{15}\}. \tag{4.11}$$

Since the selection of ξ_1 and ξ_2 for rotation was arbitrary, (4.11) implies that $(\xi_j \cdot \xi_k)^2 = \frac{1}{3}$ or $\frac{1}{15}$ for $1 \leqslant j < k \leqslant 7$.

Once again, we rerotate X, so that $\xi_1 = (1, 0, 0)$ and so $a_k = \xi_{k1} = \xi_1 \cdot \xi_k$. Since (4.7) is valid for the rotated set, upon taking the coefficient of x^4 on both sides, we find

$$1 + \sum_{k=2}^{7} a_k^4 = \frac{7}{5}. \tag{4.12}$$

By (4.11), a_k^4 can only take on the values $\frac{1}{9}$ and $\frac{1}{225}$. Suppose these occur m times and $6 - m$ times, respectively. Then (4.12) implies

$$1 + m \cdot \tfrac{1}{9} + (6 - m) \cdot \tfrac{1}{225} = \tfrac{7}{5},$$

hence $m = \frac{7}{2}$. This is a contradiction, completing the proof. ∎

The generalization to $\mathbf{R}^n$ requires two $n \times n$ determinantal identities. Let

$$D_n(A, B; \lambda) = \begin{vmatrix} A + \lambda & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & A \end{vmatrix}, \tag{4.13}$$

where the entries of the matrix in (4.13) are A on the diagonal (except for the first entry, which is $A + \lambda$), and B off the diagonal.

LEMMA 4.14.

$$D_n(A, B; \lambda) = (A - B)^{n-2} \times ((A - B)(A + (n - 1)B) + \lambda(A + (n - 2)B)). \tag{4.15}$$

Proof. Observe that $D_n(A, B; 0) = (A - B)^{n-1}(A + (n - 1)B)$. This formula is well known and can be proved either by eigenvalues, or from elementary row operations: first subtract the first row from each of the other rows, and then add each column to the first, to give an upper triangular matrix with diagonal entries $A + (n - 1)B$ followed by $n - 1$ $A - B$'s. But $D_n(A, B; \lambda) = D_n(A, B; 0) + \lambda D_{n-1}(A, B; 0)$, which gives (4.15). ■

LEMMA 4.16. *If $\alpha\gamma = \beta^2$, then the $n \times n$ determinant*

$$\begin{vmatrix} A+\alpha & B+\beta & B & \cdots & B \\ B+\beta & A+\gamma & B & \cdots & B \\ B & B & A & \cdots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & B & B & \cdots & A \end{vmatrix}$$

$$= (A - B)^{n-2}((A - B)(A + (n - 1)B)$$

$$+ (\alpha + \gamma)(A + (n - 2)B) - 2\beta B). \quad (4.17)$$

Proof. We expand the determinant, obtaining

$$D_n(A, B; 0) + \alpha D_{n-1}(A, B; 0) + \gamma D_{n-1}(A, B; 0)$$

$$- 2\beta D_{n-1}(A, B; B - A) + (\alpha\gamma - \beta^2)D_{n-2}(A, B; 0). \quad (4.18)$$

Since $\alpha\gamma - \beta^2 = 0$ and $D_{n-1}(A, B; B - A) = B(A - B)^{n-2}$ by Lemma 4.14 (with $\lambda = B - A$), (4.18) reduces to (4.17). ■

We now repeat the entire discussion of the first part of this section, following the same reasoning. What is omitted in detailed explanation is more than compensated for in the complication of the algebra!

Let $p(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^2$, $n \geqslant 3$. We index $\mathscr{I}(n, 2)$ as follows:

$$(2, 0, \ldots, 0), \ldots, (0, 0, \ldots, 2), (1, 1, \ldots, 0), (1, 0, 1, \ldots,), \ldots .$$

Then $\mathscr{H}(p)$ can be written as before: the nonzero entries of this $\binom{n+1}{2} \times \binom{n+1}{2}$ matrix consist of the upper left $n \times n$ block

$$\begin{pmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{3} \\ \frac{1}{3} & 1 & \cdots & \frac{1}{3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{3} & \frac{1}{3} & \cdots & 1 \end{pmatrix}$$

and $\binom{n}{2}$ diagonal entries of $\frac{1}{3}$.

Suppose now that X is a tight spherical 5-design in $\mathbf{R}^n$. Then X consists of $N := \binom{n+1}{2} = n(n+1)/2$ antipodal pairs, $\{\pm \xi_k\}$, and by (1.3),

$$\sum_{k=1}^{N} (\xi_{k1} x_1 + \cdots + \xi_{kn} x_n)^4$$

$$= N \frac{1 \cdot 3}{n(n+2)} \left(x_1^2 + \cdots + x_n^2\right)^2 = \frac{3(n+1)}{2(n+2)} \left(x_1^2 + \cdots + x_n^2\right)^2. \quad (4.19)$$

We may rotate X so that $\xi_1 = (1, 0, \ldots, 0)$. Since the catalecticant of $q = \sum_{k=2}^{N} (\xi_k \cdot)^4$ vanishes, and the diagonal entries are unaltered, we should find that

$$\begin{vmatrix} \frac{3(n+1)}{2(n+2)} - 1 & \frac{n+1}{2(n+2)} & \cdots & \frac{n+1}{2(n+2)} \\ \frac{n+1}{2(n+2)} & \frac{3(n+1)}{2(n+2)} & \cdots & \frac{n+1}{2(n+2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n+1}{2(n+2)} & \frac{n+1}{2(n+2)} & \cdots & \frac{3(n+1)}{2(n+2)} \end{vmatrix} = 0.$$

In the notation of Lemma 4.14, this determinant is $D_n(3B, B; -1)$ for $B = (n+1)/[2(n+2)]$, or

$$(2B)^{n-2}(2B((n+2)B) - (n+1)B)$$

$$= 2^{n-2}B^{n-1}(2(n+2)B - (n+1)) = 0,$$

as we expected. The null eigenvector for the $n \times n$ block matrix is $(n+1, -1, \ldots, -1)$, hence $(n+1)\xi_{k1}^2 - \xi_{k2}^2 - \cdots - \xi_{kn}^2 = 0$ for $k \geqslant 2$, so $\xi_{k1}^2 = 1/(n+2)$. Taking the rotation into account, this implies that $\xi_j \cdot \xi_k = \pm\sqrt{1/(n+2)}$ for $j \neq k$, information already known via the Gegenbauer polynomial. This argument is continued in [R1, p. 131] to show that $n = u^2 - 2$, where u is not a multiple of 4. This is weaker than the known result (see Bannai and Damerell [B4]), which ultimately goes back to Lemmens and Seidel [L1], that $n = u^2 - 2$ for odd u.

Theorem 4.6 now generalizes, though with considerable computational complication.

THEOREM 4.20. *If $n \geqslant 3$, then there is no spherical 5-design in $\mathbf{R}^n$ of the form $X = \{\pm\xi_k : 1 \leqslant k \leqslant \binom{n+1}{2} + 1\} \subset S^{n-1}$.*

Proof. If such an X exists, then, analogously to (4.19), we would have

$$\sum_{k=1}^{N+1} (\xi_{k1}x_1 + \cdots + \xi_{kn}x_n)^4 = \frac{3(N+1)}{n(n+2)}(x_1^2 + \cdots + x_n^2)^2$$

$$= \frac{3(n^2+n+2)}{2n(n+2)}(x_1^2 + \cdots + x_n^2)^2. \quad (4.21)$$

After a rotation, we may assume that $\xi_1 = (\sigma, \tau, 0, \ldots, 0)$ and $\xi_2 = (\sigma, -\tau, 0, \ldots, 0)$ with $\sigma^2 + \tau^2 = 1$ and $\sigma^2 - \tau^2 = \xi_1 \cdot \xi_2$. Then $q = \sum_{k=3}^{N+1}(\xi_k \cdot)^4$ has vanishing catalecticant. By (4.21), this means that either the $n \times n$ block has 0 determinant, or one of the diagonal entries is 0. In the first case,

$$\begin{vmatrix} 3v - 2\sigma^4 & v - 2\sigma^2\tau^2 & v & \cdots & v \\ v - 2\sigma^2\tau^2 & 3v - 2\tau^4 & v & \cdots & v \\ v & v & 3v & \cdots & v \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v & v & v & \cdots & 3v \end{vmatrix} = 0, \tag{4.22}$$

where $v = [n^2 + n + 2]/[2n(n + 2)]$ for simplicity, In the second case, the only candidate diagonal entry corresponds to $x_1^2x_2^2$; in this case, $v - 2\sigma^2\tau^2 = 0$, or

$$\sigma^2\tau^2 = \frac{v}{2} = \frac{n^2 + n + 2}{4n(n + 2)}. \tag{4.23}$$

To analyze the first case, we let $\alpha = -2\sigma^4$, $\beta = -2\sigma^2\tau^2$, and $\gamma = -2\tau^4$ in (4.22). Then $\alpha\gamma = \beta^2$, and applying Lemma 4.16 with $A = 3v$, $B = v$, we find that

$$\begin{aligned} 0 &= (2v)^{n-2}(2v(n + 2)v - (2\sigma^4 + 2\tau^4)(n + 1)v + 4\sigma^2\tau^2 v) \\ &= 2^{n-1}v^{n-1}((n + 2)v - (n + 1)(\sigma^4 + \tau^4) + 2\sigma^2\tau^2) \\ &= 2^{n-1}v^{n-1}\left(\frac{n^2 + n + 2}{2n} - ((n + 1)(\sigma^4 + \tau^4) - 2\sigma^2\tau^2)\right), \end{aligned} \tag{4.24}$$

so $(n^2 + n + 2)/(2n) = (n + 1)\sigma^4 - 2\sigma^2\tau^2 + (n + 1)\tau^4$. Since $(n + 1)(\sigma^2 + \tau^2)^2 = n + 1$, (4.24) thus implies that

$$(2(n + 1) - (-2))\sigma^2\tau^2 = (n + 1) - \frac{n^2 + n + 2}{2n} = \frac{n^2 + n - 2}{2n}. \tag{4.25}$$

After simplification, (4.23) and (4.25) combine to show that

$$\sigma^2\tau^2 \in \left\{\frac{n^2+n\pm 2}{4n(n+2)}\right\}, \tag{4.26}$$

hence

$$\begin{aligned}(\xi_1\cdot\xi_2)^2 &= (\sigma^2-\tau^2)^2 = (\sigma^2+\tau^2)^2 - 4\sigma^2\tau^2 \\ &\in \left\{1-\frac{n^2+n\pm 2}{n(n+2)}\right\} \\ &= \left\{\frac{n\mp 2}{n(n+2)}\right\}.\end{aligned} \tag{4.27}$$

(If $n = 3$, this becomes $(\xi_1\cdot\xi_2)^2 = (3\mp 2)/15$, as in (4.11).) Let

$$c = \sqrt{\frac{n+2}{n(n+2)}}, \qquad \bar{c} = \sqrt{\frac{n-2}{n(n+2)}}. \tag{4.28}$$

Since the selection of ξ_1 and ξ_2 for rotation was arbitrary, we conclude that $\xi_j\cdot\xi_k \in \{\pm c, \pm\bar{c}\}$ for $1 \leqslant j < k \leqslant N+1$.

Once again, we rerotate X, so that $\xi_1 = (1, 0, \ldots, 0)$ and then, by replacing ξ_k with $-\xi_k$ if necessary for $2 \leqslant k \leqslant N+1$, we can assume that $\xi_1\cdot\xi_k > 0$. Since X is a 5-design, (4.21) still holds. Upon taking the coefficient of x_1^4 on both sides, we obtain the equation

$$1 + \sum_{k=2}^{N+1} \xi_{k1}^4 = \frac{3(n^2+n+2)}{2n(n+2)}. \tag{4.29}$$

But $\xi_{k1} = \xi_1\cdot\xi_k$, and by (4.27), ξ_{k1}^4 can only take on the values $c^4, \bar{c}^4$. Suppose these occur m times and $N-m$ times, respectively. Then (4.29) implies that

$$1 + m\cdot\frac{(n+2)^2}{n^2(n+2)^2} + \left(\binom{n+1}{2} - m\right)\cdot\frac{(n-2)^2}{n^2(n+2)^2} = \frac{3(n^2+n+2)}{2n(n+2)}. \tag{4.30}$$

A little algebra shows that (4.30) reduces to

$$m = \frac{n^2 + n + 2}{4}. \tag{4.31}$$

Since m must be an integer, (4.31) implies that $n \equiv 1, 2 \bmod 4$. (For $n = 3$, (4.31) reduces to $m = \frac{7}{2}$, as before.)

We perform one final rotation. Reindex X if necessary to assume that $\xi_1 \cdot \xi_2 = c$, and rotate on the last $n - 1$ coordinates, so that $\xi_2 = (c, s, 0, \dots, 0)$, where $c^2 + s^2 = 1$; this keeps $\xi_1 = (1, 0, \dots, 0)$. For $3 \leqslant k \leqslant N + 1$, there are eight possible values for $(\xi_1 \cdot \xi_k, \xi_2 \cdot \xi_k)$, namely, $(c, \pm c)$, $(c, \pm \bar{c})$, $(\bar{c}, \pm c)$ and $(\bar{c}, \pm \bar{c})$, and we have

$$\xi_1 \cdot \xi_k = \xi_{k1}, \qquad \xi_2 \cdot \xi_k = c\xi_{k1} + s\xi_{k2},$$

hence

$$\xi_{k1} = \xi_1 \cdot \xi_k, \qquad \xi_{k2} = \frac{\xi_2 \cdot \xi_k - c\xi_1 \cdot \xi_k}{s}. \tag{4.32}$$

We return to (4.21), and set $x_3 = \cdots = x_n = 0$. Let m_j, $1 \leqslant j \leqslant 8$, denote the multiplicities of the possible values $(\xi_1 \cdot \xi_k, \xi_2 \cdot \xi_k)$, and apply (4.32):

$$\begin{aligned}
&\frac{3(n^2 + n + 2)}{2n(n + 2)} \left(x_1^2 + x_2^2 \right)^2 \\
&\quad = x_1^4 + (cx_1 + sx_2)^4 + m_1\left(cx_1 + \frac{c - c^2}{s} x_2 \right)^4 \\
&\qquad + m_2\left(cx_1 + \frac{-c - c^2}{s} x_2 \right)^4 + m_3\left(cx_1 + \frac{\bar{c} - c^2}{s} x_2 \right)^4 \\
&\qquad + m_4\left(cx_1 + \frac{-\bar{c} - c^2}{s} x_2 \right)^4 + m_5\left(\bar{c}x_1 + \frac{c - c\bar{c}}{s} x_2 \right)^4 \\
&\qquad + m_6\left(\bar{c}x_1 + \frac{-c - c\bar{c}}{s} x_2 \right)^4 + m_7\left(\bar{c}x_1 + \frac{\bar{c} - c\bar{c}}{s} x_2 \right)^4
\end{aligned}$$

$$+ m_8\left(\bar{c}x_1 + \frac{-\bar{c} - c\bar{c}}{s}x_2\right)^4. \tag{4.33}$$

For $j = 1, 2$, we already know that $|\xi_j \cdot \xi_k| = c$ for m values of k, and since $\xi_1 \cdot \xi_2 = c$,

$$\begin{aligned} m_1 + m_2 + m_3 + m_4 &= m_5 + m_6 + m_7 + m_8 = m - 1 \\ &= \frac{n^2 + n - 2}{4}, \\ m_1 + m_2 + m_5 + m_6 &= m_3 + m_4 + m_7 + m_8 = m - 1 \\ &= \frac{n^2 + n - 2}{4}, \end{aligned} \tag{4.34}$$

hence

$$m_1 + m_2 = m_7 + m_8 := M, \qquad m_3 + m_4 = m_5 + m_6 = \frac{n^2 + n - 2}{4} - M. \tag{4.35}$$

Now let $u = x_1 - (c/s)x_2$ and $v = (1/s)x_2$ in (4.33). Then $x_1 = u + cv$ and $x_2 = sv$, so $x_1^2 + x_2^2 = u^2 + 2cuv + v^2$ and (4.33) becomes

$$\begin{aligned} \frac{3(n^2 + n + 2)}{2n(n + 2)}(u^2 + 2cuv + v^2)^2 &= (u + cv)^4 + (cu + v)^4 \\ &+ m_1(cu + cv)^4 + m_2(cu - cv)^4 + m_3(cu + \bar{c}v)^4 + m_4(cu - \bar{c}v)^4 \\ &+ m_5(\bar{c}u + cv)^4 + m_6(\bar{c}u - cv)^4 + m_7(\bar{c}u + \bar{c}v)^4 + m_8(\bar{c}u - \bar{c}v)^4. \end{aligned} \tag{4.36}$$

Finally, equate the coefficients of $u^j v^{4-j}$ in (4.36):

$$\frac{3(n^2+n+2)}{2n(n+2)} = 1 + c^4 + (m_1+m_2+m_3+m_4)c^4 + (m_5+m_6+m_7+m_8)\bar{c}^4, \tag{4.37a}$$

$$\frac{3(n^2+n+2)}{2n(n+2)}4c = 4\big(c + c^3 + (m_1-m_2)c^4 + (m_3-m_4)c^3\bar{c} + (m_5-m_6)c\bar{c}^3 + (m_7-m_8)\bar{c}^4\big), \tag{4.37b}$$

$$\frac{3(n^2+n+2)}{2n(n+2)}(4c^2+2) = 6\big(c^2 + c^2 + (m_1+m_2)c^4 + (m_3+m_4+m_5+m_6)c^2\bar{c}^2 + (m_7+m_8)\bar{c}^4\big), \tag{4.37c}$$

$$\frac{3(n^2+n+2)}{2n(n+2)}4c = 4\big(c^3 + c + (m_1-m_2)c^4 + (m_3-m_4)c\bar{c}^3 + (m_5-m_6)c^3\bar{c} + (m_7-m_8)\bar{c}^4\big), \tag{4.36d}$$

$$\frac{3(n^2+n+2)}{2n(n+2)} = c^4 + 1 + (m_1+m_2+m_5+m_6)c^4 + (m_3+m_4+m_7+m_8)\bar{c}^4. \tag{4.37e}$$

This is not as bad as it looks! In fact, by (4.28) and (4.34), (4.37a) becomes

$$\frac{3(n^2 + n + 2)}{2n(n + 2)} = 1 + c^4 + (m_1 + m_2 + m_3 + m_4)c^4$$

$$+ (m_5 + m_6 + m_7 + m_8)\bar{c}^4$$

$$= 1 + \frac{n^2 + n + 2}{4} \frac{(n + 2)^2}{n^2(n + 2)^2} + \frac{n^2 + n - 2}{4} \frac{(n - 2)^2}{n^2(n + 2)^2},$$

which is an identity. The same computation applies to (4.37e) since $m_3 + m_4 = m_5 + m_6$ by (4.35). Now consider (4.37c):

$$\frac{3(n^2 + n + 2)}{2n(n + 2)}(4c^2 + 2)$$

$$= 6\big(c^2 + c^2 + (m_1 + m_2)c^4 + (m_3 + m_4 + m_5 + m_6)c^2\bar{c}^2$$

$$+ (m_7 + m_8)\bar{c}^4\big)$$

$$= 12c^2 + 6M(c^4 + \bar{c}^4) + 12\left(\frac{n^2 + n - 2}{4} - M\right)c^2\bar{c}^2.$$

This equation can be solved for M; miraculously, it turns out that $M = 0$. Since $m_j \geqslant 0$, this implies that $m_1 = m_2 = m_7 = m_8 = 0$! Now, (4.37b) and (4.37d) reduce to

$$\frac{3(n^2 + n + 2)}{2n(n + 2)} 4c = 4\big(c + c^3 + (m_3 - m_4)c^3\bar{c} + (m_5 - m_6)c\bar{c}^3\big), \tag{4.38a}$$

$$\frac{3(n^2+n+2)}{2n(n+2)}4c = 4\left(c^3 + c + (m_3 - m_4)c\bar{c}^3 + (m_5 - m_6)c^3\bar{c}\right). \tag{4.38b}$$

Subtracting (4.38b) from (4.38a), we find that

$$\begin{aligned} 0 &= (m_3 - m_4 - (m_5 - m_6))(c^3\bar{c} - c\bar{c}^3) \\ &= (m_3 - m_4 - (m_5 - m_6))c\bar{c}\frac{4}{n(n+2)}, \end{aligned}$$

hence, $m_3 - m_4 = m_5 - m_6$. After dividing (4.38a) by $4c$, we find, at last, that

$$\frac{3(n^2+n+2)}{2n(n+2)} = 1 + c^2 + (m_3 - m_4)(c^2 + \bar{c}^2)\bar{c}. \tag{4.39}$$

After some more algebraic manipulation, (4.39) reduces to

$$\sqrt{(n-2)n(n+2)} = \frac{4n}{n-1}(m_3 - m_4). \tag{4.40}$$

Recall that, if $\sqrt{r} \in \mathbf{Q}$ for an integer r, then $\sqrt{r} \in \mathbf{Z}$. Thus, (4.40) implies that the integer $(n-2)n(n+2)$ is a square. If n is odd, then $n-2$, n and $n+2$ are pairwise relatively prime, and since their product is a square, each must be a square. This is impossible: positive squares do not differ by 1, 2 or 4. If n is even, write $n - 2 = 2^a u_1$, $n = 2^b u_2$ and $n + 2 = 2^c u_3$, with $a, b, c \geqslant 1$ and u_j odd. Again, the u_j's must be pairwise relatively prime, and so are squares. By taking $n \bmod 4$, we see that either two of $\{a, b, c\}$ are even or two are odd. In the first case, there are two squares among $\{n-2, n, n+2\}$; in the second case, there are two squares among $\{n/2 - 1, n/2, n/2 + 1\}$. Either case is impossible, and at long last, this completes the proof. ■

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