MINIMAL VOLUME k-POINT LATTICE d-SIMPLICES

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ABSTRACT. We show via triangulations that for $d \geq 3$ there is exactly one class (under unimodular equivalence) of nondegenerate lattice simplices in $\mathbb{R}^d$ with minimal volume and k interior lattice points.

1. INTRODUCTION

A *d-polytope* P is a polytope of dimension d. If its *vertex set* $\mathscr{V}(P)$ is a subset of $\mathbb{Z}^d$, then P is a *lattice* d-polytope. If in addition $|\mathscr{V}(P)| = d+1$, then P is a lattice *d-simplex*. The convex hull of $\mathscr{P} = \{v_1, \cdots, v_n\} \subset \mathbb{Z}^d$, denoted by conv($\mathscr{P}$), is a lattice polytope with at most n vertices and dimension at most d. This notation will be used loosely; for convenience, we use conv(P, v) to mean conv$(\mathscr{V}(P) \cup \{v\})$ when it is clear P is a polytope and v is a point. As used in [19], we say that P is *clean* if $\partial P \cap Z^d = \mathscr{V}(P)$, where ∂P is the boundary of P. If in addition int(P), the *interior* of P, contains k lattice points, then P is a clean *k-point* lattice polytope. If $k = 0$, then the polytope is *empty*. We use $\mathscr{S}^d_k$ to denote the collection of clean k-point lattice d-simplices. Unless otherwise stated, all polytopes are taken to be convex d-polytopes.

Reznick proved in [18] and [19] that any lattice tetrahedron with at least one clean face is unimodularly equivalent to some $T_{a,b,n}$, the lattice tetrahedron with vertex set

$$\{\ (0,0,0),\ (1,0,0),\ (0,1,0),\ (a,b,n)\ \}$$

where $(a, b, n) \in \mathbb{Z}^3$ and $0 < a, b < n$. Reznick also classified the set of clean 1-point tetrahedra, up to equivalence under unimodular transformations, using barycentric coordinates. Very recently, Bey, Henk, and Wills proved in [2] that if P is a lattice d-polytope, not necessarily clean, and P has k interior lattice points, then for $d \geq 1$, the volume of P satisifies

$$\mathrm{Vol}(P) \geq \frac{1}{d!}(dk + 1). \tag{1}$$

Moreover, they showed that for $k = 1$, equality holds if and only if P is unimodularly equivalent to the simplex $S_d(1)$, where

$$S_d(k) = \mathrm{conv}\left(e_1, \cdots, e_d, -k\sum_{i=1}^{d} e_i\right)$$

and e_i denotes the i-th unit point. This is not true for $d = 2$ and $k > 2$. We will show that equality holds in (1) for all $k > 0$ if and only if $d \geq 3$ and P is unimodularly equivalent to $S_d(k)$. We first prove that if $T \in \mathscr{S}^d_k$ and $\mathrm{Vol}(T) = \frac{1}{d!}(dk + 1)$, then the interior points lie on a line passing through some vertex of T. We then show that such simplices are unimodularly equivalent to $T_{a_1,\cdots,a_d}$, the d-simplex

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whose vertex set consists of the origin, the points e_i ($1 \le i \le d-1$), and the point $(a_1, \cdots, a_d)$, where $a_j = dk$ for $1 \le j < d$ and $a_d = dk+1$. Finally, we will show that $S_d(k) \in \mathscr{S}_k^d$ and $\mathrm{Vol}(S_d(k)) = \frac{1}{d!}(dk+1)$.

2. Preliminaries

The following definitions are taken from [1], [7], [16], and [24]. A *j-flat* is a j-dimensional affine subspace of $\mathbb{R}^d$. *Points*, *lines*, and *planes* are 0-flats, 1-flats, and 2-flats, respectively. The *affine hull* of a set $\mathscr{P} \subset \mathbb{R}^d$, denoted by $\mathrm{aff}(\mathscr{P})$, is the intersection of all flats containing $\mathscr{P}$. Equivalently, $\mathrm{aff}(\mathscr{P})$ is the smallest flat containing $\mathscr{P}$. We say $\mathscr{P}$ is in *general position* if no $j+2$ points of $\mathscr{P}$ lie in a j-flat, where $j < d$. A hyperplane

$$H = \{x \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

is a $(d-1)$-flat. If P is a d-polytope, then H is a *supporting hyperplane* of P if P lies entirely on one side of H. A *face* of P is an intersection $P \cap H$, where H is a supporting hyperplane. If we allow degenerate hyperplanes, then P is a face of P corresponding to $H = \mathbb{R}^d$; $\emptyset$ is also a face of P corresponding to a hyperplane that does not meet P. A *j-face* of P is a j-dimensional face of P. A $(d-1)$-face is a *facet*, a 1-face is an *edge*, and a 0-face is a *vertex*. One property of d-polytopes is that any j-face of P is contained in at least $d-j$ facets of P.

Lemma 1. [7, Section 3.1] *If $0 \le i < j \le d-1$ and if P is a d-polytope, each i-face of P is the intersection of the family of j-faces of P containing it. There are at least $j+1-i$ such faces.*

We generally use capital letters to denote d-polytopes. In particular P and Q are d-polytopes, and S and T are d-simplices. Capital script letters will generally denote sets of d-simplices. In particular, $\mathscr{P}$ is a set of points (0-simplices).

3. Triangulations and Refinements

Borrowing from [14], we let $\mathscr{P}$ denote a set of n distinct points in $\mathbb{R}^d$, where $n \ge d+1$ and $d \ge 2$. Assume $\mathscr{P}$ does not lie entirely in a hyperplane. Let $P = \mathrm{conv}(\mathscr{P})$. A *triangulation*, $\mathscr{T}$, of $\mathscr{P}$ (or of P with the dependence on $\mathscr{P}$ understood) is a set of nondegenerate d-simplices $\{T_i\}$ with the following properties.

(a) All vertices of each simplex are members of $\mathscr{P}$.
(b) The interiors of the simplices are pairwise disjoint.
(c) Each facet of a simplex is either on the boundary of P, or else is a common facet of exactly two simplices.
(d) Each simplex contains no points of $\mathscr{P}$ other than its vertices.
(e) The union of $\{T_i\}$ is $\mathscr{P}$ and the union of T_i is P.

Since each d-simplex has volume at least $\frac{1}{d!}$, one immediate consequence is

$$\mathrm{Vol}(P) \ge \frac{1}{d!} \cdot |\mathscr{T}|. \tag{2}$$

To prove (1), Bey, Henk, and Wills showed P with k interior lattice points can be decomposed into at least $dk+1$ nondegenerate d-subpolytopes. Any d-polytope must contain a d-simplex as a subpolytope, so (2) still holds if $\mathscr{T}$ is replaced by this decomposition. We will present a slight variation of their theorem and its proof by using triangulations.

Definition. Let P be a lattice d-polytope. A *lattice triangulation* $\mathscr{T}$ of P is a triangulation of some set $\mathscr{P} \subset \mathbb{Z}^d$ such that $\mathscr{V}(P) \subseteq \mathscr{P} \subseteq P \cap \mathbb{Z}^d$. Note that if $\mathscr{V}(P) \subseteq \mathscr{P}, \mathscr{P}' \subseteq P \cap \mathbb{Z}^d$ and $\mathscr{P} \neq \mathscr{P}'$, we still have $\operatorname{conv}(\mathscr{P}) = P = \operatorname{conv}(\mathscr{P}')$. On the other hand, the triangulation $\mathscr{T}$ of $\mathscr{P}$ and the triangulation $\mathscr{T}'$ of $\mathscr{P}'$ are necessarily different. Fortunately, we can reconstruct the *vertex set* of $\mathscr{T}$, denoted by $\mathscr{V}(\mathscr{T})$, by taking the union of all vertices of all $T \in \mathscr{T}$. Thus $\mathscr{V}(\mathscr{T}) = \mathscr{P}$ and $\mathscr{V}(\mathscr{T}') = \mathscr{P}'$.

Definition. Let P be a lattice polytope and $\mathscr{T}$ be a lattice triangulation P. We say $\mathscr{T}'$ is a *refinement* of P (and write $\mathscr{T} \prec \mathscr{T}'$) provided $\mathscr{T}'$ is a lattice triangulation of P, $\mathscr{V}(\mathscr{T}) \subsetneq \mathscr{V}(\mathscr{T}')$, and for all $T' \in \mathscr{T}'$ there exists $T \in \mathscr{T}$ such that $T' \subseteq T$. We say that $\mathscr{T}$ is a *full lattice triangulation* if $\mathscr{V}(\mathscr{T}) = P \cap \mathbb{Z}^d$. Otherwise we say $\mathscr{T}$ is a *partial lattice triangulation*.

Naturally, triangulations of $\mathscr{P} \subset \mathbb{R}^d$ partition $\operatorname{conv}(\mathscr{P})$ into simplices. On the other hand, there exist partitions of $\operatorname{conv}(\mathscr{P})$ that satisify all but condition (d) in the definition of a triangulation.

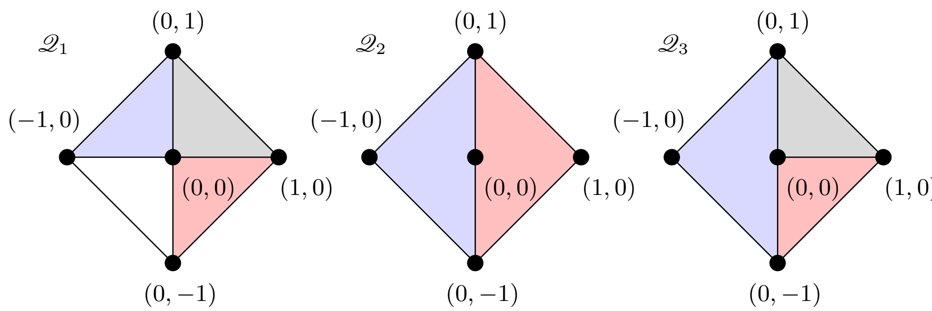


FIGURE 1. Partitions of P into triangles

Example 1. Let $\mathscr{P} = \{e_1, e_2, -e_1, -e_2\} \subset \mathbb{R}^2$ and let $P = \operatorname{conv}(\mathscr{P})$. Note that P is a 1-point lattice polygon, and its interior lattice point is the origin. Three possible partitions $\mathscr{Q}_1$, $\mathscr{Q}_2$, and $\mathscr{Q}_3$ of P into triangles are shown in Figure 1. Note that $\mathscr{Q}_1$ is a triangulation of $\mathscr{P} \cup \{(0,0)\}$ and a full lattice triangulation of P. However, $\mathscr{Q}_1$ is not a triangulation of $\mathscr{P}$. The middle partition $\mathscr{Q}_2$ is both a triangulation of $\mathscr{P}$ and a partial lattice triangulation of P (viewed as a triangulation of $\mathscr{V}(P) = \mathscr{P}$). The partition $\mathscr{Q}_3$ on the right is not a triangulation since it fails condition (d).

The following theorem, the proof of which can be found in the appendix of [1], guarantees the existence of a triangulation of the vertex set of a polytope.

Theorem 2. [1, Theorem 3.1] *Every convex polytope P can be triangulated using no new vertices. That is, there exists a triangulation of $\mathscr{V}(P)$.*

Let P be a clean, non-empty lattice d-polytope, and suppose $w \in \operatorname{int}(P) \cap \mathbb{Z}^d$. We construct a *basic lattice triangulation* $\mathscr{T}_w$ of P in the following manner. Theorem 2 guarantees that each facet F of P, as a $(d-1)$-polytope, has a (lattice) triangulation $\mathscr{T}_F$ of $\mathscr{V}(F)$. Let $\mathscr{F}$ be the set of facets of P and let $\mathscr{B}$ be the set

$$\mathscr{B} = \bigcup_{F \in \mathscr{F}} \mathscr{T}_F$$

of $(d-1)$-simplices. Finally, let $\mathscr{T}_w = \{ \text{ conv}(S, w) \ : \ S \in \mathscr{B} \ \}$. It is easy to check that $\mathscr{T}_w$ is a lattice triangulation of P. Note that

$$|\mathscr{T}_w| = |\mathscr{B}| \geq |\mathscr{F}| \geq d+1. \tag{3}$$

In particular, if $T \in \mathscr{S}_k^d$, where $k \geq 1$, and $F_1, \cdots, F_{d+1}$ are the facets of T, then a basic lattice triangulation $\mathscr{T}_w$ is simply the convex hull of the facets of T with an interior lattice point w of T. Moreover, $\mathscr{T}_w$ is a refinement of the trivial triangulation $\mathscr{T}_0 = \{T\}$. The main idea in proving the collinearity property of $\text{int}(T) \cap \mathbb{Z}^d$ is to start with the trivial triangulation $\mathscr{T}_0 = \{T\}$ and obtain a sequence $\mathscr{T}_1 \prec \cdots \prec \mathscr{T}_k$ of refinements such that $\mathscr{T}_k$ is a full triangulation of T and $|\mathscr{T}_k| \geq dk+1$, and then show that noncollinearity forces $|\mathscr{T}_k| > dk+1$. The following lemma appears as an assertion in the proof of (1) in [2]. We prove it here since it is crucial in computing $|\mathscr{T}_i| - |\mathscr{T}_{i-1}|$.

Lemma 3. *Suppose P is a clean, non-empty, lattice d-polytope. Let $\mathscr{T}$ be a partial triangulation of P. If S is a j-face of some $T \in \mathscr{T}$, and $\mathscr{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d \neq \emptyset$, then S is contained in at least $d+1-j$ simplices in $\mathscr{T}$.*

Proof. Let $w \in \mathscr{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d$. By Lemma 1, S is contained in at least $d-j$ facets of T. Moreover w is a vertex of each facet of T containing S. It follows that these facets are not contained in facets of P and are therefore shared by exactly two simplices in $\mathscr{T}$. On the other hand any two simplices in $\mathscr{T}$ intersect in at most one common facet. Thus S is contained in at least $d-j$ other simplices in $\mathscr{T}$. ■

Theorem 4. *Let P be a clean, non-empty, lattice d-polytope. Let $\mathscr{T}$ be a partial triangulation of P. For any $w \in \text{int}(P) \cap \mathbb{Z}^d \backslash \mathscr{V}(\mathscr{T})$ there exists a refinement $\mathscr{T}'$ of $\mathscr{T}$ such that $\mathscr{V}(\mathscr{T}') = \mathscr{V}(\mathscr{T}) \cup \{w\}$ and $|\mathscr{T}'| \geq |\mathscr{T}| + d$.*

Proof. Since $\mathscr{T}$ is a partial triangulation, there exists an interior lattice point w of P such that $w \notin \mathscr{V}(\mathscr{T})$. Moreover, w must lie in the relative interior of some j-face ($1 \leq j \leq d$), say S, of some simplex in $\mathscr{T}$. Note that S is in fact a j-simplex. Let $\mathscr{V}(S) = \{v_1, \cdots, v_{j+1}\}$ and consider the basic triangulation $\mathscr{T}_w$ of S into j-simplices, where

$$\mathscr{T}_w = \{ \text{ conv}(\mathscr{V}(S) \cup \{w\} \backslash \{v_i\}) \ : \ 1 \leq i \leq j+1 \ \}.$$

Clearly $\mathscr{T}_w$ is a refinement of S into j-simplices. This refinement of S induces a refinement of any d-simplex containing S. More precisely, if $T \in \mathscr{T}$ contains S, then the set

$$\{ \text{ conv}(\mathscr{V}(\mathscr{T}) \cup \{w\} \backslash \{v_i\}) \ \}$$

is a lattice triangulation of T. Since $w \in \text{int}(S)$ and P is clean, $S \not\subset \partial P$. Thus $\mathscr{V}(S) \cap \text{int}(P) \cap \mathbb{Z}^d \neq \emptyset$. By Lemma 3, there are at least $d+1-j$ simplices in $\mathscr{T}$ containing S as a j-face. Now consider $\mathscr{T}$ with all such d-simplices in $\mathscr{T}$ replaced with their respective induced triangulations and take this to be $\mathscr{T}'$. By construction, w is contained in a simplex $T' \in \mathscr{T}'$ if and only if $w \in \mathscr{V}(T')$. Hence $\mathscr{T}'$ is a refinement of $\mathscr{T}$ such that $\mathscr{V}(\mathscr{T}') = \mathscr{V}(T) \cup \{w\}$ and

$$|\mathscr{T}'| \geq |\mathscr{T}| + (d+j-1)(j+1) - (d+j-1) = |\mathscr{T}| + (d+j-1)j.$$

Finally,

$$(d+1-j)j - d = (d-j)(j-1) \geq 0 \quad \text{for} \quad 1 \leq j \leq d, \tag{4}$$

which implies $(d+1-j)j \geq d$. ■

In the proof above, it is important to note that equality in (4) holds if and only if $j = d$ or $j = 1$. Equally important is that if $\mathscr{V}(\mathscr{T}') \neq P \cap \mathbb{Z}^d$, then $\mathscr{T}'$ is again a partial triangulation.

Corollary 5. *If $T \in \mathscr{S}_k^d$ and $\mathscr{T}_0 = \{T\}$, then there exists a sequence*

$$\mathscr{T}_0 \prec \cdots \prec \mathscr{T}_k \tag{5}$$

of refinements of $\mathscr{T}_0$ such that $\mathscr{T}_k$ is a full triangulation of T and $|\mathscr{T}_k| \geq dk+1$. Moreover, $\mathrm{Vol}(T) \geq \frac{1}{d!}(dk+1)$.

Proof. For $k = 0$, then the sequence consists only of $\mathscr{T}_0$. If $k > 0$ then $\mathscr{T}_0 = \{T\}$ is a partial triangulation of T. Let $w_1, \cdots, w_k$ be an arbitrary enumeration of the interior lattice points of T. For $1 \leq i \leq k$, we refine $\mathscr{T}_{i-1}$ into $\mathscr{T}_i$, using Theorem 4 with $w = w_i$. After k refinements, $\mathscr{T}_k$ is a full triangulation of T, and $|\mathscr{T}_k| \geq |\mathscr{T}_0| + dk = dk + 1$. Lastly, (2) implies $\mathrm{Vol}(T) \geq \frac{1}{d!}(dk+1)$. ■

Corollary 6. *Suppose $T \in \mathscr{S}_k^d$ where $k \geq 2$. Let $w_1, \cdots, w_k$ be an arbitrary enumeration of* $\mathrm{int}(T) \cap \mathbb{Z}^d$*, and let (5) be the corresponding refinement sequence guaranteed by Corollary 5. If w_{i+1} lies in a j-face of a simplex in $\mathscr{T}_i$, where $i > 0$ and $1 < j < d$, then $|\mathscr{T}_{i+1}| - |\mathscr{T}_i| > d$ and* $\mathrm{Vol}(T) > \frac{1}{d!}(dk+1)$.

Proof. This follows immediately from Theorem 4 and (3), and (4). ■

In the context of Corollary 5, (4) implies that for each $\mathscr{T}_i$ in (5), $|\mathscr{T}_i| \geq di+1$. Since equality in (4) holds if and only if $j = 1$ or $j = d$, $|\mathscr{T}_i| = di + 1$ if and only if for each i, w_{i+1} lies in the relative interior of an edge in $\mathscr{T}_i$, or in the relative interior of a simplex in $\mathscr{T}_i$. Note that this must be true for any ordering of the w_i's.

Finally, to prove (1) for a general non-empty lattice d-polytope P, we start with a basic triangulation of P and obtain a refinement sequence similar to that of Corollary 5.

Corollary 7. [2, Theorem 1.2] *If P is a lattice d-polytope with $k \geq 0$ interior lattice points, then there exists a sequence*

$$\mathscr{T}_1 \prec \cdots \prec \mathscr{T}_k$$

of lattice triangulations of P such that $\mathscr{T}_1$ is a basic triangulation of P, $\mathscr{T}_k$ is a full triangulation of P and $|\mathscr{T}_k| \geq dk+1$. Moreover, $\mathrm{Vol}(P) \geq \frac{1}{d!}(dk+1)$.

Proof. If $k = 0$, take $\mathscr{T}_1$ to be the triangulation guaranteed by Theorem 2. If $k \geq 1$, then consider an arbitrary enumeration $w_1, \cdots, w_k$ of the the interior points of P. Let $\mathscr{T}_1$ be the basic triangulation $\mathscr{T}_{w_1}$ of T. Either $k = 1$ and $\mathscr{T}_1$ is a full triangulation, or we can apply Theorem 4, as in the proof of Corollary 5, to obtain a refinement sequence

$$\mathscr{T}_1 \prec \cdots \prec \mathscr{T}_k$$

such that $|\mathscr{T}_k| \geq dk+1$. Finally, (2) implies $\mathrm{Vol}(P) \geq \frac{1}{d!}(dk+1)$. ■

In general, we need not start with a basic triangulation of P. It is easy to check that so long as $\mathscr{T}$ is a partial triangulation such that

$$|P \cap \mathbb{Z}^d| - |\mathscr{V}(\mathscr{T})| = k - j \quad \text{and} \quad |\mathscr{T}| \geq dj + 1,$$

then we can still refine $\mathscr{T}$ into a full triangulation with at least $dk+1$ simplices by applying Theorem 4. This is in fact equivalent to the inductive step in [2], with triangulations replaced by decompositions into d-subpolytopes. The proof of Corollary 7 is otherwise essentially the same as that in [2].

4. Properties of $d+2$ Points in $\mathbb{R}^d$

In the context of Corollary 6, if we can show $j = d$ implies $\mathrm{Vol}(T) > \frac{1}{d!}(dk+1)$ as well, then any two interior lattice points of T must be collinear with some vertex of T. As a base case, we first consider the possible configurations of any $T \in \mathscr{S}_2^d$, where $d \geq 3$. Suppose T has interior lattice points w_1 and w_2, and let $\mathscr{V}(T) = \{v_1, \cdots, v_{d+1}\}$. The sets

$$\mathscr{P}_i = \mathscr{V}(T)\backslash\{v_i\} \cup \{w_1, w_2\} \tag{6}$$

are all sets of $d+2$ points not contained in a hyperplane. Many properties of such sets are discussed in [3], [8], [13], [14], [16], [17], and [22]. Two relevant and well known results in the theory of convex bodies are Carathéodory's theorem [4] and Radon's theorem [22].

Carathéodory's Theorem. *If $\mathscr{P} = \{v_1, \cdots, v_{d+1}\} \subset \mathbb{R}^d$ is not contained in a hyperplane, then every $x \in \mathbb{R}^d$ can be expressed as*

$$x = \sum_{i=1}^{d+1} \alpha_i v_i, \quad \textit{where} \quad \alpha_i \in \mathbb{R}, \quad v_i \in \mathscr{P}, \quad \textit{and} \quad \sum_{i=1}^{d+1} \alpha_i = 1.$$

The coefficients α_i in Carathéodory's theorem are the *barycentric coordinates* of x relative to $\mathrm{conv}(\mathscr{P})$. If $P = \mathrm{conv}(\mathscr{P})$ is a d-simplex, then the barycentric coordinates of x relative to the simplex P are the numbers $\alpha_1, \cdots, \alpha_{d+1}$ satisfying

$$\underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{d+1} \end{bmatrix}}_{1\times(d+1)} \cdot \underbrace{\begin{bmatrix} v_1 & 1 \\ v_2 & 1 \\ \vdots & \vdots \\ v_{d+1} & 1 \end{bmatrix}}_{(d+1)\times(d+1)} = \underbrace{\begin{bmatrix} x & 1 \end{bmatrix}}_{1\times(d+1)}.$$

For each i, the sign of α_i indicates the position of x relative to the hyperplane H_i containing the facet of P opposite vertex v_i. That is, $\alpha_i > 0$ when v_i and x are on the same side of H_i, $\alpha_i < 0$ if v_i and x are on opposite sides of H_i, and $\alpha_i = 0$ if x lies in H_i. The barycentric coordinates of x relative to T are all positive if and only if $x \in \mathrm{int}(P)$. Thus any point $x \in \mathscr{P}_i$ (cf. (6)), can be described in terms of its barycentric coordinates relative to the simplex $\mathrm{conv}(\mathscr{P}_i)$.

Radon's Theorem. *If $\mathscr{A}$ is a set of $k \geq d+2$ points in $\mathbb{R}^d$, then there exist a partition $\{\mathscr{A}_1, \mathscr{A}_2\}$ of $\mathscr{A}$ such that $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ and $\mathrm{conv}(\mathscr{A}_1) \cap \mathrm{conv}(\mathscr{A}_2) \neq \emptyset$.*

The partition in Radon's theorem is called a *Radon partition* of $\mathscr{A}$. A Radon partition *in* $\mathscr{A}$ is a Radon partition of a subset of $\mathscr{A}$. Let $\{\mathscr{A}_1, \mathscr{A}_2\}$ and $\{\mathscr{A}_1', \mathscr{A}_2'\}$ be Radon partitions of $\mathscr{A}$. Then $\{\mathscr{A}_1, \mathscr{A}_2\}$ *extends* $\{\mathscr{A}_1', \mathscr{A}_2'\}$ provided $\mathscr{A}_i' \subseteq \mathscr{A}_i$. In [8], Hare and Kennely introduced the notion of a *primitive Radon partition*, a Radon partition that is minimal with respect to extension. An immediate consequence is that if $\{\mathscr{A}_1, \mathscr{A}_2\}$ is a Radon partition of $\mathscr{A}$, then there exists a primitive Radon partition in $\mathscr{A}$ such that $\{\mathscr{A}_1, \mathscr{A}_2\}$ extends it. Breen proved in [3] that $\{\mathscr{A}_1, \mathscr{A}_2\}$ is a primitive Radon partition in $\mathscr{A}$ if and only if $\mathscr{A}_1 \cup \mathscr{A}_2$ is in general position in $\mathbb{R}^{|\mathscr{A}_1|+|\mathscr{A}_2|}$. Recall that $\mathscr{A}_1 \cup \mathscr{A}_2$ is in general position if no $j+2$ points in this union lie in a j-flat for all $j < |\mathscr{A}_1|+|\mathscr{A}_2|$. Peterson proved in [16] that the Radon partition of $d+2$ points in general position in $\mathbb{R}^d$ is unique, and both Breen and Peterson showed that if $\mathscr{A}_1 \cup \mathscr{A}_2$ is in general position in $\mathbb{R}^{|\mathscr{A}_1|+|\mathscr{A}_2|}$, then $\mathrm{conv}(\mathscr{A}_1) \cap \mathrm{conv}(\mathscr{A}_2)$

is a single point. Lastly, Proskuryakov proved in [17] that if $\mathscr{P} \subset \mathbb{R}^d$ is a set of $d+2$ points in general position, then two points will lie in the same component of the (unique) Radon partition of P if and only if they are separated by the hyperplane through the remaining d points. Kosmak also proved this result in [13] using affine varieties. These properties of Radon partitions are equivalent to Lawson's First and Second Theorems from [14].

Lawson's First Theorem. [14, Theorem 1] *Let $\mathscr{P} = \{v_1, \cdots, v_{d+2}\} \subset \mathbb{R}^d$ and suppose $\mathscr{P}$ does not lie entirely in any hyperplane. There is a partition of $\mathscr{P}$ into three sets $\mathscr{A}_0$, $\mathscr{A}_1$, and $\mathscr{A}_2$, and $\alpha_i \in \mathbb{R}$, satisifying*

$$\sum_{v_i \in \mathscr{A}_1} \alpha_i v_i = \sum_{v_i \in \mathscr{A}_2} \alpha_i v_i, \tag{7}$$

$$\sum_{v_i \in \mathscr{A}_1} \alpha_i = \sum_{v_i \in \mathscr{A}_2} \alpha_i = 1, \tag{8}$$

$$\alpha_i > 0 \quad \textit{if} \quad v_i \in \mathscr{A}_1 \cup \mathscr{A}_2. \tag{9}$$

The numbers α_i are uniquely determined by the set $\mathscr{P}$. We set $\alpha_i = 0$ if $v_i \in \mathscr{A}_0$. The sets $\mathscr{A}_0$ and $\{\mathscr{A}_1, \mathscr{A}_2\}$ are also unique.

Since the sets $\mathscr{A}_1$ and $\mathscr{A}_2$ in Lawson's First Theorem form the unique primitive Radon partition in $\mathscr{P}$, the point

$$\operatorname{conv}(\mathscr{A}_1) \cap \operatorname{conv}(\mathscr{A}_2) = \sum_{v_i \in \mathscr{A}_1} \alpha_i v_i = \sum_{v_i \in \mathscr{A}_2} \alpha_i v_i,$$

is in the relative interior of $\operatorname{conv}(\mathscr{A}_1)$ and $\operatorname{conv}(\mathscr{A}_2)$ by (7), (8), and (9). These sets also determine the possible triangulations of $\operatorname{conv}(\mathscr{P})$.

Lawson's Second Theorem. [14, Theorem 2] *Let $\mathscr{P} = \{v_1, \cdots, v_{d+2}\} \subset \mathbb{R}^d$, and let T_i be the simplex with vertex set $\mathscr{V}(T_i) = \mathscr{P} \backslash \{v_i\}$. There are at most two distinct triangulations of $P = \operatorname{conv}(\mathscr{P})$, namely*

$$\mathscr{T}_1 = \{T_i : v_i \in \mathscr{A}_1\} \quad \textit{and} \quad \mathscr{T}_2 = \{T_i : v_i \in \mathscr{A}_2\},$$

where the sets $\mathscr{A}_1$ and $\mathscr{A}_2$ are as defined in Lawson's First Theorem. The set $\mathscr{T}_j$ is a valid triangulation if and only if $|\mathscr{A}_j| > 1$, where $j \in \{1, 2\}$.

Corollary 8. [14, Corollary 1] *In the context of building triangulations, an enumeration of all possible configurations of $d+2$ points in $\mathbb{R}^d$, not lying in any hyperplane, is given by all of the possible ways of assigning values to $|\mathscr{A}_0|$, $|\mathscr{A}_1|$, and $|\mathscr{A}_2|$ satisfying*

$$|\mathscr{A}_2| \geq |\mathscr{A}_1| \geq 1, \tag{10}$$

$$|\mathscr{A}_2| \geq 2, \tag{11}$$

$$|\mathscr{A}_0| \geq 0, \tag{12}$$

$$|\mathscr{A}_0| + |\mathscr{A}_1| + |\mathscr{A}_2| = d+1. \tag{13}$$

Consider the possible sets of five points in $\mathbb{R}^3$ such that the convex hull is a non-degenerate polyhedron with two different triangulations. Lawson's Second Theorem implies $|\mathscr{A}_1| \geq 2$. Inequalities (10), (11), (12), and (13) imply either

$$(|\mathscr{A}_0|, |\mathscr{A}_1|, |\mathscr{A}_2|) = (0, 2, 3) \quad \text{or} \quad (|\mathscr{A}_0|, |\mathscr{A}_1|, |\mathscr{A}_2|) = (1, 2, 2).$$

Example 2. Suppose $\mathscr{P} = \{v_1, \cdots, v_5\} \subset \mathbb{R}^3$ is in general position. As shown on the left in Figures 2 and 3, let $P = \mathrm{conv}(\mathscr{P})$. In the context of Lawson's First Theorem, $\mathscr{A}_0 = \emptyset$, $\mathscr{A}_1 = \{v_1, v_5\}$, and $\mathscr{A}_2 = \{v_2, v_3, v_4\}$. The polyhedron P is a "bipyramid" with triangulations

$$\mathscr{T}_1 = \{\ \mathrm{conv}(v_1, v_2, v_3, v_4),\ \mathrm{conv}(v_2, v_3, v_4, v_5)\ \}$$

and

$$\mathscr{T}_2 = \{\ \mathrm{conv}(v_1, v_2, v_3, v_5),\ \mathrm{conv}(v_1, v_2, v_4, v_5),\ \mathrm{conv}(v_1, v_3, v_4, v_5)\ \}$$

by Lawson's Second Theorem.

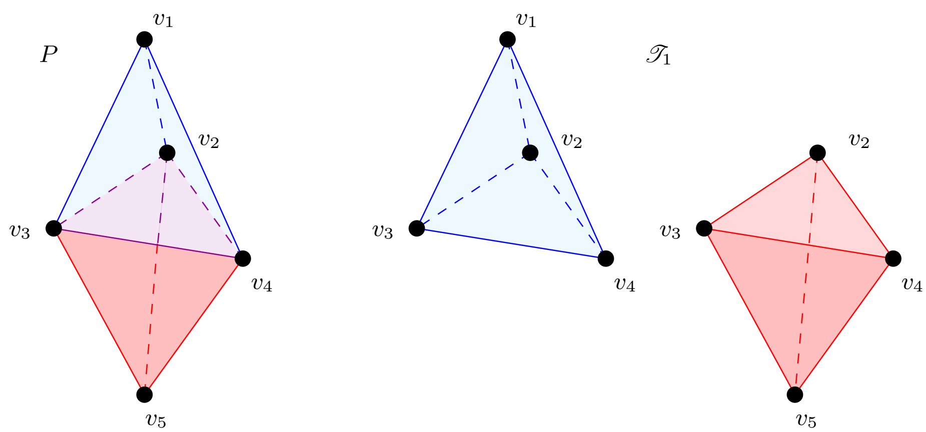


FIGURE 2. P (left) and the two simplices of $\mathscr{T}_1$ (right)

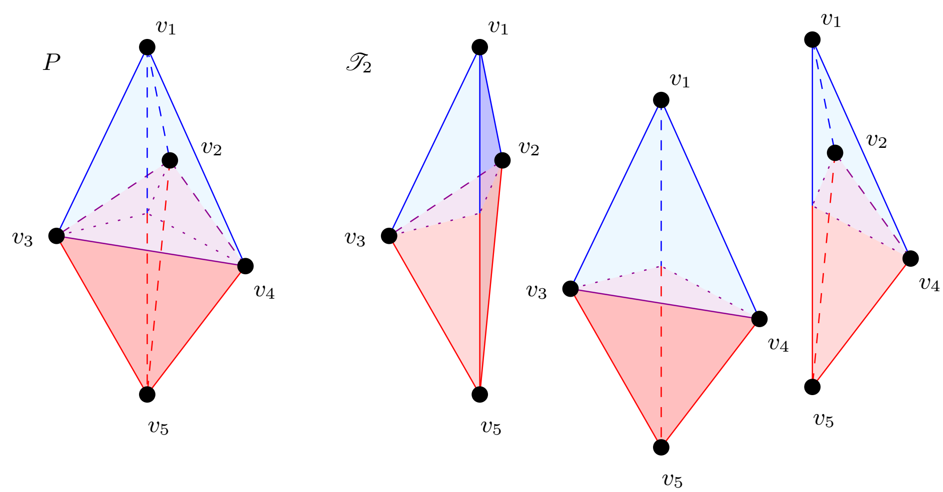


FIGURE 3. P (left) and and the three simplices of $\mathscr{T}_2$ (right)

Example 3. Consider $\mathscr{P} = \{v_1, \cdots, v_5\} \subset \mathbb{R}^3$ and $P = \operatorname{conv}(\mathscr{P})$ shown on the left in Figure 4. The polyhedron P is a nondegenerate pyramid whose base is a planar quadrilateral with vertex set $\{v_2, v_3, v_4, v_5\}$. In the context of Lawson's First Theorem, $\mathscr{A}_0 = \{v_1\}$, $\mathscr{A}_1 = \{v_2, v_4\}$ and $\mathscr{A}_2 = \{v_3, v_5\}$. Lawson's Second Theorem implies P has two triangulations

$$\mathscr{T}_1 = \{\ \operatorname{conv}(v_1, v_3, v_4, v_5),\ \operatorname{conv}(v_1, v_2, v_3, v_5)\ \}$$

and

$$\mathscr{T}_2 = \{\ \operatorname{conv}(v_1, v_2, v_4, v_5),\ \operatorname{conv}(v_1, v_2, v_3, v_4)\ \}.$$

Each triangulation is induced by the triangulation (in dimension 2) of the base.

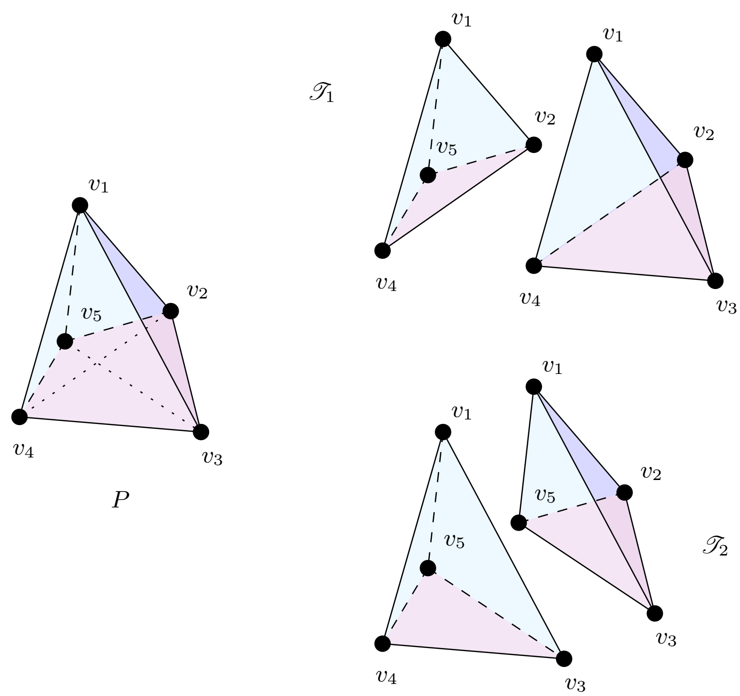


FIGURE 4. P (left) and both of its triangulations (right)

In both Examples 2 and 3, $|\mathscr{A}_1| = 2$, and $\operatorname{conv}(\mathscr{A}_1 \cup \mathscr{A}_2)$ can be viewed as two simplices joined at a common facet. Moreover, the line segment formed by the vertices not in the common facet intersects that facet at a point. Such a configuration is called a bipyramid. In general, a d-polytope P is a *d-bipyramid* if P is the convex hull of a line segment L and a $(d-1)$-simplex S such that the intersection $L \cap S$ is a single point contained in $\operatorname{int}(L) \cap \operatorname{int}(S)$. A 1-bipyramid is simply a line segment with a point in its interior (or the convex hull of three collinear points). In Example 2, P itself is a 3-bipyramid, whereas in Example 3 P contains a 2-bipyramid (its base). The following result is a direct consequence of Lawson's theorems.

Corollary 9. *Let P be a d-polytope with $d+2$ vertices. If there exists a subset $\mathscr{P} \subseteq \mathscr{V}(P)$ such that $\operatorname{conv}(\mathscr{P})$ is a j-bipyramid, where $2 \le j \le d$, then $\mathscr{V}(P)$ has exactly two triangulations, one with cardinality 2 and other with cardinality j.*

5. Existence of Bipyramids in $T \in \mathscr{S}_k^d$

For $k \geq 2$, under what conditions will $T \in \mathscr{S}_k^d$ contain a bipyramid? Consider the following construction of possible bipyramids from a d-simplex. Let S be a d-simplex, and let $\mathscr{V}(S) = \{v_1, \cdots, v_{d+1}\}$. For any point x exterior to S, let (α_i) be the barycentric coordinates of x relative to S. By the definition of a bipyramid, $P = \text{conv}(S, x)$ is a bipyramid if and only if exactly one α_i is negative, and the remainder are positive. If we allow for, say, j of the α_i to be zero, then P will contain a $(d-j)$-bipyramid.

The geometric interpretation is as follows. Let H_i be the hyperplane containing the facet of S opposite v_i. Then P is a d-bipyramid if x and S are on opposite sides of exactly one H_i, and x is on the same side of the remaining hyperplanes as S. If x is contained in j of these hyperplanes, then P contains a $(d-j)$-bipyramid.

Theorem 10. *Suppose $T \in \mathscr{S}_k^d$, where $k \geq 2$. Let w_1 and w_2 be any two interior lattice points of T, and let $\mathscr{T}_{w_1} = \{T_i\}$ be the basic triangulation of T with respect to w_1. If w_2 lies in the relative interior of a simplex in $\mathscr{T}_{w_1}$, say T_n, then there exist numbers α_i such that*

$$w_2 = \alpha_1 w_1 + \alpha_n v_n + \sum_{\substack{i=2 \\ i \neq n}}^{d+1} \alpha_i v_i, \qquad \sum_{i=1}^{d+1} \alpha_i = 1,$$

$\alpha_1 > 0$, $\alpha_n < 0$, and $\alpha_i \geq 0$ otherwise.

Proof. Let $\mathscr{V}(T) = \{v_1, \cdots, v_{d+1}\}$ and F_i be the facet of T opposite the vertex v_i. We may assume without loss of generality that $T_i = \text{conv}(F_i, w_1)$ and $n = d+1$ (i.e. $w_2 \in \text{int}(T_{d+1})$). Let L be the line through v_{d+1} and w_1.

Case 1: If w_2 lies on L, then w_1 is necessarily between v_{d+1} and w_2. There exists $\alpha > 1$ such that

$$w_2 = v_{d+1} + \alpha(w_1 - v_{d+1}) = \alpha w_1 + (1-\alpha) v_{d+1}.$$

Choose $\alpha_1 = \alpha$, $\alpha_{d+1} = 1 - \alpha$, and $\alpha_i = 0$ for $2 \leq i \leq d+1$.

Case 2: Suppose w_2 does not lie on L. Let $x = L \cap F_{d+1}$. If $x \notin \text{int}(F_{d+1})$, then the line segment $v_{d+1}x$, which contains w_1, would be contained in a facet of T and contradict the cleanliness of T. Thus $x \in \text{int}(F_{d+1})$ and consequently x cannot be a lattice point. Since $x \in \text{int}(F_{d+1})$ if and only if there exists β_i such that

$$x = \sum_{i=1}^{d} \beta_i v_i, \quad \beta_i > 0, \quad \text{and} \qquad \text{and} \quad \sum_{i=1}^{d} \beta_i = 1,$$

the set $\mathscr{P} = \mathscr{V}(T_{d+1}) \cup \{x\} \not\subset \mathbb{Z}^d$ can be partitioned into

$$\mathscr{A}_0 = \{w_1\}, \quad \mathscr{A}_1 = \{x\}, \quad \text{and} \quad \mathscr{A}_2 = \mathscr{V}(F_{d+1})$$

according to Lawson's First Theorem. By Lawson's Second Theorem, $\mathscr{P}$ has exactly one (non-lattice) triangulation

$$\mathscr{T} = \{ \ \text{conv}(\mathscr{V}(T_{d+1}) \backslash \{v_i\} \cup \{x\}) \ : \ 1 \leq i \leq d \ \}.$$

Since $T_{d+1} = \text{conv}(\mathscr{P})$ and $w_2 \in \text{int}(T_{d+1})$, w_2 must be in some (non-lattice) simplex in $\mathscr{T}$. Without loss of generality, suppose

$$w_2 \in \text{conv}(v_2, \cdots, v_d, w_1, x).$$

Then there exist γ_i such that $\gamma_i \geq 0$ for $1 \leq i \leq d+1$,

$$\sum_{i=1}^{d+1} \gamma_i = 1, \quad \text{and} \quad w_2 = \gamma_1 w_1 + \sum_{i=2}^{d} \gamma_i v_i + \gamma_{d+1} x.$$

Since w_2 cannot lie in any face of T, $\gamma_1 > 0$. Similarly, $\gamma_{d+1} > 0$ since w_2 is not in any face of T_{d+1}. Furthermore, the assumption that w_2 does not lie on L implies one of the remaining γ_i $(2 \leq i \leq d)$ must also be positive. Since w_1 lies between v_{d+1} and x on L, there exists $\mu > 1$ such that

$$x = v_{d+1} + \mu(w_1 - v_{d+1}) = \mu w_1 + (1-\mu)v_{d+1}.$$

It follows that

$$w_2 = (\gamma_1 + \mu\gamma_{d+1})w_1 + \sum_{i=2}^{d} \gamma_i v_i + (1-\mu)\gamma_{d+1} v_{d+1}.$$

Let

$$\alpha_1 = \gamma_1 + \mu\gamma_{d+1}, \quad \alpha_{d+1} = (1-\mu)\gamma_{d+1}, \quad \text{and} \quad \alpha_i = \gamma_i \quad \text{for} \quad 2 \leq i \leq d.$$

Note that

$$\sum_{i=1}^{d+1} \alpha_i = [\mu + (1-\mu)]\gamma_{d+1} + \sum_{i=1}^{d} \gamma_i = 1,$$

$\alpha_1 > 0$, and $\alpha_{d+1} < 0$. The remaining α_i are nonnegative, and at least one is positive since at least one of the γ_i is positive for $2 \leq i \leq d$. ∎

The geometric interpretation of Theorem 10 is that for some simplex $T_m \neq T_n$ in $\mathscr{T}_{w_1}$, $\operatorname{conv}(T_m, w_2)$ contains a j-bipyramid P, and $\mathscr{V}(P)$ includes w_1 and w_2. In terms of triangulations, $\mathscr{V}(T) \cup \{w_1, w_2\}$ has two triangulations, provided $j > 1$.

Corollary 11. *Suppose $T \in \mathscr{S}_k^d$, where $k \geq 2$. If w_2 lies in the relative interior of a simplex in the basic triangulation $\mathscr{T}_{w_1}$ of T, then the set $\mathscr{V}(T) \cup \{w_1, w_2\}$ has two triangulations provided w_1 and w_2 are not collinear with any $v \in \mathscr{V}(T)$.*

Proof. Let $\mathscr{T}_1 = \mathscr{T}_{w_1} = \{T_i\}$. Without loss of generality, suppose $w_2 \in \operatorname{int}(T_{d+1})$. Let $\mathscr{T}_{w_2}$ be the basic triangulation of T_{d+1} with respect to w_2. One triangulation of $\mathscr{V}(T) \cup \{w_1, w_2\}$ is the refinement $\mathscr{T}_2$ of $\mathscr{T}_1$ guaranteed by Theorem 4 (with $w = w_2$), where

$$\mathscr{T}_2 = \mathscr{T}_{w_1} \backslash \{T_{d+1}\} \cup \mathscr{T}_{w_2}.$$

Let S_i be the facet of T_{d+1} opposite v_i for $1 \leq i \leq d$, and let S_{d+1} be the facet of T_{d+1} opposite w_1. Theorem 10 guarantees there exists $1 \leq m \leq d$ such that

$$\operatorname{conv}(T_m, w_2) = \operatorname{conv}\left(T_m \bigcup \operatorname{conv}(S_m, w_2)\right).$$

contains a j-bipyramid. Note that both T_m and $\operatorname{conv}(S_m, w_2)$ are simplices in T_{w_2}. Since w_1 and w_2 are not collinear with any $v \in \mathscr{V}(T)$, T is clean, and no three vertices of T are collinear, it follows that $j > 1$. Corollary 9 implies $\mathscr{V}(T_m) \cup \{w_2\}$ has two triangulations, one of which is contained in $\mathscr{T}_2$. That is,

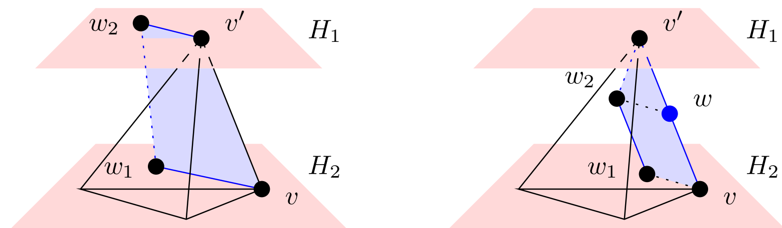
$$\operatorname{conv}(T_m, w_2) = T_m \bigcup \operatorname{conv}(S_m, w_2).$$

Thus $\mathscr{V}(T) \cup \{w_1, w_2\}$ has two triangulations. ∎

Among all possible d-bipyramids that have two triangulations, 2-bipyramids are unique in that their triangulations have the same cardinality. The existence of lattice 2-bipyramids within lattice d-simplices has further implications. Note that 2-bipyramids are simply convex planar quadrilaterals.

Lemma 12. *Suppose $T \in S_k^d$ where $k \geq 2$. Let w_1 and w_2 be any two of the interior points of T. Let $Q = \operatorname{conv}(w_1, w_2, v, v')$ where $v, v' \in \mathscr{V}(T)$. If Q is a planar quadrilateral, then the opposing edges of Q cannot be parallel.*

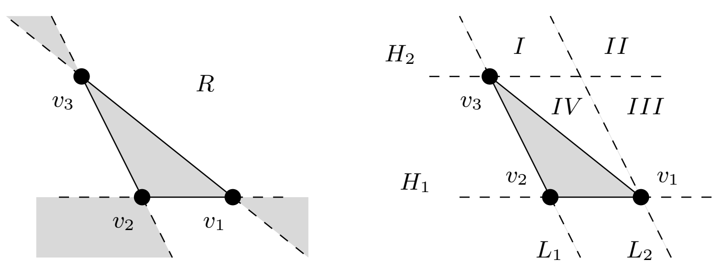
Proof. Without loss of generality, suppose $w_1 v$ and $w_2 v'$ are edges of Q. Let H_1 be the hyperplane containing the facet of T opposite v', and let H_2 be the hyperplane parallel to H_1 and containing v'. Since w_1 is an interior point of T, it must lie between H_1 and H_2. If $w_1 v \parallel w_2 v'$, then w_2 must be opposite of w_1 relative to both



H_1 and H_2. This is impossible since w_2 would lie outside of T as shown in the left figure above. If $w_1 w_2 \parallel vv'$, edge $w_1 w_2$ of Q must be shorter than edge vv' since $w_1 w_2$ is in the interior of T. Then $w = v' - (w_2 - w_1)$, or $w = v + (w_2 - w_1)$, is a lattice point on the edge vv', as shown above on the right, which is also impossible as T is clean. ■

Lemma 13. *Suppose $Q = \operatorname{conv}(v_1, v_2, v_3, v_4)$ is a planar lattice quadrilateral. If the opposing edges of Q are not parallel, then interior of Q contains a lattice point $w \neq v_i$ for $1 \leq i \leq 4$. Moreover, w lies in the interior of a triangle whose edge set is a subset of the edges and diagonals of Q.*

Proof. Consider the pairs of adjacent edges of Q. Without loss of generality, suppose $v_1 v_2$ and $v_2 v_3$ are edges of Q such that the triangle $\Delta = \operatorname{conv}(v_1, v_2, v_3)$ has minimal area. The lines containing each edge of Δ partition the plane into 7 different regions as shown below (left). Since Q is a convex quadrilateral, v_4 can



only lie in the three unshaded regions. Without loss of generality, suppose v_4 lies in region R as shown above. Let H_1 be the line through $v_1 v_2$ and H_2 be the line through v_3 such that $H_2 \parallel H_1$. Similarly, let L_1 be the line through $v_2 v_3$ and L_2

be the line through v_1 such that $L2 \parallel L1$. (See right-side figure above.) These four lines divide R into four regions. Since Q has no parallel edges, v_4 cannot lie on any one of these four lines. If v_4 lies in region I, then $\text{conv}(v_2, v_3, v_4)$ is a triangle of smaller area than Δ (they both share the same leg v_2v_3, whereas the distance between v_4 and L_1 is shorter than the distance between v_1 and L_1). Similarly, v_4 cannot lie in regions III or IV. Thus v_4 must lie in region II. Then $w = v_1 + (v_3 - v_2)$ is a lattice point contained in interior of $\text{conv}(v_1, v_3, v_4)$. ∎

6. Collinearity Property of Minimal Volume $T \in \mathscr{S}_k^d$

We can now prove the collinearity property of $\text{int}(T) \cap \mathbb{Z}^d$, where $T \in \mathscr{S}_k^d$ and $\text{Vol}(T) = \frac{1}{d!}(dk+1)$.

Theorem 14. *Suppose $T \in \mathscr{S}_k^d$, where $d \geq 3$ and $\text{Vol}(T) = \frac{1}{d!}(dk+1)$. Any two interior lattice points of T are collinear with a vertex of T.*

Proof. The claim is vacuously true for $k \leq 1$, so suppose $k \geq 2$. If $k = 2$ then let w_1 and w_2 be the two interior lattice points of T. Otherwise let w_1 and w_2 be any two interior lattice points of T. Suppose w_1 and w_2 are not collinear with any $v \in \mathscr{V}(T)$. That is, suppose there does not exist a lattice 1-bipyramid consisting of w_1, w_2, and any $v \in \mathscr{V}(T)$. Let $\mathscr{T}_1 = \mathscr{T}_{w_1} = \{T_i\}$, where $\mathscr{T}_{w_1}$ is the basic triangulation of T with respect to w_1. By assumption, w_2 must lie in the relative interior of either a simplex in $\mathscr{T}_1$, or a j-face of a simplex in $\mathscr{T}_1$, where $1 < j < d$.

Case 1: If w_2 lies in the relative interior of a j-face of a simplex in $\mathscr{T}_1$, where $1 < j < d$, then Corollary 6 implies $\text{Vol}(T) > \frac{1}{d!}(dk+1)$, a contradiction.

Case 2: If w_2 lies in the relative interior of a simplex in $\mathscr{T}_1$, then we may assume without loss of generality that $w \in \text{int}(T_{d+1})$. Let $\mathscr{T}_{w_2}$ be the basic triangulation of T_{d+1} with respect to w_2. The refinement $\mathscr{T}_2$ of $\mathscr{T}_1$ obtained by applying Theorem 4 with $w = w_2$ is

$$\mathscr{T}_2 = \mathscr{T}_{w_1} \backslash \{T_{d+1}\} \cup \mathscr{T}_{w_2}.$$

Theorem 10 implies T contains a lattice j-bipyramid P, where $2 \leq j \leq d$ and $\{w_1, w_2\} \subset \mathscr{V}(P)$. If $j = 2$, then $P = \text{conv}(v, v', w_1, w_2)$ is convex planar lattice quadrilateral, where $\{v, v'\} \subset \mathscr{V}(T)$. Lemmas 12 and 13 imply P contains a lattice point $w_3 \notin \{w_1, w_2\}$. This is a contradiction if $k = 2$. If $k > 2$, then the second part of Lemma 13 implies w_3 lies in the relative interior of some triangle Δ_i, where

$$\begin{aligned} \Delta_1 &= \text{conv}(w_1, w_2, v), & \Delta_2 &= \text{conv}(w_1, w_2, v'), \\ \Delta_3 &= \text{conv}(w_1, v, v'), & \Delta_4 &= \text{conv}(w_2, v, v'). \end{aligned}$$

Since $d \geq 3$, and each Δ_i a 2-face of simplices in $\mathscr{T}_2$, $\text{Vol}(T) > \frac{1}{d!}(dk+1)$ by Corollary 6. This contradicts the mimal volume property of T. On the other hand, if $j > 2$, then $\mathscr{V}(T) \cup \{w_1, w_2\}$ has two possible triangulations $\mathscr{T}_2$ and $\mathscr{T}_2'$ where

$$|\mathscr{T}_2'| = |\mathscr{T}_1| + d - 2 + j > 2d + 1.$$

Subsequent applications of Theorem 4, starting with $\mathscr{T}_2'$, and (2) imply

$$\text{Vol}(T) > \frac{1}{d!}(dk+1).$$

This is again a contradiction. Thus any two points in $\text{int}(T) \cap \mathbb{Z}^d$ must be collinear with some $v \in \mathscr{V}(T)$. ∎

Corollary 15. *Suppose $T \in \mathscr{S}_k^d$, where $d \geq 3$ and $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$. The points $\mathrm{int}(T) \cap \mathbb{Z}^d$ are collinear with some $v \in \mathscr{V}(T)$.*

Proof. The claim is vacuously true for $k \leq 1$. Theorem 14 implies the claim also holds for $k = 2$. Suppose $k \geq 3$. It suffices to show the claim holds for any three points in $\mathrm{int}(T) \cap \mathbb{Z}^d$. Let w_1, w_2, and w_3 be any three points in $\mathrm{int}(T) \cap \mathbb{Z}^d$. Theorem 14 implies there exists $v \in \mathscr{V}(T)$ such that w_1, w_2, and v are collinear Similarly, there exists $v' \in \mathscr{V}(T)$ such that w_1, w_3, and v' are collinear. We need to show $v = v'$. Suppose $v \neq v'$. Let L be the line through w_1, w_2 and v, and let L' be the line through w_1, w_3, and v'. Without loss of generality, suppose w_2 is between w_1 and v on L.

Case 1: If w_3 lies between w_1 and v' on L', then $P = \mathrm{conv}(v, v', w_2, w_3)$ is a convex planar quadrilateral contained within $\mathrm{conv}(w_1, v, v')$. Lemmas 12 and 13 imply $\mathrm{int}(P)$ contains a lattice point $w_4 \notin \{w_1, w_2, w_3\}$. Since $P \subset \mathrm{conv}(w_1, v, v')$ and $d \geq 3$, Corollary 6 implies $\mathrm{Vol}(T) > \frac{1}{d!}(dk+1)$.

Case 2: If w_1 lies between w_3 and v' on L', then $w_2 \in \mathrm{int}(\mathrm{conv}(w_3, v, v'))$. Again, since $d \geq 3$, Corollary 6 implies $\mathrm{Vol}(T) > \frac{1}{d!}(dk+1)$. Thus $v = v'$, and all points in $\mathrm{int}(T)$ and v are collinear. ■

Corollary 16. *Suppose $T \in \mathscr{S}_k^d$ and $d \geq 3$. Let L be the line through $\mathrm{int}(T) \cap \mathbb{Z}^d$. The consecutive points on L are evenly spaced.*

Proof. Let $v = L \cap \mathscr{V}(T)$ and let $w \in \mathrm{int}(T)$ be the lattice point closest to v. Since any line can be described as a linear combination of two points, any $w' \in L$ can be expressed as

$$w' = v + \alpha(w - v), \tag{14}$$

where $\alpha \in \mathbb{R}$. It suffices to show that for any $w' \in L \cap T \cap \mathbb{Z}^d$, the α in (14) is an integer and $0 \leq \alpha \leq k$. If $\alpha < 0$, then $v \in \mathrm{int}(\mathrm{conv}(w, w')) \subset \mathrm{int}(T)$, which is impossible. If $\alpha = 0$ then $w' = v$. Let $[a]$ and $\{\alpha\}$ denote the integer and fractional parts of α, respectively. If $\{\alpha\} \neq 0$, then

$$x = w' - [\alpha](w - v) = v + \{\alpha\}(w - v) \in \mathbb{Z}^d$$

lies between v and w on L, which contradicts our assumption that w is closest to v on L. Hence $\alpha \in \mathbb{Z}$. Finally, if $\alpha > k$, then T has more than k interior points, which is aslo impossible. ■

Corollary 17. *If $T \in \mathscr{S}_k^d$, $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$, and $d \geq 3$, then $T \cap \mathbb{Z}^d$ has a unique triangulation.*

Proof. Let $\mathscr{V}(T) = \{v_1, \cdots, v_{d+1}\}$. Without loss of generality, suppose v_{d+1} is collinear with $\mathrm{int}(T) \cap \mathbb{Z}^d$. Let w_1 be the point in $\mathrm{int}(T) \cap \mathbb{Z}^d$ closest to v_{d+1} and

$$\mathscr{W} = \{\ w_i\ :\ w_i = v_{d+1} + i(w_1 - v_{d+1}),\ 0 \leq i \leq k\ \}.$$

Let $S_{k+1,d} = \mathrm{conv}(v_1, \cdots, v_d, w_k)$. For $1 \leq i \leq k$ and $1 \leq j \leq d$, let

$$S_{i,j} = \mathrm{conv}(\mathscr{V}(T) \backslash \{v_j, v_{d+1}\} \cup \{w_i, w_{i-1}\}).$$

It is easy to check that $\mathscr{T} = \{S_{i,j}\}$ is a triangulation of $T \cap \mathbb{Z}^d$. In fact, this triangulation corresponds to the refinement sequence in Corollary 5.

Let $\mathscr{T}'$ be another full triangulation of T. Consider any $S \in \mathscr{T}'$. Since $\mathscr{T}'$ is a full triangulation, S is necessarily empty and clean. These two conditions force

$$1 \leq |\mathscr{V}(S) \cap \mathscr{W}| \leq 2.$$

Case 1: If $|\mathscr{V}(S) \cap \mathscr{W}| = 1$, then there exist $v \in \mathscr{V}(T)$ and $w \in \mathscr{W}$ such that

$$S = \mathrm{conv}(\mathscr{V}(T)\backslash\{v\} \cup \{w\}).$$

Since S is empty, it follows that $w = w_k$, $v = v_{d+1}$, and $S = S_{k+1,d}$.

Case 2: If $|\mathscr{V}(S) \cap \mathscr{W}| = 2$, then there exist $\{v, v'\} \subset \mathscr{V}(T)$ and $\{w, w'\} \subset \mathscr{W}$ such that

$$S = \mathrm{conv}(\mathscr{V}(T)\backslash\{v, v'\} \cup \{w, w'\}).$$

Without loss of generality, suppose w is closer to w_0 than w' is to w_0. Since S is clean,

$$(w, w') = (w_{i-1}, w_i) \tag{15}$$

for some $1 \leq i \leq k$. Moreover, w_0 $(= v_{d+1})$ is a vertex of S if and only if $i = 1$ in (15) since S must also be empty. Thus $S = S_{i,j}$ for some appropriate j, and $\mathscr{T}' \subseteq \mathscr{T}$. Consequently, $\mathscr{T}' = \mathscr{T}$. ■

7. Unimodular Transformations

A *unimodular transformation* $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is an affine map of the form

$$f(v) = v \cdot \mathbf{M} + u,$$

where $u, v \in \mathbb{Z}^d$, $\mathbf{M}$ is a $d \times d$ matrix with entries in $\mathbb{Z}$, and $\det(\mathbf{M}) = \pm 1$. A *translation* is a unimodular transformation in which $\mathbf{M} = \mathbf{I}_d$, where $\mathbf{I}_d$ is the $d \times d$ identity matrix. If $\det(\mathbf{M}) = \pm 1$, $\mathbf{M}$ is invertible, then $\mathbf{M}^{-1}$ is unimodular, and f is one-to-one. Thus P is a lattice d-polytope if and only if $f(P)$ is a lattice d-polytope with the same number of vertices. Using barycentric coordinates, it is easy to check that $f(w)$ is an interior point of $f(P)$ if and only if w is an interior point of P. Similarly, $f(w)$ is a boundary point of $f(P)$ if and only if w is a boundary point of P (cf. [19]). We say P_1 and P_2 are *equivalent* lattice d-polytopes (and write $P_1 \simeq P_2$) if there exists a unimodular transformation f such that $P_1 = f(P_2)$.

Lemma 18. *Let T be a d-simplex with* $\mathrm{Vol}(T) = \frac{1}{d!}$. *Then T is equivalent to $T_{1,\cdots,1}$, the d-simplex whose vertex set consists of the origin, e_i for $1 \leq i \leq d-1$, and the point* $(1, \cdots, 1)$.

Proof. Let $v_1, \cdots, v_{d+1}$ be an enumeration of the vertices of T. If necessary, translate T so that one of its vertices is the origin. Without loss of generality, suppose $v_{d+1} = \mathbf{0}$. Let

$$P = \mathrm{conv}\left(v_1, \cdots, v_{d+1}, \sum_{i=1}^{d} v_i\right).$$

Note that P is a parallelepiped containing T, and $\mathrm{Vol}(P) = \pm\det(\mathbf{A})$, where $\mathbf{A}$ is a $d \times d$ matrix whose entry $a_{i,j}$ is the j-th coordinate of v_i. Since

$$\frac{1}{d!} = \mathrm{Vol}(T) = \frac{1}{d!} \cdot \mathrm{Vol}(P) = \frac{1}{d!} \cdot |\det(\mathbf{A})|,$$

$\det(\mathbf{A}) = \pm 1$ and $\mathbf{A}$ is invertible. Thus, there exists a unique, unimodular, $d \times d$ matrix $\mathbf{M}$ such that $\mathbf{AM} = \mathbf{I}_d$. Thus $f : \mathbb{Z}^d \to \mathbb{Z}^d$ defined by $f(v) = v \cdot \mathbf{M}$ is a unimodular transformation such that and $f(v_i) = e_i$ for $1 \leq i \leq d$ and $f(v_{d+1}) = \mathbf{0}$. Let $g : \mathbb{Z}^d \to \mathbb{Z}^d$ be the unimodular transformation given by $g(v) = v \cdot \mathbf{M}'$ where

$$\mathbf{M}' = \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ 1 & 1 \end{bmatrix}.$$

It is easy to check that $g(e_i) = e_i$ for $1 \le i \le d-1$, $g(e_d) = (1, \cdots, 1)$, and $g(\mathbf{0}) = \mathbf{0}$. Then the composition $g \circ f$, where $(g \circ f)(v) = v \cdot \mathbf{M} \cdot \mathbf{M}'$, is also a unimodular transformation, and $(g \circ f)(T) = T_{1,\cdots,1}$. ■

Let $T_{a_1,\cdots,a_d}$ denote the d-simplex whose vertex set consists of the origin, the unit points $e_1, \cdots, e_{d-1}$, and the point $(a_1, \cdots, a_d) \in \mathbb{Z}^d$. What can we say about $T_{a_1,\cdots,a_d}$? If $a_d < 0$, then we can apply $f : \mathbb{Z}^d \to \mathbb{Z}^d$, where

$$f(v) = v \cdot \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ 0 & -1 \end{bmatrix}. \tag{16}$$

Note that f is simpy a reflection in the d-th coordinate. By reflection, if necessary, we can take a_d to be nonnegative. Moreover, a_d is determined by the volume of $T_{a_1,\cdots,a_d}$ since

$$0 < d! \cdot \mathrm{Vol}(T_{a_1,\cdots,a_d}) = \pm \det \begin{bmatrix} a_1 & \cdots & a_d \\ & \mathbf{I}_{d-1} & 0 \end{bmatrix} = |a_d|. \tag{17}$$

For the remaining a_i, consider the integers b_i such that such that

$$0 \le a_i + b_i a_d < a_d,$$

and the transformation $g : \mathbb{Z}^d \to \mathbb{Z}^d$, where

$$g(v) = v \cdot \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbf{I}_{d-1} & & \vdots \\ & & & 0 \\ \hline b_1 & \cdots & b_{d-1} & 1 \end{array}\right]. \tag{18}$$

All the vertices of $T_{a_1,\cdots,a_d}$ excluding $(a_1, \cdots, a_d)$ are fixed under g. On the other hand, g sends $(a_1, \cdots, a_d)$ to $(a'_1, \cdots, a'_d)$ where $a'_d = a_d$ and $a'_i = a_i + b_i a_d$ for $1 \le i \le d-1$. Since $T_{a_1,\cdots,a_d}$ is not contained in any hyperplane, $a_i > 0$ and $a'_i > 0$. Thus the class of all $T_{a_1,\cdots,a_d} \in \mathscr{S}^d_k$ is represented by $T_{a'_1,\cdots,a'_d}$ where

$$0 < a'_i < a'_d = a_d. \tag{19}$$

Lastly, let $w \in \mathrm{int}(T_{a_1,\cdots,a_d})$ and let (λ_i) be the barycentric coordinates of w relative to $T_{a_1,\cdots,a_d}$, where

$$w = \lambda_{d+1} \cdot \mathbf{0} + \lambda_d \cdot (a_1, \cdots, a_d) + \sum_{i=1}^{d-1} \lambda_i e_i.$$

The d-th coordinate of w is $\lambda_d \cdot a_d$, and for $1 \le i \le d-1$, the i-th coordinate is

$$\lambda_i + \lambda_d \cdot a_i = \lceil \lambda_d \cdot a_i \rceil. \tag{20}$$

We can conclude that every point in $\mathrm{int}(T_{a_1,\cdots,a_d}) \cap \mathbb{Z}^d$ is completely determined by its d-th coordinate. Equivalently, no two interior lattice points can have the same d-th coordinate.

Theorem 19. *Let $d \ge 3$ and $k \ge 1$. If $T \in \mathscr{S}^d_k$, and $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$, then $T \simeq T_{a_1,\cdots,a_d}$, where $(a_1, \cdots, a_{d-1}, a_d) = (dk, \cdots, dk, dk+1)$.*

Proof. Let $\mathscr{V}(T)$, v_{d+1}, $\mathscr{W}$, $S_{i,j}$, and $\mathscr{T}$ be as in the proof of Corollary 17. For $S_{1,d} \in \mathscr{T}$, Lemma 18 implies there exists a unimodular transformation f such that $f(v_{d+1}) = \mathbf{0}$, $f(w_1) = (1, \cdots, 1)$, and $f(v_j) = e_j$ for $1 \le j \le d-1$. Let

$$f(v_d) = (a_1, \cdots, a_d).$$

We may assume $0 < a_j \leq a_d$ by (19). Since $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$ and unimodular transformations preserve volume, (17) implies $a_d = dk + 1$. Unimodular transformations also preserve interior lattice points, so $f(w_1) \in \mathrm{int}(f(T)) \cap \mathbb{Z}^d$, and $f(w_1)$ is collinear with $\mathbf{0}$. Corollary 16 implies $f(w_i) = (i, \cdots, i)$. If $(\lambda_{i,j})$ are the barycentric coordinates of $f(w_i)$, where

$$f(w_i) = \lambda_{i,d+1} \cdot \mathbf{0} + \lambda_{i,d} \cdot (a_1, \cdots, a_d) + \sum_{j=1}^{d-1} \lambda_{i,j} \cdot e_j,$$

then (20) implies

$$\lambda_{i,j} + \lambda_{i,d} \cdot a_j = i.$$

Note that $0 < \lambda_{i,j} < 1$, and

$$f(w_i) \in \mathrm{int}(f(T)) \cap \mathbb{Z}^d \iff w_i \in \mathrm{int}(T) \cap \mathbb{Z}^d \iff 1 \leq i \leq k.$$

Since

$$\lambda_{i,d} \cdot a_j = (i-1) + 1 - \lambda_{i,j} > (i-1)\lambda_{i,d} + 1 - \sum_{\substack{j=1 \\ j \neq d}}^{d+1} \lambda_{i,j} > i \cdot \lambda_{i,d},$$

$a_j > i$ for all i, which implies $a_j > k \geq 1$. The final step is to show that $a_j = dk$.

Consider the interior point $f(w_k) = (k, k, \cdots, k)$ and let (λ_j) be the barycentric coordinates of $f(w_k)$. Clearly $\lambda_d = \frac{k}{dk+1}$. By (20), we have

$$\lambda_j = k - a_j \cdot \lambda_d = k - \frac{a_j \cdot k}{dk+1}.$$

Since $\displaystyle\sum_{j=1}^{d+1} \lambda_j = 1$, it follows that

$$0 < \sum_{j=1}^{d} \lambda_j = (d-1)k - \frac{k}{dk+1}(a_1 + \cdots + a_{d-1} - 1) < 1.$$

Using elementary algebraic manipulation, we obtain

$$1 \leq (kd+1)(d-1) - (a_1 + \cdots + a_{d-1} - 1) \leq d,$$

which implies

$$a_1 + \cdots + a_{d-1} \geq kd(d-1). \tag{21}$$

Finally, $a_j < a_d = dk + 1$ and (21) imply $a_j = dk$ for $1 \leq j \leq d$. ∎

Corollary 20. *If $T \in \mathscr{S}_k^d$, then $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$ if and only if $T \simeq S_d(k)$.*

Proof. We first show that $\mathrm{Vol}(S_d(k)) = \frac{1}{d!}(dk+1)$. Using determinants,

$$\mathrm{Vol}(S_d(k)) = \frac{1}{d!} \cdot |\det(\mathbf{A})|,$$

where

$$\mathbf{A} = \begin{bmatrix} e_2 - e_1 \\ e_3 - e_1 \\ \vdots \\ e_{d-1} - e_1 \\ e_d - e_1 \\ (-k, -k, \cdots, -k) - e_1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|ccc} -1 & & & \\ -1 & & \mathbf{I}_{d-1} & \\ -1 & & & \\ \hline -k-1 & -k & \cdots & -k \end{array}\right]}_{d\times d}.$$

Let $\mathbf{A}'$ be the matrix obtained by adding k times the sum of the first $d-1$ rows of $\mathbf{A}$ to row d of $\mathbf{A}$. That is,

$$\mathbf{A}' = \left[\begin{array}{c|ccc} -1 & & & \\ -1 & & \mathbf{I}_{d-1} & \\ -1 & & & \\ \hline -dk-1 & 0 & \cdots & 0 \end{array}\right].$$

It is easy to check that $|\det(\mathbf{A})| = |\det(\mathbf{A}')| = dk+1$. If $T \simeq S_d(k)$, then $\mathrm{Vol}(T) = \mathrm{Vol}(S_d(k)) = \frac{1}{d!}(dk+1)$. On the other hand, if $\mathrm{Vol}(T) = \frac{1}{d!}(dk+1)$, then $T \simeq S_d(k)$ by Theorem 19 provided $S_d(k)$ is clean. Since $\mathrm{Vol}(S_d(k)) = \frac{1}{d!}(dk+1)$ and any lattice triangulation of $\mathscr{V}(S_d(k)) \cup \mathrm{int}(S_d(k)) \cap \mathbb{Z}^d$ contains at least $dk+1$ simplices, (3) and Corollary 5 imply $S_d(k)$ cannot have any lattice points on its boundary except for its vertices. ■

Let $f : \mathbb{Z}^d \to \mathbb{Z}^d$ be the translation

$$f(v) = v \cdot \mathbf{I}_d + k \cdot \sum_{i=1}^{d} e_i = v + (k, k, \cdots, k),$$

and let $g : \mathbb{Z}^d \to \mathbb{Z}^d$ be the transformation $g(v) = v \cdot \mathbf{M}$ where

$$\mathbf{M} = \underbrace{\begin{bmatrix} 1-k & -k & -k & \cdots & -k & -k \\ -k & 1-k & -k & \cdots & -k & -k \\ -k & -k & 1-k & \cdots & -k & -k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k & -k & -k & \cdots & 1-k & -k \\ (d-1)k & (d-1)k & (d-1)k & \cdots & (d-1)k & (d-1)k+1 \end{bmatrix}}_{d\times d}.$$

That is, if $m_{i,j}$ is the entry in the row i and column j of $\mathbf{M}$, then

$$m_{i,j} = \begin{cases} 1-k, & i = j \neq d, \\ (d-1)k, & j = d,\ i \neq d \\ (d-1)k+1, & i = j = d \\ -k, & \text{otherwise.} \end{cases}$$

Consider the image of $\mathscr{V}(S_d(k))$ under the composition $g \circ f$. For $v \in \mathscr{V}(S_d(k))$, the j-th coordinate of v under $g \circ f$ is simply the dot product of $v + (k, \cdots, k)$ with the j-th column of $\mathbf{M}$. It is easy to check that

$$(g \circ f)(v) = \begin{cases} \mathbf{0}, & v = -k \sum_{i=1}^{d} e_i, \\ e_i, & v = e_i,\ 1 \leq i \leq d-1, \\ (a_1, \cdots, a_d), & v = e_d. \end{cases}$$

The composition $g \circ f$ is in fact a unimodular transformation which maps $S_d(k)$ to $T_{a_1,\cdots,a_d}$, where the a_i are as in Theorem 19.

8. Counterexamples in $\mathbb{R}^2$

The collinearity property of the interior lattice points does not hold for all clean triangles in $\mathbb{R}^2$. Pick's theorem states that if P is a lattice polygon, then

$$\mathrm{Vol}(P) = k + \frac{b}{2} - 1 \tag{22}$$

where $k = |\operatorname{int}(P) \cap \mathbb{Z}^2|$ and $b = |\partial P \cap \mathbb{Z}^2|$. For clean triangles, (22) reduces to

$$\mathrm{Vol}(P) = k + \frac{1}{2}.$$

Thus all clean lattice triangles satisfy the minimal volume condition. In [18], Reznick proved that any $T \in S_k^2$ is equivalent to some $T_{a,2k+1}$, where $0 < a < 2k+1$. Using this representation for clean lattice triangles, Reznick then showed that the number of equivalence classes of clean k-point lattice triangles increases with k. Thus there exist clean lattice triangles whose interior points are not collinear.

Consider the triangle $\Delta_{p,q}$ in $\mathbb{R}^2$ with vertex set $\{\ (-1,0),\ (0,q),\ (p,-1)\ \}$, where $1 \le p \le q$ and $\gcd(p, q+1) = 1$. It is easy to check that $\Delta_{p,q}$ is clean. We first compute the number of interior points of $\Delta_{p,q}$. Using determinants,

$$\mathrm{Vol}(\Delta p, q) = \frac{q}{2}(p+1) + \frac{1}{2}.$$

Pick's theorem implies

$$|\operatorname{int}(\Delta p, q) \cap \mathbb{Z}^2| = \frac{q}{2}(p+1). \tag{23}$$

For $p = 1$, the interior points of $\Delta_{p,q}$ are all on the line $x = 0$. However, for $p > 1$, this is not the case as they are covered by the lines $x = i$ $(0 \le i \le p-1)$. Since these lines are parallel, they cannot be contained within any one line. Incidentally, this collection of counterexamples is related to the following summation identity.

Proposition 21. *If* $(p, q) \in \mathbb{Z}_+^2$ *and* $\gcd(q+1, p) = 1$*, then*

$$\sum_{i=0}^{p-1} \left\lceil q - \frac{i(q+1)}{p} \right\rceil = \frac{q}{2}(p+1). \tag{24}$$

Proof. This is a simple counting argument using (23), the fact that the lines $x = i$ $(0 \le i \le p-1)$ cover $\operatorname{int}(\Delta_{p,q}) \cap \mathbb{Z}^2$, and the fact that there are

$$\left\lceil q - \frac{i(q+1)}{p} \right\rceil$$

lattice points on the line $x = i$ and $\operatorname{int}(\Delta_{p,q})$. ■

The counterexample above can be generalized to a triangle with vertex set

$$\{\ (-r, 0),\ (0, q),\ (p, -1)\ \},$$

where p, q, and r are positive integers and $\gcd(r, q) = \gcd(p, q+1) = 1$. The corresponding identity below is similar to (24).

Proposition 22. *If* $(p,q,r) \in \mathbb{Z}_+^3$ *and* $\gcd(q,r) = \gcd(q+1,p) = 1$*, then*

$$\sum_{i=1}^{r} \left\lceil \frac{iq}{r} \right\rceil + \sum_{i=0}^{p-1} \left\lceil q - \frac{i(q+1)}{p} \right\rceil = \frac{q(p+r)+(r-1)}{2}.$$

Pick's theorem itself is a counterexample since only in $\mathbb{R}^2$ is the volume of a lattice polytope P an equality in terms of the number of lattice points on ∂P and in $\operatorname{int}(P)$. Reeve's tetrahedra [20] have vertices

$$\{\ (0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,n)\ \},$$

where $n \in \mathbb{Z}_+$. These tetrahedra are clearly clean and empty, yet their volume increases with n. Reeve concluded that the volume of lattice of a polyhedron P cannot be expressed solely in terms of $b = |\partial P \cap \mathbb{Z}^3|$ and $k = |\operatorname{int}(P) \cap \mathbb{Z}^3|$. However, Reeve was able to find and analogue to Pick's theorem by considering sublattices.

Let L_n^d denote the lattice consisting of all points in $\mathbb{R}^d$ whose coordinates are multiples of $\frac{1}{n}$. Note that $L_1^d = \mathbb{Z}^d$. For a lattice d-polytope P, let

$$b_n = b_n(P) = |\partial P \cap L_n^d| \quad \text{and} \quad k_n = k_n(P) = |\operatorname{int}(P) \cap L_n^d|.$$

Reeve showed in [21] that

$$2n(n^2-1) \cdot \operatorname{Vol}(P) = b_n - nb_1 + 2(k_n - nk_n) + (n-1)[2\chi(P) - \chi(\partial P)], \tag{25}$$

where $n \geq 2$ and χ is the Euler characteristic. Reeve also conjectured a similar formula for $d = 4$. Soon after, Macdonald [15] not only confirmed Reeve's conjecture, but generalized Reeve's formulas to

$$\begin{aligned}(d-1)d! \cdot \operatorname{Vol}(P) = \sum_{i=1}^{d-1} (-1)^{i-1} \binom{d-1}{i-1} (b_{d-i} + 2k_{d-i}) \\ +(-1)^{d-1}[2\chi(P) - \chi(\partial P)].\end{aligned} \tag{26}$$

In 1996, Kolodzieczyk [11] showed that (26) has the following alternate form in which only the numbers k_n of interior lattice points in the sublattice L_n are required.

$$d! \cdot \operatorname{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + (-1)^d[\chi(P) - \chi(\partial P)]. \tag{27}$$

For a d-polytope P all of whose vertices join d edges, $\chi(P) - \chi(\partial P) = (-1)^d$ and (27) assumes the form

$$d! \cdot \operatorname{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + 1. \tag{28}$$

Recently, Kolodzieczyk showed in [12] that Pick-type formulas exist even when considering only the points in $L_n \cap P$ whose coordinates are odd multiples of $\frac{1}{n}$.

Though no formula for the volume of a lattice polytope in terms of the numer of its boundary and interior lattice points exist, there does exist a Pick-type inequality with sharp bounds on the volume of a polyhedron. Any polyhedron in $\mathbb{R}^3$ that does not intersect itself has an associated planar graph. We can use elementary graph theory and Euler's formula to obtain a lower bound on the volume of lattice polyhedra.

Proposition 23. *Let $P \subset \mathbb{R}^3$ be a convex lattice polyhedron with b lattice points on the boundary and $k \geq 1$ interior lattice points. Then*

$$\operatorname{Vol}(P) \geq \frac{2b + 3k - 7}{6}.$$

Proof. By (2), it suffices to show P has a lattice triangulation $\mathscr{T}$ satisfying

$$|\mathscr{T}| \geq 2b + 3k - 7.$$

Any refinement sequence starting with any basic triangulation of P will suffice. Consider the graph G whose vertex set $\mathscr{V}(G)$ is $\partial P \cap \mathbb{Z}^3$ and whose edge set consists of the edges of ∂P. Since P is convex, G is planar. (Just embed G into $\mathbb{S}^2$ and then map onto $\mathbb{R}^2$.) Thus every triangulation of G has $2 \cdot |\mathscr{V}(G)| - 4$ faces, implying ∂P can be triangulated into $2b - 4$ triangles. Since $k \geq 1$, P can be triangulated into $2b - 4$ subtetrahedra using any interior lattice point. The remaining interior points refine this triangulation, via Theorem 4, into a full triangulation having at least $2b - 4 + 3(k - 1)$ subtetrahedra. ■

This bound is sharp (consider any clean, non-empty lattice tetrahedron). A similar result is proved (independently) in [5, Lemma 3.5.5] by De Loera, Ramburg, and Leal. While their proof does not assume that P is nonempty, it does assume that $P \cap \mathbb{Z}^3$ is in general position. They also showed that any polyhedron with n vertices has a triangulation consisting of at most $2n - 7$ tetrahedra.

9. Future Research

What can we say about an upper bound for the volume of polyhedra? Hensley proved in [9] that the volume of any lattice d-polytope is bounded above by a function in terms of d and the number of interior lattice points. Ziegler improved Hensley's results and showed in [23] that the volume of a k-point lattice d-polytope P satisifies

$$\operatorname{Vol}(P) \leq k \cdot [7(k+1)]^{d \cdot 2^{d+1}}.$$

In $\mathbb{R}^3$, a few simple computations seem to indicate that the upper bound on the volume of clean tetrahedra is linear in k. Moreover, maximal-volume tetrahedra also appear to have collinear interior lattice points. Unlike the minimal-volume tetrahedra, however, the line containing the interior points does not pass through any vertex. Based on these computations, we make the following conjectures.

Conjecture 1. If $T \in \mathscr{S}_k^3$, then $\operatorname{Vol}(T) \leq \frac{1}{6!}(12k + 8)$.

Conjecture 2. If $T \in \mathscr{S}_k^3$ and $\operatorname{Vol}(T) = \frac{1}{6!}(12k + 8)$, then $T \simeq T_{2k+1,4k+3,12k+8}$. If $k > 1$, then $\operatorname{int}(T) \cap \mathbb{Z}^3$ is a set of collinear points that does not pass through any vertex of T.

These conjectures are true for $k = 1$, as proved independently by Kasprzyk [10] and Reznick [19]. It may be possible to gain some new insight on the relationship between lattice points of a lattice d-polytope and its volume using Ehrhart theory. For example, it is well known that every convex lattice d-polytope is associated with an Ehrhart polynomial of degree d. Moreover, the leading coefficient of the associated polynomial is precisely the volume of the d-polytope (cf. [1], [6]).

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