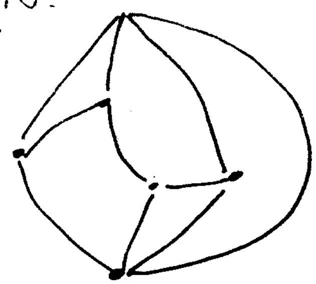
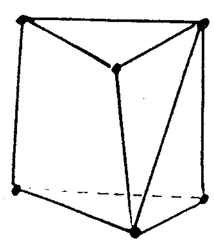
GIVEN:



a combinatorial type of 3-polytope (convex)
(= a 3-connected planar graph)

[+ additional data]

"CONSTRUCT:"



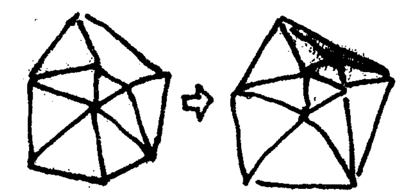
a geometric realization of the polytope.

GIVEN: combinatorial type of a convex 3-polytope (a 3-connected planar graph)

FIND: **a geometric realization of the polytope with:**
integer coordinates that are not too large

TWO APPROACHES:

1. INDUCTIVE
start with the simplest polytopes and make local modifications



- Steinitz (1922): **coordinates** $\le 2^{\exp(n)}$
- Das and Goodrich (1995) for <u>simplicial</u> polytopes (triangulations): coordinates $\le 2^{\text{poly}(n)}$

2. DIRECT
obtain the polytope as a result of
- a system of equations
- an optimization problem
- an existential proof

- Onn and Sturmfels (1994): $\le n^{169n^3}$
- this talk $\le 2^{20n^2}$

and with triangle as a face $\le n^3$ [illegible]

3.

GIVEN: combinatorial type of a convex 3-polytope
(a 3-connected planar graph)

FIND: a geometric realization of the polytope
with:

all vertices on the
unit sphere ("inscribed polytope")

cf: Delaunay triangulation

2. DIRECT
obtain the polytope as a result of
- a system of equations
- an optimization problem
- an existential proof

Test inscribability in polynomial time
(Rivin, Hodgson, Smith 1993)

GIVEN: combinatorial type of a convex 3-polytope
(a 3-connected planar graph)

FIND: a geometric realization of the polytope
with:

all edge lengths rational

(OPEN)

4.

GIVEN: combinatorial type of a convex 3-polytope
(a 3-connected planar graph)

FIND: a geometric realization of the polyto
with:
all edges tangent to the
unit sphere ("midscribed polytope"

2. DIRECT
obtain the polytope as a result of
- a system of equations
- an optimization problem
- an existential proof ← Koebe-Andreev-Thursto theorem

Colin de Verdière

Thurston's algorithm, Brightwell-Scheinerman

GIVEN: ~~combinatorial type~~ face normals and face areas of a convex 3-polytope

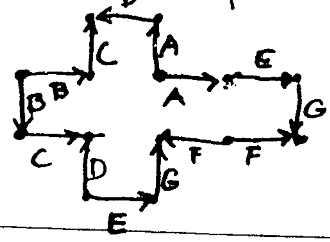
FIND: a geometric realization of the polyt
with:
these face areas and face normals

2. DIRECT
obtain the polytope as a result of
- a system of equations
- an optimization problem
- an existential proof Minkowski ~189

$\boxed{5.}$

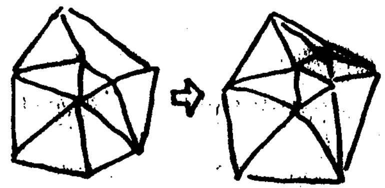
GIVEN: ~~combinatorial type~~ metric on the surface of a con 3-polytope
(~~a 3-connected planar graph~~) (a "net")

FIND: a geometric realization of the polytope with:



TWO APPROACHES:

1. INDUCTIVE
start with the simplest polytopes and make local modifications

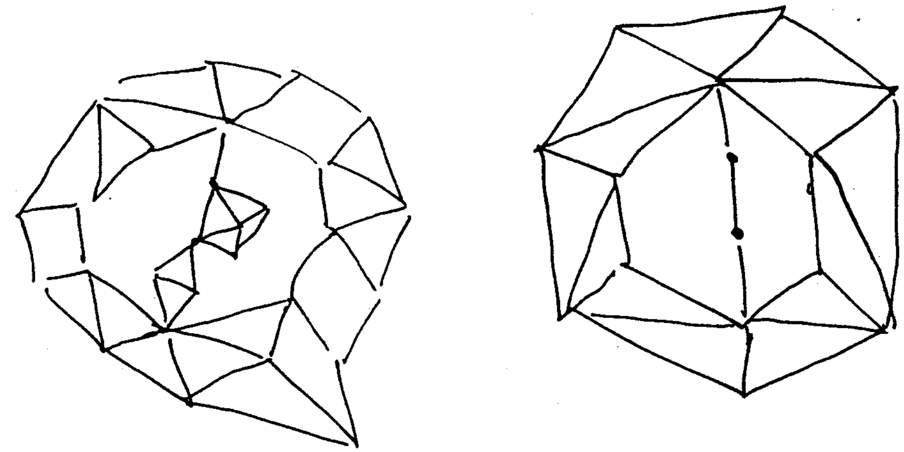


2. DIRECT
obtain the polytope as a result of
- a system of equations
- an optimization problem
- an existential proof ← Alexandrov (~1930)

$\boxed{6.}$

The graph of a 3-polytope is 3-connected.

(Removing 2 vertices does not disconnect the graph.)

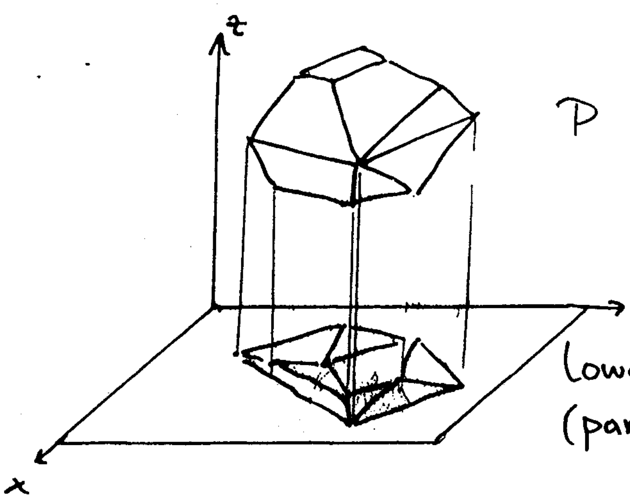


The intersection of two faces
is an edge,
a vertex, or
empty.

Theorem of Steinitz.

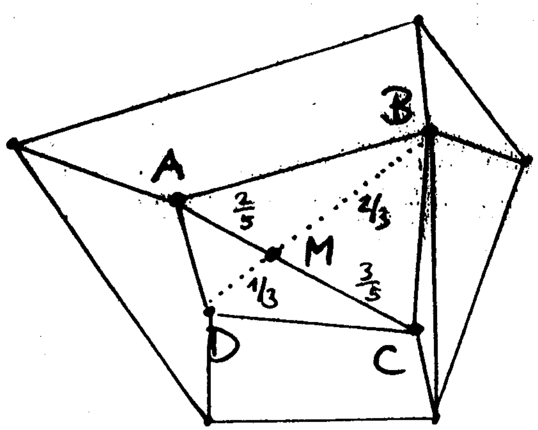
Every 3-connected planar graph is the graph of a 3-polytope.

7.



lower projection of P (parallel)

SIMILARLY: perspective projections (from a point)



IS THIS DIAGRAM A LOWER PROJECTION OF A POLYTOPE ?

$$A = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} \leftarrow \text{GIVEN}$$

$AM : MC : AC = \frac{2}{5} : \frac{3}{5} : 1$

$DM : MB : DC = \frac{1}{3} : \frac{2}{3} : 1$

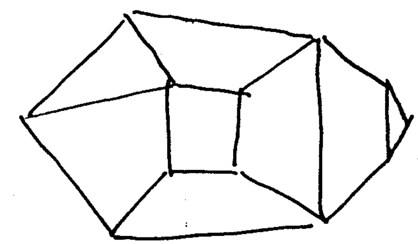
$(z_M =)$ $\boxed{\frac{3}{5} z_A + \frac{2}{5} z_C < \frac{1}{3} z_B + \frac{2}{3} z_D}$

- One homogeneous strict inequality for each edge
+ homogeneous equations for non triangular faces

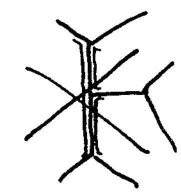
8.

GIVEN: A picture that "looks like" a lower projection of a polytope

= a decomposition of a convex polygon into convex polygons



side-to-side:



= a <u>2-diagram</u>

9.

Maxwell (1864)

Whiteley (& Crapo 1982) 1977

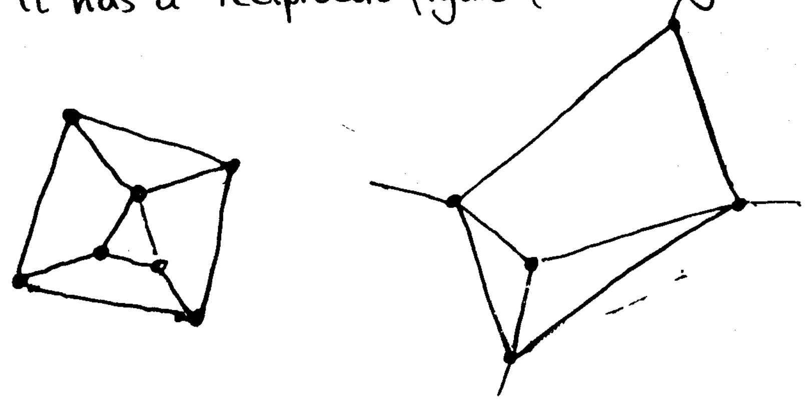
Aurenhammer, 1987

McMullen, 1992

Huffman, Mackworth, 1973

A plane graph is the lower projection of a 3-polytope

- IFF it has a "reciprocal figure" (an "orthogonal dual")

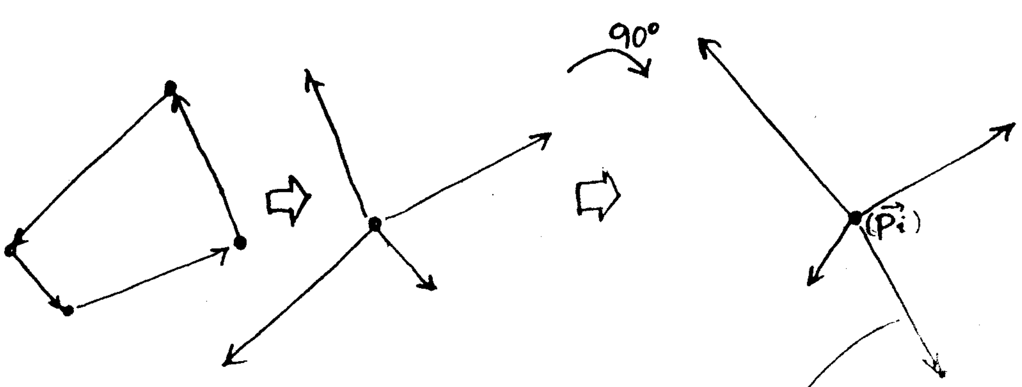


Maxwell (1864) Crapo & Whiteley 1982

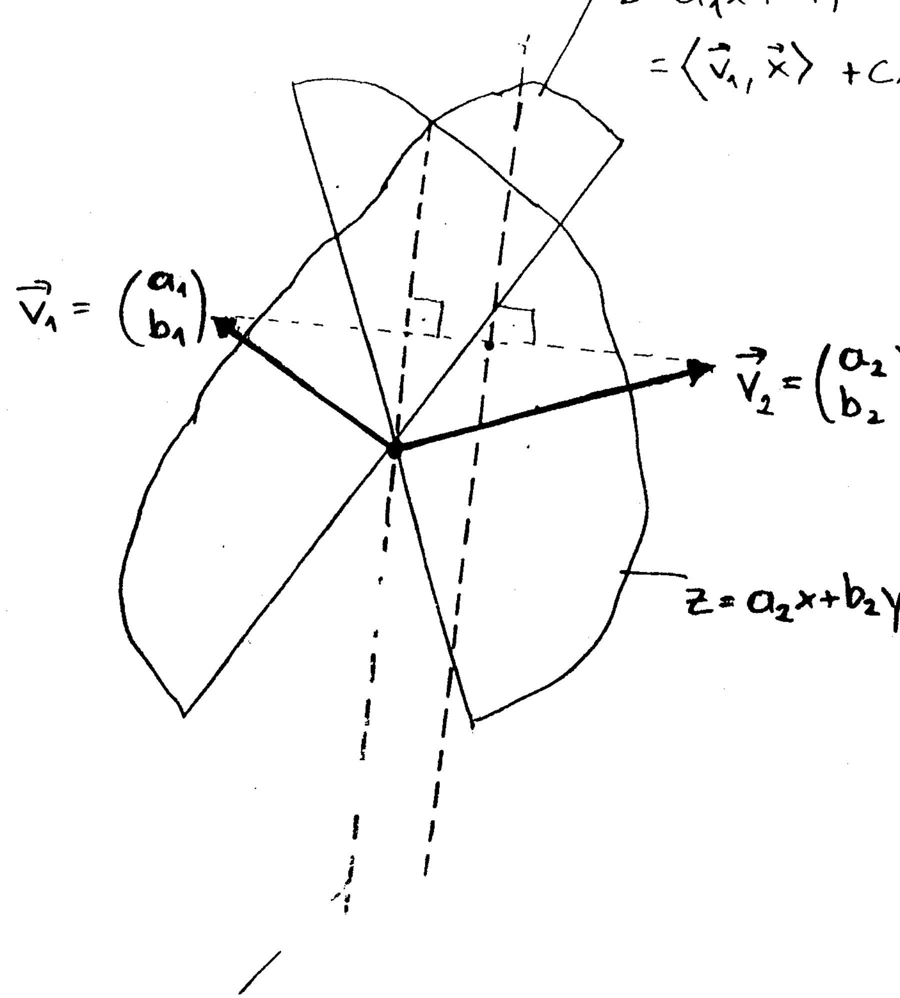
- IFF it has a positive "self-stress" $w_{ij} = w_{ji}$ on the edges:

$$\forall \text{ interior vertices } \vec{p}_i: \quad \sum_{j\sim i} w_{ij}(\vec{p}_j - \vec{p}_i) = 0$$

(EQUILIBRIUM)



10.



$$\langle \vec{v}_1, \ldots = z = \langle \vec{v}_2, \vec{x}\rangle$$

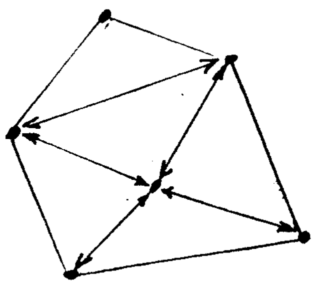
$$\langle \vec{v}_1 - \vec{v}_2, \vec{x}\rangle = 0$$

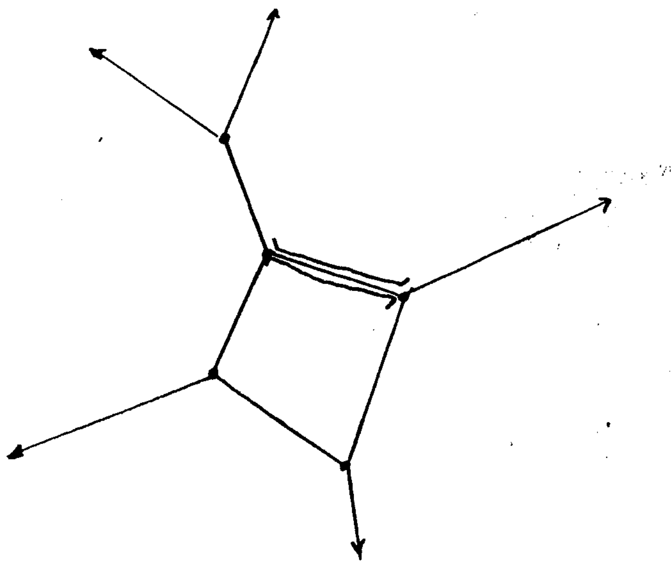
$$\vec{x} \perp \vec{v}_1 - \vec{v}_2$$

The ~~segment~~ between the gradient vectors of two adjacent faces is perpendicular to the projection of the edge between the faces

$\boxed{11.}$

stress $\Rightarrow$ reciprocal diagram

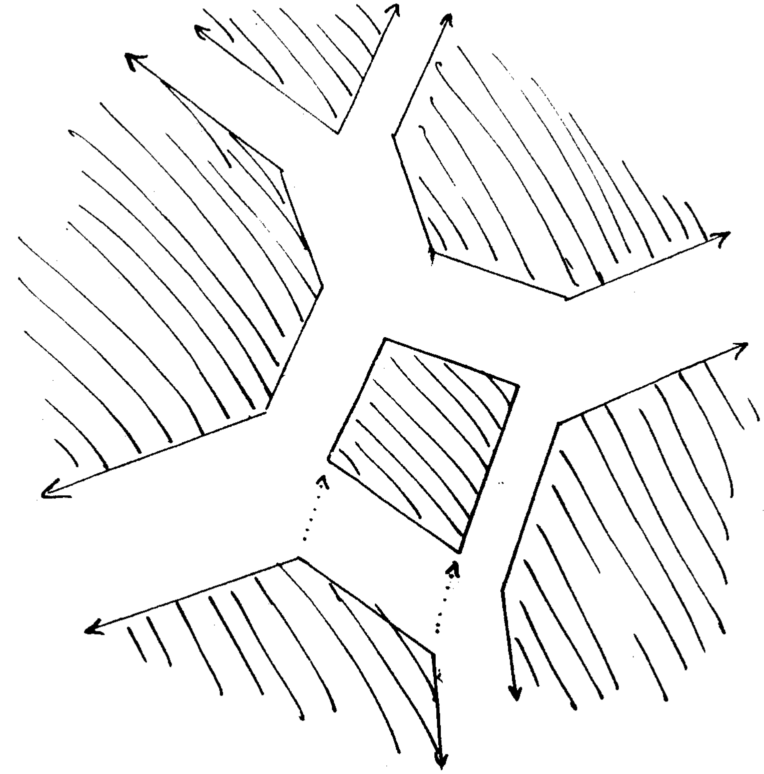




WANT:

convex polygonal regions that cover the whole plane

$\boxed{12.}$

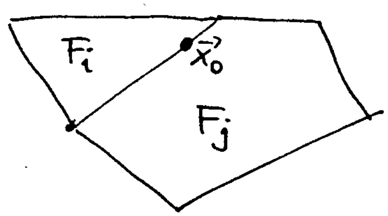


convex polygonal regions that fit togethe Locally!

13.

reciprocal diagram $\rightarrow$ projection of 3-polytope.

$\vec{v}_i = \binom{a_i}{b_i}$ given $\rightarrow$ Face F_i: $z = a_i x + b_i y + c_i = \langle \vec{v}_i, \vec{x} \rangle$



$\vec{x}_0 \in F_i \cap F_j$

$z(\vec{x}_0) = \langle \vec{v}_i, \vec{x}_0 \rangle + c_i = \langle v_j, \vec{x}_0 \rangle + c_j$

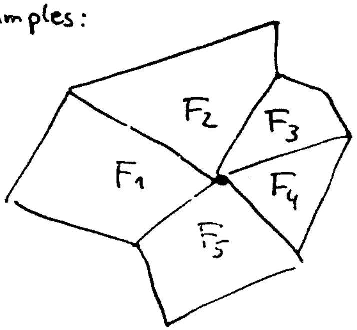
$\boxed{c_j - c_i = \Delta_{ij}} := \langle \vec{v}_i - \vec{v}_j, \vec{x}_0 \rangle \quad \forall$ adjacent F_i, F_j

- If $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_k} \neq 0$

  then $\Delta_{i_1 i_2} + \Delta_{i_2 i_3} + \dots + \Delta_{i_{k-1} i_k} + \Delta_{i_k i_1} = 0$

  $\langle \vec{v}_{i_1} - \vec{v}_{i_2}, \vec{x}_0 \rangle + \langle \vec{v}_{i_2} - \vec{v}_{i_3}, \vec{x}_0 \rangle + \langle \vec{v}_{i_3} - \vec{v}_{i_4}, \vec{x}_0 \rangle + \dots$

Examples:



$\Delta_{12} + \Delta_{23} + \Delta_{34} + \Delta_{45} + \Delta_{51} = 0$

$\Delta_{13} + \Delta_{35} + \Delta_{52} + \Delta_{23} + \Delta_{31} = 0$

$\Delta_{12} + \Delta_{21} = 0$

14.

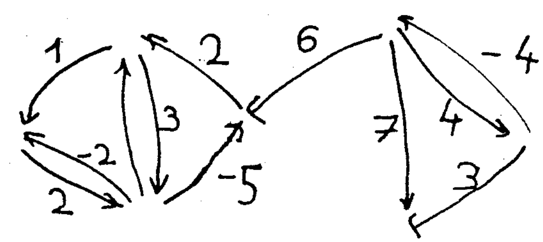
Lemma:

For a graph with arc weights $\Delta_{ij} = -\Delta_{ji}$ („voltages", „potential differences")

there are node weights c_i with $c_j - c_i = \Delta_{ij}$ („potentials")

if and only if every directed cycle K has weight 0:

$$\sum_{ij \in K}' \Delta_{ij} = 0$$

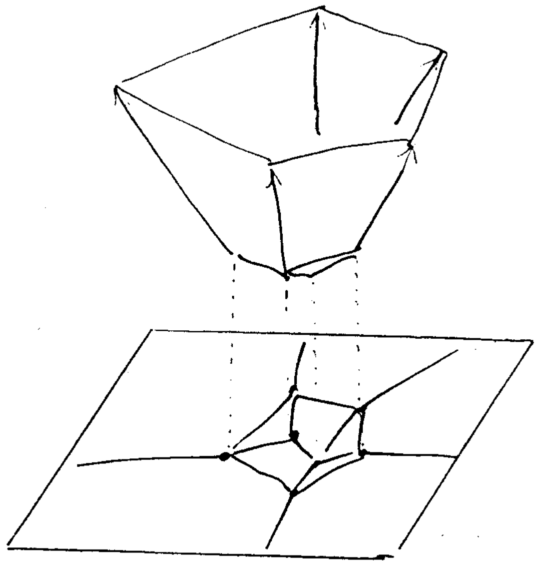


If the graph is connected, the c_i are unique up to an addition of a constant to every c_i.

MAXWELL POLARITY

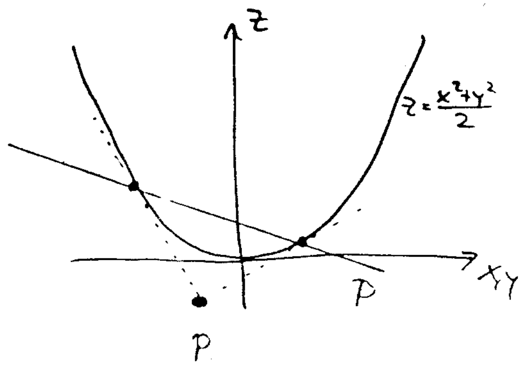
17.

The reciprocal diagram is also the lower projection of an (unbounded) 3-polytope.



The polytope with the given diagram and the polytope with the reciprocal diagram are related by a polarity with respect to the paraboloid of revolution $z = \frac{x^2+y^2}{2}$

point p		plane P
(A,B,Z)	$\Leftrightarrow$	$z = Ax + By - C$



18.

If the boundary cycle is a face of the polytope, the stress can be extended to have equilibrium in all vertices, with $w_{ij} < 0$ on the boundary

PROOF OF STEINITZ'S THEOREM

IDEA:

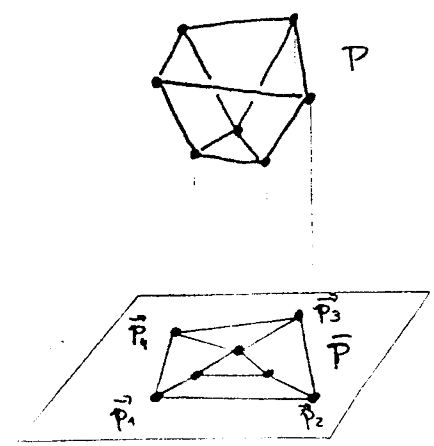
- keep exterior vertices $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_k$ fixed
- all edges are springs with some given arbitrary elasticity constants $w_{ij} = w_{ji} > 0$ obeying Hooke's law — e.g. $w_{ij} \equiv 1$
- compute equilibrium [Tutte 1961]

→ Lower projection $\bar{P}$ of a polytope P, with exterior face $\vec{p}_1, \dots, \vec{p}_k$.

→ orthogonal dual

→ face directions

→ vertices of P



For $k=3$, we are done.

- Every stress for the interior vertices can be extended to a stress on the full graph, with $w_{..}, w_{23}, w_{31} < 0$.

every i: $\sum_{j\sim i} w_{ij}(\vec{p}_j-\vec{p}_i)=0$ $\quad \left(\sum_{j\sim i} w_{ij}\right)\vec{p}_i = \sum_{j\sim i}\left(w_{ij}\cdot\vec{p}_j\right)$

$\vec{p}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ ... two separate systems for x_i and for y_i

(weighted)
Laplace matrix

$$L = \begin{pmatrix} \ddots & & 0 \quad -w_{ij} \\ 0 & \ddots & 0 \\ -w_{ij} \quad 0 & & \ddots \end{pmatrix}$$

$$l_{ii} = \sum_{j\sim i} w_{ij}$$

unweighted $L = -(\text{adjacency matrix})$ with degrees d_i on the main diagonal

For each $v_1,\dots,v_k$ delete row (equation) and column (variable) corresponding to v_i

$$\bar{L}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix} \qquad \text{festgelegt}$$

$$\bar{L}\cdot\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$$

solution by Cramer's rule x_i or $y_i = \dfrac{D_i}{D}$

a subdeterminant of L

solutions $\hat{=}$ potentials (voltages)

w_{ij} = electrical conductivity = $\dfrac{1}{\text{resistance}}$

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$ — vertex degree d_i

Hadamard's inequality:

$$|a_1, a_2, \dots, a_n| \le \|a_1\|\cdot\|a_2\|\cdot\dots\|a_n\|$$

$$\|a_i\| = \sqrt{d_i^2 + \underbrace{1+1+\dots+1}_{d_i}} = \sqrt{d_i(d_i+1)}$$

$$|\det L| \le \sqrt{\prod_{i=1}^{n} d_i(d_i+1)} \le \sqrt{(6\cdot 7)^n} = \left(\sqrt{42}\right)^n$$

$$\left(\sum_{i=1}^{n} d_i \le 6n\right)$$

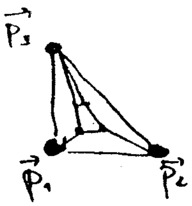
holds for any subdeterminant of L

det (submatrix of L) =

spanning trees (tree-like structures) of the graph.

- If the graph contains a triangle $p_1 p_2 p_3$:

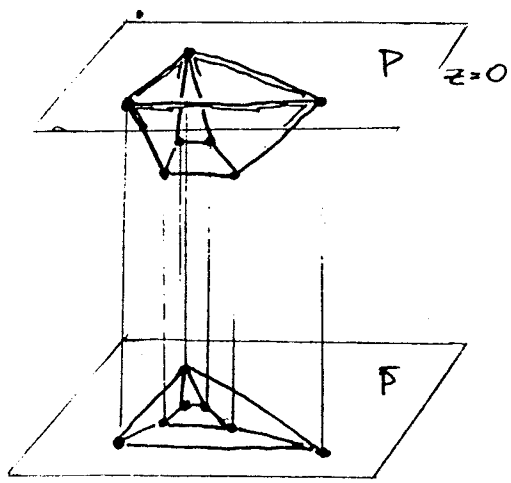
Take $\vec{p}_1 = \binom{0}{0}$ $\vec{p}_2 = \binom{1}{0}$ $\vec{p}_3 = \binom{1}{0}$



→ All $x_i, y_i = \frac{D_i}{D}$

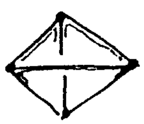
Multiply everything by D

→ integer coordinates in the range $[0..D]$



gradient of "boundary" faces bounded by $n.D$

→ P contained in



→ $z_i \in \left[-nD^2 \cdot \frac{1}{3} .. 0\right]$

THEOREM: A 3-polytope which contains a triangle can be realized with integer vertex coordinates (x_i, y_i, z_i) with

$$|x_i|, |y_i| \le (\sqrt{42})^n$$
$$|z_i| \le n.42^n$$

no triangle

no 4-face → P has a 3-valent vertex

use polarity: P^* has a triangle

move origin into interior

face $ax+by+cz = 1$ → vertex $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

a,b,c is the solution of

$$a x_1 + b y_1 + c z_1 = 1$$
$$a x_2 + b y_2 + c z_2 = 1$$
$$a x_3 + b y_3 + c z_3 = 1$$

$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ vertices of P^*

$a = \frac{A}{D}$ $b = \frac{B}{D}$, $c = \frac{C}{D}$ $|A|, |B|, |C|, |D| \le \sqrt{27} \cdot \max\{x_i, y_i, z_i\}^3$

Gives vertices of P $\vec{p}_i = \left(\frac{A_i}{D_i}, \frac{B_i}{D_i}, \frac{C_i}{D_i}\right)$

multiply by $\prod_{i=1}^{n} D_i$ → integer coordinates

A 3-polytope can be realized with integer coordinates at most $\left(\sqrt{27}\,(n.42^n)^3\right)^{n+1}$

$\le 2^{20n^2}$

Koebe-Andreyev-Thurston theorem

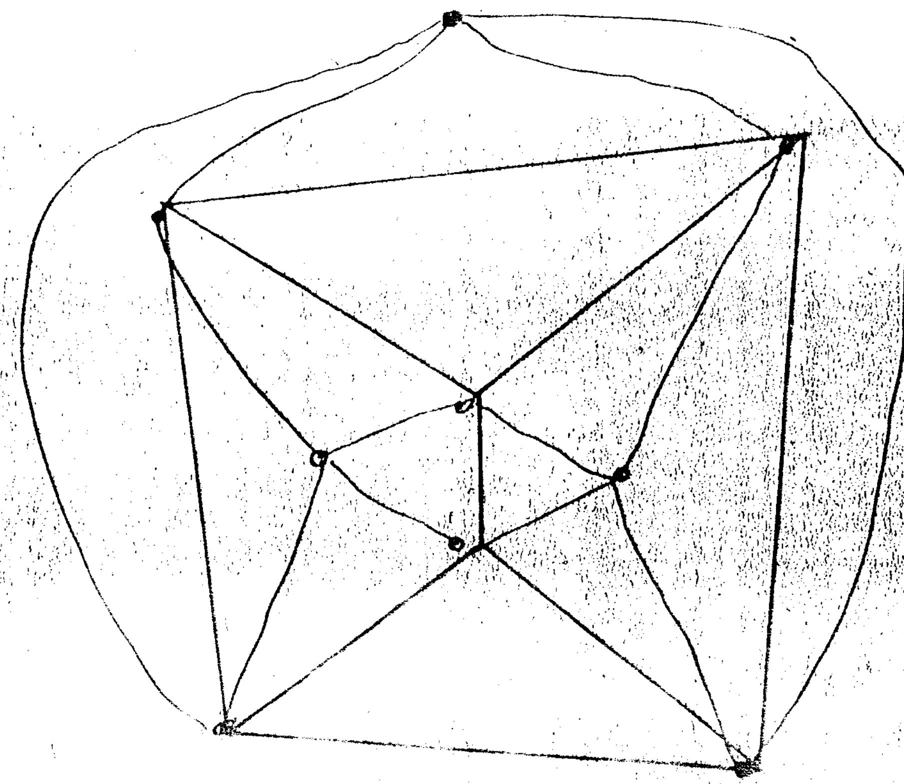
> Every planar graph can be realized as the "touching graph" of non-overlapping disks.

Andreyev (1970) → Thurston (≈ 1975/78)

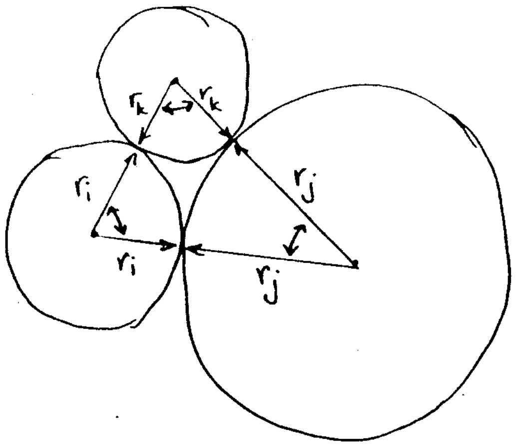
etc.
etc.
⋮

Koebe (1936)

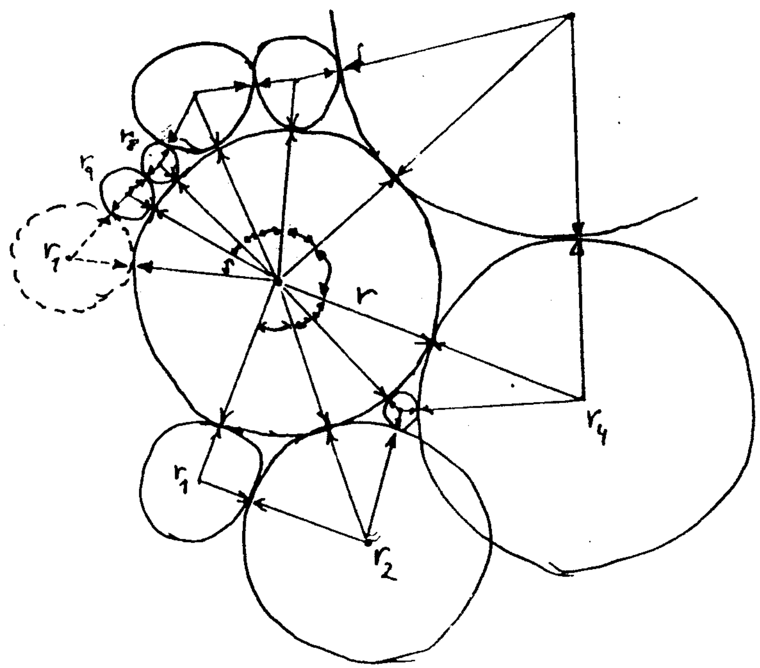
TRIANGULATE!



27.



3 radii
⇒ triangle
⇒ angle



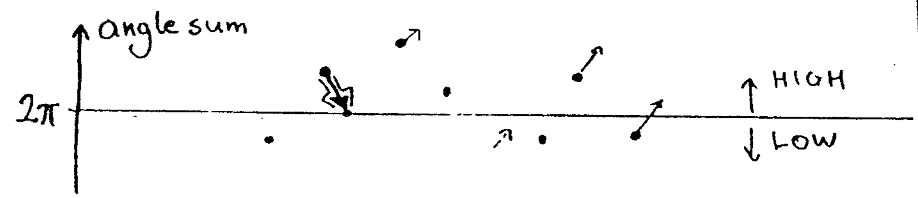
angle too small .... radius too large
total angle too large ... radius too small → increase !

28.

Algorithm: (Thurston)

```
Set all r_i := 1
Loop forever
    for every i do:
        Compute sum of angles
        around vertex i, using current radii.
        if too large
            then [ increase r_i until
                 [ sum of angles becomes 2π.
```

radii INCREASE monotonically → UNBOUNDED (→∞) / → CONVERGENT



a HIGH vertex will never become LOW.

total angle sum $= 2\pi \cdot n$

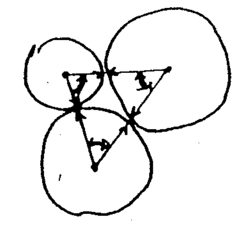
⇒ one vertex will always remain LOW

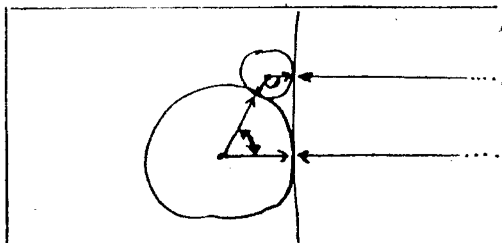
⇒ $r_i = 1$

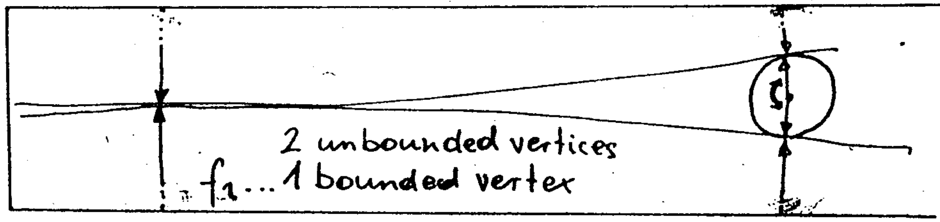
⇒ At least one radius is CONVERGENT.

Some Observations:

- radii only INCREASE.
- if an angle$^{\text{sum}}$ converges, the limit is $\leq 2\pi$
- All angles at bounded vertices converge.







G ... a connected component of BOUNDED vertices

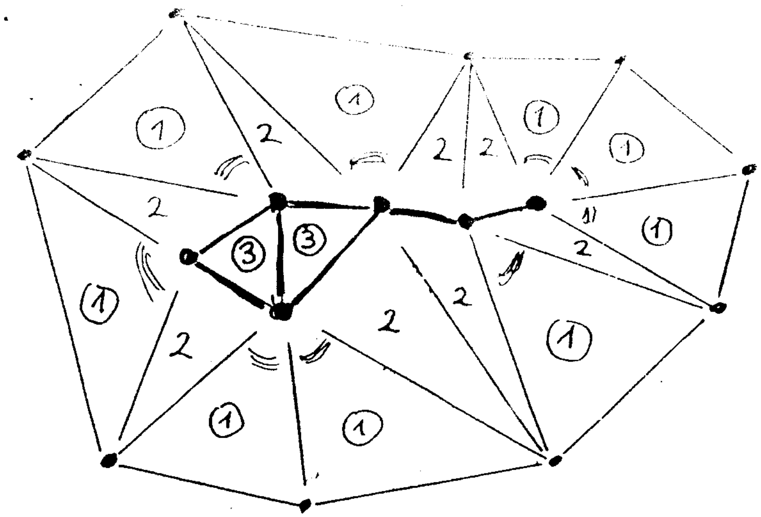
n_0 vertices, m_0 edges

f_3, f_2, f_1 incident faces

- each face contributes π to the Limiting angles at vertices of G
- all angles at G converge $\Rightarrow$ to $\leq 2\pi$

$$\left.\begin{array}{l}\text{each face contributes } \pi \text{ to the limiting angles at vertices of } G \\ \text{all angles at } G \text{ converge} \Rightarrow \text{to} \leq 2\pi\end{array}\right\} (f_1+f_2+f_3)\cdot\pi \leq n_0 \cdot 2\pi$$

$$\boxed{2n_0 \geq f_1+f_2+f_3}$$



G and its UNBOUNDED neighbors

# of UNBOUNDED neighbors $\leq f_1$

. . . . . . ≥ 3 (three-connected)

$\boxed{f_1 \geq 3}$

↑(special treatment if G contains all but two vertices)

uler's formula: planar?

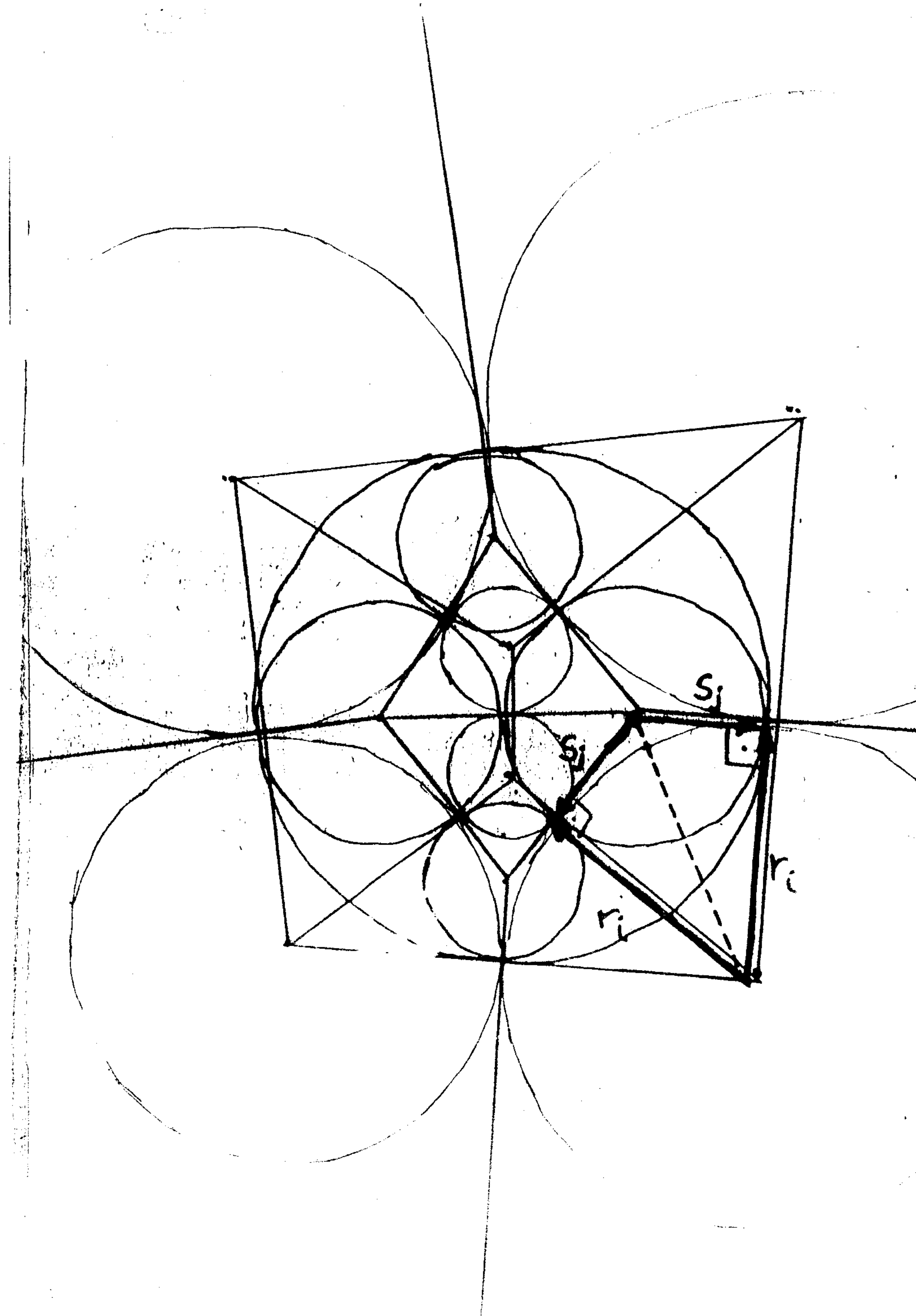
$$\underbrace{\text{Vertices}}_{n_0} + \underbrace{\text{Faces}}_{\geq f_3+1} = \underbrace{\text{Edges}}_{m_0} + 2$$

$$\boxed{n_0 + f_3 \leq m_0 + 1}$$

ach edge of G has two sides:

$$\boxed{2m_0 = 3f_3 + f_2}$$

→ contradiction



LOWER BOUNDS ?

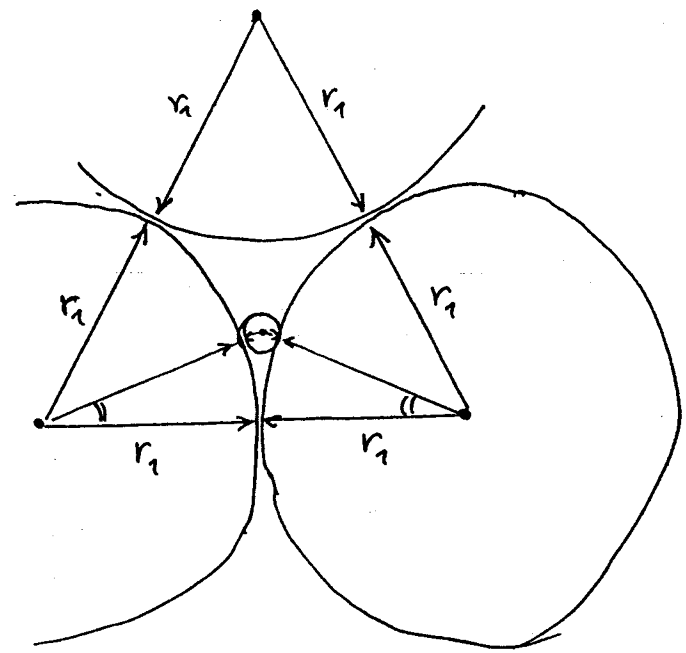
Thiele (1991) [Jarník 1922]

An n-gon needs an integer grid of side length

$$\frac{2\pi}{12^{3/2}} \cdot n^{3/2} + O(n \log n)$$

Summary:
- All r_i converge
- All angle sums converge to 2π
- In the limit, the triangles fit together locally.

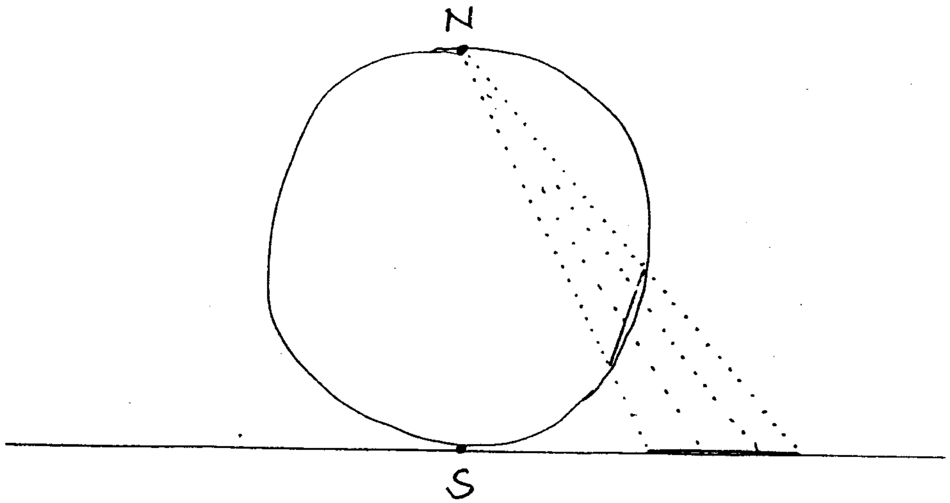
Special treatment of the exterior triangle

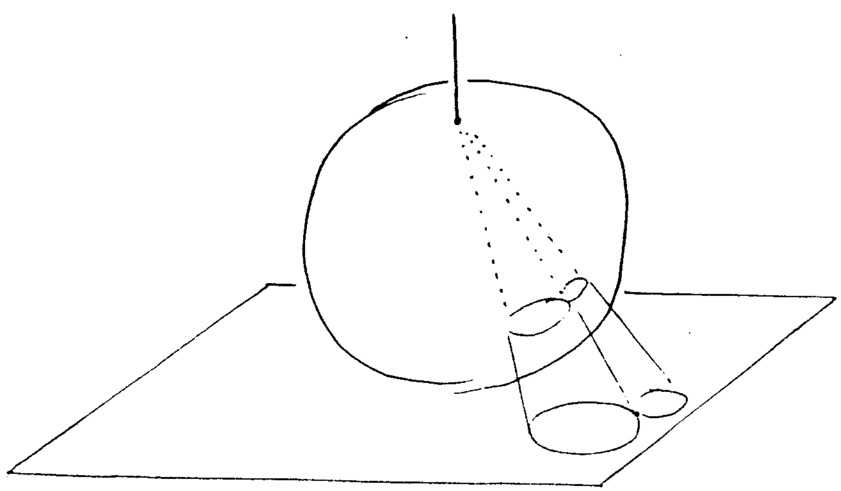


All three exterior disks have the same radius r_1.

Total angle sum at 3 vertices $= \pi$

34. Disk packings on the SPHERE by stereographic projection.





33. UNIQUENESS.

For a triangulation, the disk packing is unique up to Möbius transformations.

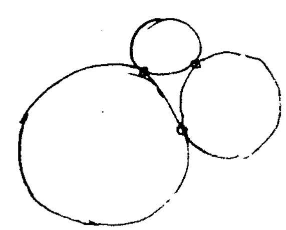
- Möbius transformations are the most general class of transformations which

  LEAVE THE CLASS OF CIRCLES INVARIANT

(straight lines are considered as special cases of circles)

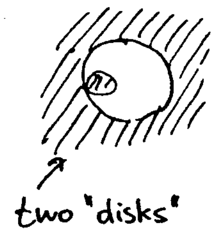
- Möbius transformations are generated by similarities and inversions.
- $z \mapsto \dfrac{az+b}{cz+d}$ or $z \to \dfrac{a\bar{z}+b}{c\bar{z}+d}$ for $z,a,b,c,d \in \mathbb{C}$

Proof idea: The packing is unique if the three exterior disks are fixed.



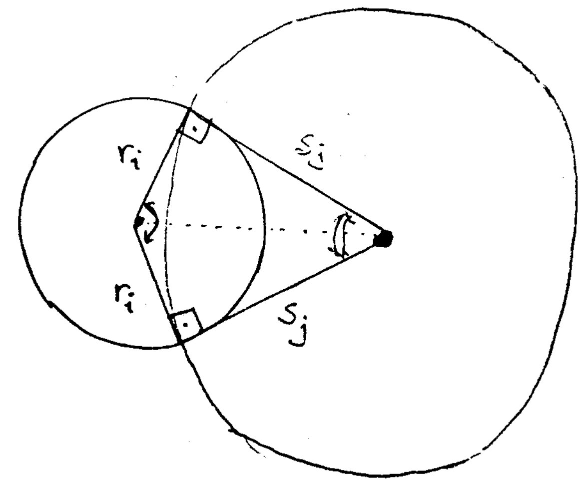
35. extension to simultaneous PRIMAL-DUAL packing.

For every 3-connected planar graph, there is a collection of blue "disks" for each vertex and red "disks" for each face, such that



- No two blue disks overlap
- No two red disks overlap
- For two adjacent VERTICES, the corresponding blue disks touch.
- For two adjacent FACES, the corresponding red disks touch.
- For every edge, the incident blue disks and the incident red disks touch in THE SAME POINT and at RIGHT ANGLES.

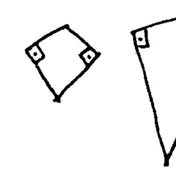
36. Two sets of radii: r_i and s_j



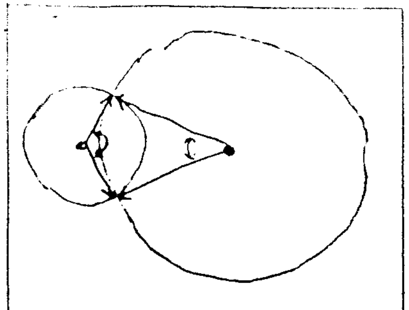
radii r_i c

$\Rightarrow$ angles

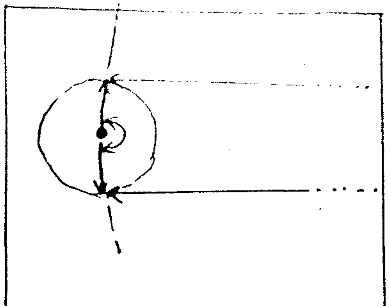
- convergence of radii
- $\to$ convergence of angle sums to 2π
- $\to$ collection of "deltoid"-shapes that fit togethere locally.



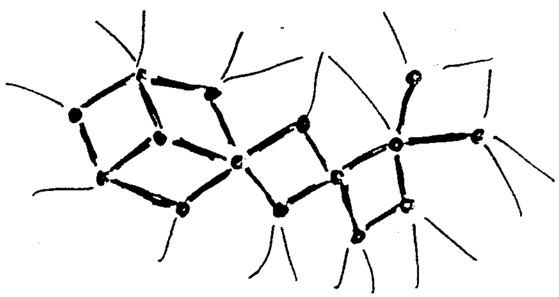
CONVERGENCE:



m_2 ... face-vertex pairs with 2 bounded radii



m_1 ... pairs with 1 unbounded radius



G: •∘
connected component of BOUNDED nodes (vertices, faces

m_2: ●━● within G

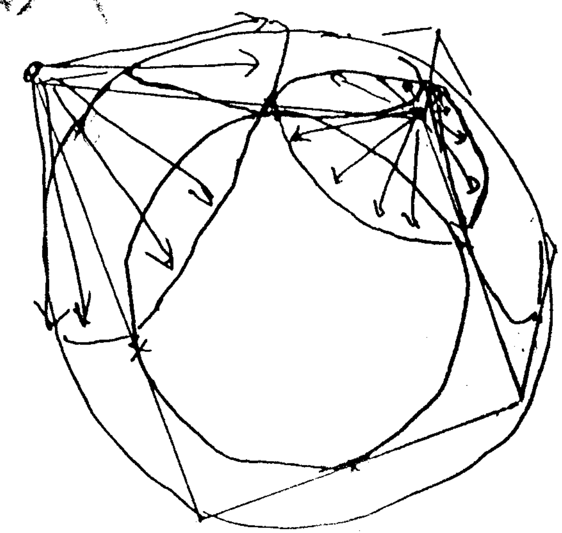
m_1: •— from G to its unbounded neighbors

Each m_1+m_2 contributes π to the angles at G

$$\boxed{m_1+m_2 \le 2n_0}$$

Every connected planar graph is the skeleton of a 1-scribable ("mid-scribable") polyhedron.

(by stereographic projection)



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n-dimensional cell complex (polytope); d-scribable in $\mathbb{R}^{n+1}$

d=0 ... inscribable

d=n ... circumscribable

n=2: d=0 → not always; $\boxed{d=1}$ ↗ (STEINITZ, 1928); d=2

n≥3, any d: not always possible

Realizations of Three-Dimensional Polytopes — Literature

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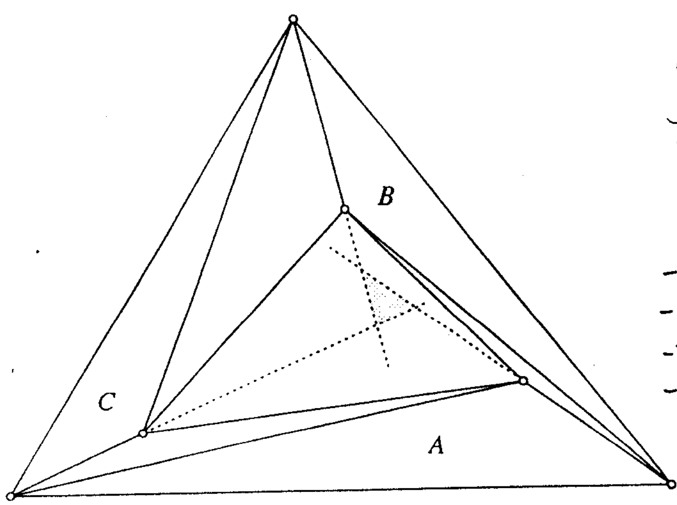
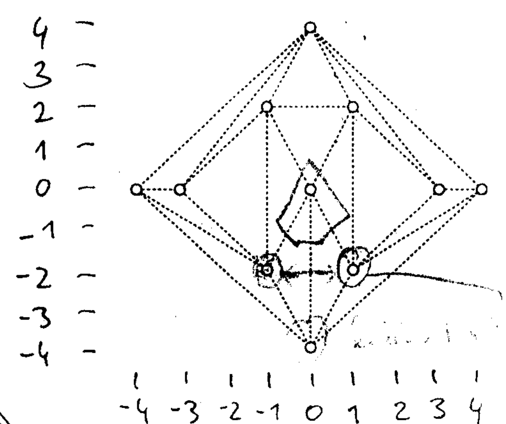
Realizations of Three-Dimensional Polytopes

1. We are given a bar framework in the plane with vertices $V = \{\vec{p}_i\}$ and scalars $w_{ij} = w_{ji}$ for the edges ij such that equilibrium holds for a subset S of vertices:
$$\sum_{j\sim i} w_{ij}(\vec{p}_j - \vec{p}_i) = 0, \quad \text{for } i \in S$$
Show that the forces which the vertices S exert on the remaining vertices $k \in V - S$ sum to zero and are torque-free:
$$\sum_{\substack{k\in V-S,\ i\in S\\ k\sim i}} w_{ik}(\vec{p}_i - \vec{p}_k) = 0, \qquad \sum_{\substack{k\in V-S,\ i\in S\\ k\sim i}} w_{ik}\cdot \vec{p}_k \times (\vec{p}_i - \vec{p}_k) = 0.$$
Here, for a point $\vec{p} = \binom{x}{y}$ and a force $\vec{f} = \binom{a}{b}$ we denote by
$$\vec{p} \times \vec{f} := \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ bx - ay \end{pmatrix}$$
the turning moment (torque) of a force $\vec{f}$ applied to a point $\vec{p}$.
Hint: Use the equation $\vec{p}_i \times (\vec{p}_j - \vec{p}_i) + \vec{p}_j \times (\vec{p}_i - \vec{p}_j) = 0$.

2. Show, by several different methods if possible, that the following two pictures are not lower projections of convex polytopes. (The vertices in the right picture have integral coordinates between -4 and 4.) For the left picture, try to generalize your answer to obtain a statement about all diagrams which contain a cyclic "windmill-like" structure.





3. If a reciprocal diagram exists, is it always unique, up to natural transformations like translations and scalings?

4. Which graphs can be the graphs of lower projections of 3-polytopes? Is the plane embedding in which such a graph can appear as the lower projection of a 3-polytope always unique?

5. How does the proof that the stress criterion implies the criterion of the reciprocal diagram fail if the given graph is not planar?

6. We are given a finite number of bounded or unbounded convex polygonal regions, with rules for matching edges and identifying vertices of different polygons. If these regions *fit together locally,* they can be arranged to cover the whole plane without overlap. (They form a *tiling* of the plane.) Find an appropriate definition of "fitting together locally" and prove the above statement.

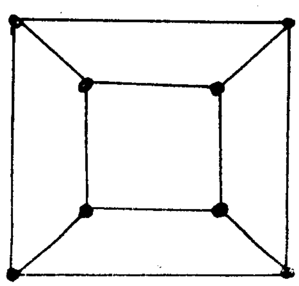
7. Try to extend (i) the criterion of the reciprocal diagram, and (ii) the criterion of the stress

   (a) to lower projections of polytopes with only one "invisible" face which would project onto the projection of the whole polytope (Schlegel diagrams)

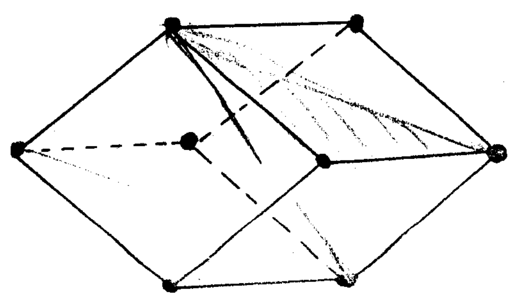
   (b) to combined upper and lower projections of polytopes.

   Hint: The reciprocal diagram will either involve crossing edges, or infinite rays and unbounded faces.

(a)



(b)



8. A plane bar framework with vertices $\vec{p}_i$ can *support* a given system of external forces $\vec{f}_i$, if there are scalars $w_{ij} = w_{ji}$ for the edges ij such that, for all vertices i,

$$\sum_{j\sim i} w_{ij}(\vec{p}_j - \vec{p}_i) + \vec{f}_i = 0.$$

   Show the following statements.

   (a) A necessary condition for a bar framework to support the forces $\vec{f}_i$ is that the total force $\sum_i \vec{f}_i$ and the total turning moment (torque) $\sum_i \vec{p}_i \times \vec{f}_i$ are both zero.

   (b) For a triangle, this condition is also sufficient.

   (c) For a quadrilateral, it is not sufficient.

9. (a) Show, using Euler's formula, that a 3-polytope without triangular faces contains at least 8 vertices of degree 3.

   (b) Show that a 3-polytope without vertices of degree 3 contains at least 8 triangular faces.

   (c) What is the minimum number of vertices of degree 3 that a 3-polytope with T triangular faces can have?

RUNDEN!