# ON THE CLASSIFICATION
# OF NON-REALIZABLE ORIENTED MATROIDS
## Part I: Generation

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# ON THE CLASSIFICATION OF NON-REALIZABLE ORIENTED MATROIDS
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**Abstract.**

*We introduce refined methods for deciding the realizability of oriented matroids. Applications of reduction theorems are tested, and new final polynomials were found. After enumerating all reorientation classes of oriented matroids with 8 points in rank 4, we provide a complete overview about all non-realizable oriented matroids in this case. We classify these orientation classes e.g. according to their mutation numbers.*

## 1. Introduction

Oriented matroids have been used as a basic tool in a variety of different fields. In looking at more than two hundred papers about oriented matroids published during the last 15 years, not to mention papers on (ordinary) matroids, it is difficult to emphasize particular applications of oriented matroids. We mention combinatorial optimization, see e.g. Bland [7],[8],[9], Bachem [4],[5]. computational synthetic geometry, see Bokowski & Sturmfels [16].

Recently, Dress used oriented matroids in the study of Penrose tilings, [18]. Of course, from a mathematical point of view, oriented matroids have been studied and investigated also in their own right, see e.g. Las Vergnas [24],[25],...,[30]. The convexity property of oriented matroids made them to form a fundamental framework in the theory of combinatorial convexity, Las Vergnas [27]. An interactive and stimulating process via the common structure *oriented matroid* as defined in various ramifications by Edmonds, Bland, Las Vergnas, Folkman, Lawrence, and many authors thereafter, is still going on, thus emphasizing the theory of oriented matroids to be of a fundamental and applicable nature.

It is essential to distinguish matroids from oriented matroids. Whereas on one hand oriented matroids differ from matroids only by the underlying field [11], on the other hand, the change of the field causes decisive better properties for oriented matroids such as convexity, [27].

The fundamental structure of oriented matroids has grown to a theory of its own right, and it is not covered in any book on matroid theory so far, but see [5],[16]. The extension of classical discrete convexity as well as other applications have always raised the following fundamental as well as naturally very general question in the theory of oriented matroids:

**Problem 1.1.** *Given a property $P(\chi)$ of an oriented matroid $\chi$ which is valid when the oriented matroid can be realized (represented) as a point configuration in a vectorspace over a field* **K**, *does this property hold in the general oriented matroid case as well?*

Many papers have made contributions to this decisive question by using small non-realizable oriented matroids. Nevertheless, we don't have many different counter-examples for this purpose. In particular, for oriented matroids with 8 points in rank 4, only two non-realizable examples were known.

It is one aim of this paper to provide new classes of non-realizable (=non-representable) oriented matroids which can be used for all variants of Problem 1.1 in various applications. Additional

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motivation for the enterprise of doing this extensive computer aided search was due to the following questions or facts, respectively.

- Is the mutation-graph of chirotopes connected in this case?
- Are there new types of final polynomials?
- Are the methods for testing realizability sufficient in these cases?
- Examples serve as bases for further investigations.
- The rank 4 case is decisive for 3-dimensional objects and might help to solve other problems.
- The $(8,4)$-case is minimal with respect to the number of points in rank 4.
- The investigation might help to find invariants for classes of oriented matroids.
- An $(8,4)$-chirotope has the chance to be self-dual.
- The classification can be used for testing isomophic chirotopes.

It is sufficient to work with reorientation classes of oriented matroids. Reorientation classes of oriented matroids are equivalence classes of oriented matroids whereby renumbering the points and reorienting the oriented matroid at any subset of points is factored out. Reorientation classes of oriented matroids are in one to one correspondence to hypersphere configurations, and in the realizable case they represent the combinatorial types of zonotopes. In rank 4 up to 7 points all (realizable) reorientation classes where determined by Goodman and Pollack, [21]. In this paper we give the precise number of (uniform) reorientation classes in rank 4 with 8 points, i.e. in the first case with non-realizable examples. We classify all of them according to various properties such as number of mutations and realizability. Additional properties, especially those of non-realizable oriented matroids, will be discussed in a forthcoming paper [13], compare Problem 1.1.

We describe the method of generating all (uniform) reorientation classes in rank 4 with 8 points in Section 2. The fundamental and difficult question of deciding the realizability of oriented matroids is (in the rational case) equivalent to Hilbert's 10th problem [39], thus a main stream of research has been in looking for conditions or algorithms to distinguish non-realizable cases from realizable ones, at least for important particular subclasses. In Section 3 we give the exact number of reorientation classes of oriented matroids with 8 points in rank 4. Moreover, in all these 2628 cases the problem of realizability was solved. The list of non-realizable oriented matroids is given in the Appendix.

We list also the distribution of the number of mutations, showing that the known example with 7 mutations is the only one with less than 8 mutations. Thus in case of 8 points, a theorem of Shannon [38] is applicable just for a single case to proof non-realizability. In Section 3 we also list our result in connection with the known classifications.

In Section 4 we discuss several reduction theorems which can be applied for deciding realizabilty. They were crucial for our classification, and some of them are very interesting in their own right. The underlying idea of several combinatorial reduction theorems is seen when looking at their original geometric counterpart. The algebraic solving technique uses the boundary structure of the realization space.

In Section 5 we discuss a new solution method which can be applied for deciding realizabilty. Again, the underlying geometric idea of the algebraic solving technique is passing to the boundary structure of the realization space.

A final polynomial is determined in Section 6. Once a final polynomial for a given oriented matroid is found, we can search for other oriented matroids with the same final polynomial structure. It turned out that 3 types of final polynomials are sufficient to characterize non-realizability of oriented matroids in rank 4 with 8 points. More details will be found in [13].

We will not concern ourselves with the natural question about the asymptotic behavior of non-realizable oriented matroids. For related results see e.g. N. Alon [1].

**2. On the Generating of All Extensions.**

According to Grünbaums's classification of pseudoline arrangements [23,p.395], there are 11 reorientation classes with 7 points in rank 3. Due to the dualizing operation we have also 11 such classes in rank 4. Now assume we are given an oriented matroid with 8 points in rank 4. Deleting a point on one hand leads to the i'th case in Grünbaum's list, and contracting the same point and taking the dual on the other hand leads to the j'th case thus defining a pair (i,j) for each vertex. We consider all these pairs. The vector of sorted pairs is invariant under reorientation. We call it the $(8,4)$-*reorientation class vector*. One can tell from this (8,4)-reorientation class vector together with the number of mutations whether a reorientation class of 8 points in rank 4 is realizable or not.

**Theorem 2.1.** The following (8,4)-reorientation class vectors extended by the number of mutations correspond to non-realizable 4-chirotopes with 8 points. The list is complete. An asterisk was used when the number of mutations is not decisive.

$$(04,04)(04,04)(04,04)(04,04)(04,04)(04,04)(07,07)(07,07) - *$$
$$(07,07)(07,07)(07,07)(07,07)(07,07)(07,07)(07,07)(07,07) - * \quad (2 types)$$
$$(07,08)(07,08)(07,08)(07,08)(08,07)(08,07)(08,07)(08,07) - *$$
$$(04,07)(04,07)(07,04)(07,04)(10,10)(10,10)(10,10)(10,10) - *$$
$$(04,08)(07,10)(08,04)(10,07)(10,10)(10,10)(10,10)(10,10) - *$$
$$(04,10)(04,10)(04,10)(07,08)(08,07)(10,04)(10,04)(10,04) - 11$$
$$(04,10)(04,10)(07,10)(10,04)(10,04)(10,07)(10,10)(10,10) - *$$
$$(07,07)(07,07)(07,07)(07,07)(10,10)(10,10)(10,10)(10,10) - *$$
$$(07,10)(07,10)(07,10)(07,10)(10,07)(10,07)(10,07)(10,07) - * \quad (2 types)$$
$$(07,10)(07,10)(07,10)(08,08)(08,08)(10,07)(10,07)(10,07) - *$$
$$(07,10)(07,10)(08,10)(08,10)(10,07)(10,07)(10,08)(10,08) - *$$
$$(08,08)(08,08)(10,10)(10,10)(10,10)(10,10)(10,10)(10,10) - *$$
$$(10,10)(10,10)(10,10)(10,10)(10,10)(10,10)(10,10)(10,10) - * \quad (5 types)$$
$$(07,11)(08,10)(08,10)(08,10)(10,08)(10,08)(10,08)(11,07) - *$$
$$(08,10)(08,10)(10,08)(10,08)(10,11)(10,11)(11,10)(11,10) - *$$
$$(08,10)(10,08)(10,10)(10,10)(10,10)(10,10)(10,11)(11,10) - *$$
$$(08,11)(08,11)(08,11)(08,11)(11,08)(11,08)(11,08)(11,08) - *$$
$$(10,10)(10,10)(10,10)(10,10)(10,10)(10,10)(11,11)(11,11) - *$$

The numbers are related to Grünbaums list [23,p.395] given in Appendix 1, the simple pseudoline arrangements of all 11 (simplicial) reorientation classes with 7 points in rank 3.

**Remark 2.2.** Note in particular that all oriented matroids with 8 points in rank 4 which are extensions of type 1,2,3,5,6, or 9 are realizable. Moreover, an $(8,4)$ reorientation class vector which is not selfdual must correspond to a realizable reorientation class.

In order to generate all reorientation classes of oriented matroids with 8 points in rank 4, we simply have to form all one point extensions for all 11 types of reorientation classes of rank

4 with 7 points. After calculating the above reorientation class vectors, we have to discard those which occur more often. Whereas from a theoretical point of view, this is easy to describe, the corresponding implementation was not at all an easy task. The main implementation was done by F.Anheuser and J.Richter.

In this section we introduced the (8, 4) reorientation class vectors. The following examples show that the orientation class is not uniquely defined by this notion.

**Example 2.3.** The following oriented matroids (given in terms of bases) both have the same (8, 4) reorientation class vector

$$(7,7)(7,7)(7,7)(7,7)(7,7)(7,7)(7,7)(7,7)$$

```
+++++---------+++++++++++------++------------+++++--++-++++++++----++-
+++++--------++++++++++----+-++++++---+--+-++++-+-------+++++++++-+---
```

Both oriented matroids are not realizable. According to our Theorem 3.2, the first one is the unique example which was studied by Roudneff, see [36] and which was the essential underlying structure showing that a certain sphere is not realizable, [2]. The second one is a former unknown example.

Example 2.4. The following oriented matroids have the same (8, 4) reorientation class vector. All of them are realizable.

$$(9,9)(9,9)(9,9)(9,9)(9,9)(9,9)(9,9)(9,9)$$

```
+++++------+-++++++++++--------+-----------+++++-++-++-+++++-++++--+++
+++++---------+++++++++------++++++--------+++---------+++++-++++-----
+++++------+-+++++++++++-------+-----------+++++-++-++-+++++-++++--+++
+++++---------+++++++++------++++++--------+++---------+++++-++++----+
```

In the six cases above, deleting a vertex, or taking the dual and contracting at a vertex afterwards, retains the same reorientation class.

## 3. Classification Results

In this section we give an overview about classifications made so far. Blackburn, Crapo, and Higgs classified all (ordinary) matroids up to 8 points [3]. The result of Goodman and Pollack about 8 point order types in rank 3, see [21], was achieved by using their method of allowable sequences, and it settled the problem in this case including all nonuniform cases. Halsey and independently Canham did the uniform case only.

J. Richter's result in rank 3 up to 9 points extends this result to the rank 3 case, [33]. The case with 8 points in rank 4 appears to be a gap in these investigations when looking at the following table of reorientation classes for various ranks and point numbers.

**Table 3.1. Reorientation classes for given rank and number of points.**

**total number : number of non-realizable ones**

| rank \ points | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|
| 7 | | | | | 1 : 0 |
| 6 | | | | 1 : 0 | 4382 : 1 |
| 5 | | | 1 : 0 | 135 : 0 | = (9 pts, rank 4) |
| 4 | | 1 : 0 | 11 : 0 | 2628 : 24<br>Bokowski/Richter | non-real. examples<br>by extensions |
| 3 | 1 : 0 | 4 : 0 | 11 : 0<br>Grünbaum | 135 : 0<br>Goodman/Pollack<br>(Halsey/Canham) | 4382 : 1<br>Richter |

We summarize a main result in the following.

**Theorem 3.2.** There are precisely 2628 reorientation classes of oriented matroids with 8 points in rank 4. Exactly 24 of them are not realizable. The distribution of the number of mutations is as follows.

| # mut. | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # r.cl. | 1 | 44 | 241 | 493 | 487 | 547 | 413 | 228 | 98 | 51 | 12 | 6 | 2 | 5 |

**Remark 3.3.** Due to the various notational ramifications, the assertion can be formulated as follows. There are 2628 (pseudo-) hyperplane configurations. The number of combinatorial different zonotopes formed out of 8 line segments in dimension 4 is 2604. The main result is the complete overview of all non-realizable oriented matroids in this case.

**Remark 3.4.** Deciding the realizability of an oriented matroid is still a challenging open problem. Realizations can be found by applying the solvability sequence method, see [13]. Non-realizability can be established by *final polynomials* , see [14], and linear programming can be applied for finding large classes of final polynomials, see [12]. Additional combinatorial ideas (especially in the rank 3 case) lead to decision techniques for extended classes, see Richter [33],[34]. But a general and at the same time practical method which would have promised that our classification (Theorem 3.2) would succeed remains unknown.

From a theoretical point of view we have a satisfactory answer. A consequence of a real version of Hilbert's Nullstellensatz shows that there is either a set of coordinates or a final polynomial, due to Dress [unpublished], and Sturmfels, see [40] as well as [16]. Our results formulated in Theorem 3.2 were found when combining known methods with several additional methods described later in this paper and in Richter [34]. Recall that even a single decision can be very time consuming in general. Thus we not only applied known methods but refined existing methods. In the following sections we describe both new refined solution methods in the realizable case and a reduction which leads to a final polynomial and therefore to a non-realizable case.

□

## 4. Reduction Theorems for Oriented Matroids

A general practical algorithm for deciding the realizability of a given oriented matroid is considered to be an important open problem. Only when restricting the problem to certain subclassses, there seems to be hope for providing such algorithms. This section is devoted to this very question. One general idea when dealing with such questions has always been in reducing the problem in various ways. We will use the notion *reduction theorem* for assertions of the following form. An oriented matroid $\chi$ with property $P$ is realizable provided realizability can be shown for a (partial) oriented matroid $\chi'$ derived from $\chi$ in a prescribed manner.

We emphasize this approach of solving realizability by using a sequence of reduction theorems until one arrives at a (partial) oriented matroid which is known to be realizable. We further extend the list of known *reduction theorems* by several new theorems which were successfully applied in the classification mentioned in the last section. A typical example of a reduction theorem is the following

**Theorem 4.1.** An oriented matroid $\chi$ is realizable if and only if its dual $\chi^*$ is realizable.

**Remark 4.2.** An example where realizability is known is the case of rank 3 oriented matroids with 8 points, see [21]. Combining these results yields an assertion about realizability of oriented matroids not only in rank 3, see e.g. Table 3.1.

In order to formulate additional reduction theorems, we will define the following notions.

**Definition 4.3.** Let $\chi$ be a $d$-chirotope over $E := \{1, 2, \ldots, n\}$. The set $\mathcal{R}(\chi)$ of $n$-element (ordered) point sets R also regarded as matrices $M$ with $n$ rows and $d$ columns (homogeneous coordinates) such that the sign of all $d$-minors of $M$ gives back the chirotope property of $\chi$ will be called the *realization space of* $\chi$.

**Definition 4.4.** An oriented matroid $\chi$ is *reducible by a point p* (w.l.o.g. assume $p$ to be the last point) if any $R \in \mathcal{R}(\chi \setminus p)$ can be extended by a point $x_p$ such that $R \cup x_p \in \mathcal{R}(\chi)$.

**Definition 4.5.** Let $\chi$ be a (partial) $d$-chirotope over $E := \{1, 2, \ldots, n\}$ and let $\mathbf{C} \in \mathcal{O}^*(\chi)$ be a set of cocircuits. A pair $(e, f) \in E \times E$ is called *a sign-invariant pair with respect to* $(\chi, \mathbf{C})$ if $\{-1, 1\}$ is not a subset of $\{C^e \cdot C^f | C \in \mathbf{C}\}$. If $(e, f)$ is sign-invariant with respect to $(\chi, \mathcal{O}^*(\chi))$, we call $(e, f)$ a *sign-invariant pair in* $\chi$.

In case of a partial chirotope the multiplication is taken in the corresponding fuzzy ring over the elements $\{-1, 0, +1, *\}$.

A well known reduction theorem, see e.g. [35], describes the relation between sign-invariant pairs and reducebility:

**Lemma 4.6.** Let $\chi$ be a 3-chirotope over $E := \{1, 2, \ldots, n\}$, and let $(p, e) \in E \times E$ be a sign-invariant pair in $\chi$, then $\chi$ is reducible by $p$.

A generalization of this theorem due to the second author [34] will be formulated in Theorem 4.8., it provides much better conditions for testing reducebility of a given oriented matroid. It uses the notion of isolated points:

**Definition 4.7.** Let $\chi$ be a simplicial $d$-chirotope over $E := \{1, 2, \ldots, n\}$. Let $p \in E$ and let $\Pi := \{C_1, \ldots, C_d\} \subset \mathcal{O}_p^*(\chi)$ be a set of $d$ pairwise different cocircuits containing $p$ in their support. We say that $\Pi$ *isolates the point p in* $\chi$ if there is no element $e \in E$ such that $(p, e)$ is a sign-invariant pair with respect to $(\chi, \Pi)$. We call *p isolated in* $\chi$ if there exists such a set $\Pi$ of cocircuits that isolated $p$ in $\chi$.

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We emphasize this approach of solving realizability by using a sequence of reduction theorems until one arrives at a (partial) oriented matroid which is known to be realizable. We further extend the list of known *reduction theorems* by several new theorems which were successfully applied in the classification mentioned in the last section. A typical example of a reduction theorem is the following

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In order to formulate additional reduction theorems, we will define the following notions.

**Definition 4.3.** Let $\chi$ be a $d$-chirotope over $E := \{1, 2, \ldots, n\}$. The set $\mathcal{R}(\chi)$ of $n$-element (ordered) point sets R also regarded as matrices $M$ with $n$ rows and $d$ columns (homogeneous coordinates) such that the sign of all $d$-minors of $M$ gives back the chirotope property of $\chi$ will be called the *realization space of* $\chi$.

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**Definition 4.5.** Let $\chi$ be a (partial) $d$-chirotope over $E := \{1, 2, \ldots, n\}$ and let $\mathbf{C} \in \mathcal{O}^*(\chi)$ be a set of cocircuits. A pair $(e, f) \in E \times E$ is called *a sign-invariant pair with respect to* $(\chi, \mathbf{C})$ if $\{-1, 1\}$ is not a subset of $\{C^e \cdot C^f | C \in \mathbf{C}\}$. If $(e, f)$ is sign-invariant with respect to $(\chi, \mathcal{O}^*(\chi))$, we call $(e, f)$ a *sign-invariant pair in* $\chi$.

In case of a partial chirotope the multiplication is taken in the corresponding fuzzy ring over the elements $\{-1, 0, +1, *\}$.

A well known reduction theorem, see e.g. [35], describes the relation between sign-invariant pairs and reducibility:

**Lemma 4.6.** Let $\chi$ be a 3-chirotope over $E := \{1, 2, \ldots, n\}$, and let $(p, e) \in E \times E$ be a sign-invariant pair in $\chi$, then $\chi$ is reducible by $p$.

A generalization of this theorem due to the second author [34] will be formulated in Theorem 4.8., it provides much better conditions for testing reducibility of a given oriented matroid. It uses the notion of isolated points:

**Definition 4.7.** Let $\chi$ be a simplicial $d$-chirotope over $E := \{1, 2, \ldots, n\}$. Let $p \in E$ and let $\Pi := \{C_1, \ldots, C_d\} \subset \mathcal{O}_p^*(\chi)$ be a set of $d$ pairwise different cocircuits containing $p$ in their support. We say that $\Pi$ *isolates the point* $p$ *in* $\chi$ if there is no element $e \in E$ such that $(p, e)$ is a sign-invariant pair with respect to $(\chi, \Pi)$. We call $p$ *isolated in* $\chi$ if there exists such a set $\Pi$ of cocircuits that isolated $p$ in $\chi$.

**Theorem 4.8.** (J.Richter, [34])
Let $\chi$ be a simplicial $d$-chirotope over $E := \{1, 2, \ldots, n\}$ and let $p \in E$ be a point which is neither isolated in $\chi$ nor isolated in the contraction $\chi/A$ for any subset $A \subset E \setminus \{p\}$. Then $\chi$ is reducible by $p$.

By applying Theorems 4.8 and 4.1 in the case of simplicial 4-chirotopes with 8 points and by using the fact that 4-chirotopes with 7 points in rank 4 are realizable, we are left with only 82 examples of reorientation classes which have to be treated by different methods.

**Corollar 4.9.** At least 2546 of all 2628 reorientation classes of rank 4 with 8 points are realizable.

**Proof.** In all these cases either for the corresponding chirotope $\chi$ or for its dual $\chi^*$, there exists a point $p$ that is not isolated and that is not isolated in any minor. So $\chi$ or $\chi^*$, respectively, are reducible by Theorem 4.8 and hence realizable because 4-chirotopes with 7 points in rank 4 are realizable. The test for deciding whether there exists such a point for a given chirotope or its dual, was done by an exhaustive computer search.

□

So after applying Theorem 4.8, we are left with only 82 reorientation classes for which the realizability could not be settled in this way. There are precisely 58 cases among these remaining 82 reorientation classes which are realizable. In the following, we provide two additional reduction theorems which were sufficient in all remaining cases.

For a chirotope $\chi$ and a basis $\lambda$ in the set of all bases $\Lambda(E, d)$, we define a map $\chi^\lambda$ as follows.

$$\chi^\lambda(\lambda') := \chi(\lambda') \text{ if } \lambda' \neq \lambda \quad \text{and} \quad \chi^\lambda(\lambda') := 0 \text{ if } \lambda' = \lambda.$$

**Theorem 4.10.**
Let $\chi$ be a $d$-chirotope over $E := \{1, 2, \ldots, n\}$ and let $\lambda := (\lambda_1, \ldots, \lambda_d) \in \Lambda(E, d)$ be a basis of this chirotope with the properties
(i) $\chi^\lambda$ is a chirotope
(ii) there is a point $p \in \{\lambda_1, \ldots, \lambda_d\}$ that is not contained in any non-basis of $\chi$.
In this case, $\chi^\lambda$ being realizable implies the same for $\chi$.

**Proof.** Let $(x_1, \ldots, x_n) := R \in \mathcal{R}(\chi^\lambda)$ be a realization of $\chi^\lambda$. We prove that this realization can be transformed into a realization of $\chi$. W.l.o.g. it may be assumed that $(\lambda_1, \ldots, \lambda_{d-1}) := (1, \ldots, d-1)$ and $p := n$. Let $y \in R^d$ be a vector orthogonal to the plane $\{x \in R^d \mid det(x_1, \ldots, x_{d-1}, x) = 0\}$ such that

$$sign(det(x_1, \ldots, x_{d-1}, y)) := \chi(1, \ldots, d-1, n).$$

We choose an $\epsilon > 0$ sufficiently small such that for all $(\lambda_1, \ldots, \lambda_{d-1}) \in \Lambda(n-1, d-1) \setminus \{(1, \ldots, d-1)\}$

$$sign(det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, x_n)) := sign(det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, x_n + \epsilon \cdot y)).$$

Since

$$\begin{aligned} & sign(det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, x_n + \epsilon \cdot y)) \\ &= sign(det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, x_n) + \epsilon det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, y)) \\ &= sign(0 + \epsilon det(x_{\lambda_1}, \ldots, x_{\lambda_{d-1}}, y)) = \chi(\lambda_1, \ldots, \lambda_{d-1}, n), \end{aligned}$$

the points $(x_1, \ldots, x_{n-1}, x_n + \epsilon \cdot y)$ form a realization of $\chi$.

□

The next reduction theorem deals with partial chirotopes. It is a combinatorial formulation stimulated by techniques which were applied in [15].

For a partial $d$-chirotope $\chi$ over $E$, we abbreviate

$$B_\chi := \{\lambda \in \Lambda(E,d) \mid \chi^2(\lambda) = 1\},$$
$$NB_\chi := \{\lambda \in \Lambda(E,d) \mid \chi(\lambda) = 0\},$$
$$NDEF_\chi := \{\lambda \in \Lambda(E,d) \mid \chi(\lambda) = *\}.$$

$B_\chi, NB_\chi$, and $NDEF_\chi$ are the sets of bases, nonbases and not defined elements, respectively. We define

$$A(a,b) := \{\lambda \in \Lambda(E,d) \mid a \in \lambda \text{ and } b \notin \lambda\}.$$

For a $d$-tupel $\lambda := (\lambda_1, \ldots, a, \ldots, \lambda_d) \in A(a,b)$, we define $((\lambda - a) \cup b) := (\lambda_1, \ldots, b, \ldots, \lambda_d)$ to be the $d$-tupel where the point $a$ is replaced with the point $b$.

**Theorem 4.11.** Let $\chi$ be a partial $d$-chirotope over $E := \{1, \ldots, n\}$, $d < n$ and let $(a,b) \in \Lambda(E,2)$ be a pair of points with
(i) $(a,b)$ is a sign-invariant-pair in $\chi$.
(ii) $B_\chi \cap A(b,a) \neq \emptyset$
(iii) $NB_\chi \cap A(a,b) = \emptyset$.

If the partial chirotope $\chi'$

$$\chi'(\lambda) := \chi(\lambda) \text{ if } \lambda \notin A(a,b) \text{ and } \chi'(\lambda) := * \text{ if } \lambda \in A(a,b)$$

is realizable, so is $\chi$.

**Proof.** We may assume w.l.o.g. that $b := 1$ and $a := n$. Because there is at least one basis that contains 1 but does not contain n, we can assume $\chi(1, \ldots, d) = 1$. Let $R$ be a realization of $\chi'$. We can assume that $R$ is represented by a standard representative matrix of the form

$$R = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ x_{d+1,1} & x_{d+1,2} & \ldots & x_{d+1,d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,d} \end{pmatrix}$$

Now let $R'(y) \in R^{nd}$ be the matrix which is formed out of $R$ by replacing $x_{n,1}$ with an arbitrary element $y$.

$$R'(y) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ x_{d+1,1} & x_{d+1,2} & \ldots & x_{d+1,d} \\ \vdots & \vdots & \ddots & \vdots \\ y & x_{n,2} & \ldots & x_{n,d} \end{pmatrix}$$

The set of determinants depending on the variable y is at most

$$\{det(x_{\lambda_1},\ldots,x_{\lambda_d}) \mid (\lambda_1,\ldots,\lambda_d) \in A(n,1)\}.$$

Since $\chi'|_{A(n,1)} = *$, we have that for any choice of y, the matrix $R'(y)$ is a realization of $\chi'$. We are going to prove that for a suitable choice of $y$, the matrix $R'(y)$ is also a realization of $\chi$. The only determinants we have to consider are of the form

$$\{det(x_{\lambda_1},\ldots,x_{\lambda_d}) \mid (\lambda_1,\ldots,\lambda_d) \in A(n,1) \setminus NDEF_\chi\}.$$

Since $A(n,1) \cap NB_\chi = \emptyset$, (iii), each of these determinants is a basis. In the final part of the proof, we can assume w.l.o.g. that $(n,1)$ is a covariant pair. For

$$(\lambda_1,\ldots,\lambda_{d-1},n) \in A(n,1) \setminus NDEF_\chi$$

consider the determinant expansion of the last row

$$\begin{aligned} det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_n) = \quad & det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_1) \cdot y \\ & +det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_2) \cdot x_{n,2} \\ & \vdots \\ & +det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_d) \cdot x_{n,d}. \end{aligned}$$

Only the first summand depends on our variable $y$. We can write

$$det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_n) = det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_1) \cdot y + r(\lambda_1,\ldots,\lambda_{d-1}) \qquad (*)$$

where $r(\lambda_1,\ldots,\lambda_{d-1})$ stands for the remainder not depending on the variable $y$. Since (n,1) is a covariant pair, we have for any

$$(\lambda_1,\ldots,\lambda_{d-1},n) \in A(n,1) \setminus NDEF_\chi$$

$$\chi(\lambda_1,\ldots,\lambda_{d-1},n) = \chi(\lambda_1,\ldots,\lambda_{d-1},1).$$

Hence in a realization of $\chi$ we must have

$$sign\, det(\chi_{\lambda_1},\ldots,\chi_{\lambda_{d-1}},x_n) = sign\, det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_1).$$

Using the relation (*) this gives us a set of inequalities that restrict the value of $y$. For any

$$(\lambda_1,\ldots,\lambda_{d-1},n) \in A(n,1) \setminus NDEF_\chi$$

we have

$$y > -\frac{r(\lambda_1,\ldots,\lambda_{d-1})}{det(x_{\lambda_1},\ldots,x_{\lambda_{d-1}},x_1)}.$$

$R'(y)$ becomes a realization of $\chi$ provided $y$ is chosen sufficiently large.

□

In using Theorem 4.10 and Theorem 4.11, additional 58 chirotopes were found to be realizable. The method was to apply Theorem 4.11 several times until one is left with a non-simplicial chirotope that fulfils the assumptions of Theorem 4.10. Afterwards Theorem 4.11. was applied several times until a (partial) 4-chirotope with seven points was reached. But again: all these chirotopes are realizable.

## 5. Algebraic Reduction Technique and a Solution Method

In this section we extend the algebraic solving technique. In passing to the boundary of the realization space, we get additional information which very often can be applied successfully. To see the advantage of this method, we consider an example with 20 mutations.

The oriented matroid in terms of bases is given below.

+-+++-+--+++---+++----+----++++---+--+-++++--+-+--+++-+-+-+---+++-+---+

We assume that there are homogeneous coordinates for our oriented matroid. The orientation of [1,2,3,4] is positive. For the corresponding matrix $M$ of homogeneous coordinates, we can assume the following form (multiplication of the inverse matrix of the first 4 rows from the right if neccessary):

$$\begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a & b & c & d\\ e & f & g & h\\ i & j & k & l\\ m & n & o & p \end{pmatrix}$$

We observe that the algebraic inequality system for the remaining 16 variables such that all 4 × 4-determinants have the prescribed sign, contains essentially only subdeterminants of the submatrix of the last four rows of this matrix $M$. This submatrix is sometimes called *Tuckermatrix* or *standard representative matrix.*

The next step lies in looking for a partial oriented matroid which still determines all other orientations. We try to use only 1 × 1- or 2 × 2-determinants. First of all, there are 20 mutations which must be part of this reduced system. These mutations are

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1235 | 1246 | 1256 | 1278 | 1348 | 1358 | 1367 | 1457 | 1468 | 1567 |
| 2347 | 2357 | 2368 | 2458 | 2467 | 2568 | 3456 | 3478 | 3678 | 4578. |

We use these mutations and all 1 × 1-determinants and delete those bases the signs of which are consequences from these signs. We have 33 such signs.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1234 | +1 | 1235 | −1 | 1236 | +1 | 1237 | +1 | 1238 | +1 |
| 1245 | −1 | 1246 | +1 | 1247 | −1 | 1248 | −1 | 1256 | +1 |
| 1278 | −1 | 1345 | +1 | 1346 | +1 | 1347 | +1 | 1348 | −1 |
| 1358 | −1 | 1367 | +1 | 1457 | −1 | 1468 | +1 | 1567 | −1 |
| 2345 | −1 | 2346 | −1 | 2347 | +1 | 2348 | −1 | 2357 | +1 |
| 2368 | −1 | 2458 | +1 | 2467 | −1 | 2568 | +1 | 3456 | −1 |
| 3478 | −1 | 3678 | +1 | 4578 | −1 | | | | |

We abbreviate a Grassmann Plücker polynomial by

$$\{ab/cdef\} := [abcd][abef] - [abce][abdf] + [abcf][abde] = 0.$$

{12/3457} determines the sign of 1257 to be positive

$$\underbrace{[1234]}_{+1}[1257] - \underbrace{[1235]}_{-1}\underbrace{[1247]}_{-1} + \underbrace{[1237]}_{+1}\underbrace{[1245]}_{-1} = 0$$

In the same manner we find (in this ordering)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| {12/3458} | → | 1258 | +1 | {12/3467} | → | 1267 | −1 |
| {12/3468} | → | 1268 | −1 | {16/2735} | → | 1356 | −1 |
| {17/2635} | → | 1357 | −1 | {18/2436} | → | 1368 | −1 |
| {13/2478} | → | 1378 | −1 | {16/2345} | → | 1456 | −1 |
| {18/2345} | → | 1458 | +1 | {17/2546} | → | 1467 | +1 |
| {17/2348} | → | 1478 | +1 | {18/2356} | → | 1568 | −1 |
| {18/2357} | → | 1578 | −1 | {17/2368} | → | 1678 | +1 |
| {35/1426} | → | 2356 | +1 | {28/1635} | → | 2358 | +1 |
| {36/1827} | → | 2367 | +1 | {38/1627} | → | 2378 | −1 |
| {56/1324} | → | 2456 | +1 | {45/1827} | → | 2457 | −1 |
| {28/1546} | → | 2468 | −1 | {48/1527} | → | 2478 | +1 |
| {27/1356} | → | 2567 | +1 | {58/1427} | → | 2578 | −1 |
| {78/1326} | → | 2678 | +1 | {57/1234} | → | 3457 | +1 |
| {58/1234} | → | 3458 | −1 | {67/1234} | → | 3467 | +1 |
| {68/1234} | → | 3468 | −1 | {67/1235} | → | 3567 | −1 |
| {68/1235} | → | 3568 | +1 | {78/1435} | → | 3578 | +1 |
| {57/1246} | → | 4567 | −1 | {58/1246} | → | 4568 | +1 |
| {78/1346} | → | 4678 | −1 | {78/1356} | → | 5678 | +1 |

The partial oriented matroid, we started with, determines the oriented matroid. Apart from 1234, we have 16 (1 × 1)-determinants, 12 (2 × 2)-determinants, and 4 (3 × 3)-determinants.

$$a > 0 \ [2345]- \quad b > 0 \ [1345]+ \quad c > 0 \ [1245]- \quad d < 0 \ [1235]-$$

$$e > 0 \ [2346]- \quad f > 0 \ [1346]+ \quad g < 0 \ [1246]+ \quad h > 0 \ [1236]+$$

$$i < 0 \ [2347]+ \quad j > 0 \ [1347]+ \quad k > 0 \ [1247]- \quad l > 0 \ [1237]+$$

$$m > 0 \ [2348]- \quad n < 0 \ [1348]- \quad o > 0 \ [1248]- \quad p > 0 \ [1238]+$$

$$ch > dg \ [1256]+ \quad kp < lo \ [1278]-$$

$$bp > dn \ [1358]- \quad fl < hj \ [1367]+$$

$$bk < cj \ [1457]- \quad fo > gn \ [1468]+$$

$$al > di \ [2357]+ \quad ep < hm \ [2368]-$$

$$ao < cm \ [2458]+ \quad ek > gi \ [2467]-$$

$$af < be \ [3456]- \quad in < jm \ [3478]-$$

$$\begin{vmatrix} b & c & d \\ f & g & h \\ j & k & l \end{vmatrix} < 0 \ [1567]- \qquad \begin{vmatrix} a & c & d \\ e & g & h \\ m & o & p \end{vmatrix} < 0 \ [2568]+$$

$$\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} > 0 \ [3678]+ \qquad \begin{vmatrix} a & b & c \\ i & j & k \\ m & n & o \end{vmatrix} > 0 \ [4578]-$$

In order to solve this system of inequalities, for convienience sake, we replace all variables with their absolute values ( $a = A, ..., d = -D, etc.$).

The following Table 5.1 can be viewed as the origin of some of the reduction theorems described in Section 4. This table is useful not only in the realizable case. It provides us with valuable informations about the realization space. In particular, we see what part of the boundary will be violated when increasing or decreasing any variable. We assume that we start at a solution point $(A, B, \ldots, P)$. For all variables we list those inequalities which are not violated when increasing(+) (or decreasing(−)) this variable. An overview about the system is given in the following table.

**Table 5.1. Gradient Information of a possible solution point**

| | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ | $K$ | $L$ | $M$ | $N$ | $O$ | $P$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1256 | | | + | − | | | − | + | | | | | | | | |
| 1278 | | | | | | | | | | | − | + | | | + | − |
| 1358 | | + | | − | | | | | | | | | | − | | + |
| 1367* | | | | | | − | | + | | + | | − | | | | |
| 1457* | | − | + | | | | | | | + | − | | | | | |
| 1468 | | | | | | + | − | | | | | | | − | + | |
| 2357 | + | | | − | | | | | − | | | + | | | | |
| 2368* | | | | | − | | | + | | | | | + | | | − |
| 2458* | − | | + | | | | | | | | | | + | | − | |
| 2467 | | | | | + | | − | | − | | + | | | | | |
| 3456 | − | + | | | + | − | | | | | | | | | | |
| 3478 | | | | | | | | | − | + | | | + | − | | |
| 1567 | | + | − | + | | + | + | − | | − | + | + | | | | |
| 2568 | + | | − | + | + | | + | − | | | | | − | | + | + |
| 3678 | | | | | + | + | | − | + | − | | + | − | − | | + |
| 4578 | + | + | − | | | | | | − | − | + | | − | + | + | |
| | 3 : 2 | 4 : 1 | 3 : 3 | 2 : 3 | 4 : 1 | 3 : 2 | 2 : 3 | 3 : 3 | 1 : 4 | 3 : 3 | 3 : 2 | 4 : 1 | 3 : 3 | 1 : 4 | 4 : 1 | 3 : 2 |
| | | * | | | * | | | | | | | * | | | * | |

Those variables $(B, E, L, O)$ marked by "*" can be increased until equality is reached in those 2x2 determinants marked above and listed below:

$$FL = HJ, \;\; BK = CJ, \;\; AO = CM, \;\; GI = EK.$$

We pass to the boundary of our realization space and observe that the $3 \times 3$ determinants are fulfilled, e.g. we have:

$$-D(FK + GJ) + L(-BG - CF) < 0 \implies [1567]-$$

Thus we have to find positive variables so that the following remaining system holds:

$$DG < CH \; [1256]+ \quad KP < LO \; [1278]-$$

$$DN < BP \; [1358]- \quad FL = HJ \; [1367]$$

$$BK = CJ \; [1457] \quad GN < FO \; [1468]+$$

$$DI < AL \; [2357]+ \quad EP < HM \; [2368]-$$

$$AO = CM \; [2458] \quad GI = EK \; [2467]$$

$$AF < BE \; [3456]- \quad IN < JM \; [3478]-.$$

We emphasize that in contrast to an application of a reduction theorem, we might still find either a contradiction or a solution of our original problem. The algebraic reduction in our concrete example leads to a solution as follows.

Without loss of generality, we are still free to choose $C = G = K = O = 1$, as well as $I = J = K = L = 1$. ( otherwise multiply rows or columns appropriately )

In using this, we are left with the system $B = 1$, $E = 1$, $A = M$, $F = H$, and

$$D < F,\ D < A,$$

$$N < F,\ N < A,$$

$$DN < P < AF < 1$$

which has even a solution for $D = N$, $A = F$, and $0 < DD < P < AA < 1$.

The given oriented matroid is realizable. Coordinates can be found by going all the steps backwards.

□

## 6. Bi-quadratic Final Polynomials

In this section we consider a former unknown non-realizable oriented matroid. We show in detail how a final polynomial was determined. The example under consideration is also remarkable in the following sense. Delete a vertex and take its dual or contract at a vertex. The result is in all cases the same reorientation class of an oriented matroid, independent of the choice of the vertex. We provide the oriented matroid in terms of bases. It is different from all former unknown non-realizable oriented matroids with 8 points in rank 4 since it has 12 mutations.

| | | | | | |
|---|---|---|---|---|---|
| 1237 | 1248 | 1256 | 1345 | 1368 | 1467 |
| 2358 | 2457 | 2678 | 3478 | 3567 | 4568 |

+++++---+--+-++----------++++++-+++++++++++-----+--+--++++++--+++--+++

(10, 10)(10, 10)(10, 10)(10, 10)(10, 10)(10, 10)(10, 10)(10, 10)

A first step lies in calculating a reduced system for the oriented matroid. We provide a *final polynomial* for this example thus proving that the oriented matroid is not realizable.

**Theorem 5.1.** An oriented matroid with the following partial structure (used in the following proof) is not realizable.

Proof: In the realizable case, the following second order syzygy holds:

$$\begin{aligned}
&+[4,5,7,8][1,5,6,7][1,4,5,8]\{1,7|2,3,5,8\}\\
&+[4,5,7,8][1,5,6,7][1,3,5,7]\{1,8|2,4,5,7\}\\
&-[4,5,7,8][1,4,7,8][1,3,5,7]\{1,5|2,6,7,8\}\\
&-[1,2,5,7][1,4,7,8][1,3,5,7]\{5,8|1,4,6,7\}\\
&-[1,4,5,8][1,2,5,7][1,4,7,8]\{5,7|1,3,6,8\}\\
&-[1,5,6,7][1,4,5,8][1,2,5,7]\{7,8|1,3,4,5\} \quad = 0.
\end{aligned}$$

In deleting pairwise equal bracket monomials. we find that the above expression of zero is equal to the following polynomial multiplied by [1,5,7,8].

$$-[1,2,3,7][4,5,7,8][1,5,6,7][1,4,5,8]$$

$$+[1,2,4,8][4,5,7,8][1,5,6,7][1,3,5,7]$$

$$+[1,2,5,6][4,5,7,8][1,4,7,8][1,3,5,7]$$

$$-[4,5,6,8][1,2,5,7][1,4,7,8][1,3,5,7]$$

$$+[3,5,6,7][1,4,5,8][1,2,5,7][1,4,7,8]$$

$$-[3,4,7,8][1,5,6,7][1,4,5,8][1,2,5,7]$$

The signs of all these 6 monomials are equal, we arrive at a contradiction. □

**Remark 5.2.** The combinatorial structure of the above final polynomial corresponds to the bundle condition in projective geometry. This will be worked out in detail in [13].

Here we only state a main result of our investigations.

**Theorem 5.3.** There are 3 projective incidence theorems (oriented versions) which imply realizability when they are required to hold for uniform oriented matroids in rank 4 with 8 points.

**Remark 5.4.** Among these requirements mentioned in Theorem 5.3, we have the bundle condition as a special case (Type 1 in Appendix 2).

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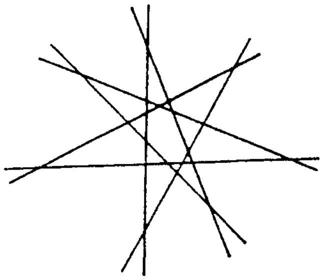
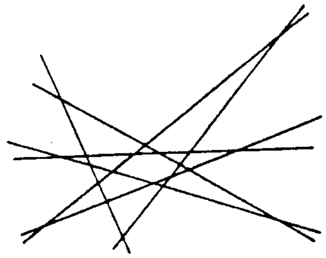
## Appendix 1
## Line arrangements representing all reorientation classes of uniform oriented matroids in rank 3 with 7 points



1



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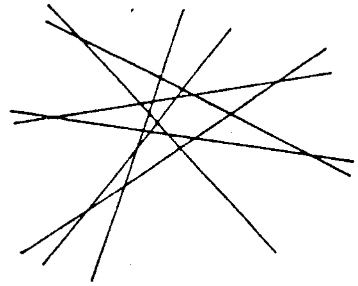
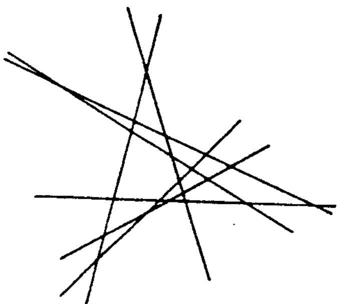
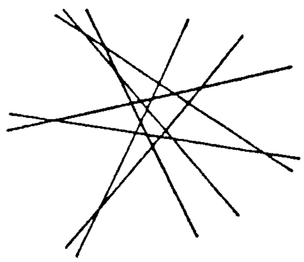


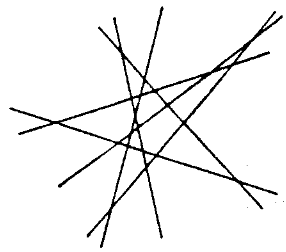
3



4



5



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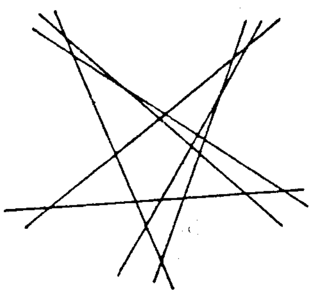

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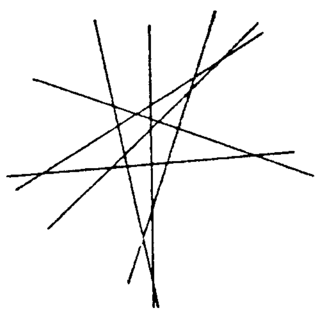

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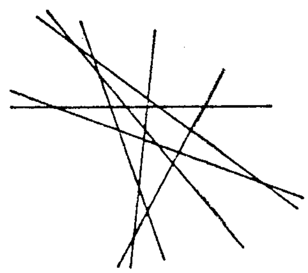

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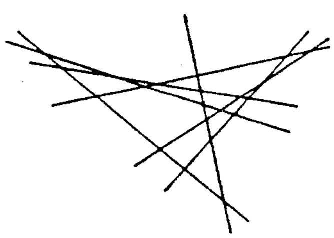

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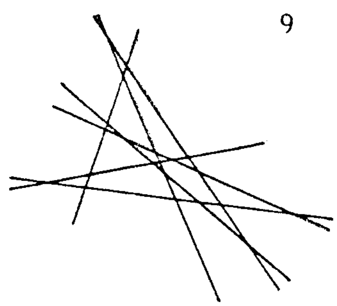

11

## Appendix 2
## Representatives of all uniform reorientation classes of non-realizable oriented matroids in rank 4 with 8 points

```
+++++----------++++++-+----+-++++++------+-++++-+-------++++-++++----+   type1
+++++---------+++++++++++------++------------++++++--++-++++++++---++-   type1
+++++--------++++++++++----+-++++++---+-+-++++-+--------+++++++++-+---   type2
++-+++------+++---+-+++++--+-------++-++--+-++++++-++++++-+--++++-----   type1
--++++---+++-+++-----+---++++++---+-+++--++--++-+++---+++-+++---+---+-   type1
++----++++++--+-++++++---+--++++++-+------++--++--+---++++-+++++----+-   type1
+-++-+--+-++-+-+--+---++++++-----+--++--+--+++--+-+--+-+-++--++-+--+--   type1
-++--++---+++++------+++++-++++-++++--+++------++++++++-++++++++-++---   type1
+--++----++++++--+-+++++-+-+--+++--+-+++--+-++-----+++++-++-+++-+-+--+   type3
+--++---+++++++++--+--++++++++-+++++++++-----++-+--++++-++++-+++++++++   type3
--+-----+--++-+---+--++++--+------+-+-----++-+---+----++--------+---++   type3
--++-++--+--+-++-+--+--+++--++-++-++++++++--------+--+--+-++-++-+--+++   type1
-+--+-+-+-+--++--++++-+++-+-+-+-+++++++--+-+++++-+++---+++++-----++--+   type1
++++----+++--+++++-++-+-+--+---+---+++----+--+-+-+++---------+----++++   type1
++++--------------+--+-+-++-+------+++-++++++-+++++++-++-+-+----------   type2
++++----+--+-+----+--+-+-++-+++-+-++++-++-+----++-++-++++++++-+++---++   type3
++++--------------+--+-+-++-+------+++-++++++-+++++++-++-+-+------+---   type1/2
++++--------------+--+---++-+------+++-++-+++-+++++++-++-+-+------+---   type1
+++++---+--+-++---------+++++++-++++++++++-----+--+--++++++--++++--+++   type1/2
-+--++++++--++++-+---++-++++-++---+--++++-++++-----+++++--++-+-+---++-   type1
++++----+--++-+---+--++++--++-----+---+--++---+++-++++--++++++++-+++-+   type1
----++++---+-+----+-+-+-++---+-+-++---+--++-+++-+--+-+++-+-++-+--+-+++   type1
----+-+++++++++-+--++----+--++++++---+-++-+-+---++-+++-++++--+--+++-++   type1
+++++--++-+++++-----++++--++++-++--++--+----++----+++++-++++-++-+++-++   type1
```