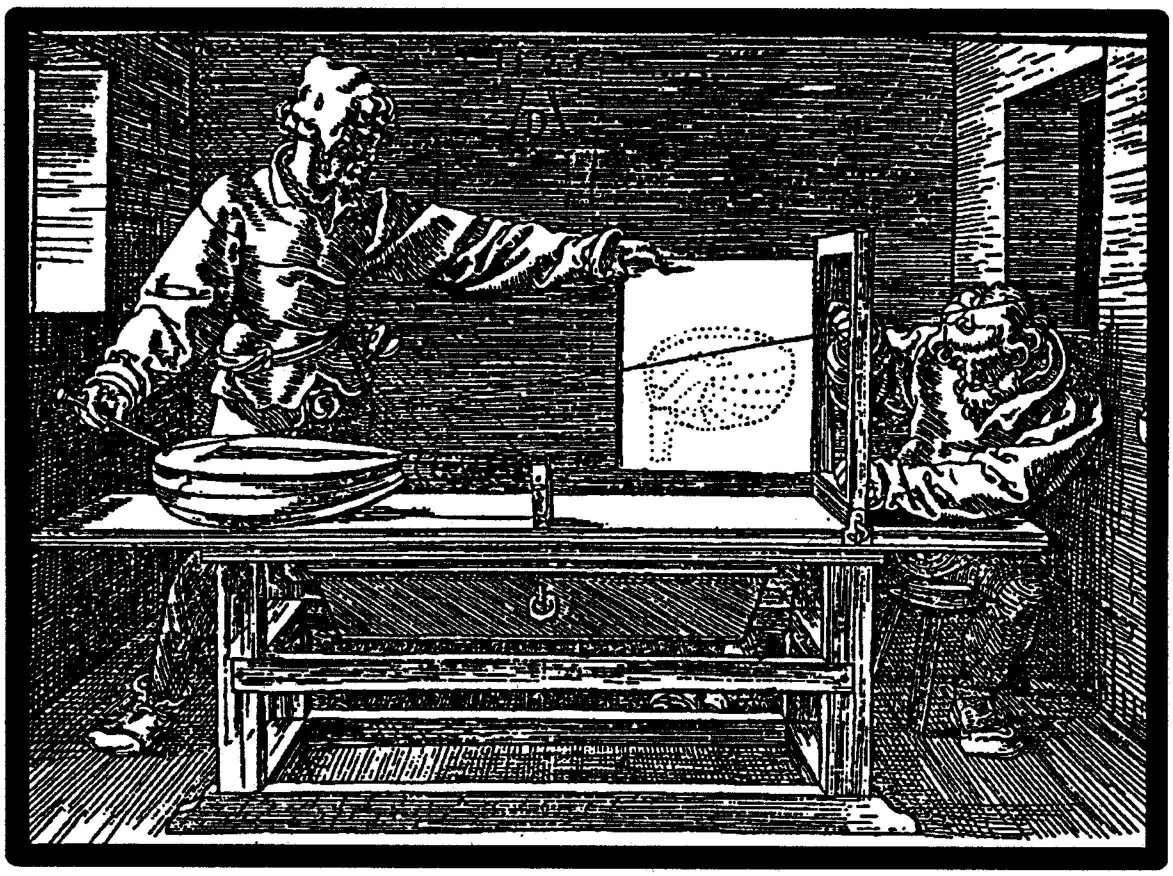
Geometry and Computation

Equinoctial School · ETH Zürich · 1997

Primitives for Geometric Operations

by Jürgen Richter–Gebert



Part I: Projective Geometry

by Jürgen Richter-Gebert, ETH Zürich

Data and Algorithms

The objects $\Longrightarrow$ data:

- points,
- lines,
- circles,
- conics,
- lines in space,
- planes in space.

What is a good data format for points, lines, circles?

The operations $\Longrightarrow$ algorithms:

- intersection,
- line through two points,
- circles by center and radius,
- angle measurement,
- distance measurement,
- intersection of lines and planes in space.

What are good ways to process the data?

Intersecting Line and Conic

Before thinking:

```
public boolean intersectWithLine
  (HCoord l, HCoord p1, HCoord p2) {

   if (l.x==0 && l.y==0) {
     double qc= yy;
     double qb= xy;
     double qa= xx;
     double ww=(qb/qa)/2;
     double dis=ww*ww-qc/qa;

     if (dis>=0) {
       double dissq=Math.sqrt(dis);
       p1.x=-ww+dissq;
       p1.y=1;
       p1.z=0;
       p2.x=-ww-dissq;
       p2.y=1;
       p2.z=0;
       return true;
     } else {
       return false;
     }
   }


  if (l.x*l.x>l.y*l.y){
     double qc = l.z*l.z*xx -
                 l.z*l.x*xz +
                 l.x*l.x*zz;

     double qb=  2*l.y*l.z*xx -
                 l.z*l.x*xy -
                 l.y*l.x*xz +
                 l.x*l.x*yz;

     double qa=  l.y*l.y*xx -
                 l.y*l.x*xy +
                 l.x*l.x*yy;

     double ww=(qb/qa)/2;
     double dis=ww*ww-qc/qa;

     if (dis>=0) {
       double dissq=Math.sqrt(dis);
       p1.y=-ww+dissq;
       p1.x=-(l.y*p1.y+l.z)/l.x;
       p1.z=1;
       p2.y=-ww-dissq;
       p2.x=-(l.y*p2.y+l.z)/l.x;
       p2.z=1;
       return true;
     } else {
      return false;
     }
   }
  else {
    double qc = l.z*l.z*yy -
                l.z*l.y*yz +
                l.y*l.y*zz;

    double qb=  2*l.x*l.z*yy -
                l.z*l.y*xy -
                l.x*l.y*yz +
                l.y*l.y*xz;

    double qa=  l.x*l.x*yy -
                l.x*l.y*xy +
                l.y*l.y*xx;

    double ww=(qb/qa)/2;
    double dis=ww*ww-qc/qa;

    if (dis>=0) {
      double dissq=Math.sqrt(dis);
      p1.x=-ww+dissq;
      p1.y=-(l.x*p1.x+l.z)/l.y;
      p1.z=1;
      p2.x=-ww-dissq;
      p2.y=-(l.x*p2.x+l.z)/l.y;
      p2.z=1;
      return true;
    } else {
      return false;
    }
   }
  }
```

Intersecting Line and Conic

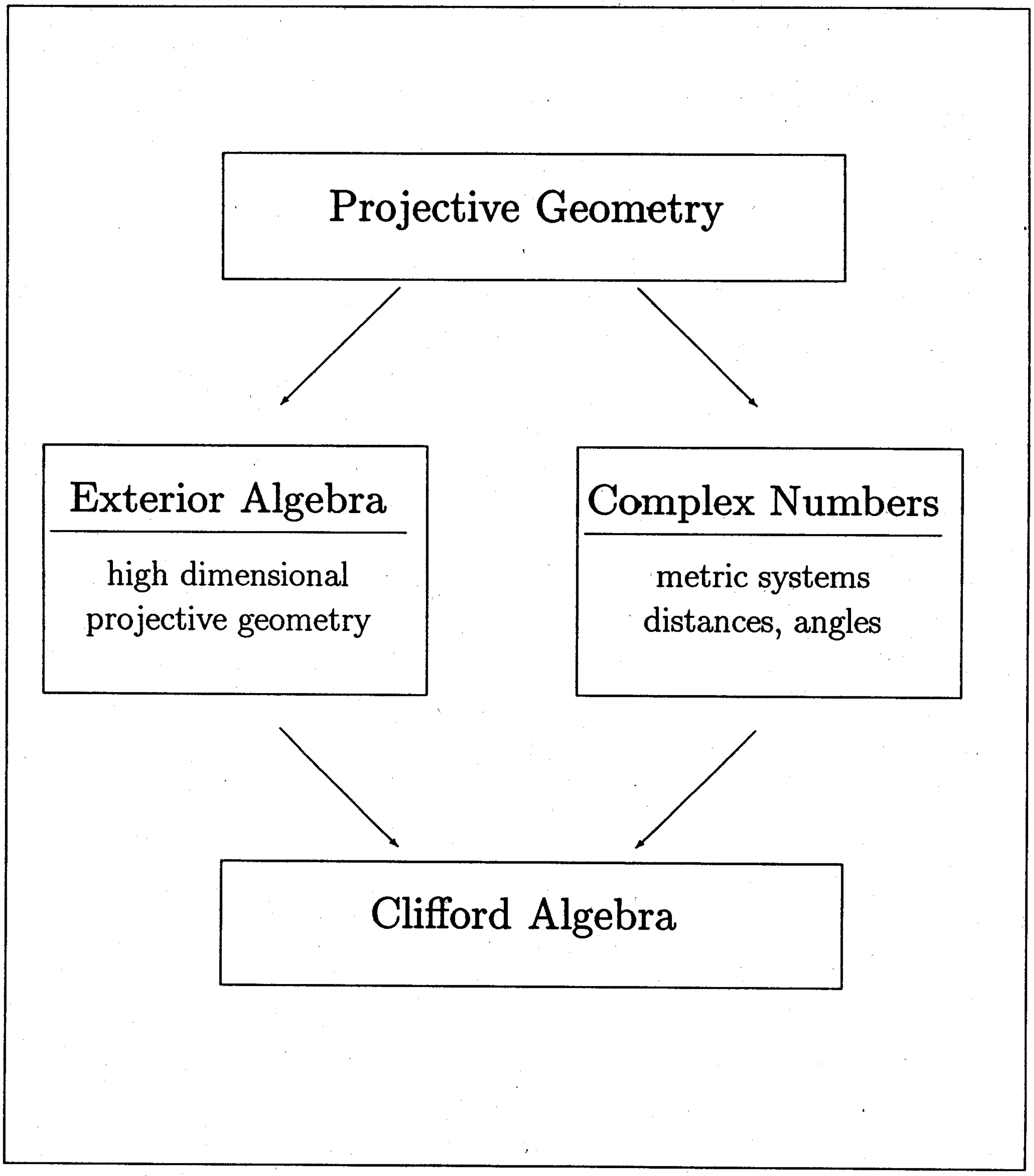
After thinking:

```
public HConicCoord sqr(HCoord a){
  return new HConicCoord(a.x*a.x,a.y*a.y,a.z*a.z,
                         a.x*a.y,a.x*a.z,a.y*a.z)
}

public void tangentConic(HConicCoord a,
                         HCoord b) {

  HConicCoord s=b.sqr();
  xx=-a.yy*s.zz-a.zz*s.yy+a.yz*s.yz;
  yy=-a.xx*s.zz-a.zz*s.xx+a.xz*s.xz;
  zz=-a.xx*s.yy-a.yy*s.xx+a.xy*s.xy;
  xy=a.xy*s.zz+2*a.zz*s.xy-a.yz*s.xz-a.xz*s.yz;
  xz=a.xz*s.yy+2*a.yy*s.xz-a.yz*s.xy-a.xy*s.yz;
  yz=a.yz*s.xx+2*a.xx*s.yz-a.xz*s.xy-a.xy*s.xz;
}

public int sign(double x){
  return (x>0)?+1:-1; }

public boolean intersectWithLine(HCoord l, HCoord p1, HCoord p2) {

  deg.tangentConic(this,l);
  double m3=Math.sqrt(-(deg.xx*deg.yy-deg.xy*deg.xy/4))*sign(l.z);
  double m2=-Math.sqrt(-(deg.xx*deg.zz-deg.xz*deg.xz/4))*sign(l.y);
  double m1=Math.sqrt(-(deg.zz*deg.yy-deg.yz*deg.yz/4))*sign(l.x);
  p1.move(deg.xx,deg.xy/2-m3,deg.xz/2-m2);
  p2.move(deg.xx,deg.xy/2+m3,deg.xz/2+m2);
  return true;
}
```

Design Principles

- General approaches
  - similar elements $\rightarrow$ similar algebraic representations
  - similar operations $\rightarrow$ similar algebraic formulation
  - similar geometries $\rightarrow$ similar algebraic model
- No special cases
  - unified treatment of parallelity
  - unified treatment of distances and angles
- Coordinate free approaches
  - independence of choice of the coordinate system
  - "high level" algebraic operations
- Algebraic consistency
  - no unmotivated behavior
  - consistent orientation properties

Algebra & Geometry



Projective Geometry

The projective plane:

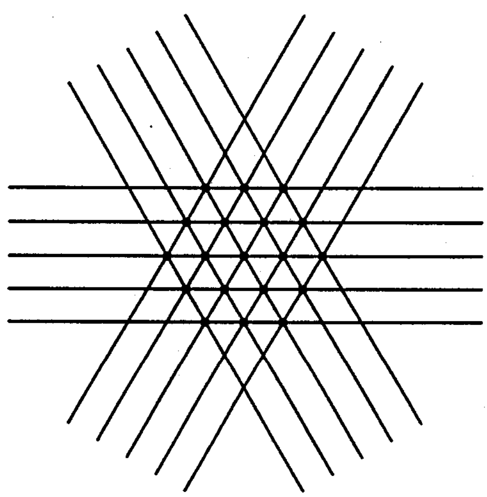
A system of **points** and **lines** such that:

(A1) For any two points there is a connecting line.
(A2) For any two lines there is an intersection point.
(A3) There are at least four non-collinear points.

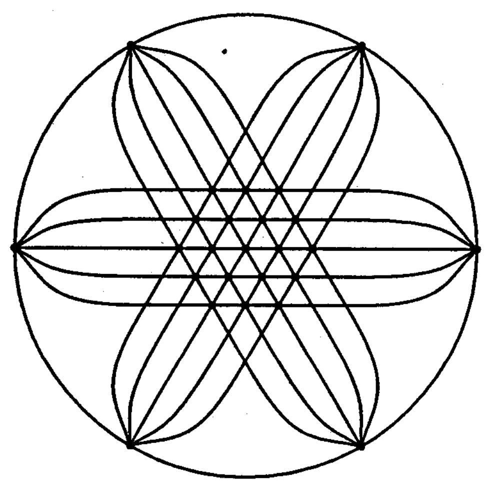
A model:

The usual plane,
plus **line at infinity,**
plus **an infinite point for every direction.**

A configuration in ...



... the usual plane



... the projective plane

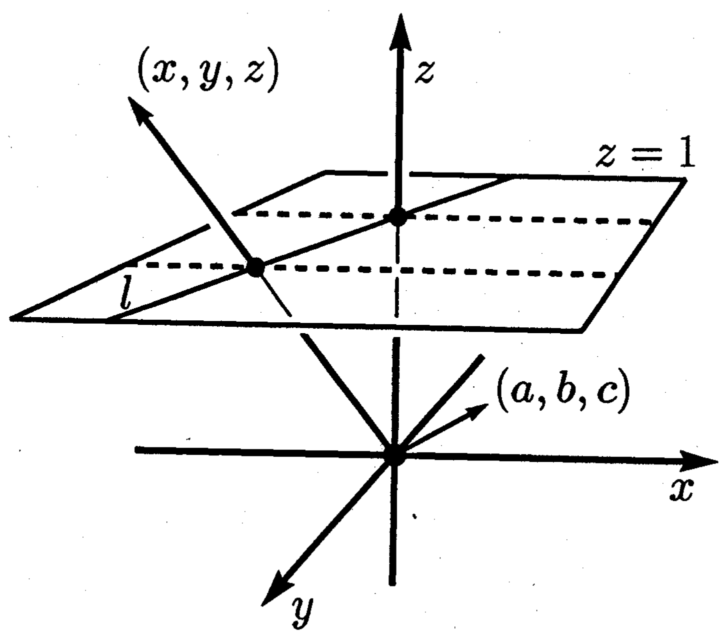
Real Projective Geometry

Homogeneous coordinates:

Points: $(x, y) \mapsto (x, y, 1)$.
$(x, y, 0)$ is *a point at infinity.*

Lines: $ax + by + c = 0 \mapsto (a, b, c)$.
$(0, 0, 1)$ is *the line at infinity.*

Identify scalar (positive) multiples of vectors in $\mathbb{R}^3$.



A point (x, y, z) is on a line (a, b, c)

$$\Longleftrightarrow$$

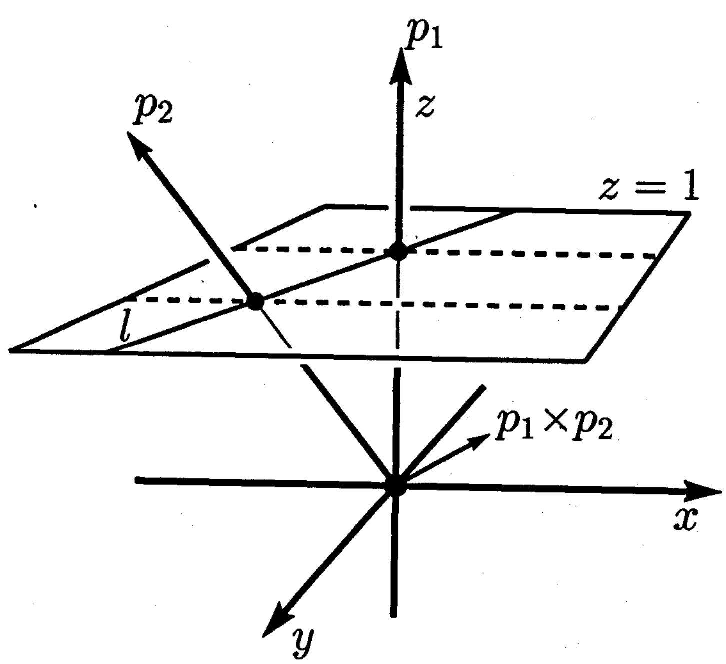
$$ax + by + cz = 0.$$

Join and Meet

A point (x, y, z) is on a line (a, b, c)

$$\Longleftrightarrow$$

$$ax + by + cz = 0$$



- **Join** of points p_1, p_2 is $l = p_1 \times p_2$.
- **Meet** of lines l_1, l_2 is $p = l_1 \times l_2$.

- p_1, p_2, p_3 are collinear if $\det(p_1, p_2, p_3) = 0$.
- l_1, l_2, l_3 are concurrent if $\det(l_1, l_2, l_3) = 0$.
- p is on l if $\langle p, l \rangle = 0$.

Three Dimensions

2-dimensional case:

Line l through $p_1=(x_1,y_1,z_1)$ and $p_2=(x_2,y_2,z_2)$:

$$l=(a,b,c) \text{ with } \langle l,p_1\rangle=0 \text{ and } \langle l,p_2\rangle=0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longleftrightarrow$$

$$p_1 \vee p_2 := \lambda\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right); \quad \lambda \neq 0.$$

3-dimensional case:

Plane h through $p_1=(x_1,y_1,z_1,w_1)$, $p_2=(\ldots)$ and $p_3=(\ldots)$:

$$h=(a,b,c,d) \text{ with } \langle h,p_1\rangle=0,\ \langle h,p_2\rangle=0 \text{ and } \langle h,p_3\rangle=0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Longleftrightarrow$$

$$p_1 \vee p_2 \vee p_3 := \left(\begin{vmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 & w_1 \\ x_2 & z_2 & w_2 \\ x_3 & z_3 & w_3 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \\ x_3 & y_3 & w_3 \end{vmatrix}, -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\right)$$

All points on h are of the form $p = \alpha p_1 + \beta p_2 + \gamma p_3$. We get:

$$\langle p,\ p_1 \vee p_2 \vee p_3\rangle = \det(\alpha p_1 + \beta p_2 + \gamma p_3,\ p_1,\ p_2,\ p_3) = 0.$$

Plücker Coordinates

Line l through $p_1=(x_1, y_1, z_1, w_1)$ and $p_2=(x_2, y_2, z_2, w_2)$:

$$
\begin{array}{c}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}
\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}
\vee
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}
\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}
=
\begin{matrix} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{matrix}
\begin{pmatrix}
+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \\
+\begin{vmatrix} x_1 & w_1 \\ x_2 & w_2 \end{vmatrix} \\
+\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \\
-\begin{vmatrix} y_1 & w_1 \\ y_2 & w_2 \end{vmatrix} \\
+\begin{vmatrix} z_1 & w_1 \\ z_2 & w_2 \end{vmatrix}
\end{pmatrix} \\
p_1 \quad \vee \quad p_2 \quad = \quad l
\end{array}
$$

All points on l are of the form $p = \alpha p_1 + \beta p_2$.
For arbitrary $q \in \mathbb{R}^4$ we have:

$$\det(p_1,\ p_2,\ \alpha p_1+\beta p_2,\ q) = 0$$

The Plücker coordinates for $p_1 \vee p_2$ come from the 2×2 block decomposition of this determinant.

Plücker Coordinates II

The general pattern: (Book-keeping for computers)

- k-flats in rank d are indexed by the $\binom{n}{k}$ k-subsequences $\Lambda(n,k)$ of $E = (1,\ldots,n)$
- For $\tau \subseteq E$ and $\mu \subseteq E$: $\mathbf{sign}(\tau,\mu) = (-1)^{\mathrm{parity}(\tau,\mu)}$.
- Flat P of rank k and flat Q of rank m.

Join: $R = P \vee Q$ has rank $k+m$

$$R_\lambda = \sum_{\substack{\tau\in\Lambda(n,k)\\ \mu\in\Lambda(n,m)\\ \tau\cup\mu=\lambda}} \mathbf{sign}(\tau,\mu)\cdot P_\tau \cdot Q_\mu$$

Meet: $R = P \wedge Q$ has rank $k+m-n$

$$R_\lambda = \sum_{\substack{\tau\in\Lambda(n,k)\\ \mu\in\Lambda(n,m)\\ \tau\cap\mu=\lambda}} \mathbf{sign}(\tau\backslash\mu,\mu\backslash\tau)\cdot P_\tau \cdot Q_\mu$$

Examples

Join

$$\begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} \vee \begin{matrix}12\\13\\14\\23\\24\\34\end{matrix}\begin{pmatrix}-1\\0\\-1\\1\\0\\-1\end{pmatrix} = \begin{matrix}123\\124\\134\\234\end{matrix}\begin{pmatrix}1+0+1\\0+0-1\\-1-1+0\\0+0+1\end{pmatrix} = \begin{matrix}123\\124\\134\\234\end{matrix}\begin{pmatrix}2\\-1\\-2\\1\end{pmatrix}$$

point $\vee$ **line** $=$ **plane**

Meet

$$\begin{matrix}123\\124\\134\\234\end{matrix}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} \wedge \begin{matrix}12\\13\\14\\23\\24\\34\end{matrix}\begin{pmatrix}-1\\0\\-1\\1\\0\\-1\end{pmatrix} = \begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}-1+0+0\\0+1+0\\-1+0+0\\1+0+0\end{pmatrix} = \begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$$

plane $\wedge$ **line** $=$ **point**

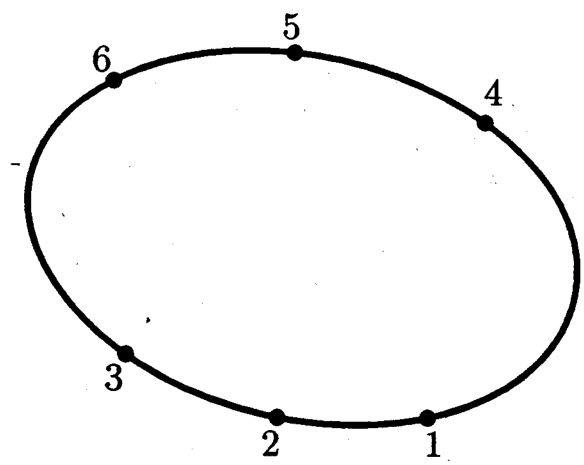
$$\begin{matrix}123\\124\\134\\234\end{matrix}\begin{pmatrix}2\\-1\\-2\\1\end{pmatrix} \vee \begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix} = 2-1-2+1=0$$

plane $\vee$ **line** $=$ **scalar**

Infinity

line at infinity	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{inf}$
points at infinity	$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$
intersection of parallel lines	$\begin{pmatrix} a \\ b \\ \tau \end{pmatrix} \times \begin{pmatrix} a \\ b \\ \mu \end{pmatrix} = \begin{pmatrix} b\mu - b\tau \\ a\tau - a\mu \\ 0 \end{pmatrix} = (\mu - \tau) \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$
parallel to l through p	$(l \times \mathbf{inf}) \times p$
translation	$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$
rotation about the origin	$\begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

Conics



General conic: all (x, y, z) with

$$a \cdot x^2 + b \cdot y^2 + c \cdot z^2 + d \cdot xy + e \cdot xz + f \cdot yz = 0$$

Parameters of the conic:

$$(a, b, c, d, e, f)$$

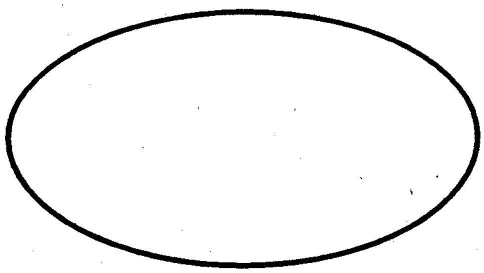
... unique up to scalar multiples.

Matrix form:

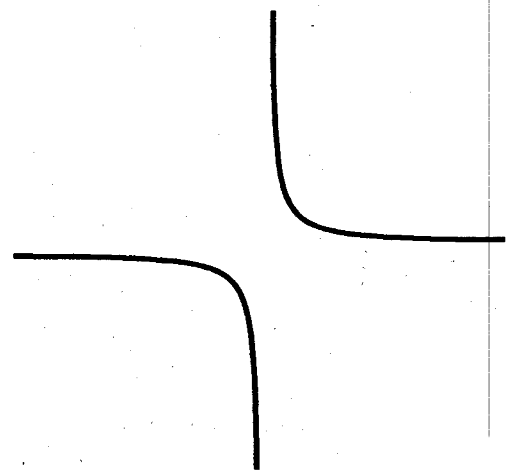
$$(x, y, z) \begin{pmatrix} a & d' & e' \\ d' & b & f' \\ e' & f' & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

with $d' = d/2, \ e' = e/2, \ f' = f/2.$

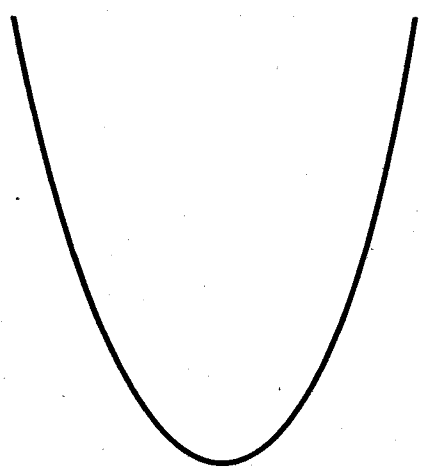
Special Conics



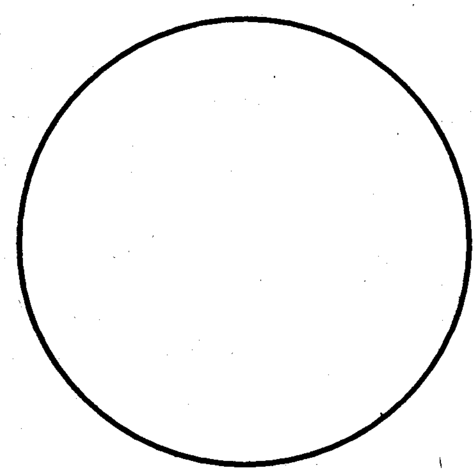
Ellipse:
No intersection with infinity



Hyperbola:
Two intersections with infinity

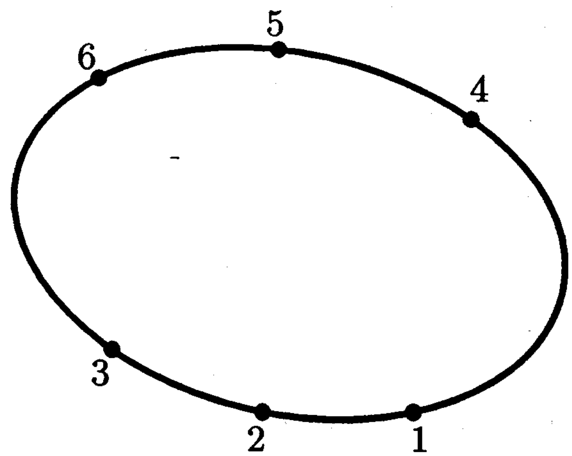


Parabola:
Double intersection with infinity



Circle:
$x^2 + y^2 + \alpha xz + \beta yz + \gamma z^2$

Points on a Conic



General conic: all (x, y, z) with

$$a \cdot x^2 + b \cdot y^2 + c \cdot z^2 + d \cdot xy + e \cdot xz + f \cdot yz = 0.$$

Six points on a conic:

$$\mathbf{det} \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & x_1z_1 & y_1z_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & x_2z_2 & y_2z_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & x_3z_3 & y_3z_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & x_4z_4 & y_4z_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & x_5z_5 & y_5z_5 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & x_6z_6 & y_6z_6 \end{pmatrix} = 0$$

With $(x, y, z)^2 = (x^2, y^2, z^2, xy, xz, yz)$ we get:

$$c = p_1^2 \vee p_2^2 \vee p_3^2 \vee p_4^2 \vee p_5^2.$$

A point p is on c if and only if $\langle c, p^2 \rangle = 0$.

Plücker's μ

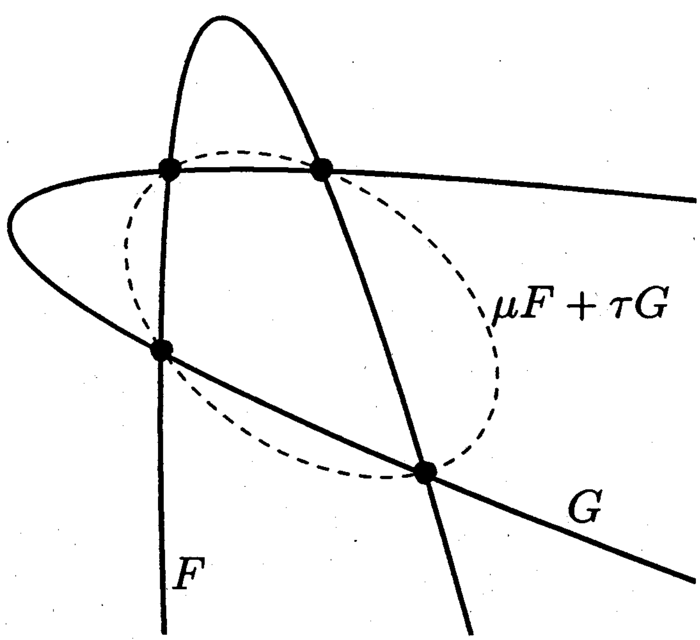
Let $\mu, \tau \in \mathbb{R}$ and let

$$F(x, y, z) = 0 \quad \text{and} \quad G(x, y, z) = 0$$

be the equations of two curves. Then

$$(\mu F + \tau G)(x, y, z)$$

passes through all common zeros of F and G.



A lovely method to adjust one parameter in an equation!

Points on a Conic II

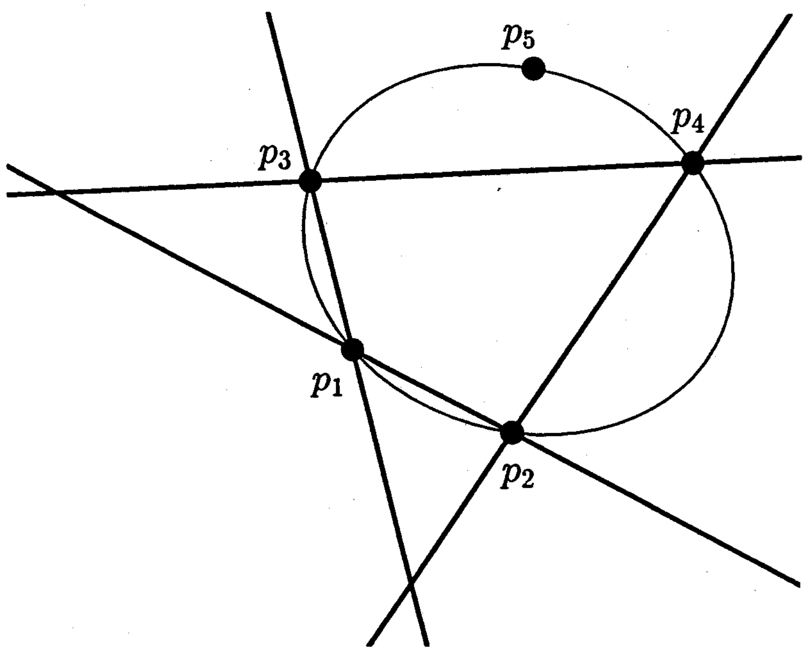
Equation for a point p on line $\overline{ab}$:

$$[a, b, p] = 0.$$

Equation for a point p either on line $\overline{ab}$ or on line $\overline{cd}$:

$$[a, b, p] \cdot [c, d, p] = 0.$$

This is a degenerate conic.



All conics through p_1, p_2, p_3, p_4:

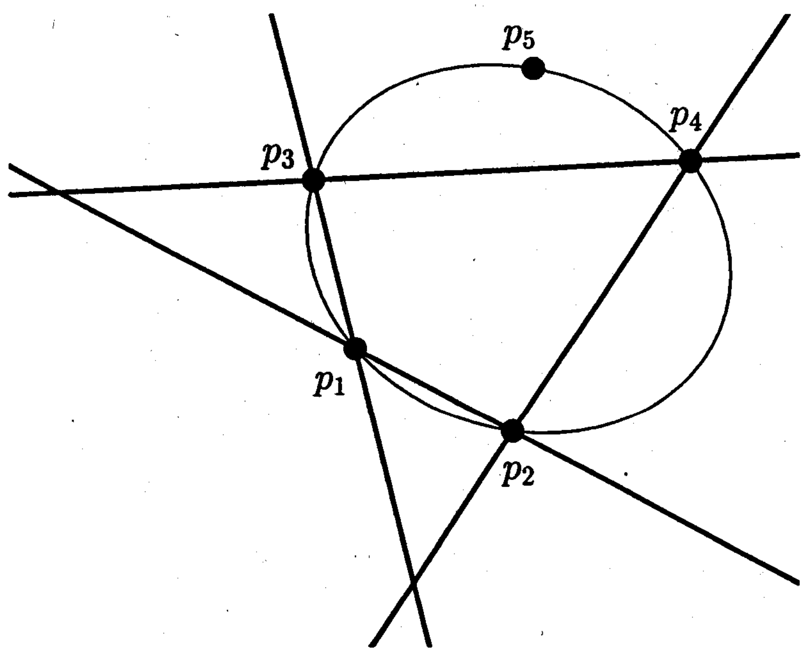
$$\tau \cdot [p_1, p_2, p] \cdot [p_3, p_4, p] + \mu \cdot [p_1, p_3, p] \cdot [p_2, p_4, p] = 0$$

Adjust parameters to pass through p_5:

$$\begin{aligned} &\underbrace{[p_1, p_3, p_5] \cdot [p_2, p_4, p_5]}_{\tau} \cdot [p_1, p_2, p] \cdot [p_3, p_4, p] \\ - &\underbrace{[p_1, p_2, p_5] \cdot [p_3, p_4, p_5]}_{-\mu} \cdot [p_1, p_3, p] \cdot [p_2, p_4, p] = 0 \end{aligned}$$

A Typical Piece of Algorithm

```
h12=cross(p1,p2);            Calculating the lines
h34=cross(p3,p4);
h13=cross(p1,p3);
h24=cross(p2,p4);
h12_34=product(h12,h34);Calculating the degenerate conics
h13_24=product(h13,h24);
q5=sqr(p5);                  Calculating (p5)^2

conic=linear_combination( h12_34,13_24,      Plücker's μ
                         -dot(q5,h13_24),
                          dot(q5,h12_34));
```



Matrices and Conics

Different matrices may represent the same conic.

For all (λ, μ, τ) the matrix

$$M = \begin{pmatrix} a & d+\tau & e-\mu \\ d-\tau & b & f+\lambda \\ e+\mu & f-\lambda & c \end{pmatrix}$$

represents the same quadratic form

$$\begin{aligned} p^T M p = & \quad ax^2 + by^2 + cz^2 \\ & + (d+\tau)xy + (d-\tau)yx \\ & + (e+\mu)xz + (e-\mu)zx \\ & + (f+\lambda)yz + (f-\lambda)zy \\ = & \quad ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz \end{aligned}$$

$\det(M) = 0 \quad \Longrightarrow \quad p^T M p = 0$ describes a degenerate conic.

Then (λ, μ, τ) can be chosen such that the rank of M is 1.

For this every 2×2 submatrix must be degenerate:

$$\begin{vmatrix} a & d{+}\tau \\ d{-}\tau & b \end{vmatrix} = 0, \quad \begin{vmatrix} a & e{-}\mu \\ e{+}\mu & c \end{vmatrix} = 0, \quad \begin{vmatrix} b & d{+}\lambda \\ d{-}\lambda & c \end{vmatrix} = 0.$$

This is solved by:

$$\tau^2 = -\begin{vmatrix} a & d \\ d & b \end{vmatrix}, \quad \mu^2 = -\begin{vmatrix} a & e \\ e & c \end{vmatrix}, \quad \lambda^2 = -\begin{vmatrix} b & d \\ d & c \end{vmatrix},$$

Matrices and Conics II

Let $p^T M p = 0$ be a **degenerate conic** with

$$M = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

Then there are (λ, μ, τ) such that:

$$M' = \begin{pmatrix} a & d+\tau & e-\mu \\ d-\tau & b & f+\lambda \\ e+\mu & f-\lambda & c \end{pmatrix}$$

has rank 1.

Simply choose

$$\tau = \pm\sqrt{-\begin{vmatrix} a & d \\ d & b \end{vmatrix}}, \quad \mu = \pm\sqrt{-\begin{vmatrix} a & e \\ e & c \end{vmatrix}}, \quad \lambda = \pm\sqrt{-\begin{vmatrix} b & d \\ d & c \end{vmatrix}},$$

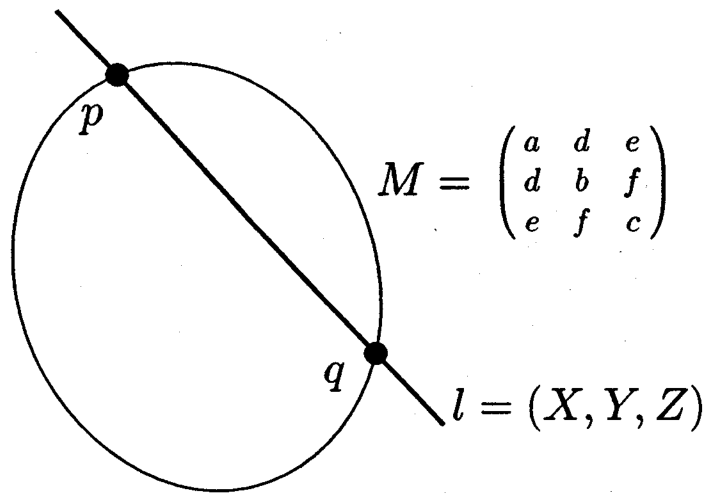
with the "correct" signs.

The homogeneous coordinates of the lines of $p^T M p = 0$ are the **rows** and **columns** of M'.

Intersecting a Line and a Conic

The intersection of a conic and a line is a

PAIR OF POINTS !!



One can calculate: $pq^T + qp^T$.

Set $L = \begin{pmatrix} 0 & Z & -Y \\ -Z & 0 & X \\ Y & -X & 0 \end{pmatrix}$, and calculate:

$$pq^T + qp^T = L^T M L$$

Observation: for a line g we have:

$$\begin{aligned} g^T(L^T M L)g &= (Lg)^T M (Lg) \\ &= (l \times g)^T M (l \times g). \end{aligned}$$

$\Longrightarrow$ $g^T(L^T M L)g{=}0$ the intersection of g and l is on the conic.

$\Longrightarrow$ $g^T(L^T M L)g{=}0$ if and only if g meets p or g meets q.

Intersecting Conic and Line

Calculating squared coordinates of l:

```
public HConicCoord sqr(HCoord a) {
    return new HConicCoord(a.x*a.x,a.y*a.y,a.z*a.z,a.x*a.y,a.x*a.z,a.y*a.z)
  }
```

Calculating $L^T M L$:

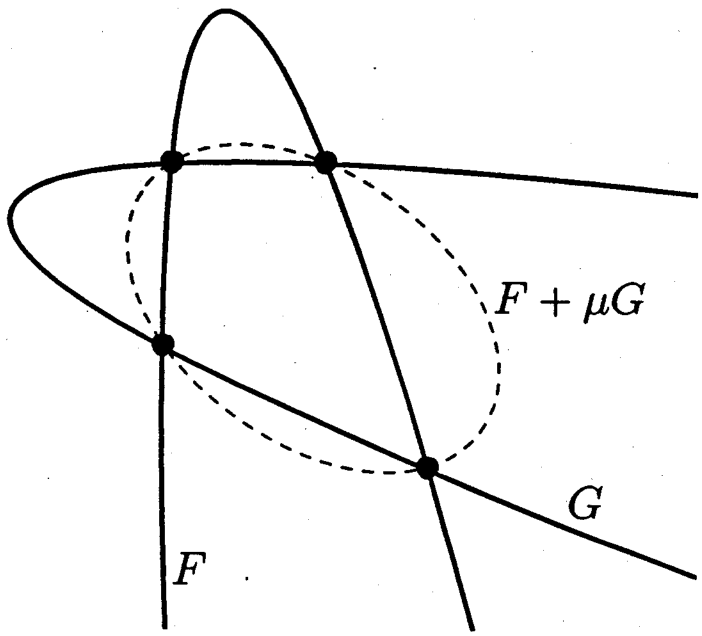
```
public void tangentConic(HConicCoord a,HCoord b) {
    HConicCoord s=b.sqr();
    xx=-a.yy*s.zz-a.zz*s.yy+a.yz*s.yz;
    yy=-a.xx*s.zz-a.zz*s.xx+a.xz*s.xz;
    zz=-a.xx*s.yy-a.yy*s.xx+a.xy*s.xy;
    xy=a.xy*s.zz+2*a.zz*s.xy-a.yz*s.xz-a.xz*s.yz;
    xz=a.xz*s.yy+2*a.yy*s.xz-a.yz*s.xy-a.xy*s.yz;
    yz=a.yz*s.xx+2*a.xx*s.yz-a.xz*s.xy-a.xy*s.xz;
  }
```

Splitting the solution:

```
public boolean intersectWithLine(HCoord l, HCoord p1, HCoord p2) {
    deg.tangentConic(this,l);
    double m3=Math.sqrt(-(deg.xx*deg.yy-deg.xy*deg.xy/4))*sign(l.z);
    double m2=-Math.sqrt(-(deg.xx*deg.zz-deg.xz*deg.xz/4))*sign(l.y);
    double m1=Math.sqrt(-(deg.zz*deg.yy-deg.yz*deg.yz/4))*sign(l.x);
    p1.move(deg.xx,deg.xy/2-m3,deg.xz/2-m2);
    p2.move(deg.xx,deg.xy/2+m3,deg.xz/2+m2);
    return (deg.xx*deg.yy-deg.xy*deg.xy/4)<0;
  }
```

Intersection Conic with Conic

Task:
Given two symmetric matrices F anf G.
Calculate the common zeros of $p^T F p = 0$ and $p^T G p = 0$



Step 1:
find real root μ_0 of

$$det(F + \mu \cdot G) = 0.$$

(solve an equation of degree 3).

Step 2:
$F + \mu_0 \cdot G$ describes two real lines l, g.
Split $F + \mu_0 \cdot G$ into l and g

Step 3:
Apply `intersectionConicLine()` to l and F (and to g and F).