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HECKE OPERATORS ON RATIONAL FUNCTIONS

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1. Introduction

We study the vector space of all linear recurrence sequences over the reals, by defining linear operators that sift out arithmetic progressions from the sequence. We call these linear operators Hecke operators, by analogy with the theory of automorphic forms, and we develop their spectral theory completely. Because the generating function of any linear recurrence sequence is a rational function (with nonzero constant term in the denominator) this study is equivalent to the action of our linear operators on rational functions. Although we borrow terminology from the traditional theory of Hecke operators on modular forms, prior knowledge of Hecke operators is not assumed here, since both the problems and the methods herein are grounded in the new context of rational functions.

To begin our study of Hecke operators, we let $\mathcal{R}$ be the vector space of all rational functions $f(x) = A(x)/B(x)$ with real coefficients such that $\deg A(x) < \deg B(x)$, and such that $B(0) \neq 0$. Given a rational function $f \in \mathcal{R}$ whose Taylor series is $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and given a positive integer $p \in \mathbb{N}$, we define the *Hecke operator* $T_p : \mathcal{R} \to \mathcal{R}$ by

$$T_p f(x) = \sum_{n=0}^{\infty} a_{pn} x^n. \tag{1.1}$$

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It turns out that the class of rational functions which are eigenfunctions of at least one of the Hecke operators defined above generate the subspace $\mathcal{R}_{qp} \subset \mathcal{R}$ of all rational functions with poles at the roots of unity.

An equivalent description of this class of rational functions can be given by noting that each rational $f = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{R}_{qp}$ has coefficients that are *quasipolynomials* in n (by the standard Theorem 2.1 below), and hence our use of the subscripts qp in $\mathcal{R}_{qp}$. That is,

$$a_n = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n),$$

where each $c_j(n) \in \mathbb{Q}$ is a periodic function on $\mathbb{Z}$. There is a large body of knowledge on Ehrhart quasi-polynomials whose generating functions give rational functions in $\mathcal{R}_{qp}$. Thus $\mathcal{R}_{qp}$ provides a rich source of functions that arise naturally in the theory of lattice point enumeration in rational polytopes and combinatorial geometry (see for example [?], [?], and [?]). We note that the reader does not, however, require any previous knowledge in this field for the analysis presented here.

In Section 3 we study the spectral properties of the Hecke operators on rational functions and show that they have discrete spectra. The first result concerning the structure of eigenfunctions is the following:

Theorem (Involution Property). *If $f(x) = A(x)/B(x)$ is an eigenfunction of T_p, then all roots of $B(x)$ are roots of unity and we have the identity*

$$x^d B(\tfrac{1}{x}) = (-1)^d B(x).$$

Moreover, if $T_p f = \lambda f$ with $\lambda \neq 1$, then

$$T_p(f(\tfrac{1}{x})) = \lambda f(\tfrac{1}{x}).$$

That is, $f(\frac{1}{x})$ is another eigenfunction of T_p with the same eigenvalue λ, and with the same denominator $B(x)$.

The involution $x \mapsto \frac{1}{x}$ on eigenfunctions plays an analogous role to the Fricke-Atkin-Lehner involutions on eigenforms. Here we uncover more properties of eigenfunctions, reducing the problem of computing eigenvalues of an infinite dimensional linear operator to the problem of computing eigenvalues of a finite matrix.

The main result of this section is described by:

Theorem (The Spectrum). *Let p be any positive integer greater than 1. Then*

$$\operatorname{spec}(T_p) = \{\pm p^k \mid k \in \mathbb{N}\} \cup \{0\}.$$

The eigenfunctions of any Hecke operator obey a rigid structure theorem, described in Section 4, that makes them appealing and easy to work with. There is far more structure in $\mathcal{R}$ due to these eigenfunctions than has been hitherto apparent.

The main structure theorem is the following:

Theorem (Structure Theorem). *Let $f(x) = A(x)/B(x) = \sum a_n x^n$ be an eigenfunction of T_p for some integer $p > 1$, associated to an eigenvalue $\lambda_p \neq 0$. If $B(x) = \prod_{j=1}^{d}(1 - \gamma_j x)$, then there is an integer κ dividing the degree d, and an integer L such that*

$$a_n = n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, e^{\frac{2\pi i \ell_j}{L} n}, \text{ for all } n \geq 0,$$

where each pole of f is given by $\gamma_j = e^{\frac{2\pi i \ell_j}{L}}$, $\ell_j \in \mathbb{N}$, and the constants $C_j \in \mathbb{C}$ are determined by the initial conditions of the linear recurrence sequence $\{a_n\}$. We note that each pole γ_j must occur with the same multiplicity κ.

When we consider rational functions that are simultaneous eigenfunctions of a family of Hecke operators, we discover that there is a natural character $\chi_f \pmod{L}$ that comes into the spectrum. We also get the curious phenomenon of "partial characters" whenever we find a rational function f that is an eigenfunction of some, but *not all* Hecke operators. Here L is called the *level* of f, and is defined as the least common multiple of the orders of all the roots of unity that comprise the poles of f. There is also a notion of the *weight* κ of an eigenfunction f, arising naturally in the structure theorem above, and defined by the common multiplicity of the poles.

In Section 5 we decompose the infinite dimensional vector space $\mathcal{R}_{pq}$ into finite dimensional subspaces, by using the weight and level of an eigenfunction as the grading parameters. We let $\mathcal{V}_{\kappa,L}(T_3, T_5)$, for example, denote the finite dimensional vector space of eigenfunctions of (at least) the Hecke operators T_3 and T_5 that have weight κ and level L.

By a further analogy with modular forms, the set of simultaneous eigenfunctions for the full Hecke algebra $\mathfrak{H}$ (that are not in the kernel of any T_p) is of special interest. In Section 6 we give a complete description of the vector space $\mathcal{V}$ spanned by all of the simultaneous eigenfunctions of $\mathfrak{H}$. The next two results handle the two separate cases when the level is $L = 1$, and $L > 1$ for simultaneous eigenfunctions.

Theorem (Simultaneous Eigenfunctions). *Let f be a simultaneous eigenfunction of $\mathfrak{H}$ such that f is not in the kernel of T_p for any p, and level$(f) = L$. That is, let f be a rational function with the property that for every p there is a $\lambda_p \neq 0$ such that $T_p f = \lambda_p f$. Then $L = 1$, and*

$$f(x) = C(x\partial_x)^k \left(\frac{1}{1-x}\right)$$

for some $k \in \mathbb{N}$ and $C \in \mathbb{C}$.

In Corollary 4.10, the following counterpart to this result is given for all simultaneous eigenfunctions that have some of its taylor coefficients equal to zero.

Theorem. *Let $L > 1$ be a given integer. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a real rational function of level L with $a_0 = 0$, and $f(x)$ is a simultaneous eigenfunction of the operators $T_2, T_3, \dots, T_L$ (i.e. $T_m f = \chi_f(m) m^{\kappa-1} f$ for every $m = 2, \dots, L$).*

Then χ_f is the real quadratic character mod L, f is in fact a simultaneous eigenfunction of all the Hecke operators T_m, and in addition we must have

$$f(x) = a_1 \sum_{n=0}^{\infty} \chi_f(n) n^{\kappa-1} x^n.$$

It is worthwhile noting that f can also be written as

$$f(x) = a_1 \, (x\partial_x)^{\kappa-1} \left(\frac{\sum_{j=1}^{L-1} \chi_f(j) x^j}{1 - x^L} \right).$$

Under the same hypothesis, except with $a_0 \neq 0$, we conclude that χ_f is the principal character and $f(x) = \frac{a_0}{1-x}$.

Here the differential operator $x\partial_x$ plays the role of the "weight-raising" operator in modular forms, because it takes eigenfunctions of weight κ to eigenfunctions of weight $\kappa + 1$.

As a curious application of the explicit characterization of $\mathcal{V}$, the vector space of simultaneous eigenfunctions, we can realize any finite Euler product of the Riemann zeta function in Section 7 as the spectral zeta function of a very explicit operator $\mathbf{T}_S$. More precisely, for any finite set of primes $S = \{p_1, \dots, p_n\}$, we define a corresponding operator $\mathbf{T}_S$ as a finite tensor product $T_{p_1} \otimes \cdots \otimes T_{p_n}$. This operator $\mathbf{T}_S$ acts on tensor products of eigenfunctions, and it turns out that we retrieve any finite piece of the Euler product for the Riemann zeta function, precisely as the spectral zeta function $\zeta_{\mathbf{T}_S}(s) = \sum_{\lambda \in \mathrm{spec}(\mathbf{T}_S)} \frac{1}{\lambda^s}$ of the operator $\mathbf{T}_S$.

Theorem (Euler product).

$$\zeta_{\mathbf{T}_S}(s) = \zeta_{T_{p_1}}(s) \cdots \zeta_{T_{p_n}}(s) = \prod_{p \in S} \frac{1}{1 - p^{-s}}.$$

To extend these ideas to infinite Euler products, we now define $\mathcal{H}^\infty$ to be the space of products $\mathbf{f} = f_1 \otimes f_2 \otimes \cdots$, where $\{f_n\}_{n \in \mathbb{N}}$ is an infinite sequence of rational functions with the following properties:

(1) There is a finite set $I \subset \mathbb{N}$ such that $f_j \in \mathcal{V}$ for every $j \in I$.
(2) $f_j = 1$ for every $j \in \mathbb{N} \setminus I$.

For $\mathbf{f} \in \mathcal{H}^\infty$ we define the operator $\mathbf{T}$ by

$$\mathbf{T}\mathbf{f} = (T_{p_{i_1}} f_{i_1}) \otimes \cdots \otimes (T_{p_{i_m}} f_{i_m}), \tag{1.2}$$

where $I = \{i_1, \dots, i_m\}$ is the finite set of positive integers associated to $\mathbf{f}$, and where p_{i_k} is the i_k'th prime number. Notice that similarly to $\mathbf{T}_S$, the operator $\mathbf{T}$ maps tensor products of rational functions into rational functions in several variables.

Theorem (Riemann zeta function). *The spectral zeta function of the operator* $\mathbf{T}$ *on* $\mathcal{H}^\infty$ *satisfies*

$$\zeta_{\mathbf{T}}(s) = \zeta(s),$$

where $\zeta(s)$ *is the Riemann zeta function.*

Finally, we conclude with an appendix that displays explicit examples of eigenfunctions, to give the reader a better feeling for the eigenfunctions and eigenspaces that arise. It is a highly non-trivial problem to compute the dimensions of the various vector spaces of eigenfunctions defined by fixing the weight and the level of admissible eigenfunctions. Indeed, even computing $\dim(\mathcal{V}_{1,L}(T_p))$, for example, involves the Artin conjecture for p being a primitive root (mod L) for infinitely many integers L.

2. Some preliminaries

We recall some standard facts about linear recurrence sequences and their generating functions. The following theorem gives a characterization of linear recurrence sequences in terms of rational functions, and gives a closed form for their Taylor coefficients. For a proof see R. Stanley's book [?, Chapter 4].

Theorem 2.1. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_d$ *be a sequence of complex numbers,* $d \in \mathbb{N}$ *and* $\alpha_d \neq 0$. *Consider the formal power series* $\sum_{n=0}^{\infty} a_n x^n$. *The following conditions on the coefficients* a_n *are equivalent:*

(i)
$$\sum_{n=0}^{\infty} a_n x^n = \frac{A(x)}{B(x)},$$
where $B(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$ *and* $A(x)$ *is a polynomial in* x *of degree less than* d.

(ii) *For all* $n \in \mathbb{N}$,
$$a_{n+d} = -\alpha_1 a_{n+d-1} - \cdots - \alpha_d a_n.$$

(iii) *For all* $n \in \mathbb{N}$,
$$a_n = \sum_{j=1}^{d} C_j n^{m_j - 1} \gamma_j^n,$$
where each $C_j \in \mathbb{C}$, m_j *is a positive integer, and*
$$1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{j=1}^{d} (1 - \gamma_j x).$$
Each m_j *is the multiplicity of the root* γ_j.

When all the poles γ_j above are roots of unity, a_n is known as a *quasi-polynomial* in n, cf. [?].

One of our primary goals is to study the spectral properties of Hecke operators acting on the vector space of rational functions $\mathcal{R}$. To this end we must first justify the definition of the Hecke operator (1.1) with a lemma. That is, we do not yet know that the image of $T_p f$ is indeed a rational function in $\mathcal{R}$.

Lemma 2.2. *Given a rational function $f \in \mathcal{R}$, $T_p f$ is again in $\mathcal{R}$. Moreover, there is a simple algorithm that constructs the rational function $T_p f$ from the roots of f. If the pole set of f is $\{\gamma_1, \dots, \gamma_d\}$, then the pole set of $T_p f$ is $\{\gamma_1^p, \dots, \gamma_d^p\}$.*

Proof. We employ the structure Theorem 2.1 to write the Taylor coefficients of $T_p f$ as

$$a_{pn} = \sum_{j=1}^{d} C_j (pn)^{m_j - 1} (\gamma_j^p)^n.$$

Thus, the defining characteristic polynomial for the sought-after linear recurrence given by $T_p f$ is

$$\prod_{j=1}^{d} (1 - \gamma_j^p x) = 1 + \beta_1 x + \cdots + \beta_d x^d. \tag{2.3}$$

It is clear that the coefficients of this polynomial are real, since the Taylor coefficients of f are real. We provide a simple algorithm for finding the polynomial (2.3).

By Newton's identities, every symmetric function of $\gamma_1, \dots, \gamma_d$ is a polynomial in the elementary symmetric functions of $\gamma_1, \dots, \gamma_d$, with integers coefficients. In particular, we can write each coefficient β_j as a polynomial over the integers in the variables $\alpha_1, \dots, \alpha_d \in \mathbb{R}$. □

Since $f(x^p) = \sum_{n=0}^{\infty} a_n x^{pn}$ is again rational, and $T_p(f(x^p)) = f(x)$, we observe that the map T_p is surjective. On the other hand, for $1 \le j < p$ the rational function $x^j f(x^p)$ is in the kernel of T_p, so T_p is not injective. Thus, the kernel of T_p is clearly infinite dimensional. Although there is no left inverse, there is a right inverse for T_p given by the map $f(x) \mapsto f(x^p)$.

It is trivial to check that our Hecke operators form a commutative algebra. In particular, $T_n = T_{p_1^{\alpha_1}} \circ \cdots \circ T_{p_m^{\alpha_m}}$ whenever $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$.

3. The spectrum of T_p

We begin by showing that the Hecke operator T_p almost commutes with the operator $x\partial_x$, up to a factor of p. This result becomes useful because it allows us to easily construct infinitely many eigenfunctions from each known eigenfunction, by iteration of the operator $x\partial_x$.

Lemma 3.1. *For every positive integer p we have*

$$T_p(x\partial_x) - p\,(x\partial_x)T_p = 0.$$

Proof. Let $f = \sum_{n=0}^{\infty} a_n x^n$. Then

$$(x\partial_x)f = \sum_{n=1}^{\infty} n a_n x^n,$$

and it follows that

$$T_p[(x\partial_x)f] = \sum_{n=1}^{\infty} (pn) a_{pn} x^n = p(x\partial_x)T_p f.$$

□

Lemma 3.2. *Let $\lambda \neq 0$ and $k \in \mathbb{N}$. If $T_p f = \lambda f$, then*

$$T_p[(x\partial_x)^k f] = (p^k \lambda)(x\partial_x)^k f.$$

In other words, if λ is an eigenvalue of T_p, then so is $p^k\lambda$ for every $k \in \mathbb{N}$, with the corresponding eigenfunction $(x\partial_x)^k f$.

Proof. We proceed by induction on k.

$$T_p[(x\partial_x)f] = p(x\partial_x)T_p f = p\lambda(x\partial_x)f$$

Assuming the statement for $k-1$, we get

$$\begin{aligned} T_p[(x\partial_x)^k f] &= T_p[(x\partial_x)(x\partial_x)^{k-1} f] = p\,(x\partial_x)T_p[(x\partial_x)^{k-1} f] \\ &= p\,(x\partial_x)[p^{k-1}\lambda(x\partial_x)^{k-1} f] \\ &= (p^k\lambda)(x\partial_x)^k f. \end{aligned}$$

□

We now give an important family of eigenfunctions that have the eigenvalues ± 1 for each T_p. The function $f(x) = \frac{1}{1-x}$ trivially satisfies $T_p f = f$ for every positive integer p. However, it is less trivial to find eigenfunctions for the eigenvalue $\lambda = -1$.

Example 3.3. For every integer $p > 1$ we explicitly give an eigenfunction f_p satisfying $T_p f_p = -f_p$. If p is even, consider

$$f_p(x) = \frac{x - x^p}{1 - x^{p+1}} = \sum_{n=0}^{\infty} x^{(p+1)n+1} - \sum_{n=0}^{\infty} x^{(p+1)n+p}. \tag{3.4}$$

Then, using the change of variables $j = \frac{(p+1)n+1}{p}$ and $k = \frac{(p+1)n}{p}$ we get

$$\begin{aligned} T_p f_p(x) &= \sum_{pn+n+1\equiv 0\ (p)} x^{\frac{(p+1)n+1}{p}} - \sum_{pn+n\equiv 0\ (p)} x^{\frac{(p+1)n}{p}+1} \\ &= \sum_{pj\equiv 1\ (p+1)} x^j - \sum_{pk\equiv 0\ (p+1)} x^{k+1} \\ &= \sum_{j\equiv p\ (p+1)} x^j - \sum_{k\equiv 0\ (p+1)} x^{k+1} \\ &= \sum_{m=0}^{\infty} x^{(p+1)m+p} - \sum_{m=0}^{\infty} x^{(p+1)m+1} = -f_p(x). \end{aligned}$$

If p is odd, then we can write $p-1 = q\ell$ with integers q and ℓ such that q is even and ℓ is odd. In this case, the function

$$f_p(x) = \frac{x}{1+x^q} = \sum_{n=0}^{\infty} (-1)^n x^{qn+1} \tag{3.5}$$

satisfies $T_p f_p = -f_p$ for $p = q\ell + 1$. In fact,

$$\begin{aligned} T_p\left(\frac{x}{1+x^q}\right) &= \sum_{qn+1\equiv 0\ (p)} (-1)^n x^{(qn+1)/p} \\ &= \sum_{pk\equiv 1\ (q)} (-1)^{(pk-1)/q} x^k. \end{aligned}$$

Since $p \equiv 1 \bmod q$ it follows that $k \equiv 1 \bmod q$, so we write $k = qn+1$ and get

$$\begin{aligned} T_p\left(\frac{x}{1+x^q}\right) &= \sum_{n=0}^{\infty} (-1)^{pn+\ell} x^{qn+1} \\ &= -\sum_{n=0}^{\infty} (-1)^n x^{qn+1} = -\frac{x}{1+x^q} \end{aligned}$$

since p and ℓ are both odd.

The following lemma gives a first glimpse into the spectrum of T_p.

Lemma 3.6. *For every integer $p \geq 2$, we have*

$$\{\pm p^k \mid k \in \mathbb{N}\} \cup \{0\} \subset \operatorname{spec}(T_p).$$

Proof. Since $x/(1-x^p)$ is in the kernel of T_p, it follows that 0 belongs to $\operatorname{spec}(T_p)$. The previous example gives eigenfunctions of T_p for $\lambda = 1$ and $\lambda = -1$. Invoking Lemma 3.2 with these eigenfunctions, the iterated operator $(x\partial_x)^k$ provides us with the eigenvalues $\lambda = \pm p^k$. □

Concerning the eigenfunctions of T_p we have the following basic equivalence condition.

Lemma 3.7. *Let $\lambda \neq 0$. $T_p f = \lambda f$ if and only if*

$$\lambda f(x^p) = \frac{1}{p}\sum_{j=0}^{p-1} f(e^{\frac{2\pi i}{p}j}x). \tag{3.8}$$

Proof. Let $f = \sum_{n=0}^{\infty} a_n x^n$ be an eigenfunction associated to λ. Then,

$$\begin{aligned}\lambda f(x^p) = (T_p f)(x^p) &= \sum_{k=0}^{\infty} a_{pk} x^{pk} \\ &= \frac{1}{p}\sum_{n=0}^{\infty} a_n (\sum_{j=0}^{p-1} e^{\frac{2\pi i}{p}nj}) x^n \\ &= \frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=0}^{\infty} a_n (e^{\frac{2\pi i}{p}j}x)^n \\ &= \frac{1}{p}\sum_{j=0}^{p-1} f(e^{\frac{2\pi i}{p}j}x).\end{aligned}$$

Here we have used

$$\sum_{j=0}^{p-1} e^{\frac{2\pi i}{p}nj} = \begin{cases} p & \text{for } n \equiv 0 \pmod p \\ 0 & \text{otherwise} \end{cases}.$$

Now let (3.8) be satisfied. Applying T_p to both sides of the equation we get

$$\lambda T_p(f(x^p)) = \frac{1}{p}\sum_{j=0}^{p-1} T_p(f(e^{\frac{2\pi i}{p}j}x)) = T_p f(x)$$

since, for every j,

$$T_p(f(e^{\frac{2\pi i}{p}j}x)) = T_p\left(\sum_{n=0}^{\infty} a_n (e^{\frac{2\pi i}{p}nj}) x^n\right) = \sum_{n=0}^{\infty} a_{pn} x^n = T_p f(x).$$

Using the identity $T_p(f(x^p)) = f(x)$, we conclude that $T_p f = \lambda f$. □

Lemma 3.9. *Let $f(x) = A(x)/B(x)$ be an eigenfunction of T_p. Then*

$$B(x^p) = \prod_{j=0}^{p-1} B(e^{\frac{2\pi i}{p}j}x). \tag{3.10}$$

Proof. We use the identity (3.8)

$$p\lambda\frac{A(x^p)}{B(x^p)} = p\lambda f(x^p) = \sum_{j=0}^{p-1} f(e^{\frac{2\pi i}{p}j}x) = \sum_{j=0}^{p-1}\frac{A(e^{\frac{2\pi i}{p}j}x)}{B(e^{\frac{2\pi i}{p}j}x)}$$

and compare the denominators. □

Theorem 3.11. *If λ is an eigenvalue of T_p and $f(x) = A(x)/B(x)$ is a corresponding eigenfunction, then λ is an eigenvalue of a $d \times d$ matrix $\mathfrak{B}$ determined by the coefficients of the polynomial $B(x)$.*

Proof. Using (3.8) and (3.10) we get the identity

$$\lambda A(x^p) = \frac{1}{p}\sum_{j=0}^{p-1}\Big(\prod_{\ell\neq j} B(\zeta_p^\ell x)\Big)A(\zeta_p^j x),$$

where $\zeta_p = e^{\frac{2\pi i}{p}}$. Writing $A(x) = \sum_{k=0}^{d-1} c_k x^k$, the identity above becomes

$$\begin{aligned}\lambda\sum_{k=0}^{d-1} c_k x^{pk} &= \frac{1}{p}\sum_{j=0}^{p-1}\Big(\prod_{\ell\neq j} B(\zeta_p^\ell x)\Big)\sum_{k=0}^{d-1} c_k(\zeta_p^j x)^k \\ &= \sum_{k=0}^{d-1}\Big(\sum_m \beta_{k,m}c_m\Big)x^{pk}\end{aligned}$$

by rearranging the sum on the right-hand side. Comparing the coefficients gives us

$$\lambda c_k = \sum_{m=0}^{d-1}\beta_{k,m}c_m$$

for every $k = 0,\dots,d-1$. Finally, we conclude that λ is an eigenvalue of the matrix $\mathfrak{B} = (\beta_{k,m})_{k,m=0,\dots,d-1}$. We observe that the $\beta_{k,m}$ are complex polynomials in the coefficients of the denominator $B(x)$. □

This theorem is very useful because it allows us to explicitly construct eigenfunctions with a given denominator by computing eigenvectors of a *finite* matrix. As a direct consequence of this theorem, we obtain a converse to Lemma 3.9.

Corollary 3.12. *Given any integer p and a denominator $B(x)$ satisfying the identity (3.10), there is a numerator $A(x)$ such that $f(x) = A(x)/B(x)$ is an eigenfunction of T_p.*

Theorem 3.13 (Involution Property). *If $f(x) = A(x)/B(x)$ is an eigenfunction of T_p, then all roots of $B(x)$ are roots of unity and we have the identity*

$$x^d B(\tfrac{1}{x}) = (-1)^d B(x).$$

Moreover, if $T_p f = \lambda f$ with $\lambda \neq 1$, then

$$T_p(f(\tfrac{1}{x})) = \lambda f(\tfrac{1}{x}).$$

That is, $f(\frac{1}{x})$ is another eigenfunction of T_p with the same eigenvalue λ, and with the same denominator $B(x)$.

Proof. Write

$$B(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{k=1}^{d} (1 - \gamma_k x).$$

The identity (3.10) yields

$$\begin{aligned}\prod_{k=1}^{d} (1 - \gamma_k x^p) &= \prod_{j=0}^{p-1} \prod_{k=1}^{d} (1 - \gamma_k (e^{\frac{2\pi i}{p} j} x)) \\ &= \prod_{k=1}^{d} \prod_{j=0}^{p-1} (1 - \gamma_k (e^{\frac{2\pi i}{p} j} x)) \\ &= \prod_{k=1}^{d} (1 - \gamma_k^p x^p)\end{aligned}$$

which implies $\{\gamma_1, \ldots, \gamma_d\} = \{\gamma_1^p, \ldots, \gamma_d^p\}$. So the second set is a permutation of the first set. This permutation breaks up into a disjoint product of cycles. Consider now a fixed cycle in this decomposition, say of length ℓ. By iterating through the cycle, we easily see that each of the roots in this cycle must satisfy the equation

$$x^{p^\ell - 1} = 1. \tag{3.14}$$

Thus they are all roots of unity (different from -1 if p is even). Since we can do this for each cycle, all of the γ_j's are in fact roots of unity.

As a consequence, we get $\prod_{k=1}^{d} \gamma_k = 1$ and $\gamma_k^{-1} = \bar{\gamma}_k$. Therefore,

$$\begin{aligned}x^d B(\tfrac{1}{x}) &= x^d \prod_{k=1}^{d} (1 - \gamma_k / x) = (-1)^d \prod_{k=1}^{d} (\gamma_k - x) \\ &= (-1)^d \prod_{k=1}^{d} \gamma_k (1 - \gamma_k^{-1} x) \\ &= (-1)^d \prod_{k=1}^{d} (1 - \gamma_k^{-1} x) = (-1)^d \prod_{k=1}^{d} (1 - \bar{\gamma}_k x) \\ &= (-1)^d \prod_{k=1}^{d} (1 - \gamma_k x) = (-1)^d B(x).\end{aligned}$$

To prove the last claim, let $f(x) = A(x)/B(x)$ be an eigenfunction of T_p with eigenvalue $\lambda \neq 1$. We will verify that the function $g(x) = f(\frac{1}{x})$ satisfies the condition (3.8). The identity $x^d B(\frac{1}{x}) = (-1)^d B(x)$ gives

$$g(x) = \frac{A(\frac{1}{x})}{B(\frac{1}{x})} = (-1)^d \, \frac{x^d A(\frac{1}{x})}{B(x)},$$

where $x^d A(\frac{1}{x})$ is a polynomial of degree less than d since $\lambda \neq 1$ implies $A(0) = 0$. Thus g belongs to $\mathcal{R}$ and has the same denominator as f. Now,

$$\sum_{j=0}^{p-1} g(\zeta_p^j x) = \sum_{j=0}^{p-1} \frac{A\left(\frac{1}{\zeta_p^j x}\right)}{B\left(\frac{1}{\zeta_p^j x}\right)} = \sum_{k=0}^{p-1} \frac{A(\zeta_p^k y)}{B(\zeta_p^k y)} = \sum_{k=0}^{p-1} f(\zeta_p^k y),$$

where $\zeta_p = e^{\frac{2\pi i}{p}}$ and $y = \frac{1}{x}$. In the latter equality we used the fact that $1/\zeta_p^j = \zeta_p^k$ with $k = p - j$. Using the condition (3.8) for f, we obtain

$$\frac{1}{p}\sum_{j=0}^{p-1} g(\zeta_p^j x) = \frac{1}{p}\sum_{k=0}^{p-1} f(\zeta_p^k y) = \lambda f(y^p) = \lambda f(\tfrac{1}{x^p}) = \lambda g(x^p).$$

Thus $g(x)$ satisfies (3.8) and $f(\frac{1}{x})$ is therefore another eigenfunction of T_p with the same eigenvalue λ. □

It is interesting to note the analogy here with the classical Fricke involutions on Riemann surfaces. The preceding theorem gives us an involution on the vector space of eigenfunctions, and supplies us with an easy construction of new eigenfunctions from known ones.

Corollary 3.15. *Let f be any rational function such that $T_p f = \lambda_p f$ for some integer p, and let L be the level of f. Then p is relatively prime to L.*

Proof. □

Definition 3.16. Fix p and suppose we have an eigenfunction f for T_p with eigenvalue λ_p. If $\lambda_p \neq 0$, we define

$$\chi_f(p) = \frac{\lambda_p}{|\lambda_p|}.$$

If $\lambda_p = 0$, then we define $\chi_f(p) = 0$.

In Section 6 we will prove that, if f is a simultaneous eigenfunction for an appropriate family $\{T_p \,|\, p \in S \subset \mathbb{N}\}$, then χ_f is a *character*.

Lemma 3.17. *For every integer p, $\operatorname{spec}(T_p) \subset \mathbb{R}$. In particular, for every f with $T_p f = \lambda f$, $\lambda \neq 0$, we have $\chi_f(p) = \pm 1$.*

Proof. Let f be an eigenfunction of T_p for $\lambda \neq 0$. Since $f(\bar\rho x) = \overline{f(\rho x)}$ for any complex number ρ, we have

$$f(e^{\frac{2\pi i}{p}(p-j)}x) = f(e^{-\frac{2\pi i}{p}j}x) = \overline{f(e^{\frac{2\pi i}{p}j}x)}$$

for every $j = 1, \ldots, p-1$. Thus $f(e^{\frac{2\pi i}{p}j}x) + f(e^{\frac{2\pi i}{p}(p-j)}x)$ is always real, which implies that the right-hand side of (3.8) is real. Consequently, λ is real as well. □

The next theorem gives us the complete structure of the spectrum for every Hecke operator T_p. We note that the proof of the main structure theorem for eigenfunctions, Theorem 4.2 in the next section, bootstraps the proof of the following theorem.

Theorem 3.18 (The Spectrum). *Let p be any positive integer greater than 1. Then*

$$\operatorname{spec}(T_p) = \{\pm p^k \mid k \in \mathbb{N}\} \cup \{0\}.$$

Proof. Let $f(x) = \sum a_n x^n$ satisfy $T_p f = \lambda f$, $\lambda \neq 0$. Thus $a_{pn} = \lambda a_n$ and by iteration

$$a_{p^k n} = \lambda^k a_n \tag{3.19}$$

for every $k \in \mathbb{N}$. Let $B(x) = \prod_{j=1}^{d}(1 - \gamma_j x)$. From Theorem 3.13 we know that the γ_j's must be roots of unity. We now compare the asymptotics of the coefficients from their closed form as a sum of polynomials in n times roots of unity. Namely, from Theorem 2.1 we know that

$$a_n = \sum_{j=1}^{\ell} c_j n^{m_j - 1} \gamma_j^n,$$

where each m_j is the multiplicity of the root γ_j. For the purposes of using asymptotics, we let κ denote the largest exponent m_j in this representation of a_n. We collect together all of the terms that correspond to this largest exponent κ, and label the remaining terms by $R(n)$. Thus we may write

$$a_n = n^{\kappa-1} \sum_{j=1}^{\ell_1} C_j \gamma_{\sigma(j)}^n + R(n), \tag{3.20}$$

for some constants C_j and some permutation σ. It follows from Equation 3.19 above (using the assumption that $\lambda \neq 0$) that

$$a_n = \frac{a_{p^k n}}{\lambda^k} = \left(\frac{p^{\kappa-1}}{\lambda}\right)^k n^{\kappa-1} \sum_{j=1}^{\ell_1} C_j \gamma_{\sigma(j)}^{p^k n} + \frac{R(p^k n)}{\lambda^k}. \tag{3.21}$$

We first claim that $|\frac{p^{\kappa-1}}{\lambda}| \leq 1$. To see this note that all of the terms in $\frac{R(p^k n)}{\lambda^k}$ contain exponential terms in k of the form $(\frac{p^{m_j-1}}{\lambda})^k$, which are strictly of smaller growth than $(\frac{p^{\kappa-1}}{\lambda})^k$. Since the left-hand side of the equation above is a_n, and in particular independent of k, we cannot have $|\frac{p^{\kappa-1}}{\lambda}| > 1$, unless perhaps the leading sum vanishes for all k. We now argue that there is a subsequence of k's for which the leading sum $\sum_{j=1}^{\ell_1} C_j \gamma_{\sigma(j)}^{p^k n}$ does not vanish.

To this end note that, by Corollary 3.15, p is relatively prime to L, where L is the least common multiple of the orders (as roots of unity) of all the poles of f. Using $p^{\phi(L)} \equiv 1 \bmod(L)$, we now let the index k approach infinity through the subsequence $k' = m\phi(L)$, where $m \in \mathbb{N}$. Thus $\gamma_{\sigma(j)}^{p^{k'}} = \gamma_{\sigma(j)}$, for all $1 \le j \le \ell_1$, and we have

$$a_n = \left(\frac{p^{\kappa-1}}{\lambda}\right)^{k'} n^{\kappa-1} \sum_{j=1}^{\ell_1} C_j \gamma_{\sigma(j)}^n + \frac{R(p^{k'} n)}{\lambda^{k'}}. \tag{3.22}$$

Thus the term with the largest exponent cannot vanish as $k \to \infty$ through the given subsequence, so that have shown that $|\frac{p^{\kappa-1}}{\lambda}| \le 1$.

On the other hand, we cannot have $|\frac{p^{\kappa-1}}{\lambda}| < 1$, for then all of the terms on the right hand side of Equation 3.21 would tend to 0 as $k \to \infty$, contradicting $a_n \ne 0$. Therefore $|\frac{p^{\kappa-1}}{\lambda}| = 1$ and thus $\lambda = \pm p^{\kappa-1}$ since λ is real by Lemma 3.17. Together with the inclusion from Lemma 3.6 we finally get the assertion. □

4. A STRUCTURE THEOREM FOR EIGENFUNCTIONS

By refining the proof of Theorem 3.18 further, we can get a very useful structure theorem for eigenfunctions. We can subsequently draw several interesting conclusions that resemble ideas from automorphic forms. In particular, we will define a *weight* and a *level* for eigenfunctions, and show in Section 5 that we have a finite dimensional eigenspace for a fixed level L *and* weight κ.

Definition 4.1. Given an eigenfunction f of T_p, we know by Theorem 3.13 that its poles are all roots of unity. We define the *level* L of f as the least common multiple of the orders of all these roots of unity; thus each pole γ_j is a root of unity $e^{\frac{2\pi i \ell_j}{L}}$ for some integer ℓ_j.

We observe that when the level of f is L, the smallest group containing all of the poles of f is simply the group μ_L of L'th roots of unity. We now give a structure theorem that simplifies the analysis of eigenfunctions.

Theorem 4.2 (Structure Theorem). *Let $f(x) = A(x)/B(x) = \sum a_n x^n$ be an eigenfunction of T_p for some integer $p > 1$, associated to an eigenvalue $\lambda_p \ne 0$. If $B(x) = \prod_{j=1}^{d}(1 - \gamma_j x)$, then there is an integer κ dividing the degree d, and an integer L such that*

$$a_n = n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, e^{\frac{2\pi i \ell_j}{L} n},$$

where each pole of f is given by $\gamma_j = e^{\frac{2\pi i \ell_j}{L}}$, $\ell_j \in \mathbb{N}$, and the constants $C_j \in \mathbb{C}$ are determined by the initial conditions of the linear recurrence sequence $\{a_n\}$. We note that each pole γ_j must occur with the same multiplicity κ.

Proof. We begin with the identity (??) derived in the proof of Theorem 3.18, namely

$$
\begin{aligned}
a_n &= \left(\frac{p^{\kappa-1}}{\lambda}\right)^k n^{\kappa-1} \sum_{j=1}^{\ell} C_j \gamma_{\sigma(j)}^{p^k n} + R(k) \\
&= \chi_f^k(p)\, n^{\kappa-1} \sum_{j=1}^{\ell} C_j\, e^{\frac{2\pi i \ell_j}{L} p^k n} + R(k) \\
&= (\pm 1)^k n^{\kappa-1} \sum_{j=1}^{\ell} C_j\, e^{\frac{2\pi i \ell_j}{L} p^k n} + R(k),
\end{aligned}
$$

where we used Lemma 3.17 to express $\chi_f(p) = \pm 1$. We now claim that $R(k)$ is identically zero. To see this, note that the sum on the right-hand side is a periodic function in k which implies that $R(k)$ is also periodic in k.

On the other hand, recall that all of the terms in $R(k)$ contain exponential terms of the form $(\frac{p^{m_j-1}}{\lambda})^k$ with $m_j < \kappa$. Since $p^{m_j-1} < p^{\kappa-1}$, we have $|\frac{p^{m_j-1}}{\lambda}| < |\frac{p^{\kappa-1}}{\lambda}| = 1$, so that $|\frac{p^{m_j-1}}{\lambda}| \to 0$. Thus $R(k) \to 0$ as $k \to \infty$ and we conclude from the periodicity of $R(k)$ that it is the zero function. □

Definition 4.3. We call the κ appearing in the previous theorem the *weight* of the eigenfunction f. Note that κ is independent of p. That is, if f is an eigenfunction of any other T_q, then $T_q f = \pm q^{\kappa-1} f$.

The motivation for this terminology comes from the weight of the classical Eisenstein series in automorphic forms. With hindsight, we call the operator $x\partial_x$ the weight-raising operator, since it takes a weight κ eigenfunction to a weight $\kappa+1$ eigenfunction.

Remark 4.4. There is a precise connection between the weight κ, the level L and the degree of the denominator of an eigenfunction $f(x) = A(x)/B(x)$. Namely, $\deg(B(x)) \in \{\kappa, 2\kappa, \ldots, L\kappa\}$.

At this stage we note that, given $p > 1$, every eigenvalue of T_p must have the form

$$
\lambda = \chi_f(p)\, p^{\kappa-1}, \tag{4.5}
$$

where $\chi_f(p) = \pm 1$. In the corollaries that follow we will show that χ_f has the properties of a multiplicative character.

Corollary 4.6. *Let $f = \sum a_n x^n$ be a rational function such that $T_p f = \lambda_p f$ for some integer p, and let L be the level of f. Then*

(i) *If $a_0 = 0$, then $\chi_f(L) = 0$ and $T_{mL}(f) = 0$ for every positive integer m. That is, $a_{nL} = 0$ for every n.*
(ii) *If $a_0 \neq 0$, then $\chi_f(p) = 1$, so that $T_p(f) = f$.*

Proof. To show (i), we use the structure theorem directly, keeping in mind that by definition of L we have $\gamma_j^L = 1$ for all of the roots γ_j.

$$a_{nL} = (nL)^{\kappa-1} \sum_j C_j \gamma_j^{nL} = (nL)^{\kappa-1} \sum_j C_j = (nL)^{\kappa-1} a_0 = 0.$$

Thus by definition of T_{mL}, we have $T_{mL}(f) = 0$.

To show (ii) we simply recall that $a_0\lambda = a_0$, so that $\lambda = 1 = \chi_f(p)p^{\kappa-1}$, and we have the required result. □

In other words any eigenfunction f, with eigenvalue $\lambda \neq 1$ and level L, must lie in the kernel of T_L and thus f has an infinite arithmetic progression of zeros among its Taylor coefficients.

Corollary 4.7. *For any positive integer n, we have the following formulas for $\chi_f(p)$:*

(i) $\chi_f(p) \sum_j C_j \gamma_j^n = \sum_j C_j \gamma_j^{pn}$.
(ii) $\chi_f(p + mL) = \chi_f(p)$ *for all positive integers m.*
(iii) *Suppose that f is a simultaneous eigenfunction for all T_p. Then f is not in the kernel of any operator T_p if and only if $\chi_f(p) = 1$ for all p.*

Proof. We begin with the identity

$$a_{pn} = (pn)^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, e^{\frac{2\pi i \ell_j}{L} pn}.$$

On the other hand,

$$a_{pn} = \chi_f(p)p^{\kappa-1}a_n = \chi_f(p)p^{\kappa-1}n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, e^{\frac{2\pi i \ell_j}{L} n}.$$

Equating both right-hand sides, we get the desired identity (i).

To prove (ii), we simply pick an n for which $0 \neq a_n = n^{\kappa-1}\sum_j C_j\gamma_j^n$, whence

$$\chi_f(p + mL) = \frac{\sum_j C_j \gamma_j^{(p+mL)n}}{\sum_j C_j \gamma_j^n} = \frac{\sum_j C_j \gamma_j^{pn}}{\sum_j C_j \gamma_j^n} = \chi_f(p).$$

To prove (iii), first assume that f is not in the kernel of any operator T_p. We observe that Corollary 3.15 implies $(p, L) = 1$. Since L is now relatively prime to *all* integers p, we must have $L = 1$ which implies $\gamma_j = 1$ for all j. Using $L = 1$ in part (i) now gives us

$$\chi_f(p) \sum_j C_j = \sum_j C_j,$$

and we conclude that either $\chi_f(p) = 1$ for each p (and we're done) or else $\sum_j C_j = 0$. To see that the latter case never occurs, we observe that

$\sum_j C_j = 0$ means that $a_0 = 0$, which by Corollary 4.6 (i) above implies that f is in the kernel of T_L, a contradiction.

To prove (iii) in the other direction, let $\chi_f(p) = 1$ for all p. Then $T_p f = \chi_f(p)p^{\kappa-1} f = p^{\kappa-1} f$, for all p, so that f is trivially not in the kernel of any Hecke operator T_p. □

Corollary 4.8. *Whenever f is a simultaneous eigenfunction of two distinct operators T_m and T_n, then f is also an eigenfunction of T_{mn} and in particular we obtain the identity*

$$\chi_f(mn) = \chi_f(m)\chi_f(n).$$

Proof. We first compute

$$T_{mn}(f) = T_m(T_n f) = T_m(\chi_f(n)n^{\kappa-1} f) = \chi_f(m)m^{\kappa-1}\chi_f(n)n^{\kappa-1} f,$$

whence f is indeed an eigenfunction of T_{mn}. From the structure theorem, we therefore obtain

$$T_{mn}(f) = \chi_f(mn)(mn)^{\kappa-1} f,$$

where we note that the same weight κ appears in both computations, due to the fact that the weight κ depends only on f and not on the Hecke operator. The result follows by comparing the two equalities above. □

Corollary 4.9. *Let f be any rational function such that $T_p f = \lambda_p f$ for some integer p, and let L be the level of f. Then*

$$T_{p+mL}(f) = \lambda_{p+mL} f$$

for every positive integer m. In addition, whenever f is an eigenfunction of a single operator T_p it is also an eigenfunction of an infinite collection of operators T_q with prime index q.

Proof. For any positive integer m, it follows from the structure theorem with $\gamma_j = e^{\frac{2\pi i \ell_j}{L}}$ that

$$\begin{aligned}
a_{(p+mL)n} &= (p+mL)^{\kappa-1} n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, \gamma_j^{(p+mL)n} \\
&= (p+mL)^{\kappa-1} n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \, \gamma_j^{pn} \\
&= (p+mL)^{\kappa-1} n^{\kappa-1} \chi_f(p) \sum_{j=1}^{d/\kappa} C_j \gamma_j^{n} \\
&= (p+mL)^{\kappa-1} \chi_f(p+mL) n^{\kappa-1} \sum_{j=1}^{d/\kappa} C_j \gamma_j^{n} \\
&= (p+mL)^{\kappa-1} \chi_f(p+mL) a_n,
\end{aligned}$$

where the third equality is part (i) of Corollary 4.7, the fourth equality is part (ii) of Corollary 4.7, and the last equality simply uses the structure theorem for eigenfunctions. Thus, by definition, we arrive at $T_{p+mL}(f) = \lambda_{p+mL} f$.

The second claim follows from Dirichlet's theorem on primes in non-trivial arithmetic progressions (see Knapp [?, p. 189]), once we know that $(p, L) = 1$ from Corollary 3.15. □

The latter result links the index p of the operator with the level L of the eigenfunction in a strong way. This connection makes L a natural candidate for grading the eigenspaces of T_p, a task we take up in the following section.

We conclude this section by showing that χ_f is in fact the real quadratic character mod L if we know that the real rational function f is a simultaneous eigenfunction of sufficiently many Hecke operators.

Theorem 4.10. *Let $L > 1$ be a given integer.*

(i) *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a real rational function of level L with $a_0 = 0$, and $f(x)$ is a simultaneous eigenfunction of the operators $T_2, T_3, \ldots, T_L$ (i.e. $T_m f = \chi_f(m) m^{\kappa-1} f$ for every $m = 2, \ldots, L$). Then χ_f is the real quadratic character mod L, f is in fact a simultaneous eigenfunction of all the Hecke operators T_m, and in addition we must have*

$$f(x) = a_1 \sum_{n=0}^{\infty} \chi_f(n) n^{\kappa-1} x^n.$$

It is worthwhile noting that f can also be written as

$$f(x) = a_1\ (x\partial_x)^{\kappa-1} \left(\frac{\sum_{j=1}^{L-1} \chi_f(j) x^j}{1 - x^L} \right).$$

Under the same hypothesis, except with $a_0 \neq 0$, we conclude that χ_f is the identity character and $f(x) = \frac{a_0}{1-x}$.

(ii) *Conversely, given any (real or complex) character χ mod L, and any positive integer κ, the rational function*

$$f(x) = \sum_{n=0}^{\infty} \chi(n) n^{\kappa-1} x^n$$

satisfies $T_p f = \chi(p) p^{\kappa-1} f$ for every p.

Remark 4.11. In other words, the previous theorem tells us that when we restrict a level L rational function f to be a simultaneous eigenfunction of the first L Hecke operators, the function f must lie in the 1-dimensional vector space generated by the given function.

Proof. To prove (i), we start with Corollary 4.7(ii), from which we know that the sequence of real values $\{\chi_f(1), \chi_f(2), \chi_f(3), \ldots, \chi_f(L)\}$ extends to all of $\mathbb{N}$ by the periodicity of χ_f mod L. We also know from Corollary 4.8 that these values are multiplicative. Thus we have a real character mod L. The fact that f is a simultaneous eigenfunction of all the Hecke operators follows from the previous Corollary 4.9, which tells us that it suffices to only consider those Hecke operators T_p with p less than or equal to the level of f.

Now let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ be an eigenfunction satisfying the hypothesis. Then $a_p = \chi_f(p)\, p^{\kappa-1} a_1$ for every $p = 2, \ldots, L$. Thus for $n > L$, we have $a_n = a_{p+jL} = \chi_f(p + jL)(p + jL)^{\kappa-1} a_1$ by Corollary 4.9, and so

$$f(x) = a_1 \sum_{n=1}^{\infty} \chi_f(n)\, n^{\kappa-1} x^n.$$

In the case that $a_0 \neq 0$, we use the fact that $\lambda_p a_0 = a_0$ for each $2 \leq p \leq L$ to get $1 = \lambda_p = \chi_f(p)p^{\kappa-1}$. Thus $\kappa = 1$ and $\chi_f(p) = 1$ for each such p. By the periodicity of χ_f, we obtain $\chi_f(n) = 1$ for all positive integers n, and therefore

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} \chi_f(n) a_1 x^n \\ &= a_0 + a_1 \sum_{n=1}^{\infty} x^n = \frac{a_0 + (a_1 - a_0)x}{1 - x} = \frac{a_0}{1 - x}, \end{aligned}$$

since $a_1 = a_0$, a conclusion that follows from the fact that the degree of the numerator must be smaller than the degree of the denominator ($f \in \mathcal{R}$).

To prove (ii), we just compute

$$\begin{aligned} T_p f(x) &= \sum_{n=1}^{\infty} \chi(pn)(pn)^{\kappa-1} x^n \\ &= \chi(p)p^{\kappa-1} \sum_{n=1}^{\infty} \chi(n) n^{\kappa-1} x^n \\ &= \chi(p)p^{\kappa-1} f(x). \end{aligned}$$

Finally, we apply the weight-raising operator $(x\partial_x)$ to get

$$f(x) = (x\partial_x)^{\kappa-1}\Big(\sum_{n=1}^{\infty} \chi(n) x^n\Big).$$

Thus the second representation of $f(x)$ follows from the identity

$$\begin{aligned} \sum_{n=1}^{\infty} \chi(n) x^n &= \sum_{j=1}^{L-1} \sum_{m=0}^{\infty} \chi(j + mL) x^{j+mL} \\ &= \sum_{j=1}^{L-1} \chi(j) x^j \sum_{m=0}^{\infty} x^{mL} \\ &= \frac{\sum_{j=1}^{L-1} \chi(j) x^j}{1 - x^L}, \end{aligned}$$

where we used the property $\chi(j + mL) = \chi(j)$. □

Example 4.12. We give an example of some rational functions of level 7 that are simultaneous eigenfunctions of exactly two Hecke operators, but not of all of them. Due to the periodicity property $T_{p+mL}(f) = \lambda_{p+mL} f$ of Corollary 4.9 it suffices to consider only Hecke operators T_p with $2 \leq p \leq 7$. This illustrates the interesting fact that although the values χ_f are multiplicative, they can be restricted away from actually being the full character mod L, and we thus get "partial" characters.

Let $f(x) = \frac{x+x^2+x^4}{1-x^7}$, and let $g(x) = \frac{x^3+x^5+x^6}{1-x^7}$. It is clear that both f and g are of weight 1 and level 7, since the poles of each function are in fact all the distinct 7'th roots of unity. Furthermore, we have

$$f(x) = x + x^2 + x^4 + x^8 + x^9 + x^{11} + x^{15} + x^{16} + x^{18} + \cdots$$

and

$$g(x) = x^3 + x^5 + x^6 + x^{10} + x^{12} + x^{13} + x^{17} + x^{19} + x^{20} + \cdots$$

From the power series, it is trivial to check that $T_2 f = f$, $T_4 f = f$, $T_2 g = g$, and $T_4 g = g$. Thus in this example $\chi_f(2) = \chi_f(4) = 1$ and $\chi_g(2) = \chi_g(4) = 1$, but $\chi_f(p)$ is not even defined for other values of p mod 7. For all other p mod 7, it is easy to see from their Taylor series that the functions f and g are not eigenfunctions of any other T_p. We note that the numerators of eigenfunctions are in general non-trivial polynomials and it is an interesting (and in general difficult) problem to compute them. Furthermore this example illustrates the involution property from Theorem 3.13. Indeed, $f(\frac{1}{x}) = -g(x)$.

5. A DECOMPOSITION INTO FINITE DIMENSIONAL EIGENSPACES

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{R}_{qp}$ be any real rational function in $\mathcal{R}$ whose poles are roots of unity. In this section we first show that f lies in the real span of some very simple Hecke eigenfunctions, of the same level L. We then define some finite dimensional vector spaces of eigenfunctions that have fixed weight and level, again by analogy with automorphic forms.

Theorem 5.1. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be any rational function in $\mathcal{R}_{qp}$. Then f lies in the real span of the following eigenfunctions of T_{L+1}, each eigenfunction having level L:*

$$(x\partial_x)^k \left(\frac{x^j}{1-x^L} \right),$$

for all non-negative integers k and $0 \leq j < L$.

Proof. From the standard Theorem 2.1 we know that a_n is a quasi-polynomial in n. That is, we have $a_n = \sum_{j=1}^{L} n^{m_j} b_j(n)$, where the b_j's are periodic, real-valued functions of n. Let L be the least common multiple of all the periods of the b_j's. We first observe that for any infinite periodic sequence $b(n)$ of period L, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)x^n &= \frac{b(0)}{1-x^L} + \frac{b(1)x}{1-x^L} + \cdots + \frac{b(L-1)x^{L-1}}{1-x^L} \\ &= \frac{b(0)+b(1)x+\cdots+b(L-1)x^{L-1}}{1-x^L}.\end{aligned}$$

Thus, for each index j we obtain

$$\sum_{n=0}^{\infty} n^{m_j} b_j(n)x^n = (x\partial_x)^{m_j}\left(\frac{b_j(0)+\cdots+b_j(L-1)x^{L-1}}{1-x^L}\right),$$

and consequently

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\sum_{j=1}^{L} n^{m_j} b_j(n)x^n \\ &= \sum_{j=1}^{L} (x\partial_x)^{m_j}\left(\frac{b_j(0)+\cdots+b_j(L-1)x^{L-1}}{1-x^L}\right) \\ &= \sum_{j=1}^{L}\left(b_j(0)\,(x\partial_x)^{m_j}\left(\tfrac{1}{1-x^L}\right)+\cdots+b_j(L-1)\,(x\partial_x)^{m_j}\left(\tfrac{x^{L-1}}{1-x^L}\right)\right).\end{aligned}$$

We now note that each rational function $\frac{x^j}{1-x^L}$ on the right-hand of the last equation is an eigenfunction of T_{L+1}, which follows easily from its Taylor series. By Lemma 3.2, the same statement holds for $(x\partial_x)^{m_j}\left(\frac{x^{L-1}}{1-x^L}\right)$. In conclusion, we have expressed f as a finite linear combination of real eigenfunctions of the operator T_{L+1}. □

The vector space generated by all of the eigenfunctions given in this theorem is infinite dimensional. It is natural to ask how we can decompose it into finite dimensional vector spaces, so that we can do analysis on each finite dimensional piece with greater ease. When we fix the weight κ and the level L of admissible eigenfunctions, we obtain a finite-dimensional vector space (Theorem 5.3 below) of eigenfunctions. We note that this grading is quite natural, given the structure theorem. It also plays an analogous role to the grading of the finite-dimensional vector spaces of cusp forms and Eisenstein series that arise in automorphic forms.

We now define the relevant notions that are used in the afore-mentioned grading.

Definition 5.2. We denote by

$$\mathcal{V}_{\kappa,L}(T_p)$$

the vector space of all real rational functions with fixed weight κ and fixed level L, that are eigenfunctions of the Hecke operator T_p. Given a set of integers $S = \{p_1, \dots, p_n\}$, we let

$$\mathcal{V}_{\kappa,L}(S) = \mathcal{V}_{\kappa,L}(T_{p_1}, \dots, T_{p_n})$$

denote the vector space over $\mathbb{R}$ of all real rational functions with fixed weight κ and fixed level L that are simultaneous eigenfunctions of the collection of Hecke operators $T_{p_1}, \dots, T_{p_n}$.

We remark that when $\{p_1, p_2, p_3, \dots, p_n\}$ is the set of all integers between 2 and L inclusively, the corresponding vector space of simultaneous eigenfunctions for $T_2, T_3, \dots, T_L$ is 1-dimensional, generated by the function

$$\sum_{n=0}^{\infty} \chi(n) n^{\kappa-1} x^n,$$

as we saw in Theorem 4.10 of the previous section (with χ being the real character mod L). We further define

$$\mathcal{S}_{\kappa,L}(S) = \mathcal{S}_{\kappa,L}(T_{p_1}, \dots, T_{p_n}) = \{f \in \mathcal{V}_{\kappa,L}(S) \,|\, a_0 = 0\}.$$

It is clear that $\mathcal{S}_{\kappa,L}(S)$ is a vector space over $\mathbb{R}$, and is T_{p_j}-invariant for each $p_j \in S$.

Theorem 5.3. *For a fixed weight κ and fixed level L, $\mathcal{V}_{\kappa,L}(S)$ is a finite-dimensional vector space. Considered as a vector space over $\mathbb{C}$, it has the basis*

$$\{f_{\chi_1}, f_{\chi_2}, \dots, f_{\chi_{\phi(L)}}\},$$

where $\phi(L)$ is the Euler ϕ-function of L, and where

$$f_\chi(x) = \sum_{n=0}^{\infty} \chi(n) n^{\kappa-1} x^n.$$

Proof. We make the easy observation that for each fixed denominator $B(x)$ of an eigenfunction there are at most finitely many possible numerators, each numerator being an eigenfunction of the corresponding matrix defined in Section 3. Note that the degree of $B(x)$ must be less than or equal to κL, by the structure theorem. Since there are at most finitely many possible denominators of degree $\le \kappa L$ whose roots are L'th roots of unity, we conclude that there are at most finitely many linearly independent eigenfunctions of level L and weight κ. The second statement concerning the basis is tantamount to doing Fourier analysis on the finite group $\mathbb{Z}/L\mathbb{Z}$ (see Knapp [?], for example), from which we know that we can expand every periodic function into a complex linear combination of the $\phi(L)$ characters χ mod L. □

The dimensions of the vector spaces defined here offer challenging combinatorial problems. Indeed, it is not clear how to compute $\dim(\mathcal{V}_{1,L}(T_2))$ even in the case when L is prime (and involves the Artin conjecture for primitive roots mod L).

Example 5.4. We note that the space $\mathcal{V}_{1,L}$ always has the eigenfunction

$$f(x) = \frac{2+x}{1 - 2\cos(\frac{2\pi}{L})x + x^2},$$

of weight 1 and level L. The Taylor coefficients of $f = \sum_{n=0}^{\infty} a_n x^n$ are given by $a_n = e^{\frac{2\pi i n}{L}} + e^{\frac{-2\pi i n}{L}} = 2\cos(\frac{2\pi n}{L})$. Equivalently, the Taylor coefficients satisfy the linear recurrence

$$a_n = 2\cos(\tfrac{2\pi}{L})a_{n-1} - a_{n-2}.$$

Indeed, we have $T_{L-1}f = f$. We single out this class of eigenfunctions for being eigenfunctions of T_{L-1}, but of no other Hecke operator T_p with $p < L$. To wit,

$$a_{pn} = e^{\frac{2\pi i p n}{L}} + e^{\frac{-2\pi i p n}{L}} = \lambda_p a_n$$

for some λ_p, only when $p = L-1$ (assuming $p < L$).

6. Simultaneous eigenfunctions

Consider the algebra $\mathfrak{H} = \{T_p \mid p \in \mathbb{N},\ p \geq 2\}$ of all Hecke operators acting on rational functions. We are interested in the intersection of the spectra and in the set of common eigenfunctions. Recall that $\lambda = 1$ belongs to $\mathrm{spec}(T_p)$ for every p, and $\frac{1}{1-x}$ is a common eigenfunction for the full algebra $\mathfrak{H}$. In this section we give a precise description of all possible common eigenvalues and eigenfunctions for the whole algebra $\mathfrak{H}$.

We first show that the simultaneous spectrum of any two Hecke operators T_m and T_n is trivialized, if m and n are relatively prime. However, it turns out that simultaneous eigenspaces are in general non-trivial.

Lemma 6.1. *For any two relatively prime integers m and n, we have*

$$\mathrm{spec}(T_m) \cap \mathrm{spec}(T_n) = \{0, \pm 1\}.$$

Proof. We already know that 0, 1 and -1 are always contained in the spectrum of T_p for every p, from Lemma 3.6. Thus we only need to show the other inclusion. In fact, if $\lambda \in \mathrm{spec}(T_m) \cap \mathrm{spec}(T_n)$, then either $\lambda = 0$ or Theorem 3.18 implies $\lambda = \pm m^k$ and $\lambda = \pm n^\ell$ for some $k, \ell \in \mathbb{N}$. But this implies $k = \ell = 0$ since $(m,n) = 1$, that is, $\lambda = \pm 1$. □

Lemma 6.2. *Let f be such that $T_p f = \lambda f$ for some p, $\lambda \neq 0$. If $f(x) = x^m \tilde{f}(x)$ for some positive integer m and some $\tilde{f}$ with $\tilde{f}(0) \neq 0$, then $p \nmid m$.*

Proof. Using (3.8) we get

$$\lambda\, x^{pm} \tilde{f}(x^p) = \frac{1}{p} \sum_{j=0}^{p-1} e^{\frac{2\pi i m}{p} j} x^m \tilde{f}(e^{\frac{2\pi i}{p} j} x)$$

which implies

$$\lambda\, x^{pm-m} \tilde{f}(x^p) = \frac{1}{p}\sum_{j=0}^{p-1} e^{\frac{2\pi i m}{p} j} \tilde{f}(e^{\frac{2\pi i}{p} j} x).$$

Evaluating at $x = 0$ gives

$$0 = \frac{1}{p}\sum_{j=0}^{p-1} e^{\frac{2\pi i m}{p} j} \tilde{f}(0)$$

and therefore $p \nmid m$. □

The following theorem completely describes the set of simultaneous eigenfunctions for the algebra $\mathfrak{H}$ of all Hecke operators on rational functions. Among others, our description reveals the importance of the operator $x\partial_x$.

Theorem 6.3 (Simultaneous Eigenfunctions). *Let f be a simultaneous eigenfunction of $\mathfrak{H}$ such that f is not in the kernel of T_p for any p. That is, let f be a rational function with the property that for every p there is a $\lambda_p \neq 0$ such that $T_p f = \lambda_p f$. Then $L = 1$, and*

$$f(x) = C(x\partial_x)^k \left(\frac{1}{1-x}\right)$$

for some $k \in \mathbb{N}$ and $C \in \mathbb{C}$. Consequently, $\lambda_p = p^k$.

Proof. We consider the two cases $f(0) \neq 0$ and $f(0) = 0$ separately. If $f(0) \neq 0$, then we plug in $x = 0$ into (3.8) and get $\lambda_p = 1$, so $T_p f = f$ for every p. If we write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then we must have $a_p = a_1$ for every p, so

$$\begin{aligned} f(x) &= a_0 + a_1 x \sum_{n=0}^{\infty} x^n \\ &= a_0 + \frac{a_1 x}{1-x} = \frac{a_0 + (a_1 - a_0)x}{1-x} \end{aligned}$$

which implies $a_1 = a_0$, in other words, $f(x) = a_0(\frac{1}{1-x})$ as claimed.

Suppose now $f(0) = 0$ and write $f(x) = x^m \tilde{f}(x)$ with $\tilde{f}(0) \neq 0$. By Lemma 6.2, f cannot be an eigenfunction of T_m. Since f is assumed to be a common eigenfunction of $\mathfrak{H}$, it follows that $m = 1$. The structure theorem shows that if $T_p f = \lambda_p f$ with $\lambda_p \neq 0$, then $\lambda_p = \chi_f(p)\, p^k$ for some integer k and $\chi_f(p) = \lambda_p/|\lambda_p|$. Write $f(x) = \sum_{n=1}^{\infty} a_n x^n$ with $a_1 \neq 0$. Then

$a_p = \chi_f(p)\, p^k a_1$ holds for every p, thus

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \chi_f(n)\, n^k a_1 x^n \\ &= a_1 \sum_{n=1}^{\infty} n^k x^n \\ &= a_1 (x\partial_x)^k \left(\frac{1}{1-x}\right) \end{aligned}$$

since $\chi_f(n) = 1$ for every n by Corollary 4.7(iii). □

We denote by $\mathcal{V}$ the vector space over $\mathbb{C}$ spanned by the functions

$$\phi_k(x) = (x\partial_x)^k \left(\frac{1}{1-x}\right) \tag{6.4}$$

for $k \in \mathbb{N}$. That is,

$$\mathcal{V} = \operatorname{span}_{\mathbb{C}}\{\phi_0, \phi_1, \phi_2, \dots\}. \tag{6.5}$$

Remark 6.6. Notice that although every ϕ_k is a simultaneous eigenfunction of $\mathfrak{H}$, the sum of two of them $\phi_i + \phi_j$ is not, simply because their weights are different ($i \neq j$). However, if $f = \sum c_k \phi_k \in \mathcal{V}$, then $T_p f = \sum c_k T_p \phi_k = \sum c_k p^k \phi_k \in \mathcal{V}$. Thus the space $\mathcal{V}$ is T_p-invariant for every p.

Every function ϕ_k can obviously be written as

$$\phi_k(x) = \frac{A_k(x)}{(1-x)^{k+1}}$$

where

$$A_k(x) = (1-x)^{k+1} \sum_{n=1}^{\infty} n^k x^n.$$

It is easy to check the identity

$$A(x) = \sum_{\ell=0}^{k} S(k,\ell)\, \ell!\, x^\ell (1-x)^{k-\ell} \tag{6.7}$$

where $S(k,\ell)$ are the well-known Stirling numbers of the second kind.

The polynomials A_k are known as Eulerian polynomials, cf. [?]. Here are the first few:

$$\begin{aligned} &A_1(x) = x, \quad A_2(x) = x + x^2, \\ &A_3(x) = x + 4x^2 + x^3, \\ &A_4(x) = x + 11x^2 + 11x^3 + x^4, \\ &A_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5, \\ &A_6(x) = x + 57x^2 + 302x^3 + 302x^4 + 57x^5 + x^6. \end{aligned}$$

Lemma 6.8. *The family of functions $\{\phi_k \mid k \in \mathbb{N}\}$ from (6.4) is a linearly independent system. In particular, $\mathcal{V}$ is an infinite dimensional vector space.*

Proof. For any given integer n we will prove that the functions $\phi_0, \dots, \phi_n$ are linearly independent. For arbitrary constants $c_0, \dots, c_n$, we have

$$\begin{aligned}\sum_{k=0}^{n} c_k \phi_k(x) &= \sum_{k=0}^{n} c_k \frac{A_k(x)}{(1-x)^{k+1}} \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{k} c_k \alpha_\ell \frac{x^\ell}{(1-x)^{\ell+1}}\end{aligned}$$

(where $\alpha_\ell = S(k,\ell)\,\ell!$ from (6.7))

$$\begin{aligned} &= \sum_{\ell=0}^{n}\sum_{k=\ell}^{n} c_k \alpha_\ell \frac{x^\ell}{(1-x)^{\ell+1}} \\ &= \frac{1}{(1-x)^{n+1}} \sum_{\ell=0}^{n} \alpha_\ell \Big(\sum_{k=\ell}^{n} c_k \Big) x^\ell (1-x)^{n-\ell}.\end{aligned}$$

If $\sum_{k=0}^{n} c_k \phi_k(x) = 0$, then $\alpha_\ell \sum_{k=\ell}^{n} c_k = 0$ for every ℓ; that is,

$$\begin{pmatrix} \alpha_0 & \alpha_0 & \alpha_0 & \dots & \alpha_0 \\ 0 & \alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \vdots & \ddots & \alpha_2 & \dots & \alpha_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \alpha_n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

Thus $c_0 = c_1 = \dots = c_n = 0$ since the α_ℓ are all different from zero. □

Lemma 6.9. *If $f \in \mathcal{V}$ is an eigenfunction for some T_p, then it is a simultaneous eigenfunction of the whole Hecke algebra $\mathfrak{H}$.*

Proof. Let $0 \neq f = \sum c_j \phi_j \in \mathcal{V}$ satisfy $T_p f = \chi_f(p) p^k f$ for some p and some k. We also have

$$T_p f = \sum c_j T_p \phi_j = \sum c_j p^j \phi_j$$

so that we get the identity

$$\sum c_j (p^j - \chi_f(p) p^k) \phi_j = 0.$$

The linear independence of the ϕ_j implies $c_j = 0$ for every $j \neq k$ and therefore $f = c_k \phi_k$, a simultaneous eigenfunction of $\mathfrak{H}$. □

7. A FIRST APPLICATION: TENSOR PRODUCTS OF HECKE OPERATORS AND THE RIEMANN ZETA FUNCTION

In this section we consider tensor products of Hecke operators study their spectral properties. Let $\mathcal{V}$ be the vector space introduced in (6.5) and let

$$\mathcal{H}^n = \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{n \text{ times}}.$$

For any finite set $S = \{p_1, \ldots, p_n\}$ of prime numbers with $p_j \neq p_k$ if $j \neq k$, we define the operator

$$\mathbf{T}_S = T_{p_1} \otimes \cdots \otimes T_{p_n} : \mathcal{H}^n \to \mathcal{R}(x_1, \ldots, x_n)$$

by

$$\mathbf{T}_S(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = (T_{p_1} f_1)(x_1) \cdots (T_{p_1} f_n)(x_n), \tag{7.1}$$

where every T_{p_k} is a Hecke operator and the multiplication on the right-hand side of (7.1) is the usual multiplication of rational functions.

Lemma 7.2. *An element $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^n$ is an eigenfunction of $\mathbf{T}_S$ if and only if each f_j is a simultaneous eigenfunction of the Hecke algebra $\mathfrak{H}$.*

Proof. First, let each f_j be a simultaneous eigenfunction of $\mathfrak{H}$. Thus $\chi_{f_j} = 1$ (the identity character) and we have

$$\begin{aligned} \mathbf{T}_S(f_1 \otimes \cdots \otimes f_n) &= (T_{p_1} f_1) \ldots (T_{p_1} f_n) \\ &= (p_1^{k_1} f_1) \ldots (p_n^{k_n} f_n) \\ &= (p_1^{k_1} \ldots p_n^{k_n}) f_1 \otimes \cdots \otimes f_n. \end{aligned} \tag{7.3}$$

Therefore $f_1 \otimes \cdots \otimes f_n$ is an eigenfunction of $\mathbf{T}_S$ with eigenvalue $p_1^{k_1} \ldots p_n^{k_n}$ for some integers $k_1, \ldots, k_n$.

To prove the converse we now suppose that

$$\mathbf{T}_S(f_1 \otimes \cdots \otimes f_n) = \lambda f_1 \otimes \cdots \otimes f_n$$

for some λ, and some $f_1, \ldots, f_n \in \mathcal{V}$ such that no f_j is the zero function. Then we have

$$\lambda = \left(\frac{T_{p_1} f_1(x_1)}{f_1(x_1)}\right) \cdots \left(\frac{T_{p_n} f_n(x_n)}{f_n(x_n)}\right)$$

which implies that every factor $\left(\frac{T_{p_j} f_j(x_j)}{f_j(x_j)}\right)$ must be constant. Thus, for every j, f_j is an eigenfunction of T_{p_j} and consequently a simultaneous eigenfunction of $\mathfrak{H}$ by Lemma 6.9. □

Corollary 7.4. *The spectrum of $\mathbf{T}_S$ is the set*

$$\{p_1^{k_1} \cdots p_n^{k_n} \mid k_j \in \mathbb{N} \text{ for every } j\},$$

where each eigenvalue occurs with multiplicity 1.

Proof. From (7.3) it is clear that every integer $p_1^{k_1}\cdots p_n^{k_n}$ lies in the spectrum of $\mathbf{T}_S$. On the other hand, it also follows from the proof of the previous theorem that, if λ is an eigenvalue of $\mathbf{T}_S$, then $\lambda = \lambda_1\cdots\lambda_n$ with $T_{p_j}f_j = \lambda_j f_j$ for some $f_j \in \mathcal{V}$. Lemma 6.9 then implies that f_j is a simultaneous eigenfunction, so $f_j = c_j\phi_{k_j}$ for some integer k_j and some constant c_j. Thus $\lambda = \lambda_1\cdots\lambda_n = p_1^{k_1}\cdots p_n^{k_n}$.

To prove the multiplicity 1 statement, let $g_1\otimes\cdots\otimes g_n$ be another eigenfunction for λ. Then we similarly get $g_j = c_j'\phi_{\ell_j}$ and so $\lambda = p_1^{\ell_1}\cdots p_n^{\ell_n}$. Finally, by the unique factorization theorem for $\mathbb{Z}$, it follows that $k_j = \ell_j$ for every j. Hence each g_j is a multiple of f_j and the multiplicity of λ is 1. □

Definition 7.5. For an operator A with nonnegative discrete spectrum, let

$$\zeta_A(s) = \sum_{\lambda\in\operatorname{spec}_+(A)} \frac{m(\lambda)}{\lambda^s},$$

where $\operatorname{spec}_+(A) = \operatorname{spec}(A)\setminus\{0\}$ is the set of positive eigenvalues of A and $m(\lambda)$ is their multiplicity.

Example 7.6. For a Hecke operator T_p acting on $\mathcal{V}$ we have

$$\zeta_{T_p}(s) = \sum_{j=0}^{\infty} \frac{1}{p^{js}} = \frac{1}{1-p^{-s}}.$$

The following theorem is an interesting application of the spectral properties of the Hecke operators acting on the vector space $\mathcal{V}$ spanned by the simultaneous eigenfunctions of $\mathfrak{H}$. More precisely, the spectrum of $\mathbf{T}_S$ forges a natural link to the Riemann zeta function.

Theorem 7.7 (Euler product). *Let $S = \{p_1,\dots,p_n\}$ be a set of prime numbers. Then*

$$\zeta_{\mathbf{T}_S}(s) = \zeta_{T_{p_1}}(s)\cdots\zeta_{T_{p_n}}(s) = \prod_{p\in S}\frac{1}{1-p^{-s}}.$$

In other words, the zeta function of the operator $\mathbf{T}_S$ acting on $\mathcal{H}^n$ (cf. (7.1)) is a finite Euler product of the Riemann zeta function $\zeta(s)$.

Proof. From the definition of the zeta function of $\mathbf{T}_S$ and because the multiplicity $m(\lambda) = 1$ for all $\lambda \in \operatorname{spec}(\mathbf{T}_S)$ (by Corollary 7.4), we have

$$\begin{aligned}\zeta_{\mathbf{T}_S}(s) &= \sum_{\lambda \in \operatorname{spec}(\mathbf{T}_S)} \frac{1}{\lambda^s} \\ &= \sum_{k_1,\dots,k_n \in \mathbb{N}} \frac{1}{(p_1^{k_1} \cdots p_n^{k_n})^s} \\ &= \sum_{k_1=0}^{\infty} \left(\frac{1}{p_1^s}\right)^{k_1} \cdots \sum_{k_n=0}^{\infty} \left(\frac{1}{p_n^s}\right)^{k_n} \\ &= \prod_{p \in S} \frac{1}{1-p^{-s}}.\end{aligned}$$

□

Let $\mathcal{H}^\infty$ be the space of products $\mathbf{f} = f_1 \otimes f_2 \otimes \cdots$, where $\{f_n\}_{n\in\mathbb{N}}$ is an infinite sequence of rational functions with the following properties:

(1) There is a finite set $I \subset \mathbb{N}$ such that $f_j \in \mathcal{V}$ for every $j \in I$.
(2) $f_j = 1$ for every $j \in \mathbb{N} \setminus I$.

For $\mathbf{f} \in \mathcal{H}^\infty$ we define the operator $\mathbf{T}$ by

$$\mathbf{Tf} = (T_{q_1} f_{i_1}) \otimes \cdots \otimes (T_{q_m} f_{i_m}), \tag{7.8}$$

where $I = \{i_1, \dots, i_m\}$ is the finite set of positive integers associated to $\mathbf{f}$, and for each k the number $q_k = p_{i_k}$ is the i_k-th prime number. Notice that similarly to $\mathbf{T}_S$, the operator $\mathbf{T}$ maps tensor products of rational functions into rational functions in several variables.

Given $\mathbf{f} \in \mathcal{H}^\infty$ with corresponding index set $\{i_1, \dots, i_m\}$, we let $S_\mathbf{f}$ be the set of prime numbers $\{p_{i_1}, \dots, p_{i_m}\}$ from (7.8). Clearly,

$$\mathbf{Tf} = \mathbf{T}_{S_\mathbf{f}}(f_{i_1} \otimes \cdots \otimes f_{i_m}),$$

where $\mathbf{T}_{S_\mathbf{f}}$ is the operator from (7.1). Therefore Lemma 7.2 and Corollary 7.4 apply verbatim to the operator $\mathbf{T}$.

Since $\mathbb{Z}$ is a unique factorization domain, Corollary 7.4 implies that the spectrum of $\mathbf{T}$ is exactly the set of all positive integers and each eigenvalue has multiplicity one. This leads to the following result.

Theorem 7.9 (Riemann zeta function). *The spectral zeta function of the operator* $\mathbf{T}$ *on* $\mathcal{H}^\infty$ *satisfies*

$$\zeta_\mathbf{T}(s) = \zeta(s),$$

where $\zeta(s)$ *is the Riemann zeta function.*

8. A second application: completely multiplicative coefficients

Mordell and Hecke were motivated to study simultaneous eigenforms in the context of modular forms in order to classify those forms that have multiplicative coefficients, after the discovery that the coefficients of the discriminant function $\Delta(\tau)$ indeed have those properties. In the same spirit, we now give a complete classification of those rational functions that have completely multiplicative coefficients, since our Hecke operators by definition have a completely multiplicative action on rational functions: $T_m T_n f = T_{mn} f$, for all m, n.

We use the vector spaces $\mathcal{V}$ and $\mathcal{V}_{\kappa,L}(T_2,\dots,T_L)$ of the simultaneous eigenfunctions to characterize all rational functions $f = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{R}$ whose coefficients are completely multiplicative - i.e. such that $a_{mn} = a_m a_n$ for all m, n.

Theorem 8.1. *A rational function $f = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{R}$ has completely multiplicative coefficients if and only if it has the following form:*

(i) *If $a_0 = 0$, then there exist positive integers κ and $L \geq 1$ such that*

$$f(x) = a_1\ (x\partial_x)^{\kappa-1}\left(\frac{\sum_{j=1}^{L-1}\chi_f(j)x^j}{1-x^L}\right),$$

where χ_f is the real quadratic character mod L.

(ii) *If $a_0 \neq 0$, then*

$$f(x) = \frac{a_0}{1-x}.$$

Proof. In both cases, we observe that if we *fix* m, then the assumption that $a_{nm} = a_n a_m$ holds for all n can be regarded as telling us that the coefficients a_k give us an eigenfunction of T_m, with eigenvalue $\lambda = a_n$. Furthermore, this condition is satisfied by *all* m, hence making the function $f = \sum_{n=0}^{\infty} a_n x^n$ a simultaneous eigenfunction of all the Hecke operators.

We first treat the case $L > 1$. Here the hypotheses of Theorem 4.10(i) are satisfied by the remarks made in the previous paragraph concerning simultaneous eigenfunctions, giving us the eigenfunctions in part (i) of the theorem.

For the case $L = 1$, it is trivially true that $\chi_f(p) = 1$ for all p. The conclusion of Corollary 4.7(iii) now implies that f is not in the kernel of any of the Hecke operators. Therefore the hypotheses of Theorem 6.3 are satisfied, giving us the eigenfunctions in part (i) of the theorem, where in this case $L = 1$ and $\chi_f(j)$ is the trivial character. □

9. Appendix: Explicit examples of eigenfunctions

For illustration purposes, we now focus on the finite dimensional vector spaces $\mathcal{S}_{\kappa,L}(T_2)$ of eigenfunctions with constant term equal to zero ($a_0 = 0$), in the range $\kappa L \leq 6$. This class is large enough to already exhibit some of implying

$$\lambda c_j = \sum_{k=1}^{2j} (-1)^k \alpha_{2j-k} c_k$$

for every $j = 1, \dots, d-1$. Setting $c_j := 0$ for $j > d-1$ and $\alpha_k := 0$ for $k > d$ we get that λ is an eigenvalue of the matrix $\mathfrak{B} = ((-1)^k \alpha_{2j-k})_{j,k=1,\dots,d-1}$. We now give the explicit form of $\mathfrak{B}$ for various values of d.

- $d = 3$:

$$\mathfrak{B} = \begin{pmatrix} -\alpha_1 & 1 \\ 1 & -\alpha_1 \end{pmatrix}$$

  since $\alpha_3 = -1$ and $\alpha_1 + \alpha_2 = 0$.
- $d = 4$:

$$\mathfrak{B} = \begin{pmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_1 & \alpha_2 & -\alpha_1 \\ 0 & 1 & -\alpha_1 \end{pmatrix}$$

  since $\alpha_4 = 1$ and $\alpha_1 - \alpha_3 = 0$.
- $d = 5$:

$$\mathfrak{B} = \begin{pmatrix} -\alpha_1 & 1 & 0 & 0 \\ \alpha_2 & \alpha_2 & -\alpha_1 & 1 \\ 1 & -\alpha_1 & \alpha_2 & \alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \end{pmatrix}$$

  since $\alpha_5 = -1$ and $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = 0$.
- $d = 6$:

$$\mathfrak{B} = \begin{pmatrix} -\alpha_1 & 1 & 0 & 0 & 0 \\ -\alpha_3 & \alpha_2 & -\alpha_1 & 1 & 0 \\ -\alpha_1 & \alpha_2 & -\alpha_3 & \alpha_2 & -\alpha_1 \\ 0 & 1 & -\alpha_1 & \alpha_2 & -\alpha_3 \\ 0 & 0 & 0 & 1 & -\alpha_1 \end{pmatrix}$$

  since $\alpha_6 = 1$ and $\alpha_1 - \alpha_5 = \alpha_2 - \alpha_4 = 0$.

We now list a basis of eigenfunctions of T_2 together with their corresponding eigenvalues λ for every given degree d.

- $d = 2$:

$$f_{2,1}(x) = \frac{x}{(1-x)^2}, \qquad \lambda = 2,$$
$$f_{2,2}(x) = \frac{x}{1+x+x^2}, \qquad \lambda = -1.$$

- $d = 3$:

$$f_{3,1}(x) = \frac{x+x^2}{(1-x)^3}, \qquad \lambda = 4,$$
$$f_{3,2}(x) = \frac{x+x^2}{1-x^3}, \qquad \lambda = 1.$$

- $d = 4$:

$$f_{4,1}(x) = \frac{x + 4x^2 + x^3}{(1-x)^4}, \qquad \lambda = 8,$$

$$f_{4,2}(x) = \frac{x - x^3}{(1 + x + x^2)^2}, \qquad \lambda = -2,$$

$$f_{4,3}(x) = \frac{x + 4x^2 + x^3}{(1 + x + x^2)^2}, \qquad \lambda = 2,$$

$$f_{4,4}(x) = \frac{x - x^3}{1 + x + x^2 + x^3 + x^4}, \qquad \lambda = -1.$$

- $d = 5$:

$$f_{5,1}(x) = \frac{x + 11x^2 + 11x^3 + x^4}{(1-x)^5}, \qquad \lambda = 16,$$

$$f_{5,2}(x) = \frac{x + x^2 + x^3 + x^4}{1 - x^5}, \qquad \lambda = 1.$$

- $d = 6$:

$$f_{6,1}(x) = \frac{x + 26x^2 + 66x^3 + 26x^4 + x^5}{(1-x)^6}, \qquad \lambda = 32,$$

$$f_{6,2}(x) = \frac{x - x^2 - 6x^3 - x^4 + x^5}{(1 + x + x^2)^3}, \qquad \lambda = -4,$$

$$f_{6,3}(x) = \frac{x + 7x^2 + 7x^4 + x^5}{(1 + x + x^2)^3}, \qquad \lambda = 4,$$

$$f_{6,4}(x) = \frac{2x + 2x^2 + x^3}{(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)}, \qquad \lambda = -1,$$

$$f_{6,5}(x) = \frac{x + 3x^2 - 3x^4 - x^5}{(1 + x + x^2)(1 + x + x^2 + x^3 + x^4)}, \qquad \lambda = 1,$$

$$f_{6,6}(x) = \frac{x + 2x^2 + x^3 + 2x^4 + x^5}{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}, \qquad \lambda = 1.$$

We now reorganize the eigenfunctions according to their level L, and give an explicit basis for each $\mathcal{S}_{\kappa,L}(T_2)$ in the range $\kappa L \leq 6$. Notice that by Corollary 3.15 we must have $(p, L) = (2, L) = 1$. Thus all of the vector spaces $\mathcal{S}_{\kappa,2m}(T_2)$ are empty (for all m).

- $L = 1$:

$$\mathcal{S}_{\kappa,1}(T_2) = \operatorname{span}\{f_{\kappa,1}\}$$

  for $\kappa = 1, \ldots, 6$.

- $L = 3$:

$$\mathcal{S}_{1,3}(T_2) = \operatorname{span}\{f_{2,2}, f_{3,2}\}, \quad \mathcal{S}_{2,3}(T_2) = \operatorname{span}\{f_{4,2}, f_{4,3}\}$$

- $L = 5$:

$$\mathcal{S}_{1,5}(T_2) = \operatorname{span}\{f_{4,4}, f_{5,2}\}$$

Note that the eigenfunctions that do not appear in the vector spaces above are $f_{6,2}, f_{6,3} \in \mathcal{S}_{3,3}(T_2)$, $f_{6,6} \in \mathcal{S}_{1,7}(T_2)$, and $f_{6,4}, f_{6,5} \in \mathcal{S}_{1,15}(T_2)$. In addition, we give a basis for $\mathcal{S}_{1,7}(T_2)$ by using the eigenfunctions from Example 4.12:

$$\mathcal{S}_{1,7}(T_2) = \operatorname{span}\left\{ \tfrac{x+x^2+x^4}{1-x^7}, \tfrac{x^3+x^5+x^6}{1-x^7} \right\}.$$

We end with a conjecture about the numerators of all eigenfunctions. We can always write f with a denominator of the form $\prod(1-x^{m_j})$, by multiplying $A(x)$ and $B(x)$ by some common factor if necessary. We now assume that f has this form.

Conjecture. Let $f(x) = A(x)/B(x)$ be an eigenfunction of at least one Hecke operator, written in the form given above. Then the absolute value of the nonzero coefficients of $A(x)$ form a unimodal sequence.

Throughout this section we have made extensive use of MAPLE (version 6) to compute the eigenfunctions of T_2 by finding the eigenvectors of the corresponding matrix $\mathfrak{B}$ associated to the given degree.

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