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Counting Affine Roots of Polynomial Systems via Pointed Newton Polytopes

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We give a new upper bound on the number of isolated roots of a polynomial system. Unlike many previous bounds, our bound can also be restricted to different open subsets of affine space. Our methods give significantly sharper bounds than the classical Bézout theorems and further generalize the mixed volume root counts discovered in the late 1970s. We also give a complete combinatorial classification of the subsets of coefficients whose genericity guarantees that our bound is actually an exact root count in affine space. Our results hold over any algebraically closed field.

1. INTRODUCTION

Root counting for polynomial systems can be reduced almost completely to a convex geometric computation. This approach began with the BKK bound [Kus75, Ber75, Kus76, Kho78] and was further refined in [Roj94]. We will carry these techniques a step further.

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Our results seek to give the tightest possible upper bound on the number of isolated roots lying in the complement of an arbitrary union of coordinate hyperplanes. One should note that the BKK bound corresponds to the case where one works in the complement of the union of *all* the coordinate hyperplanes. Just as in the BKK bound and its recent generalizations, the bounds we derive depend solely on which monomial terms are allowed to occur in the given polynomial system. Given only this monomial term information, our bounds are best possible for certain families of polynomial systems. In particular, we obtain previously unknown generically exact root counts in affine space. We now review some notation necessary to state our results.

Let K be any algebraically closed field (positive characteristic is allowed) and let $f_1, \ldots, f_n$ be polynomials in the ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We call $F := (f_1, \ldots, f_n)$ an *$n \times n$ polynomial system over K.* For any $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ let x^e denote the Laurent monomial $x_1^{e_1} \cdots x_n^{e_n}$. The *support* of a polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the set of $e \in \mathbb{Z}^n$ such that the coefficient of the x^e term in f is nonzero. For instance, with $n = 3$, the polynomial $3x_1 - x_2^{-2}x_3^5$ has support $\{(1, 0, 0), (0, -2, 5)\}$. The support of an n-tuple of polynomials is simply the n-tuple whose ith coordinate is the support set of the ith polynomial. We also let $\mathcal{M}(E)$ denote the n-dimensional mixed volume [Grü69, Roj94, Sch94, HS95, EC95, DGH96, VGC96] of the convex hulls of $E_1, \ldots, E_n$, whenever $E := (E_1, \ldots, E_n)$ is an n-tuple of nonempty finite subsets of $\mathbb{Z}^n$. When the support of f_i is contained in E_i for all $i \in \{1, \ldots, n\}$, we simply say that the *support of F is contained in E.* The convex hulls of the supports of the f_i are commonly known as the *Newton polytopes* of F. For further background on the theory and applications of mixed volumes and Newton polytopes we refer the reader to [Grü69, Roj94, GKZ94, HS95, Sch94, EC95, DGH96, LW96, VGC96, HS96, Roj96].

Remark 1. When one fixes the support of F, it is not always true that the isolated roots of F avoid a given coordinate hyperplane for a generic choice of the coefficients. (We give examples of this phenomenon in the next section.) Hence the ordinary BKK bound is insufficient for accurate root-counting in affine space.

For any $I \subseteq \{1, \ldots, n\}$ (possibly empty) define $\mathrm{Hyper}(I) \subset K^n$ to be the union of coordinate hyperplanes $\cup_{j \in I}\{x \in K^n \mid x_j = 0\}$ and let $\mathrm{Lin}(I) \subseteq \mathbb{R}^n$ be the coordinate subspace generated by the subset $\{\hat{e}_j \mid j \in I\}$ of the standard basis. (In particular, $\mathrm{Hyper}(\emptyset) := \emptyset$ and $\mathrm{Lin}(\emptyset) := \mathbf{O}$.) For any $a_1, \ldots, a_n \in \mathbb{Z}^n$ let a be the matrix with rows $a_1, \ldots, a_n$ and define $a \cup E := (\{a_1\} \cup E_1, \ldots, \{a_n\} \cup E_n)$. We will prove the following upper bound on the number of isolated roots of a polynomial system in certain open subsets of affine space.

AFFINE POINT THEOREM I. *Let F be an $n \times n$ polynomial system over K with support E and fix $I \subseteq \{1, . . . , n\}$. Suppose further that x_j has a negative exponent in some monomial term of F only if $j \in I$. Then for any $a_1, . . . , a_n \in \mathbb{Z}^n \cap Lin(I)$, F has at most $\mathcal{M}(a \cup E)$ isolated roots in $K^n \backslash Hyper(I)$, counting multiplicities.*

Remark 2. Note that when $I = \{1, . . . , n\}$, we can simply pick each a_i to lie in E_i. In this way, the $I = \{1, . . . , n\}$ case is just the BKK bound over a general algebraically closed field [Dan78, Roj94]. So in this case we obtain an upper bound on the number of isolated roots whose coordinate are all *non*zero.

Remark 3. Note that the $I = \emptyset$ case is quite important: Each a_i must be equal to the origin $\mathbf{O}$ and we obtain an upper bound on the number of isolated roots in K^n. Setting $(K, I) = (\mathbb{C}, \emptyset)$ we thus recover the upper bound discovered in [LW96].

Remark 4. From the monotonicity of the mixed volume, it easily follows that one can always find $a_1, . . . , a_n$ such that the above bound is at least as sharp as the shadowed mixed volume bounds in [Roj94] (cf. Theorem 3). The older bound is already at least as sharp, and frequently significantly sharper, than any of the multihomogeneous Bézout theorems [Wam92, Roj96]. How to pick the best $a_1, . . . , a_n$ for any given E and $I \neq \{1, . . . , n\}$ will be addressed in [Roj96], but a simple choice which already beats the methods of [Roj94] is $a_1 = \cdots = a_n = \mathbf{O}$.

rather chatty style

A simple explanation for the improvement these bounds give over those in [Roj94] arises from the monotonicity of the mixed volume: growing polytopes potentially increase (and never decrease) the mixed volume. Thus polynomial systems with more monomial terms should, potentially, have more roots since their corresponding support sets have potentially larger convex hulls. In particular, the "pointed" Newton polytopes we compute the mixed volumes for here are usually much smaller than the shadowed Newton polytopes used in [Roj94].

To see how often the above bound gives an exact root count we will simplify things slightly by letting $I = \emptyset$, thus restricting to the case of counting all roots in K^n. (The $I = \{1, . . . , n\}$ case is covered extensively in [Roj94].) Fix $E_1, . . . , E_n$ and suppose the coefficients of all the f_i are now (algebraically independent) indeterminates. Further suppose that the support of F is precisely E and let $\mathcal{C}_E$ be the vector consisting of all the coefficients of all the f_i. We now call F an *indeterminate* polynomial system with support E. Since we want to give a tight upper bound solely in terms of E, it would be especially nice if our formula was exactly the number of roots for generically specialized $\mathcal{C}_E$. This optimality is indeed attained by our upper bound, for a large family of supports E (cf. Corollary 3). However, expressing such an optimal bound canonically as a *single* mixed volume is an open problem for general E and I. Optimal bounds for general E and I can be found in [HS96, Roj96], but the methods there are not always as simple as just appending a single point to each support and computing a mixed volume.

We will also make the natural restriction of considering only those E such that $E_i \subset (\mathbb{N} \cup \{0\})^n$ for all i and F has only finitely many roots in K^n for generic $\mathcal{C}_E$. We call such an E *nice for* K^n [Roj96] and give a combinatorial characterization of niceness in Section 3. A related notion that will be quite useful is the following: Call $B \subset \mathbb{R}^n$ *cornered* whenever B lies in the nonnegative orthant of $\mathbb{R}^n$ and $\{(y_1, \ldots, y_n) \in B \mid y_j = 0\} \neq \emptyset$ for all $j \in \{1, \ldots, n\}$. Note that E_i being cornered is equivalent to f_i *not* being divisible by x_j for all $j \in \{1, \ldots, n\}$. We then say that E is *cornered* iff $E_1, \ldots, E_n$ are all cornered. Note that $\mathbf{O} \cup E := (\{\mathbf{O}\} \cup E_1, \ldots, \{\mathbf{O}\} \cup E_n)$ is always cornered.

Remark 5. In general, neither of the last two definitions implies the other. For example, if $n = 3$ and $E_1 = E_2 = E_3 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$, then E is cornered but *not* nice for K^3. This is because any polynomial system with support contained in E contains the coordinate axes in its zero set. Going the opposite way, it is easily checked that $(\{(0, 1), (1, 1)\}, \{(1, 0), (1, 1)\})$ is nice for K^2 but not cornered.

Given these two conditions on E, we can then deduce that the above mixed volume bound is generically the exact number of roots in K^n.

THEOREM 1. *Suppose E is an n-tuple of finite subsets of* $(\mathbb{N} \cup \{0\})^n$ *which is nice for K^n and cornered. Then a generic polynomial system with support contained in E has exactly* $\mathcal{M}(\mathbf{O} \cup E)$ *roots in K^n, counting multiplicities.*

Remark 6. This result will follow as a special case of a more general result (Theorem 7) in Section 3.

Remark 7. This complements, and makes explicit, a generic affine root count alluded to in [Dan78, DK87]. In essence, these works covered the case where $K = \mathbb{C}$ and the zero set of F generically intersects Hyper $(\{1, \ldots, n\})$ nonsingularly. However, an explicit formula in terms of mixed volume was not mentioned. One should also note that although our hypotheses and those of [Dan78, DK87] overlap, neither set of hypotheses implies the other.

Remark 8. It is worth noting that, in the notation of [HS96], Theorem 1 is equivalent to the following statement: E is nice for K^n and cornered $\Rightarrow \mathcal{SM}(E) = \mathcal{M}(\mathbf{O} \cup E)$, where $\mathcal{SM}(\cdot)$ denotes *stable* mixed volume.

Even better, when E is nice for K^n and cornered, we can get a stronger genericity result by arbitrarily fixing some of the coefficients and leaving only a few coefficients generic. To be more precise, let $c_{i,e}$ denote the indeterminate coefficient of the x^e term of f_i. If an n-tuple $D := (D_1, \ldots, D_n)$ satisfies $D_i \subseteq E_i$ for all $i \in \{1, \ldots, n\}$ then we abbreviate this simply as $D \subseteq E$. For any such D define $\mathscr{C}_D := \{c_{i,e} \mid 1 \leq i \leq n, e \in D_i\}$. We will then say that D *K^n-counts* E iff (0) $D \subseteq E$, (1) D and E are nice for K^n, and (2) for any specialization over K of the coefficients $\mathscr{C}_E \backslash \mathscr{C}_D$, a generic specialization of the remaining coefficients $\mathscr{C}_D$ suffices to make F have maximally many isolated roots lying in K^n, counting multiplicities.[1] Note that D K^n-counts $E \Rightarrow D'$ K^n-counts E, for any $D' \supseteq D$.

In Section 3 we give a complete combinatorial classification of the D which K^n-count E, whenever E is nice for K^n and cornered (cf. Theorem 7). From this classification we can then recover more information about the non-cornered case. In particular, we can strengthen Theorem 1 to determine *exactly* when its main assertion is true (cf. Corollary 3). We thus obtain that for a large family of polynomial systems, appending the origin to each support set and then computing the mixed volume of the new "pointed" Newton polytopes generically gives the exact number of roots in K^n. Our results also generalize a different version of Theorem 1 (where $K = \mathbb{C}$) due to Li and Wang [LW96]. It is interesting to note that their version has the hypothesis that a particular zero set be nonsingular at infinity, while our hypothesis is stated combinatorially. Also, their proofs rely on complex analytic techniques while ours are more algebraic in nature.

In the following section we review some background and give examples illustrating the generic behavior of roots lying in coordinate subspaces for various E. Theorem 1 and its more combinatorial extensions are proved via toric variety techniques in Section 3. The Affine Point Theorem I, proved in Section 4, follows directly from the groundwork laid in [Roj94] and an additional homotopy argument. Our homotopy proof avoids the use of Puiseaux series and thus also applies to the case where the characteristic of K is nonzero.

2. Preliminaries

Background for this paper can be found in a number of journal articles but we will primarily use the language and notation of [Roj94]. We now state a few necessary results.

Recall that a *quasi-affine* variety in K^N is a Zariski-open subset of an

[1] We would like to emphasize that K^n-counting replaces (as well as generalizes) the earlier notion of 0-counting found in [Roj94].

affine variety in K^N [Har77, Sha80, CLO92]. A *constructible* set is a finite union of quasi-affine varieties. Constructible sets are closed under complementation, projection [CLO92], and finite union and intersection. Note that a dense constructible set has nonempty interior.

PROPOSITION 1. *Any two dense constructible subsets of K^N have dense constructible intersection.*

DEFINITION 1. By a *generic specialization* of N indeterminates (or by a set of N indeterminates being *generic* or satisfying a *generic condition*) we will mean a selection of values lying in some *a priori fixed* dense constructible subset of K^N.

Alternatively, the above proposition can be restated more applicably by "any finite conjunction of generic conditions is again a generic condition." In particular, when $K = \mathbb{C}$, note that any generic condition is true with probability 1, given any continuous probability measure on the domain in question.

PROPOSITION 2. [Roj94]. *Suppose $U \subseteq K^N$ and the set*

$$\{z \in K^{N-1} \mid (\{z\} \times K) \cap U \text{ is Zariski-dense in } \{z\} \times K\}$$

is Zariski-dense in K^{N-1}. Then U is Zariski-dense.

For convenience, we will let $Z(h_1, \ldots, h_s)$ denote the subscheme of K^n defined by $h_1(x) = \cdots = h_s(x) = 0$, whenever $h_1, \ldots, h_s \in K(x_1, \ldots, x_n)$. As an application of Proposition 2 we will prove the following useful lemma.

LEMMA 1. *Let $Y \subseteq K^n$ be a constructible subset of codimension e. Also let $f_1, g_1, \ldots, f_k, g_k : K^n \dashrightarrow K$ be rational functions such that $g_1, \ldots, g_k$ never vanish at any point of Y. Then for generic $(c_1, \ldots, c_k) \in K^k$,*

$$\operatorname{codim}(Y \cap Z(f_1 + c_1 g_1, \ldots, f_k + c_k g_k)) \geq e + k.$$

In particular, $k > n \Rightarrow Y \cap Z(f_1 + c_1 g_1, \ldots, f_k + c_k g_k) = \emptyset$ for generic $(c_1, \ldots, c_k)$.

Proof. Let $\hat{f}_i := f_i + c_i g_i$ for all i, and pick points $\{z_l\}$, one in each irreducible component of Y. As long as $\hat{f}_1(z) \neq 0$ for all $z \in \{z_l\}$, it is clear that codim $(Y \cap Z(\hat{f}_1)) \geq e + 1$. Since g_1 never vanishes within Y and $\{z_l\}$ is finite, it is clear that the set of $c_1 \in K$ such that codim $(Y \cap Z(\hat{f}_1)) \geq e + 1$ is cofinite. Thus a generic choice of $c_1 \in K$ suffices to keep this lower bound true.

Next we proceed by induction. Assume $i > 1$ and that for generic $(c_1, \ldots, c_i) \in K^i$,

$$\operatorname{codim}(Y \cap Z(\hat{f}_1, \ldots, \hat{f}_i)) \geq e + i.$$

Then, mimicking the preceding paragraph, pick points $\{z_i'\}$, one in each irreducible component of $Y \cap Z(\hat{f}_1, \ldots, \hat{f}_i)$. As long as $\hat{f}_{i+1}(z) \neq 0$ for all $z \in \{z_i'\}$, it is clear that

$$\operatorname{codim}(Y \cap Z(\hat{f}_1, \ldots, \hat{f}_{i+1})) \geq e + i + 1.$$

Since g_{i+1} never vanishes within Y and $\{z_i'\}$ is finite, it is clear that (for generic $(c_1, \ldots, c_i) \in K^i$) the set of $c_{i+1} \in K$ such that $\operatorname{codim}(Y \cap Z(\hat{f}_1, \ldots, \hat{f}_{i+1})) \geq e + i + 1$ is cofinite. So by Proposition 2 the set of $(c_1, \ldots, c_{i+1}) \in K^{i+1}$ such that this lower bound is true is Zariski-dense. Moreover, since constructible sets are closed under projection, it easily follows that the set of $(c_1, \ldots, c_{i+1}) \in K^{i+1}$ such that $\operatorname{codim}(Y \cap Z(\hat{f}_1, \ldots, \hat{f}_{i+1})) \geq e + i + 1$ is also constructible. Hence a generic choice of $(c_1, \ldots, c_{i+1}) \in K^{i+1}$ suffices to maintain this lower bound. Having completed our inductive construction, the lemma follows. ■

Let $K^* := K \backslash \{0\}$ and recall the following version of the BKK bound.

THEOREM 2. [Roj94] *Suppose F is an $n \times n$ system of nonzero polynomials over K. Let E be the support of F. Then F has at most $\mathcal{M}(E)$ isolated roots in $(K^*)^n$, counting multiplicities. Furthermore, a generic polynomial system with support contained in E has exactly $\mathcal{M}(E)$ roots in $(K^*)^n$, counting multiplicities.*

Remark 9. The $K = \mathbb{C}$ case is originally due to Bernshtein [Ber75]. The generalization to any algebraically closed K appears in [Dan78] but is not stated explicitly.

Remark 10. It is useful to recall, to clarify our upcoming examples, that for $n = 2$, $\mathcal{M}(E) = \text{Area}(\text{Conv}(E_1 + E_2)) - \text{Area}(\text{Conv}(E_1)) - \text{Area}(\text{Conv}(E_2))$.

The following is a more recent upper bound on the number of affine roots which is as sharp as the Affine Point Theorem I in some cases, but not all.

THEOREM 3. [Roj94]. *Suppose F is an $n \times n$ system of nonzero polynomials over K with no negative exponents in its monomial terms. Let E be the support of F and fix $r \in \{0, \ldots, n\}$. Then F has at most $\mathcal{M}_r(E)$ isolated roots in $(K^*)^r \times K^{n-r}$, counting multiplicities, where $\mathcal{M}_r(\cdot)$ denotes r-shadowed mixed volume.*

Remark 11. The definition of $\mathcal{M}_r(\cdot)$ appears in [Roj94]. For the present discussion it suffices to know only that $\mathcal{M}_0(E)$ is simply the mixed volume of the n-tuple where each E_i is replaced by the union of E_i and the images of all its coordinate projections.

EXAMPLE 1. Let $n = 2$ and consider the bivariate polynomial system

$$f_1(x, y) = a_1x + a_2y + a_3xy^2 + a_4x^2y$$
$$f_2(x, y) = b_1x + b_2y + b_3xy^2 + b_4x^2y$$

where the a_j and b_j are constants in K. Then setting

$$E_1 = E_2 = \{(1, 0), (0, 1), (1, 2), (2, 1)\},$$

the BKK bound implies that F has at most $\mathcal{M}(E) = 4$ isolated roots in $(K^*)^2$. Remark 11 tells us that $\mathcal{M}_0(E) = \mathcal{M}(P, P)$, where

$$P = \text{Conv}(\{(0,0), (2, 0), (0, 2), (2, 1), (1, 2)\}).$$

So Theorem 3 implies that F has at most seven isolated roots in K^2. However, the Affine Point Theorem I implies that F has at most $\mathcal{M}(\mathbf{O} \cup E) = 5$ roots in K^2. Better still, Theorem 1 tells us that this bound is best possible. Note that the root (0, 0) persists for any choice of the a_j and b_j. Also note that this is the only root of F lying on the coordinate axes, for generic $\{a_j\}$ and $\{b_j\}$.

EXAMPLE 2. Let $n = 2$ and

$$f_1(x, y) = a_1y + a_2y^2 + a_3xy^3$$
$$f_2(x, y) = b_1x + b_2x^2 + b_3x^3y.$$

Then setting $E_1 = \{(0, 1), (0, 2), (1, 3)\}$ and $E_2 = \{(1, 0), (2, 0), (3, 1)\}$, the BKK bound implies that F has at most $\mathcal{M}(E) = 3$ isolated roots in $(K^*)^2$. Remark 11 tells us that $\mathcal{M}_0(E) = \mathcal{M}([0, 1] \times [0, 3], [0, 3] \times [0, 1])$. So Theorem 3 implies that F has at most ten isolated roots in K^2. The Affine Point Theorem I implies that F has at most eight isolated roots in K^2. However, our new upper bound is *not* optimal for this example: F generically has only six roots in K^2. This system and the preceding fact are quoted from [HS96], but we make two additional observations: (1) For generic $\{a_j\}$ and $\{b_j\}$, F has exactly one root lying in each of following sets: $\{(0, 0)\}$, $K^* \times \{0\}$, and $\{0\} \times K^*$; and (2) E is nice for K^2 but *not* cornered. In fact, the way in which E fails to be cornered affects the generic number of roots

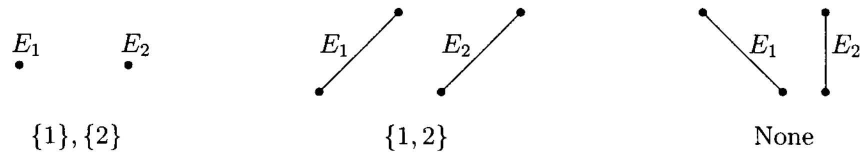

FIG. 1. The essential subsets for three different pairs of plane polygons.[3]

in the sets mentioned in (1). This is explored further in the next section and [Roj96].

3. Optimality in Affine Space

In discussing when the Affine Point Theorem I is exact, it will be helpful to review $(K^*)^n$*-counting* and some related concepts.

Let $\mathcal{S}^{n-1} \subset \mathbb{R}^n$ denote the unit $(n-1)$-sphere centered at the origin. For any compact $B \subset \mathbb{R}^n$ and any $w \in \mathbb{R}^n$, define B^w to be the set of $x \in B$ where the inner product $x \cdot w$ is minimized. (Thus B^w is the intersection of B with its supporting hyperplane in the direction w.) We then define $E^w := (E_1^w, \ldots, E_n^w)$ and $D \cap E^w := (D_1 \cap E_1^w, \ldots, D_n \cap E_n^w)$. We will also use $|\cdot|$ for set cardinality, and the *dimension* of a finite subset $B \subseteq \mathbb{R}^n$ is simply the dimension of its convex hull, $\mathrm{Conv}(B)$.

DEFINITION 2. Suppose $C := (C_1, \ldots, C_n)$ is an n-tuple of polytopes in $\mathbb{R}^n$ or an n-tuple of finite subsets of $\mathbb{R}^n$. We will allow any C_i to be empty and define $\mathrm{Supp}(C) := \{i \mid C_i \neq \emptyset\}$. We will also say that a nonempty subset $J \subseteq \{1, \ldots, n\}$ is *essential* for C (or *C has essential subset J*) $\Leftrightarrow$ (0) $\mathrm{Supp}(C) \supseteq J$, (1) $\dim(\Sigma_{j \in J} C_j) = |J| - 1$, and (2) $\dim(\Sigma_{j \in J'} C_j) \geq |J'|$ for all nonempty *proper* $J' \subset J$.[2]

Alternatively, J is essential for $C \Leftrightarrow$ the $|J|$-dimensional mixed volume of the $|J|$-tuple $(\mathrm{Conv}(C_j) \mid j \in J)$ is 0 and no proper subset of J has this property [Oda88, Roj94]. Figure 1 shows some simple examples of essential subsets for E, for various E in the case $n = 2$.

A basic fact about mixed volumes is that $\mathcal{M}(E) = 0 \Leftrightarrow E$ has an essential subset, whenever $\mathrm{Supp}(E) = \{1, \ldots, n\}$. This begins to hint at the deep connections between convex geometry and polynomial systems. In particu-

[2] Note that condition (0) is new and slightly generalizes the original definition found in [Stu94].

[3] For aesthetic reasons, we have actually drawn the *convex hulls* of the E_i. Also, the segments in the middle pair are meant to be parallel.

lar, we can define $(K^*)^n$-*niceness* in the most natural way (paralleling the definition of K^n-niceness) and apply essentiality as follows.

LEMMA 2. [Roj94, Sect. 2.5]. *An n-tuple* $E := (E_1, \ldots, E_n)$ *of finite subsets of* $\mathbb{Z}^n$ *is nice for* $(K^*)^n \Leftrightarrow$ *one of the following exclusive conditions holds:*

1. *E has an essential subset, or*
2. $\mathrm{Supp}(E) = \{1, \ldots, n\}$ *and* $\dim(\Sigma_{j \in J'} E_j) \geq |J'|$, *for all nonempty* $J' \subseteq \{1, \ldots, n\}$.

Moreover, $\mathcal{M}(E) > 0 \Leftrightarrow$ *condition 2 holds.*

Essentiality arose from the study of the combinatorics of the sparse resultant [Stu94]. Closer to our focus, one can define $(K^*)^n$-counting by paralleling the definition of K^n-counting. The notion of $(K^*)^n$-counting was first considered in [CR91] and completely characterized in [Roj94] where it was called "counting." Essentiality came into play as follows.

THEOREM 4 [Roj94, Sect. 2.5] *Suppose D and E are n-tuples of finite subsets of* $\mathbb{Z}^n$ *such that* $D \subseteq E$ *and E is nice for* $(K^*)^n$. *Then D* $(K^*)^n$*-counts E* $\Leftrightarrow$ *one of the following exclusive conditions holds:*

1. $\mathrm{Supp}(D)$ *contains a subset essential for E, or*
2. $\mathcal{M}(E) > 0$ *and for all* $w \in \mathcal{S}^{n-1}$, $\mathrm{Supp}(D \cap E^w)$ *contains a subset essential for* E^w.

Remark 12. One certainly need not check infinitely many w in the second case. In fact, we need only check one w (just pick any inner normal) for each face of the polytope $\Sigma\ \mathrm{Conv}(E_i)$.

Remark 13. Note that we immediately obtain that E $(K^*)^n$-counts E. Better still, it follows just as easily that E is $(K^*)^n$-counted by D when D_i is the vertex set of $\mathrm{Conv}(E_i)$ for all i. In fact, one can sometimes get away with using far fewer points for D [Roj94].

The crux of the relationship between generic root counts in $(K^*)^n$ and mixed volumes is the following definition and theorem: D *fills* $E \Leftrightarrow$ (0) $D \subseteq E$ and (1) $\mathcal{M}(D) = \mathcal{M}(E)$.

THEOREM 5. [Roj94]. *Following the notation of Theorem 4, D fills E* $\Leftrightarrow$ *D* $(K^*)^n$*-counts E.*

Remark 14. When some E_i is empty and E has an essential subset, we make the natural extension of defining $\mathcal{M}(E) := 0$.

Our combinatorial results for K^n-counting and K^n-niceness will then follow easily upon partitioning K^n in a special way.

DEFINITION 3. For any $J \subseteq \{1, \ldots, n\}$, let $O_J := \{x \in K^n \mid x_j \neq 0 \Leftrightarrow j \in J\}$. (In particular, $O_\emptyset := \mathbf{O}$.) We call O_J an *orbit* and it is clear that $K^n = \bar{O}_{\{1,\ldots,n\}} = \coprod O_J$, where the union is disjoint and ranges over all subsets of $\{1, \ldots, n\}$.

It will also be useful to refine K^n-niceness in the following way.

DEFINITION 4. Suppose $W \subseteq K^n$ is a finite union of orbits and E is an n-tuple of finite subsets of $(\mathbb{N} \cup \{0\})^n$. We then call E *null* for $W \Leftrightarrow$ a generic polynomial system with support contained in E has *no* roots in W.

For any $J \subseteq \{1, \ldots, n\}$, define $E \cap \text{Lin}(J) := (E_1 \cap \text{Lin}(J), \ldots, E_n \cap \text{Lin}(J))$. We can then simplify our analysis of counting and niceness with the following immediate corollary of Proposition 1.

COROLLARY 1. *Suppose E is an n-tuple of finite subsets of* $(\mathbb{N} \cup \{0\})^n$. *Then E is* nice (*resp. null*) *for $K^n \Leftrightarrow E$ is nice* (*resp. null*) *for O_J for all* $J \subseteq \{1, \ldots, n\}$.

To conclude our analysis of generic conditions on coefficients, we will need the following final definition.

DEFINITION 5. We say that C has an almost essential subset $J \Leftrightarrow$ (0) $\text{Supp}(C) \supseteq J$, (1) $\dim(\Sigma_{j\in J} C_j) = |J|$, and (2) $\dim(\Sigma_{j\in J'} C_j) \geq |J'|$ for all nonempty $J' \subseteq J$. Also, $\emptyset$ is defined to be almost essential for C iff Supp $(C) = \emptyset$.[4]

In particular, note that the $|J|$-dimensional mixed volume of (Conv $(C_j) \mid j \in J)$ is positive whenever J is nonempty and almost essential for C. Also, it is clear that $J \cup \{j\}$ is essential for $E \Rightarrow J$ is almost essential for E, provided $|J \cup \{j\}| > |J| > 0$. Almost essentiality arose from the study of affine root counts [Roj94] and also turns out to be quite useful for characterizing when E is nice for O_J.

COROLLARY 2. *Following the notation of Corollary* 1, *E is nice for $O_J \Leftrightarrow E \cap$* Lin$(J)$ *has an almost essential subset of cardinality $|J|$ or an essential subset. In particular, E is null for $O_J \Leftrightarrow E \cap$* Lin$(J)$ *has an essential subset.*

Proof. Let F be an indeterminate polynomial system with support E and coefficient vector $\mathscr{C}_E$. Setting the variables $\{x_j \mid j \notin J\}$ equal to 0, we get a new polynomial system in $\leq |J|$ variables which we will call F_J. Clearly then, F_J has support $E \cap \text{Lin}(J)$ and the vector of indeterminate coefficients of F_J is a subvector of $\mathscr{C}_E$.

[4] Note that condition (0) is new and this version slightly generalizes the original definition found in [Roj94].

Note that F generically has a positive finite number of roots in $O_J \Leftrightarrow F_J$ generically has a positive finite number of roots in O_J. Since $\dim(E_i \cap \text{Lin}(J)) \leq |J|$ for all i, it is then easy to see that the latter condition holds iff the vector $(\text{Conv}(E_i \cap \text{Lin}(J)) \mid E_i \cap \text{Lin}(J) \neq \emptyset)$ is a $|J|$-tuple with positive $|J|$-dimensional mixed volume (by Theorem 2). But then the last condition is true iff $E \cap \text{Lin}(J)$ has an almost essential subset of cardinality $|J|$ (by Lemma 2). Similarly, F generically has no roots in $O_J \Leftrightarrow E \cap \text{Lin}(J)$ has an essential subset. ■

We can then combine Corollaries 1 and 2 into the following lemma.

LEMMA 3. *An n-tuple E of finite subsets of* $(\mathbb{N} \cup \{0\})^n$ *is nice for* $K^n \Leftrightarrow$ *for all* $J \subseteq \{1, \ldots, n\}$, $E \cap \text{Lin}(J)$ *has an almost essential subset of cardinality* $|J|$ *or an essential subset. In particular, E is null for* $K^n \Leftrightarrow$ *for all* $J \subseteq \{1, \ldots, n\}$, $E \cap \text{Lin}(J)$ *has an essential subset.*

Remark 15. It is easily checked (from the above lemma or from first principles) that $\mathbf{O} \cup E$ is always nice for K^n.

Before we state and prove our classification of K^n-counting, we will need the following more abstract version of Theorem 2. From here on, we will implicitly assume familiarity with some basic facts about intersection theory on toric varieties, e.g., parts of [Ful93, GKZ94]. Let $T := (K^*)^n$.

THEOREM 6. [Roj96]. *Suppose F is an* $n \times n$ *polynomial system over K with support contained in an n-tuple* $\mathscr{P} := (P_1, \ldots, P_n)$ *of integral polytopes. Let* $P \subset \mathbb{R}^n$ *be any n-dimensional rational polytope compatible with* $P_1, \ldots, P_n$ *and let* $\mathscr{T}_P$ *be the toric variety over K corresponding to P. Finally, for all* $i \in \{1, \ldots, n\}$, *let* $\mathscr{E}_i$ *be the T-invariant Cartier divisor of* $\mathscr{T}_P$ *corresponding to* P_i, and define $\mathscr{D}_i := \mathscr{E}_i + \text{Div}(f_i)$. *Then*

1. $\mathscr{T}_P$ *is an n-dimensional, normal, and complete algebraic variety. In particular,* $\mathscr{T}_P$ *is compact when* $K = \mathbb{C}$.

2. $\mathscr{D}_1, \ldots, \mathscr{D}_n$ *are effective and their intersection product* $\mathscr{D}$ *has cycle class degree* $\mathscr{M}(\mathscr{P})$. *Furthermore,* $\mathscr{D} \cap (K^*)^n$ *is precisely the zero scheme of F in* $(K^*)^n$.

3. *If* $\mathscr{D}$ *is zero-dimensional or empty, then* $\mathscr{D}$ *has exactly* $\mathscr{M}(\mathscr{P})$ *irreducible components in* $\mathscr{T}_P$, *counting multiplicities.*

4. *If* $\mathscr{D}$ *is positive-dimensional and* $\mathscr{M}(P) > 0$, *then* $\mathscr{D}$ *has strictly less than* $\mathscr{M}(\mathscr{P})$ *zero-dimensional irreducible components in* $\mathscr{T}_P$, *counting multiplicities.*

Although we prove our classification via toric variety methods, its statement is pure combinatorial geometry.

THEOREM 7. *Suppose D and E are n-tuples of finite subsets of* $(\mathbb{N} \cup$

$\{0\})^n$ *such that* $D \subseteq E$ *and* E *is nice for* K^n *and cornered. Then a polynomial system with support contained in* E *generically has exactly* $\mathcal{M}(\mathbf{O} \cup E)$ *roots in* K^n, *counting multiplicities. Furthermore,* D K^n*-counts* $E \Leftrightarrow$ *one of the following conditions holds:*

1. $\mathcal{M}(\mathbf{O} \cup E) = 0$ *and for all* $J \subseteq \{1, \ldots, n\}$, $\mathrm{Supp}(D \cap \mathrm{Lin}(J))$ *contains a subset essential for* $E \cap \mathrm{Lin}(J)$, *or*

2. $\mathcal{M}(\mathbf{O} \cup E) > 0$ *and, for all* $w \in \mathbb{R}^n$ *lying outside the nonnegative orthant,* $\mathrm{Supp}(D \cap E^w)$ *contains a subset essential for* E^w.

Remark 16. Theorem 7 was derived by the first author after this paper was submitted for publication. A more general version is proved in [Roj96] where further background is also developed. For completeness, we have included here Theorem 7 and a sketch of its proof.

Remark 17. Similar to Theorem 4, the number of n-tuples that one must check in case 2 is finite and is precisely the number of faces, with an inner normal lying outside the nonnegative orthant, of the polytope Σ $\mathrm{Conv}(\{\mathbf{O}\} \cup E_i)$.

Remark 18. Assuming E is nice for K^n and cornered, we immediately obtain that E K^n-counts $\mathbf{O} \cup E$. Better still, under the same assumptions on E, it follows almost as easily that $\mathbf{O} \cup E$ is K^n-counted by D when $\mathcal{M}(\mathbf{O} \cup E) > 0$ and each D_i is the set of vertices of $\mathrm{Conv}(E_i)$ having an inner normal lying outside the nonnegative orthant.

Proof. Let F be an indeterminate $n \times n$ polynomial system with support E and coefficient vector $\mathscr{C}_E$.

Note that the $\mathcal{M}(\mathbf{O} \cup E) = 0$ case follows almost tautologously: by Proposition 1, the Affine Point Theorem I and our hypotheses immediately imply that F generically has no roots at all within K^n, i.e., E is null for K^n. As for condition 1, since E is null for K^n, it easily follows from Proposition 1 that D K^n-counts $E \Leftrightarrow D$ O_J-counts E for all $J \subseteq \{1, \ldots, n\}$. But then the last condition is equivalent to condition 1 by Theorem 4 and the fact that F has a root in $O_J \Leftrightarrow F_J$ has a root in $(K^*)^n$ (following the notation of the proof of Corollary 2).

So let us now assume that $\mathcal{M}(\mathbf{O} \cup E) > 0$. Here we will apply Theorem 6, but first we must define an appropriate P in terms of the input data $E_1, \ldots, E_n$. To do this, we begin by defining $P_i = \mathrm{Conv}(\mathbf{O} \cup E_i)$ for all i and $P' := \Sigma P_i$. By Lemma 2, it is clear that $\dim P' = n$. Now let $\sigma \subset \mathbb{R}^n$ be the nonnegative orthant, $\varepsilon \in \mathbb{Q}^n \cap \sigma$, and let $\mathscr{F}_\varepsilon$ be the normal fan of $(\varepsilon + \sigma) \cap P'$. Then it is easily checked that, for $\|\varepsilon\|$ sufficiently small, (a) $\mathscr{F}_\varepsilon$ contains σ as one of its cones and (b) $\mathscr{F}_\varepsilon$ is a refinement of the normal fan of P'. So let $P := (\varepsilon + \sigma) \cap P'$ for such an ε. In particular, P is rational, n-dimensional, and compatible with $P_1, \ldots, P_n$.

The resulting $\mathcal{T}_P$ from Theorem 6 has two very nice properties: (I) it has a naturally embedded copy of K^n, and (II) $\mathcal{D}_i \cap K^n = Z(f_i)$ for all i. Property (I) follows easily from (a), and property (II) follows from the definition of the $\mathcal{E}_i$ and the fact that E is cornered.

Now let O_w denote the T-orbit of $\mathcal{T}_P$ corresponding to the face P^w of P and consider the following condition on $\mathcal{D}$:

A: $\mathcal{D} \cap O_w = \emptyset$ for all $w \in \mathbb{R}^n$ lying *outside* the nonnegative orthant.

It then follows (e.g., from Corollary 2 of [Roj96]) that Condition A holds iff the relativized initial term system $(\mathrm{in}_{w,P_1}(f_1), \ldots, (\mathrm{in}_{w,P_n}(f_n))$ has no roots in $(K^*)^n$, for all w lying outside of the nonnegative orthant. In particular, the latter condition holds for generic $\mathcal{C}_D$ (for any specialization of $\mathcal{C}_E \backslash \mathcal{C}_D$) iff condition 2 of our present theorem holds (by condition 1 of Theorem 4).

However, condition A also turns out to be equivalent of $\mathcal{D}$ being zero-dimensional and lying entirely within K^n. That condition A is equivalent to $\mathcal{D}$ lying entirely within K^n follows almost immediately from the definition of the orbits O_w. However, establishing the wonderful coincidence that $\dim \mathcal{D} = 0$ as well takes a little more work. The argument, which we will omit, reduces to the observation that toric varieties of the form $\mathcal{T}_{P+\mathrm{Conv}(\mathbf{O},\hat{e}_j)}$ have canonical proper morphisms onto $\mathcal{T}_P$ and $\mathbb{P}^1_K$. In particular, using the fact that the jth coordinate projection defined on $(K^*)^n$ extends to the proper morphism from $\mathcal{T}_{P+\mathrm{Conv}(\mathbf{O},\hat{e}_j)}$ to $\mathbb{P}^1_K$, one can show that a positive dimensional component of $\mathcal{D}$ which intersects K^n must intersect some O_w with w outside the nonnegative orthant.

In any case, we have just shown that (i) condition A holds for generic $\mathcal{C}_D$ (for any specialization of $\mathcal{C}_E \backslash \mathcal{C}_D$) iff condition 2 holds, and (ii) condition A is equivalent to $\mathcal{D}$ being 0-dimensional and equal to $Z(F)$. So we can now quickly conclude by Theorem 6: Parts 3 and 4 imply that (ii) is true iff F has exactly $\mathcal{M}(\mathbf{O} \cup E)$ roots in K^n, counting multiplicities. The latter half of Theorem 6 also tells us that this must indeed be the maximal number of isolated roots in K^n, counting multiplicities. So by (i) and (ii) (and the $\mathcal{M}(E) = 0$ case, which we have already proved), D K^n-counts E iff condition 2 holds. In particular, setting $D = E$, we see that F generically has exactly $\mathcal{M}(\mathbf{O} \cup E)$ roots in K^n, counting multiplicities. ∎

Remark 19. An immediate consequence of our proof above is a quick toric variety proof of the $I = \emptyset$ case of the Affine Point Theorem I. (Of course, to avoid circularity, one must do a tiny bit of additional work in the case $\mathcal{M}(\mathbf{O} \cup E) = 0$.) However, we will give another more elementary argument in the next section.

Theorem 1 is, of course, contained in Theorem 7 and generalizes in a different way to the following result.

COROLLARY 3. *Suppose E is an n-tuple of finite subsets of* $(\mathbb{N} \cup \{0\})^n$ *such that E is nice for K^n. Then a generic polynomial system with support contained in E has exactly $\mathcal{M}(\mathbf{O} \cup E)$ roots, counting multiplicities, in $K^n \Leftrightarrow$ one of the following condition holds:*

1. $\mathcal{M}(\mathbf{O} \cup E) = 0$ *and, for all* $J \subseteq \{1, \ldots, n\}$, $\mathrm{Supp}(E \cap \mathrm{Lin}(J))$ *contains a subset essential for* $(\mathbf{O} \cup E) \cap \mathrm{Lin}(J)$, *or*

2. $\mathcal{M}(\mathbf{O} \cup E) > 0$ *and, for all $w \in \mathbb{R}^n$ lying outside the nonnegative orthant*, $\mathrm{Supp}(E \cap (\mathbf{O} \cup E)^w)$ *contains a subset essential for* $(\mathbf{O} \cup E)^w$.

Proof. Obviously, $\mathbf{O} \cup E$ is cornered. Also, in Remark 15 we already observed that $\mathbf{O} \cup E$ is nice for K^n. So by Theorem 7, it is clear that the conditions of our present corollary are equivalent to E K^n-counting $\mathbf{O} \cup E$. By the Affine Point Theorem I (and the definition of K^n-counting) we are done. ■

EXAMPLE 3. One might believe (as the first author did, for a little while) that the above criterion can be simplified further. For example, one might conjecture that E is null for $K^n \Leftrightarrow \mathcal{M}(\mathbf{O} \cup E) = 0$. However, even this simple statement is false: counterexamples already crop up for $n = 2$. One of the simplest is $E = (\{(0, 1)\}, \{\mathbf{O}, (1, 1)\})$.

EXAMPLE 4. It is easy to check that the criterion from the above corollary is violated by Example 2 when $w = (-1, 1)$. This confirms the fact that the Affine Point Theorem I overcounts the generic number of roots of this particular example.

Remark 20. It is worth noting that in the notation of [HS96], the above corollary is equivalent to a complete classification of the E such that $\mathcal{SM}(E) = \mathcal{M}(\mathbf{O} \cup E)$, where $\mathcal{SM}(\cdot)$ denotes *stable* mixed volume. Alternatively, by Remark 8, we also have a classification of all E such that $\mathcal{SM}(E) = \mathcal{SM}(\mathbf{O} \cup E)$.

Thus we now know exactly when $\mathcal{M}(\mathbf{O} \cup E)$ is an optimal upper bound on the number of isolated roots in affine space.

4. AN ALGEBRAIC HOMOTOPY PROOF OF THE AFFINE POINT THEOREM I

Let $b = (b_1, \ldots, b_n) \in K^n$ and $H(x, t) := F(x) + t(b_1 x^{a_1}, \ldots, b_n x^{a_n})$, where t is a new variable. Note that $H|_{t=0} = F$. Our proof is essentially a reduction to some basic facts about the degree of a proper morphism between algebraic curves over an algebraically closed field, e.g., [Sil86, Chap. 2]. The one possibly nonstandard technicality of our proof is the use of a special toric variety $\mathcal{T}_P$ (containing $K^n\setminus\text{Hyper}(I)$) in order to compactify our curves and make our morphisms well defined. We will confine the construction of $\mathcal{T}_P$ to a remark immediately following our proof.

By Lemma 1, and since $a \in \text{Lin}(I)$, it is easily checked that a generic choice of b suffices to make $\dim(((K^n\setminus\text{Hyper}(I)) \times K^*) \cap \{x_1 \cdots x_n = 0\} \cap Z(H)) \leq 0$. So let us fix b so that this condition on $Z(H)$ holds. We will then call any one-dimensional irreducible component of $((K^n\setminus\text{Hyper}(I)) \times K) \cap Z(H)$, whose image under projection onto the last coordinate is dense, a *good curve*. Note that by our assumption on b, the intersection of any good curve with $(K^n\setminus\text{Hyper}(I)) \times \{c\}$ lies entirely within $(K^*)^n \times \{c\}$, for all but finitely many $c \in K$.

Clearly, for *all* $c \in K$, if $z \in K^n\setminus\text{Hyper}(I)$ is any isolated root of $H|_{t=c}$ then (z, c) must lie on some good curve. (First, (z, c) must lie in some one-dimensional irreducible component of $Z(H)$ *not* contained in $(K^n\setminus\text{Hyper}(I)) \times \{c\}$, since $\dim Z(H) > 0$ and $\dim\{z\} = 0$. Second, this curve must be good since any nonconstant rational map between algebraic curves (over an algebraically closed field) has dense image.) In particular, since

$$K[t][x_j \mid j \notin I][x_j^{\pm 1} \mid j \in I]/\langle H, t-c\rangle \cong K[x_j \mid j \notin I][x_j^{\pm 1} \mid j \in I]/\langle H|_{t=c}\rangle$$

for all $c \in K$, we see that the number of isolated roots of $H|_{t=c}$ in $K^n\setminus\text{Hyper}(I)$, counting multiplicities, is bounded above by

$$S(c) := \sum_{C} \sum_{z \in C \cap ((K^n\setminus\text{Hyper}(I)) \times \{c\})} \mu(z; \langle H, t-c\rangle)$$

for all $c \in K$, where the outer sum ranges over all good curves and $\mu(\cdot)$ denotes intersection multiplicity [Ful84, Chap. 2]. Therefore to complete our proof it suffices to show that $S(0) \leq \mathcal{M}(a \cup E)$.

Now the projection of a good curve C onto the $(n+1)$st coordinate is a nonconstant rational map of algebraic curves $\phi_C: C \to K$. Let $\bar{C}$ be the closure of C in $\mathcal{T}_P \times \mathbb{P}^1_K$ and note that $\bar{C}\setminus C$ is 0-dimensional. Since the natural projection from $(K^n\setminus\text{Hyper}(I)) \times K$ onto the last coordinate naturally extends to a proper morphism $\varphi : \mathcal{T}_P \times \mathbb{P}^1_K \to \mathbb{P}^1_K$ [Ful93, Chap. 2], ϕ_C can be extended to a proper morphism $\varphi_C : \bar{C} \to \mathbb{P}^1_K$. Now by [Ful84, Examples 4.3.7 and 7.1.15], $\Sigma_{z \in \bar{C} \cap (\mathcal{T}_P \times \{c\})}\, \mu(z; \langle H, t-c\rangle)$ is precisely the degree of the morphism φ_C, for all $c \in \mathbb{P}^1_K$. In particular, for all $c \in K$, $S(c) \leq T$, where $T := \Sigma_C \deg \varphi_C$ and the sum is over all good curves. But by construction, the fiber $\varphi_C^{-1}(c) \subset \mathcal{T}_P$ actually lies entirely within $(K^*)^n$ for all but finitely many $c \in K$. So for all but finitely many $c \in K$, T is precisely the number of isolated roots in $(K^*)^n$, counting multiplicities, of the polynomial system $(H, t-c)$.

To conclude, we thus obtain by Theorem 2 that $T \le \mathcal{M}(a \cup E, \{\mathbf{O}, \hat{e}_{n+1}\}) = \mathcal{M}(a \cup E)$. The last equality follows almost immediately from the definition of the mixed volume. So $\mathcal{M}(a \cup E) \ge T \ge S(0)$ and we are done. ■

Remark 21. The toric variety $\mathcal{T}_P$ used in the above proof was constructed as follows: Similar to the Proof of Theorem 7, define $P_i := \mathrm{Conv}(\{a_i\} \cup E_i)$ for all i and let $P' := \Sigma P_i$. Now define σ_I^v to be the intersection of half-spaces $\bigcap_{j\in\{1,\ldots,n\}\setminus I}\{y \in \mathbb{R}^n \mid y_j \ge 0\}$. (In particular, $\sigma_{\{1,\ldots,n\}}^v := \mathbb{R}^n$.) Also let $\varepsilon \in \mathbb{Q}^n \cap (\sigma_I^v + \Sigma a_i)$ and let $\mathcal{F}_\varepsilon$ be the normal fan of $(\varepsilon + \sigma_I^v) \cap P'$. Then it is easily checked that for $\|\varepsilon - \Sigma a_i\|$ sufficiently small, (a) $\mathcal{F}_\varepsilon$ contains the dual cone of σ_I^v as one of its cones, and (b) $\mathcal{F}_\varepsilon$ is a refinement of the normal fan of P'. So let $P := (\varepsilon + \sigma_I^v) \cap P'$ for such an ε. In particular, P is rational, n-dimensional, and compatible with $P_1, \ldots, P_n$. Now the resulting $\mathcal{T}_P$ has two very nice properties: (I) it has a naturally embedded copy of $K^n \setminus \mathrm{Hyper}(\mathrm{I})$, and (II) $\mathcal{D}_i \cap (K^n \setminus \mathrm{Hyper}(I)) = Z(f_i) \cap (K^n \setminus \mathrm{Hyper}(I))$ for all i. Property (I) follows easily from (a), and property (II) follows from the definition of the $\mathcal{E}_i$ and the fact that $a \cup E$ is cornered relative to σ_I.

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