

ON TRIANGULATIONS OF THE CONVEX HULL OF n POINTS

B. L. ROTHSCHILD and E. G. STRAUS

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A set S of n points in Euclidean d -space determines a convex hull which can be triangulated into some number m of simplices using the points of S as vertices. We characterize those sets S for which all triangulations minimize m . This is used to characterize sets of points maximizing the volume of the smallest non-trivial simplex.

1. Introduction

Consider a set S of n points in d -dimensional Euclidean space E^d . If no hyperplane contains S we call S d -dimensional. We consider the d -dimensional simplex T of smallest positive volume with vertices in S . Then we consider the ratio of the volume of T to the volume of $C(S)$, the convex hull of S . We are interested in $f_d(n)$, the maximum value of this ratio for all sets S of n points. In Section 5 we show that it is exactly $(n-d)^{-1}$, and we characterize the sets S which achieve this bound. To do this we use the characterization of sets S with only 'minimal triangulations' (see sections 3 and 4 below).

For $d=2$ this is related to the Heilbronn conjecture, recently shown false by Komlós, Pintz and Szemerédi [4]. In the Heilbronn conjecture the points of S are required to be in general position (no three on a line) and the ratio under consideration is that of the area of the minimum triangle of S to the area of the smallest circle containing S . For this situation (the points in general position) the maximum value of the ratio, call it $g(n)$, is at most $O(n^{-\mu})$ for some $\mu > 1$, and at least $O(n^{-2} \log n)$. (See [3], and also Problem 6.4 in section 6 below.) (If $f^*(n)$ is the maximum value of the ratio of the minimum area of a triangle of S to the area of $C(S)$ for sets S with points in general position then $f^*(n) = O(g(n))$. See [2].) Erdős, Purdy and Straus [2] considered the closely related problem of the ratio of the area of the smallest to the area of the largest triangle in S . Since the ratio of the area of the largest triangle to that of $C(S)$ is bounded below by a constant

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(see [2]), this problem also yields a ratio of order $O(1/n)$. (The exact value given in [2] is $1/[\frac{1}{2}(n-1)]$.) (For a suggestion on the higher dimensional versions of this, see Problem 6.5 below.)

We use the notion of triangulations \mathcal{T} of $C(S)$, establish some of their properties, characterize certain special ones, and use them in characterizing the sets S which give the extreme values for the ratio $f_d(n)$ (Theorems 5.1 and 5.2). The characterization in Theorem 4.1 of sets S with only minimal triangulations we find interesting in itself.

Let S be a d -dimensional set of n points in E^d (S is not contained in a hyperplane). Let $C(S)$ be the convex hull of S . Then a *triangulation* of $C(S)$ is a set of nondegenerate simplices $\mathcal{T} = \{T_1, T_2, \dots, T_N\}$ with the following properties

1. $\cup_{\mathcal{T}} T_i = C(S)$;
2. $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ if $i \neq j$;
3. $T_i \cap S$ is the set of vertices of T_i for each $T_i \in \mathcal{T}$.
4. $T_i \cap T_j$ is a simplex with vertices in S , for all i, j .

We are interested in the possible cardinalities of $N = |\mathcal{T}|$ for the different triangulations, especially the lower bounds. We also characterize those sets S for which all triangulations are minimal.

With each triangulation \mathcal{T} of $C(S)$ we associate the *triangulation graph* $G(\mathcal{T})$ whose vertices are the simplices T_1, \dots, T_N and two vertices T_i, T_j are connected by an edge if and only if $T_i \cap T_j$ is a $(d-1)$ -simplex.

We are grateful to Victor Klee for pointing out the use of the triangulation graph in obtaining lower bounds for $|\mathcal{T}|$. The proof of Theorem 3.2 was supplied by him.

For $d=2$ the number of triangles in a triangulation of $C(S)$ is uniquely determined by S . In fact

$$(1.1) \quad |\mathcal{T}| = 2(n-1) - k$$

where k is the number of points of S on the boundary of $C(S)$. In particular there for

$$(1.2) \quad |\mathcal{T}| \cong n - 2$$

in all cases, with equality when S lies on the boundary of its convex hull.

For $d > 2$ it is no longer true that S determines $|\mathcal{T}|$. However in Section 3 we investigate the generalizations of inequality (1.2) and characterize those S for which equality holds.

2. An upper bound on $|\mathcal{T}|$

The triangulation \mathcal{T} is very nearly (in a sense indicated below) a simplicial d -sphere. Thus we can take advantage of the bounds known for d -sphere, in particular the 'Upper Bound Conjecture', to obtain similar bounds for $|\mathcal{T}|$. We state the relevant result below as Theorem 2.5. For a good treatment of the subject see [5].

In $(d+1)$ -space consider the 'moment curve' (x, x^2, \dots, x^{d+1}) , x a real parameter. Let $C(n+1, d+1)$ denote the convex polytope determined by any $n+1$ points on the curve ($n \geq d+1$). That is $C(n+1, d+1)$ is the convex hull of the $n+1$ points, and has faces of dimensions $d, d-1, \dots, 1, 0$. These faces form a

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 we associate the triangulation graph $G(\mathcal{T})$ with \mathcal{T} , and two vertices T_i, T_j are connected by an edge if they share a $(d-1)$ -simplex.
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 boundary of its convex hull. The triangulation of $C(S)$ determines $|\mathcal{T}|$. However in Section 4 we determine the number of triangulations of $C(S)$ (Theorem 1.2) and characterize those sets S for which $|\mathcal{T}|$ is minimal.

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 in a sense indicated below) a theorem which gives the bounds known for d -spheres. We also obtain similar bounds for $|\mathcal{T}|$ for a general set S . A good treatment of the subject is given in [1].
 curve' (x, x^2, \dots, x^{d+1}) , x a real number. The convex polytope determined by the convex hull of $(n+1, d+1)$ is the convex hull of $(0, d-1, \dots, 1, 0)$. These faces

simplicial d -sphere, the 'boundary complex' of $C(n+1, d+1)$, denoted by $\Delta(n+1, d+1)$. For any simplicial d -sphere \mathcal{S} , let $f_i(\mathcal{S})$ denote the number of i -faces, $0 \leq i \leq d$.

For $f_i = f_i(\mathcal{S})$, $0 \leq i \leq d$, $f_{-1} = 1$, we define $h_i = h_i(\mathcal{S})$ by

$$(2.1) \quad h_i = \sum_{j=0}^i \binom{d+1-j}{d+1-i} (-1)^{i-j} f_{j-1} \quad 0 \leq i < d+1$$

The following relations hold ([5]):

$$(2.2) \quad f_i = \sum_{j=0}^{d+1} \binom{d+1-j}{d-i} h_j \quad 0 \leq i \leq d$$

$$(2.3) \quad h_i = h_{d+1-i} \quad 0 \leq i \leq d+1$$

Finally, for $\mathcal{S} = \Delta(n+1, d+1)$,

$$(2.4) \quad f_i(\Delta(n+1, d+1)) = \binom{n+1}{i+1} \quad 0 \leq i \leq \left\lfloor \frac{1}{2}(d+1) \right\rfloor - 1.$$

The remaining values of $f_i(\Delta(n+1, d+1))$ $i \geq \lfloor (d+1)/2 \rfloor$ can be determined from (2.1)–(2.4).

Theorem 2.1 (Upper Bound Conjecture). [6] *for any simplicial d -sphere \mathcal{S} ,*

$$f_i(\mathcal{S}) \leq f_i(\Delta(n+1, d+1)) \quad 0 \leq i \leq d. \quad \blacksquare$$

In order to use Theorem 2.1 we imbed \mathcal{S} in a d -sphere in E^{d+1} . Let the points of S be $\bar{x}_i = (x_{i1}, \dots, x_{id}, 0)$, $1 \leq i \leq n$. Let S_d be the unit sphere in E^{d+1} with center at the origin. Let $\epsilon > 0$, and form a set S' by replacing each point \bar{x}_i lying in the interior of $C(S)$ by the 'lifted' point $\bar{x}'_i = (x_{i1}, \dots, x_{id}, \epsilon)$. Let S'' be the projection from the origin of S' onto S_d . S'' has the property that the points (x_1, \dots, x_{d+1}) on the 'equator', $x_{d+1} = 0$, are exactly the projections of the points of S on the boundary of $C(S)$. The other points of S'' are in the 'northern hemisphere' $x_{d+1} > 0$. S'' determines a d -sphere where the faces correspond to the simplices of \mathcal{S} , together with the (possibly) non-simplicial face in the equatorial plane $x_{d+1} = 0$. This face can be subdivided by adjoining to S'' the 'south pole', $(0, \dots, 0, 1)$.

For each $(d-1)$ face on the equator (the projections of the boundary faces of \mathcal{S}) we adjoin $(0, \dots, 0, 1)$ to form a d -face. The resulting complex is a simplicial d -sphere \mathcal{S}' .

We can now apply Theorem 2.1 to \mathcal{S}' and obtain

$$(2.5) \quad |\mathcal{S}'| \leq f_d(\Delta(d+1, d+1)).$$

But from our construction $|\mathcal{S}'| = |\mathcal{S}| + l$, where l is the number of $(d-1)$ -faces on the boundary of $C(S)$. Since the number is at least $d+1$, we get

$$(2.6) \quad |\mathcal{S}| = |\mathcal{S}'| - l \leq |\mathcal{S}'| - (d+1).$$

Theorem 2.2. $|\mathcal{S}| \leq f_d(\Delta(n+1, d+1)) - (d+1). \quad \blacksquare$

Corollary 2.3. $|\mathcal{F}| = O(n^{[(d+1)/2]}).$

Proof. If $\mathcal{S} = \Delta(n+1, d+1)$ we have for $0 \leq i < [(d+1)/2] - 1$ $f_i = O(n^{[(d+1)/2]})$ by (2.4). Then $h_i = O(n^{[(d+1)/2]})$ for $i \leq [(d+1)/2]$ by (2.1) and for $i > [(d+1)/2]$ by (2.3). Then (2.2) implies $f_i = O(n^{[(d+1)/2]})$ also for $i \geq [(d+1)/2]$. Hence in particular $f_d = O(n^{[(d+1)/2]})$, and by Theorem 2.2, $|\mathcal{F}| = O(n^{[(d+1)/2]}).$ ■

As an example we consider $d=3$. By applying (2.1)–(2.4) and Theorem 2.2 we get

$$|\mathcal{F}| \leq \frac{(n+1)(n-2)}{2} - 4.$$

For $n=4, 5, 6$, we get upper bounds respectively of 1, 5, 10. For $n=4$ the bound is exact. For $n=5$, the maximum value of $|\mathcal{F}|$ is 4, and for $n=6$ the maximum value $|\mathcal{F}|$ can have is 8.

If we remove any vertex from a simplicial d -sphere, the resulting polytope can be thought of as a triangulation of a set in E^d (by using stereographic projection onto E^d). So if we take the maximum case, which is $\Delta(n+1, d+1)$, then this has $(n+1)(n-2)/2$ d -simplices for $d=3$ using (2.1)–(2.4). Consider a vertex v of minimum 'degree', that is a vertex meeting a minimum number of d -simplices. Since each simplex has 4 vertices, and there are $n+1$ vertices, the average degree is $((n+1)(n-2)/2) \cdot 4/(n+1)$. Thus v has degree no larger than that, and removing v will yield a triangulation with at least

$$\left(\frac{(n+1)(n-2)}{2} \right) \left(1 - \frac{4}{n+1} \right) = \frac{(n-3)(n-2)}{2}$$

simplices. For $n=4, 5, 6$, this gives 1, 3, 6, respectively. These are lower bounds for the maximum of $|\mathcal{F}|$. That is, the maximum T_n of $|\mathcal{F}|$ satisfies

$$\frac{(n-3)(n-2)}{2} \leq T_n \leq \frac{(n+1)(n-2)}{2} - 4,$$

for $d=3$.

3. Sets with minimal triangulations

We first observe that since our hypothesis states that $C(S)$ has a nonempty interior, it follows that $C(S)$ remains connected if we delete all $(d-2)$ -simplices whose vertices are points of S .

Call this deleted set $C_0(S)$. Now for any triangulation \mathcal{F} and any two simplices $T_i, T_j \in \mathcal{F}$ there exists a path in $C_0(S)$ which joins an interior point of T_i to an interior point of T_j . This path must correspond to a path of $G(\mathcal{F})$. We have thus proved

Lemma 3.1. *The triangulation graph $G(\mathcal{F})$ is connected.* ■

Since $G(\mathcal{F})$ is connected we can order the vertices T_1, T_2, \dots, T_N so that each subgraph induced by $\{T_1, T_2, \dots, T_m\}$ is connected, $1 \leq m \leq N$.

Theorem 3.2. *We have*

$$(3.1) \quad |\mathcal{T}| \cong n - d$$

with equality if and only if $G(\mathcal{T})$ is a tree.

Proof. Let $\mathcal{T} = \{T_1, T_2, \dots, T_N\}$ be a triangulation so that the subgraph induced by $\{T_1, T_2, \dots, T_m\}$ is connected for every $m \leq N$. Then the set of vertices V_m which is the union of the vertices of T_1, T_2, \dots, T_m satisfies

$$(i) \quad |V_1| = d + 1$$

$$(ii) \quad |V_{m+1}| \leq |V_m| + 1$$

since T_{m+1} has at least d vertices in V_m . Thus

$$n = |V_N| \leq (d + 1) + (N - 1) = d + |\mathcal{T}|$$

which proves (3.1).

If $G(\mathcal{T})$ is a tree, then T_{m+1} is joined to exactly one T_i , $1 \leq i \leq m$ and therefore $|V_{m+1}| = |V_m| + 1$ for all m and (3.1) becomes an equality.

If $G(\mathcal{T})$ contains a cycle and T_{m+1} is the last vertex in that cycle, then all the vertices of T_{m+1} lie in V_m and $|V_{m+1}| = |V_m|$. Thus inequality (3.1) is strict in this case. ■

We can now characterize all trees which arise as triangulation graphs.

Theorem 3.3. *A tree τ is a triangulation graph, if and only if it has $n - d$ vertices and no vertex has valence greater than $d + 1$.*

Proof. The necessity of the first condition was proved in Theorem 3.2 and the necessity of the second condition is obvious since a d -simplex has only $d + 1$ faces. Now order the vertices T_1, \dots, T_{n-d} of τ so that $\{T_1, \dots, T_m\}$ induces a connected subtree for each $m \leq n - d$. We can now construct a set S so that $C(S)$ has a triangulation $\{T_1, \dots, T_{n-d}\}$ with graph τ as follows:

- (i) Pick the vertices of a simplex T .
- (ii) Assume the vertices of the simplices T_1, \dots, T_m have been picked so that they all lie on the boundary of their convex hull C_m and that all faces of C_m are faces of these simplices. Now if T_{m+1} is connected to T_i , $1 \leq i \leq m$ then one of the faces F_i of T_i is a face of C_m and we choose a point v_{m+1} exterior to C_m but so close to the centroid of F_i that the convex hull C_{m+1} of $C_m \cup \{v_{m+1}\}$ has all the faces of C_m with the exception of F_i . We now identify T_{m+1} as the convex hull of $F_i \cup \{v_{m+1}\}$. Then the triangulation graph of C_{m+1} corresponds to the induced subgraph $\{T_1, \dots, T_{m+1}\}$ of τ . This completes the proof by induction. ■

Next we characterize those triangulations \mathcal{T} of $C(S)$, if any, for which $G(\mathcal{T})$ is a tree.

Lemma 3.4. *The graph $G(\mathcal{T})$ of a triangulation $\mathcal{T} = \{T_1, T_2, \dots, T_N\}$ is a tree, if and only if no $(d - 2)$ -face of any of the T_i intersects the interior of $C(S)$.*

Proof. Assume that $G(\mathcal{F})$ contains a cycle $\{T_1, T_2, \dots, T_1\}$. This cycle can be represented by a polygonal path P in the interior of $C(S)$ whose vertices are the centroids of the faces $T_i \cap T_{i+1}$ (where $T_{i+1} = T_1$), and whose edges are the line segments joining successive vertices. If we consider the hyperplane H determined by $T_1 \cap T_2$, then we see that P must meet both the interior of $T_1 \cap T_2$ and the exterior of $T_1 \cap T_2$ in H since P is closed. Now if P shrinks to a point it remains in the interior of $C(S)$. But at some stage it will intersect one of the boundary $(d-2)$ -faces of $T_1 \cap T_2$. Thus this $(d-2)$ -face meets the interior of $C(S)$.

Conversely, let F be a $(d-2)$ -face of a simplex $T \in \mathcal{F}$ and let p be in the intersection of the relative interior of F with the interior of $C(S)$. Consider a circle C with center p lying in the 2-plane perpendicular to the plane of F and radius so small that it is interior to $C(S)$ and its relative interior does not intersect any $(d-2)$ -face other than F . Then the simplices T_i , which intersect C in cyclic order form a cycle of $G(\mathcal{F})$. ■

Corollary 3.5. *If $C(S)$ has a triangulation with $n-d$ simplices, then S lies on the boundary of $C(S)$.*

Proof. If a point $s \in S$ lies in the interior of $C(S)$ then every triangulation of $C(S)$ has a simplex T with vertex s , and all $(d-2)$ -faces of T which contain s meet the interior of $C(S)$. ■

Corollary 3.5 is by no means a sufficient condition for the existence of a triangulation \mathcal{F} for which $G(\mathcal{F})$ is a tree. For example, every simplex T which is a leaf of $G(\mathcal{F})$ must contain a vertex v of $C(S)$ so that all the incident 1-faces of $C(S)$ are 1-faces of T . In particular the number of 1-faces at v must be d . Thus for example, the regular octahedron has no tree triangulation since all vertices have 4 edges. (All the graphs are 4-cycles).

As a consequence of Lemma 3.4 we get a characterization of those S where all triangulations of $C(S)$ contain $n-d$ simplices.

Theorem 3.6. *All triangulations of $C(S)$ contain $n-d$ simplices if and only if all $(d-2)$ -simplices with vertices in S lie on the boundary of $C(S)$.*

Proof. This requires only the observation that any $(d-2)$ -simplex of S contains a $(d-2)$ -face of one of the simplices of a triangulation of $C(S)$. We see this inductively, starting with $n=d+1$, for which there is only one simplex. For $n>d+1$ let D' be any set of $d-1$ points of S forming a $(d-2)$ -simplex. $C(D')$ then contains a subset D of $d-1$ points of S forming a $(d-2)$ -simplex which is minimal in the sense that $C(D) \cap S = D$ ($D=D'$ is, of course, possible). Let $x \in S - D$ be an extreme point of $C(S)$ (i.e. $x \notin C(S - \{x\})$). By induction there will be a triangulation \mathcal{F}' of $C(S - \{x\})$ including D as a $(d-2)$ -face of one simplex. Now we simply adjoin to \mathcal{F}' all simplices exterior to $C(S - \{x\})$ which are formed by x together with a $(d-1)$ -face of a simplex of \mathcal{F}' . ■

As mentioned in (1.2), for $d=2$ the condition in Theorem 3.6 simply means that S lies in the boundary of $C(S)$. For $d=3$, Theorem 3.6 means that every two points of S can 'see' one another on the boundary of $C(S)$. According to a theorem of Buchman and Valentine [1] this means that $C(S)$ is either a cone whose base is a convex $(n-1)$ -gon (not necessarily strictly convex) or a convex

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How about
Stackable
decompositions
of polytopes
they all
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property!!!

and having
a degree
3 vertex is
not a
guarantee
either!
Dodecahedron

triangular prism (in the projective sense; that is, the triangular bases are not necessarily parallel, and the lateral edges lie on parallel lines or on concurrent lines) and $n-6$ points (other than the 6 corners) lie on the lateral edges of the prism. (There is a degenerate case of the prism where one of the bases is a single point.)

4. Characterization of sets S for which $G(\mathcal{T})$ is a tree for every triangulation of $C(S)$, $d \geq 3$

From Theorems 3.2 and 3.6 we know that the equivalent property that S must satisfy so that all triangulations are trees is:

$F(d)$: S is a d -dimensional set with all of its $(d-2)$ simplices on the boundary of $C(S)$.

Then the generalization from the $d=3$ case mentioned above is:

Theorem 4.1. For $d \geq 3$ S satisfies $F(d)$ if and only if S has one of the two following structures:

- (A_d): $C(S)$ is a 'prism' in the projective sense: $U = \{u_1, \dots, u_d\}$, $V = \{v_1, \dots, v_d\}$ are two 'bases' and $C(S)$ is the convex hull of $U \cup V$, where the lines $u_i v_i$ are either all parallel or all concurrent at a single point. In this case all the points in $S - (U \cup V)$ lie on the 'ribs' of the prism. (The degenerate case of $|U|=1$ or $|V|=1$ is possible.)
- (B_d): $C(S)$ is a 'cone': all points of S except one lie in a hyperplane H , and for $d > 3$ $H \cap S$ satisfies A_{d-1} or B_{d-1} , and for $d=3$ $C(H \cap S)$ is a convex polygon with $H \cap S$ on its boundary.

Corollary 4.2. A necessary and sufficient condition that every triangulation \mathcal{T} of $C(S)$ be a tree is that S satisfies A_d or B_d .

Proof of Theorem 4.1. The proof will be by induction on d . For $d=3$ the result is due to Buchman and Valentine [1], as mentioned above. Thus we assume $d > 3$ and that the theorem holds for all $d' < d$. The sufficiency of conditions A_d and B_d is easy to see. First assume A_d holds. Then any $(d-2)$ -simplex of S (i.e., formed by $d-1$ points of S) lies on at most $d-1$ of the ribs $u_i v_i$ of $C(S)$, and is thus on a boundary hyperplane. If B_d holds, then any $(d-2)$ -simplex of S either lies in the 'base' $H \cap S$, and is thus on the boundary of $C(S)$, or includes the 'apex' p and meets the base in a $(d-3)$ -simplex. But induction (applied to $H \cap S$ and using $F(d-1)$; see Lemma 4.4 below) implies that this $(d-3)$ -simplex is on the (relative) boundary of the base, and thus, when p is adjoined, the resulting $(d-2)$ -simplex is on the boundary of $C(S)$.

Before proving the necessity of A_d or B_d we require a few lemmas.

Lemma 4.3. Let $k \geq 3$, and suppose K is a set with $|K| \geq k+2$ and satisfying $F(k)$. Then some $k+1$ points of k lie on a common hyperplane.

Proof. Let P_1, \dots, P_{k+1} determine a k -simplex D . Then the $k+1$ face-hyperplanes of D divide E^k into $2^{k+1} - 1$ regions. Let Q be another point of K . If Q lies on any of the face-hyperplanes, then K has $k+1$ points in that hyperplane, and the con-

clusion of the lemma holds. We can assume then that Q is in the interior of one of the regions. In particular D itself is such a region. If Q is in D , then any $(k-2)$ -simplex containing Q is not on the boundary, violating $F(k)$. Thus Q is in the interior of one of the other regions R .

R can be described in a nice way. Let $I \subseteq \{1, \dots, k+1\}$ be the set of indices such that for $i \in I$, the face-hyperplane not containing P_i separates P_i and R . Let F_I denote the face of D determined by the $P_i, i \in I$, and let G_I denote the face determined by the $P_i, i \notin I$. Then (by linear algebra) for each point $r \in R$ there are unique points $f \in F_I$ and $g \in G_I$ so that the line extending from f through g meets r . If r is the interior point Q of R , then f is interior to F_I or $|I|=1$, and similarly g is interior to G_I or $|I|=k$. In any case, the line fg contains interior points of D , and hence g is on the interior of $C(K)$.

Now both the faces determined by G_I and by $F_I \cup \{Q\}$ contain g , and one of $|G_I|$ and $|F_I \cup \{Q\}|$ is no larger than $[k/2]+1$. Thus any $(k-2)$ -face containing it will contain g and thus violate $F(k)$. Hence Q is not an interior point of R , and the lemma is proved. ■

We note that we only used for this lemma what we might call $F'(k)$: S is a k -dimensional set with all its $[k/2]$ -dimensional simplices on the boundary of $C(S)$.

Now let S be a k -dimensional set. Let H be a hyperplane determined by k of the points of S .

Lemma 4.4. *If S satisfies $F(k)$, then $H \cap S$ satisfies $F(k-1)$ in $H, k \geq 3$.*

Proof. Suppose $k-2$ points of $S \cap H$ determine a $(k-3)$ -simplex D containing a point interior to $C(S \cap H)$, say q . Let p be a point of $S - (S \cap H)$. Then the line pq must have points interior to $C(S)$, contradicting the assumption that $C(D \cup \{p\})$ must be on the boundary of $C(S)$. ■

Lemma 4.5. *Consider the following cases:*

(i) S is a k -dimensional set, $k \geq 4$, and there is a $(k-2)$ -space H_{k-2} such that $S - (H_{k-2} \cap S) \supseteq \{a, b, c\}$, where a, b, c are not collinear, and the three lines they determine are all skew to H_{k-2} . Moreover, there is a set $Q \subseteq H_{k-2} \cap S$ disjoint from the plane of a, b, c such that Q is the vertex set of a $(k-2)$ -simplex.

(ii) S is a 5-dimensional set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, where each of the sets $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ are 2-dimensional and form strictly convex quadrilaterals.

(iii) S is a 4-dimensional set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$ where a_1, a_2, a_3, a_4 are coplanar and form a strictly convex quadrilateral, and the points b_1, b_2, b_3 are collinear.

Then in cases (i), (ii), (iii) respectively S does not satisfy $F(k), F(5), F(4)$.

Proof. In case (i) assume $F(k)$. We derive a contradiction. Applying Lemmas 4.3 and 4.4 repeatedly to $Q \cup \{a, b, c\}$, we must eventually obtain a 2-dimensional subset F of four points (satisfying $F(2)$). F must contain all three of a, b, c for otherwise it would be a subset of one of the sets $Q \cup \{a, b\}, Q \cup \{a, c\}, Q \cup \{b, c\}$

... then that Q is in the interior of ...
 ... a region. If Q is in D , then any ...
 ... boundary, violating $F(k)$. Thus Q is in ...
 ... let $I \subseteq \{1, \dots, k+1\}$ be the set of indices ...
 ... not containing P_i separates P_i and R . Let ...
 ... the $P_i, i \in I$, and let G_I denote the face ...
 ... near algebra) for each point $r \in R$ there ...
 ... that the line extending from f through ...
 ... then f is interior to F_I or $|I|=1$, and ...
 ... in any case, the line fg contains interior ...
 ... or of $C(K)$.
 ... G_I and by $F_I \cup \{Q\}$ contain g , and one ...
 ... $[k/2]+1$. Thus any $(k-2)$ -face containing ...
 ... since Q is not an interior point of R , and

... lemma what we might call
 ... $[k/2]$ -dimensional simplices on the bound

Let H be a hyperplane determined by

... satisfies $F(k-1)$ in $H, k \geq 3$.
 ... determine a $(k-3)$ -simplex D containing
 ... be a point of $S - (S \cap H)$. Then the
 ... contradicting the assumption that $C(D \cup S)$

... and there is a $(k-2)$ -space H_{k-2}
 ... a, c are not collinear, and the three
 ... cover, there is a set $Q \subseteq H_{k-2} \cap S$ disjoint
 ... vertex set of a $(k-2)$ -simplex.

... $a_3, a_4, b_1, b_2, b_3, b_4$, where each of
 ... 2-dimensional and form strictly convex
 ... $a_2, a_3, a_4, b_1, b_2, b_3$ where a_1, a_2, b_1, b_2
 ... quadrilateral, and the points b_1, b_2, b_3
 ... does not satisfy $F(k), F(5), F(4)$.

... a contradiction. Applying Lemma
 ... must eventually obtain a 2-dimen
 ... F must contain all three of $a, b,$
 ... the sets $Q \cup \{a, b\}, Q \cup \{a, c\}, Q \cup$

all of which are simplices, since ab, ac, bc are skew to H_{k-2} . So $F = \{a, b, c, q\}$,
 $q \in Q$. This contradicts the assumption that Q was disjoint from the plane abc .
 In case (ii) we may take the points to be:

$$\begin{aligned} a_1 &= (00000) & b_1 &= (00100) \\ a_2 &= (00001) & b_2 &= (01000) \\ a_3 &= (00010) & b_3 &= (10100) \\ a_4 &= (00011) & b_4 &= (11000) \end{aligned}$$

Then the 3-face determined by a_2, a_3, b_2, b_3 contains the point $(1/4, 1/4, 1/4, 1/4, 1/4)$.
 an interior point of $C(S)$. Thus $F(5)$ is not satisfied.

In case (iii) we may take the points to be:

$$\begin{aligned} a_1 &= (0001) & b_1 &= (1000) \\ a_2 &= (0010) & b_1 &= (0100) \\ a_3 &= (0011) & b_3 &= (x, 1-x, 0, 0) \\ a_4 &= (0000) \end{aligned}$$

Then the 2-plane determined by a_1, a_2, b_3 contains the point $(x/3, (1-x)/3, 1/3, 1/3)$,
 an interior point of $C(S)$, violating $F(4)$.

We now proceed to prove the necessity of conditions A_d and B_d . Recall
 that $d \geq 4$. Assume that S satisfies $F(d)$, and let H be a hyperplane so that $H \cap S$
 has the maximum possible number of points of S . By Lemma 4.4 $H \cap S$ satisfies
 $F(d-1)$ in H . There are two possibilities:

- (a) $H \cap S$ satisfies A_{d-1}
- (b) $H \cap S$ satisfies B_{d-1} .

In either case, if $S - (S \cap H)$ has only one point, then S satisfies B_d , and the con-
 clusion of the theorem is satisfied. So we can assume $S - (S \cap H)$ contains at least
 two points x and y .

Consider (b) first. Let p be the apex of the cone $H \cap S$, and let H_{d-2} be the
 $(d-2)$ -space containing the base $B = H_{d-2} \cap S$. $B \cup \{x, y\}$ must be d -dimensional,
 or any hyperplane containing $B \cup \{x, y\}$ would violate the maximality of H . B
 must be $(d-2)$ -dimensional (not less) for the same reason. Thus xy is skew to H_{d-2} .
 Similarly px and py are skew to H_{d-2} . Either p, x, y are collinear, or they determine
 a plane P .

Assume first pxy is a plane P . Then by Lemma 4.5 (b) this leads to a contra-
 diction unless the plane P also contains points of B , and $B - P$ is contained in
 some $(d-3)$ -space H_{d-3} . $B \cap P$ actually contains exactly one point, q . For if it
 contains two points, then $P \cap H_{d-2}$ contains a line L . Not all of xy, yp, xp can
 avoid L , and thus must meet H_{d-2} . This would violate the fact that these lines
 are skew to H_{d-2} . So P contains x, y, p, q . Further, $xypq$ forms a strictly convex

quadrilateral. To see this, apply Lemmas 4.3 and 4.4 repeatedly to the set $\{x, y, p, q\} \cup B'$, where $B' \cup \{q\}$ is a $(d-2)$ -simplex in B . We ultimately get a 2-dimensional set F satisfying $F(2)$. Since $B' \cup \{q, x, y\}$, $B' \cup \{q, x, p\}$, $B' \cup \{q, p, y\}$ are simplices, the only possibility for the set F is $\{x, y, p, q\}$. So F is convex, by $F(2)$, and F is strictly convex since xy, xp, yp are skew to B , and x, y, p are assumed not to be collinear. Now in H_{d-3} suppose there are only $d-2$ points of S . Then if $r \in H_{d-3}$, there is a hyperplane determined by $[(S \cap H_{d-3}) - \{r\}] \cup \{x, y, p, q\}$, which violates the maximality of H . Hence H_{d-3} contains at least $(d-1)$ points of S . Then by repeated application of Lemmas 4.3 and 4.4, if $d \geq 5$, then H_{d-3} must contain a coplanar set of 4 points, satisfying $F(2)$, and hence convex, say a_1, a_2, a_3, a_4 . But if this set is strictly convex, Lemma 4.5 (ii) applies to $\{x, y, p, q, a_1, a_2, a_3, a_4\}$, which then does not satisfy $F(5)$. This leads to a contradiction from repeated application of Lemma 4.4 to S . On the other hand, if a_1, a_2, a_3, a_4 is not strictly convex, then three are collinear, say a_1, a_2, a_3 . But now Lemma 4.5 (iii) applies to $\{x, y, p, q, a_1, a_2, a_3\}$, leading to the contradiction that this set does not satisfy $F(4)$, as in the previous case. There remains the case $d=4$. Here H_{d-3} is a line with at least $(4-1)=3$ points, and once again Lemma 4.5 (iii) applies and leads to a contradiction. This completes the case where p, x, y determine a plane.

Now assume pxy is a line, which is skew to H_{d-2} . By induction and Lemma 4.4, B is either a cone or a prism. If it is a cone with apex p' , then $(B - \{p'\}) \cup \{x, y, p\}$ is a $(d-1)$ -dimensional set, violating maximality of H . So B is a prism and not a cone. Then any of its 2-dimensional faces (determined by two of its parallel (or concurrent) ribs) has four points forming a strictly convex quadrilateral. This quadrilateral, together with x, y, p must satisfy $F(4)$, by repeated use of Lemma 4.4. But this contradicts Lemma 4.5 (iii). This completes the case where pxy is a line, and thus case (b), providing a contradiction for every case with $S - (S \cap H) \supseteq \{x, y\}$.

Now consider case (a): $H \cap S$ satisfies A_{d-1} . Let $U = \{u_1, \dots, u_{d-1}\}$ be one of the base simplices of the prism $H \cap S$. If $H \cap S - \{u_i\}$ is $(d-2)$ -dimensional, then case (b) above is applicable. Thus we may assume the prism has two disjoint simplicial bases, U and $V = \{v_1, \dots, v_{d-1}\}$, with possibly, but not necessarily, one more point at the common intersection of $u_i v_i$ (if it exists).

Now we may assume that x and y are chosen so that the line xy is skew to at least one of the lateral faces of the prism. For if not, then all possible choices of x and y are on a line including a point common to all faces of the prism. But then all points of $S - (S \cap H)$ must be collinear and together with $S \cap H$ form a d -dimensional prism, satisfying A_d . Let the face F skew to xy be determined by $\{u_2, \dots, u_{d-1}\} \cup \{v_2, \dots, v_{d-1}\}$. At least one of the two triples x, y, u_1 or x, y, v_1 will not be collinear, since xy is skew to F , say x, y, u_1 . Now xu_1 and yu_1 are also skew to F , since F and u_1 are in H whereas x and y are not. Then the $(d-2)$ -space containing F , together with x, y and u_1 satisfy Lemma 4.5 (i). This implies S does not satisfy $F(d)$, a contradiction. This completes the proof of Theorem 4.1. ■

5. Maximizing the volume of the minimal simplex determined by n points S with given volume of $C(S)$

As before we consider n -tuples S of points in E^d not all on a hyperplane and now define

$$f(S) = \min_T \text{vol}(T) / \text{vol}(C(S))$$

where T ranges all over nondegenerate simplices with vertices in S . Set $f_d(n) = \sup_S f(S)$. From Theorem 3.2 we get

Theorem 5.1. $f_d(n) = 1/(n-d)$

Proof. The inequality $f_d(n) \leq 1/(n-d)$ is an immediate consequence of Theorem 3.2. To see that the upper bound is attained let S be the vertices of a simplex, T , plus $n-d-1$ equally spaced points on one of the edges E of T . Here $C(S)$ has exactly one triangulation $\mathcal{T} = \{T_1, \dots, T_{n-d}\}$ where each T_i contains the $(d-2)$ -face F opposite to E in T and two consecutive vertices on E . ■

We can actually characterize all d -dimensional sets S of n points so that $f(S) = f_d(n)$. Let S be such a set, and let \mathcal{T} be any triangulation of $C(S)$. By Lemma 3.2 and Theorem 5.1, \mathcal{T} has exactly $n-d$ simplices, all equal in volume. It is easy to see that for $d=1$ S must consist of n points equally spaced on a line, and for $d=2$ either S consists of points equally spaced on two parallel lines, with the spacing the same on both lines, or else S is the set S_6 of six points formed by the vertices of a triangle together with the midpoints of the edges. This can be generalized to arbitrary d as follows:

Theorem 5.2. Let S be a d -dimensional set of n points. Then $f(S) = f_d(n)$ if and only if one of the following conditions holds.

(A_d): $C(S)$ is a prism with parallel ribs (the bases are not necessarily parallel) and S divides these ribs all into equal length segments. That is, if $S' = \{u_1, \dots, u_d\}$, $S'' = \{v_1, \dots, v_d\}$ are the two bases, then $C(S)$ is the convex hull of $S' \cup S''$. The ribs $u_i v_i$ are all parallel. Let l_i be the length of $u_i v_i$, and m_i the number of points of S on $u_i v_i$. Then $l_i / (m_i - 1) = l_j / (m_j - 1)$ for all i, j with $m_i, m_j > 1$, and the m_i points are equally spaced on $u_i v_i$. We allow the degenerate cases where $m_i = 1$, $l_i = 0$, $u_i = v_i$, and $u_i v_i$ is considered parallel to any line.

(B_d): $C(S)$ is a cone, where S consists of S_6 together with $d-2$ other points.

Proof. For $d=1, 2$ we already saw that A_d or B_d holds, and they are clearly sufficient for $f(S) = f_d(n)$. Since for any \mathcal{T} we saw that there are $n-d$ simplices, all of equal volume, we can invoke Lemma 3.6 and Theorem 4.1 to conclude for $d \geq 3$ that S satisfies A_d or B_d. First consider the case A_d. It is easy to see that in this case the ribs of the prism must be parallel, and all are subdivided by S into the same length equal segments. For if not there will be two 'basic' d -simplices of different volumes, where a d -simplex T of S is 'basic' if its $d+1$ vertices are the only points of S it contains.

Since any basic simplex is part of some triangulation \mathcal{T} (by the same argument as in Theorem 3.6), this would violate the observations about \mathcal{T} above. Thus A_d is satisfied.

why does it follow from

On the other hand, suppose B_d holds. We prove by induction on d that A'_d or B'_d holds. For $d \geq 3$ we have a cone with apex p and a base hyperplane H so that $S - (S \cap H) = \{p\}$. Then every basic $(d-1)$ -simplex of $S \cap H$ must have the same volume, or by adjoining p we would obtain basic d -simplices of different volumes. Then $H \cap S$ satisfies A'_{d-1} or B'_{d-1} by induction. If A'_{d-1} is satisfied by $H \cap S$, then S satisfies A'_d , where p is a degenerate rib of the prism. If $H \cap S$ satisfies B'_{d-1} , then S satisfies B'_d .

Conversely, it is easy to see that A'_d and B'_d imply that all basic simplices T have the same volume. This gives $f(S) = f_d(n)$ and completes the proof. ■

6. Further Problems

6.1 Characterize triangulation graphs which are not trees.

6.2 Characterize the different sets of triangulations that arise from a single set S . In the case where all triangulations are trees, there are only two possibilities by Theorem 4.1. First, if S consists of some points determining an l -dimensional prism, together with $d-l$ points in general position, then only one tree occurs, which is the path of $n-d$ points. On the other hand, the only other alternative is that S consists of $n-d+2$ points on a convex polygon together with $d-2$ points in general position. By Theorem 3.3 then the trees arising in this case all have $n-d$ vertices and vertex degrees at most 3. It may be that all such trees are in the set, or only some subset of them depending on how many of the $n-d+2$ points are actually vertices of the polygon, for instance.

6.3 Characterize sets with a unique triangulation graph. Besides the examples mentioned in the previous problem, there is the example of the vertices of the regular octahedron, which determine only the 4-cycle C_4 as a triangulation graph.

6.4 Determine how close to general position the points of S may be and still have $f(S) = O(n^{-1})$. For $d=2$, we know that if we don't allow three points colinear, then the ratio for the Heilbronn problem is $O(n^{-\mu})$, $\mu > 1$. If we allow \sqrt{n} points on a line, then considering the $\sqrt{n} \times \sqrt{n}$ square lattice we see that $f(S) = O(n^{-1})$. We conjecture that if we allow no more than $n^{1/2-\epsilon}$ points to be colinear, then $f(S) = o(n^{-1})$. In general, we suspect that $n^{((d-1)/d)}$ is the corresponding critical number for points on a hyperplane in d -dimensions.

6.5 Generalize the results of [2] on the ratio of the areas of maximal and minimal triangles to d -dimensions. This is given as problem 1 [2]: Let S be a set of n points not all on a hyperplane in E^d . Let $g(S)$ denote the ratio of the minimal to the maximal volume of nondegenerate simplices with vertices in S . Find $g_d(n) = \sup g(S)$.

The conjecture, by analogy, is that

$$g_d(n) = \frac{1}{[(n-1)/d]}.$$

Note. The authors wish to thank the referee for several careful observations of errors in the original version of the manuscript.

We prove by induction on d that if S is a set of n points in an apex p and a base hyperplane H , then the $(d-1)$ -simplex of $S \cap H$ must have d basic d -simplices of different orientations. If A_{d-1} is satisfied, then the d -simplex of the rib of the prism. If $H \cap S$

and B'_d imply that all basic simplices are of different orientations. $f_d(n)$ and completes the proof.

problems

are not trees.

relations that arise from a single set of points, there are only two possibilities for the points determining an l -dimensional simplex in a given position, then only one tree occurs. On the other hand, the only other alternative is a convex polygon together with $d-2$ points. The trees arising in this case all have $n-d$ leaves. It may be that all such trees are in the same class. How many of the $n-d+2$ points are on the boundary?

triangulation graph. Besides the examples mentioned above, there is an example of the vertices of the regular simplex C_d as a triangulation graph.

the points of S may be and still have $f_d(n)$ if we don't allow three points colinear. For $d=2$, $f_2(n) = O(n^{-\mu})$, $\mu > 1$. If we allow \sqrt{n} points to be colinear, we see that $f_2(S) = O(n^{-\mu})$. For $d=3$, we see that $f_3(S) = O(n^{-\mu})$ if we allow $n^{1/3-\epsilon}$ points to be colinear, then $f_3(S) = O(n^{-(d-1)/d})$ is the corresponding critical exponent in d dimensions.

of the areas of maximal and minimal simplices. Problem 1 [2]: Let S be a set of n points in the plane. Let $g_d(n)$ be the ratio of the minimal to the maximal area of a simplex with vertices in S . Find $g_d(n) = \sup \{ \frac{\text{min area}}{\text{max area}} \}$.

$\frac{1}{(d-1)/d}$

for several careful observations. See the appendix.

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B. L. Rotschild and E. G. Straus

Department of Mathematics
U.C.L.A.
Los Angeles, CAL 90024
U.S.A.