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# Sylvester Waves in the Coxeter Groups*

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**Abstract.** A new recursive procedure of the calculation of partition numbers function $W(s, \mathbf{d}^m)$ is suggested. We find its zeroes and prove a lemma on the function parity properties. The explicit formulas of $W(s, \mathbf{d}^m)$ and their periods $\tau(G)$ for the irreducible Coxeter groups and a list for the first twelve symmetric group $\mathcal{S}_m$ are presented. A *least common multiple* $\mathsf{lcm}(m)$ of the series of the natural numbers $1, 2, \ldots, m$ plays a role in the period $\tau(\mathcal{S}_m)$ of $W(s, \mathbf{d}^m)$ in $\mathcal{S}_m$.

**Key words:** partitions, asymptotic of arithmetic functions, Coxeter groups

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## 1. Introduction

More than hundred years ago J.J. Sylvester stated [11, 12] and proved [13] a theorem about restricted partition number $W(s, \mathbf{d}^m)$ of positive integer $s$ with respect to the $m$-tuple of positive integers $\mathbf{d}^m = \{d_1, d_2, \ldots, d_m\}$:

**Theorem.** *The number $W(s, \mathbf{d}^m)$ of ways in which s can be composed of (not necessarily distinct) m integers $d_1, d_2, \ldots, d_m$ is made up of a finite number of waves*

$$W(s, \mathbf{d}^m) = \sum_q^{max\ q} W_q(s, \mathbf{d}^m), \quad W_q(s, \mathbf{d}^m) = \sum_k^{max\ k} W_{p_k|q}(s, \mathbf{d}^m), \tag{1}$$

*where q run over all distinct factors in $d_1, d_2, \ldots, d_m$ and $W_{p_k|q}(s, \mathbf{d}^m)$ denotes the coefficient of $t^{-1}$ in the series expansion in ascending powers of t of*

$$F(s, \mathbf{d}^m, k; t) = e^{sw_k} \prod_{r=1}^m \frac{1}{1 - e^{d_r u_k}}, \quad w_k = 2\pi i\, \frac{p_k}{q} + t, \quad u_k = 2\pi i\, \frac{p_k}{q} - t, \tag{2}$$

*and $p_1, p_2, \ldots, p_{max\ k}$ are all numbers (unity included) less than q and prime to it.*

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$W(s, \mathbf{d}^m)$ is also a number of sets of positive integer solutions $(x_1, x_2, \ldots, x_m)$ of the equation $\sum_r^m d_r x_r = s$. It is known that $W(s, \mathbf{d}^m)$ is equal to the coefficient of $t^s$ in the expansion of generating function

$$M(\mathbf{d}^m, t) = \prod_{r=1}^{m} \frac{1}{1 - t^{d_r}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)\, t^s. \tag{3}$$

If the exponents $d_1, d_2, \ldots, d_m$ become the series of integers $1, 2, 3, \ldots, m$, the number of waves is $m$ and $W(s, \mathbf{d}^m)$ of $s$ is usually referred to as a restricted partition number $\mathcal{P}_m(s)$ of $s$ into parts none of which exceeds $m$.

Another definition of $W(s, \mathbf{d}^m)$ comes from the polynomial invariant of finite reflection groups. Let $M(\mathbf{d}^m, t)$ is a Molien function of such a group $G$, $d_r$ are the degrees of basic invariants, and $m$ is the number of basic invariants [9]. Then $W(s, \mathbf{d}^m)$ gives a number of algebraic independent polynomial invariants of the $s$-degree for group $G$.

Throughout his papers J.J. Sylvester gave different names for $W(s, \mathbf{d}^m)$: *quotity, denumerant, quot-undulant* and *quot-additant*. Sometime after he discarded some of them. Because of the wide usage of $W(s, \mathbf{d}^m)$ not only as a partition number we shall call $W(s, \mathbf{d}^m)$ a *Sylvester wave*.

The Sylvester theorem is a very powerful tool not only in the trivial situation when $m$ is finite but also it was used for the purposes of asymptotic evaluations of $\mathcal{P}_m(s)$, as well as for the main term of the Hardy-Ramanujan formulas for unrestricted partition number $\mathcal{P}(s)$ [14].

Recent progress in the self-dual problem of effective isotropic conductivity in two-dimensional three-component regular checkerboards [5] and its further extension on the $m$-component anisotropic cases [6] have shown the existence of algebraic equations with permutation invariance with respect to the action of the finite group $G$ permuting $m$ components. $G$ is a subgroup of symmetric group $\mathcal{S}_m$ and the coefficients in the equations are built out of algebraic independent polynomial invariants for group $G$. Here $W(s, \mathbf{d}^m)$ measures a degree of non-universality of the algebraic solution with respect to the different kinds of $m$-color plane groups.

Several proofs of Sylvester theorem are known [3, 13]. All of them make use of the Cauchy's theory of residues. The recursion relations imposed on $W(s, \mathbf{d}^m)$ provide a combinatorial version of Sylvester formula. The classical example for the elementary (complex-variable-free) derivation was shown by Erdös [4] for the main term of the Hardy-Ramanujan formula. Recently an elementary derivation of Szekeres' formula for $W(s, \mathbf{d}^m)$ based on the recursion satisfied by $W(s, \mathbf{d}^m)$ was elaborated in [2]. In this paper we give a new derivation of the Sylvester waves based on the recursion relation for $W(s, \mathbf{d}^m)$. We find also its *zeroes* and prove a lemma on parity properties of the Sylvester waves. Finally we present a list of the first twelve Sylvester waves $W(s, \mathcal{S}_m), m = 1, \ldots, 12$ for symmetric groups $\mathcal{S}_m$ and for all Coxeter groups.

## 2. Recursion relation for $W(s, \mathbf{d}^m)$

We start with a recursion that follows from (3)

$$M(\mathbf{d}^m, t) - M(\mathbf{d}^{m-1}, t) = t^{d_m} M(\mathbf{d}^m, t), \quad (4)$$

and after inserting the series expansions into the last equation we arrive at

$$W(s, \mathbf{d}^m) = W(s, \mathbf{d}^{m-1}) + W(s - d_m, \mathbf{d}^m), \quad d_m \le s, \quad (5)$$

where $s$ is assumed to be real. We apply now the recursive procedure (5) several times

$$W(s, \mathbf{d}^m) = \sum_{p=0}^{r_m} W(s - p \cdot d_m, \mathbf{d}^{m-1}) + W(s - (r_m + 1) \cdot d_m, \mathbf{d}^m). \quad (6)$$

Let us consider *the generic form* of $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m),\ s < \tau\{\mathbf{d}^m\}$ where $k,\ s$ and $\tau\{\mathbf{d}^m\}$ are the independent positive integers. We will choose them in such a way that

$$k \cdot \tau\{\mathbf{d}^m\} + s - (r_m + 1) \cdot d_m = (k - 1) \cdot \tau\{\mathbf{d}^m\} + s, \quad \Rightarrow \tau\{\mathbf{d}^m\} = (r_m + 1) \cdot d_m. \quad (7)$$

Thus the relation (6) reads

$$W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) = W(\,(k - 1) \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) + \sum_{p=0}^{\delta_m - 1} W(k \cdot \tau\{\mathbf{d}^m\} - p \cdot d_m + s, \mathbf{d}^{m-1}), \delta_m = \frac{\tau\{\mathbf{d}^m\}}{d_m}. \quad (8)$$

As follows from (7), in order to return via the recursive procedure from $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$ to $W((k - 1) \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$ we must use $\tau\{\mathbf{d}^m\}$ which have $d_m$ as a divisor. Due to the arbitrariness of $d_m$ it is easy to conclude that all exponents $d_1, d_2, \ldots, d_m$ serve as the divisors of $\tau\{\mathbf{d}^m\}$. In other words $\tau\{\mathbf{d}^m\}$ is the *least common multiple* lcm of the exponents $d_1, d_2, \ldots, d_m$

$$\tau\{\mathbf{d}^m\} = \mathrm{lcm}(d_1, d_2, \ldots, d_m). \quad (9)$$

Actually $\tau\{\mathbf{d}^m\}$ does play a rôle in the "*period*" of $W(s, \mathbf{d}^m)$. But strictly speaking it is not a periodic function with respect to the integer variable $s$ as could be seen from (8). The rest of the paper clarifies this hidden periodicity.

As we have mentioned above, $W(s, \mathbf{d}^m)$ gives a number of algebraic independent polynomial invariants of the $s$-degree for the group $G$. The situation becomes more transparent if we deal with the irreducible Coxeter group where the degrees $d_r$ and the number of basic invariants $m$ are well known.

The periods $\tau$ of the irreducible Coxeter groups are given in Table 1.

*Table 1.* The "*periods*" $\tau(G)$ of $W(s, \mathbf{d}^m)$ for the irreducible Coxeter groups.

| $G$ | $A_m$ | $B_m$ | $D_m$ | $G_2$ | $F_4$ | $E_6$ |
|---|---|---|---|---|---|---|
| $\tau(G)$ | $\mathrm{lcm}(m+1)$ | $2\,\mathrm{lcm}(m)$ | $2\,\mathrm{lcm}(m)$ | 6 | 24 | 360 |
| $G$ | $E_7$ | $E_8$ | $H_3$ | $H_4$ | $I_2(2m)$ | $I_2(2m+1)$ |
| $\tau(G)$ | 2520 | 2520 | 30 | 60 | $2m$ | $2\,(2m+1)$ |

where $\mathrm{lcm}(m)$ is the *least common multiple* of the series of the natural numbers $1, 2, \ldots, m$.

$\mathrm{lcm}(m)$ can be viewed as $\tau(\mathcal{S}_m)$ for symmetric group $\mathcal{S}_m$ or, in other words, as a "*period*" of the restricted partition number $\mathcal{P}_m(s)$. $\mathrm{lcm}(m)$ is a very fast growing function: $\mathrm{lcm}(10) = 2520$, $\mathrm{lcm}(20) = 232792560$, $\mathrm{lcm}(30) = 2329089562800$ etc. Actually $\frac{\ln \mathrm{lcm}(m)}{m}$ oscillates infinitely many times around 1 and according to Landau [15] the function $\mathrm{lcm}(m)$ grows exponentially with the asymptotic law

$$\ln \mathrm{lcm}(m) = m + O(\sqrt{m}\,\ln m). \tag{10}$$

## 3. Polynomial representation for $W(s, \mathbf{d}^m)$

Making use of the relations (8, 9) we obtain the exact formula for $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$ for different $\mathbf{d}^m$. We will treat it in an ascending order in the number $m$ of exponents. The first steps are simple and they yield

$\mathbf{d}^1 = (d_1)$, $\tau\{\mathbf{d}^1\} > s \geq 0$

$$W(k \cdot d_1 + s, \mathbf{d}^1) = W(s, \mathbf{d}^1) = \Psi_{d_1}(s) = \begin{cases} 1, & s = 0 \pmod{d_1} \\ 0, & s \neq 0 \pmod{d_1} \end{cases} \tag{11}$$

$\Psi_{d_1}(s)$ may be represented as a sum of prime roots of unit of degree $d_1$:

$$\Psi_{d_1}(s) = \frac{1}{d_1}\sum_{k=0}^{d_1-1} \exp\left(\frac{2\pi i k s}{d_1}\right) = \frac{1}{d_1}\begin{cases} 1 + \cos \pi s + 2\sum_{k=1}^{d_1/2-1} \cos \frac{2\pi k s}{d_1}, & \text{even } d_1 \\ 1 + 2\sum_{k=1}^{(d_1-1)/2} \cos \frac{2\pi k s}{d_1}, & \text{odd } d_1 \end{cases}.$$

$\mathbf{d}^2 = (d_1, d_2)$, $\tau\{\mathbf{d}^2\} > s \geq 0$

$$W(k \cdot \tau\{\mathbf{d}^2\} + s, \mathbf{d}^2) = W(s, \mathbf{d}^2) + k \cdot \sum_{p=0}^{\delta_2 - 1} W(|s - p\, d_2|, \mathbf{d}^1). \tag{12}$$

$\mathbf{d}^3 = (d_1, d_2, d_3)$, $\tau\{\mathbf{d}^3\} > s \geq 0$

$$W(k \cdot \tau\{\mathbf{d}^3\} + s, \mathbf{d}^3) = W(s, \mathbf{d}^3) + k \cdot \sum_{p=0}^{\delta_3 - 1} W(|s - p\, d_3|, \mathbf{d}^2) + \frac{k(k+1)}{2}\,\frac{\tau\{\mathbf{d}^3\}}{\tau\{\mathbf{d}^2\}} \sum_{p=0}^{\delta_3-1}\sum_{q=0}^{\delta_2-1} W(|s - p\, d_3 - q\, d_2|, \mathbf{d}^1). \tag{13}$$

Now it is simple to deduce by induction that in the general case $W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$ has a polynomial representation with respect to $k$

$$W(k \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) = A_{m-1}^m(s)k^{m-1} + A_{m-2}^m(s)k^{m-2} + \cdots + A_1^m(s)k + A_0^m(s, \mathbf{d}^m), \quad (14)$$

where $A_{m-r}^m(s)$ is based on the $\tau\{\mathbf{d}^r\}$-periodic functions as well as the entire $W(s, \mathbf{d}^m)$ is based on the $\tau\{\mathbf{d}^m\}$-periodic functions. The coefficient of the leading term can be written in a closed form

$$A_{m-1}^m(s) = \frac{1}{(m-1)!} \cdot \frac{\tau^{m-2}\{\mathbf{d}^m\}}{\tau\{\mathbf{d}^2\} \cdot \tau\{\mathbf{d}^3\} \cdot \cdots \cdot \tau\{\mathbf{d}^{m-1}\}} \times \sum_{p=0}^{\delta_m - 1} \sum_{q=0}^{\delta_{m-1}-1} \cdots \sum_{v=0}^{\delta_2 - 1} W(|s - p\, d_m - q\, d_{m-1} - \cdots - v\, d_2|, \mathbf{d}^1). \quad (15)$$

With $d_1 = 1$ we have $W(|s - p\, d_m - q\, d_{m-1} - \cdots - v\, d_2|, 1) = 1$, which makes $A_{m-1}^m(s)$ independent of $s$ and gives an asymptotics of $W(s, \mathbf{d}^m)$ for $s \gg m$

$$A_{m-1}^m(s) = \frac{\tau^{m-1}\{\mathbf{d}^m\}}{(m-1)!\, m!}, \qquad W(s, \mathbf{d}^m) \overset{s\to\infty}{\simeq} \frac{s^{m-1}}{(m-1)!\, m!}. \quad (16)$$

Now we are ready to prove the statement about splitting of $W(s, \mathbf{d}^m)$ into periodic and non-periodic parts.

**Lemma 3.1.** *The Sylvester wave $W(s, \mathbf{d}^m)$ can be represented in the following way*

$$W(s, \mathbf{d}^m) = Q_m^m(s) + \sum_{j=1}^{m-1} Q_j^m(s) \cdot s^{m-j}, \quad (17)$$

*where $Q_j^m(s)$ is a periodic function with the period $\tau\{\mathbf{d}^j\} = \mathrm{lcm}(d_1, d_2, \ldots, d_j)$.*

**Proof:** We start with the identity for the polynomial representation for $W(k\cdot\tau\{\mathbf{d}^m\} + s, \mathbf{d}^m)$

$$W((k+1) \cdot \tau\{\mathbf{d}^m\} + s, \mathbf{d}^m) = W(k \cdot \tau\{\mathbf{d}^m\} + s + \tau\{\mathbf{d}^m\}, \mathbf{d}^m),$$

that can be transformed, using (14), into

$$\begin{aligned} &A_{m-1}^m(s)\,(k+1)^{m-1} + A_{m-2}^m(s)\,(k+1)^{m-2} + \cdots + A_1^m(s)\,(k+1) + W(s, \mathbf{d}^m) \\ &\quad = A_{m-1}^m(s + \tau\{\mathbf{d}^m\})\,k^{m-1} + A_{m-2}^m(s + \tau\{\mathbf{d}^m\})\,k^{m-2} + \cdots + A_1^m(s + \tau\{\mathbf{d}^m\})\,k \\ &\qquad + W(s + \tau\{\mathbf{d}^m\}, \mathbf{d}^m). \end{aligned} \quad (18)$$

The last identity generates a finite number of coupled difference equations for the coefficients $A_r^m(s)$

$$A_{m-r}^m(s + \tau\{\mathbf{d}^m\}) = \sum_{j=1}^{r} C_{m-j}^{m-r} \cdot A_{m-j}^m(s), \quad 1 \le r \le m, \quad (19)$$

where $C_n^k$ denotes a binomial coefficient. The first equation ($r = 1$)

$$A_{m-1}^m(s + \tau\{\mathbf{d}^m\}) = A_{m-1}^m(s)$$

declares that $A_{m-1}^m(s)$ is an arbitrary $\tau\{\mathbf{d}^m\}$-periodic function. We can specify the last statement taking into account (14) that actually $A_{m-1}^m(s)$ is $\tau\{\mathbf{d}^1\}$-periodic function which will be denoted as $Q_1^m(s)$. The second equation ($r = 2$)

$$A_{m-2}^m(s + \tau\{\mathbf{d}^m\}) = A_{m-2}^m(s) + (m-1) \cdot A_{m-1}^m(s)$$

can be solved completely

$$A_{m-2}^m(s) = Q_2^m(s) + (m-1) \cdot s \cdot Q_1^m(s), \tag{20}$$

where $Q_2^m(s + \tau\{\mathbf{d}^2\}) = Q_2^m(s)$. Continuing this procedure, it is not difficult to prove by induction that for any $r$ we have

$$A_{m-r}^m(s) = \sum_{j=1}^{r} C_{m-j}^{m-r} \cdot Q_j^m(s) \cdot s^{r-j}, \tag{21}$$

where $Q_j^m(s + \tau\{\mathbf{d}^j\}) = Q_j^m(s)$. Since $W(s, \mathbf{d}^m) = A_0^m(s)$ we arrive finally at (17) by inserting $r = m$ into Eq. (21), that splits $W(s, \mathbf{d}^m)$, in accordance with the Sylvester theorem, into periodic and non-periodic parts. □

## 4. Partition identities and zeroes of $W(s, \mathbf{d}^m)$

In this section we assume that the variable $s$ has only integer values.

Consider a new quantity

$$V(s, \mathbf{d}^m) = W(s - \xi\{\mathbf{d}^m\}, \mathbf{d}^m), \quad \xi\{\mathbf{d}^m\} = \frac{1}{2}\sum_{i=1}^{m} d_i. \tag{22}$$

**Lemma 4.1.** *$V(s, \mathbf{d}^m)$ has the following parity properties*:

$$V(s, \mathbf{d}^{2m}) = -V(-s, \mathbf{d}^{2m}), \quad V(s, \mathbf{d}^{2m+1}) = V(-s, \mathbf{d}^{2m+1}). \tag{23}$$

**Proof:** The basic recursion relation (5) can be rewritten for $V(s, \mathbf{d}^m)$

$$V(s, \mathbf{d}^m) - V(s - d_m, \mathbf{d}^m) = V\left(s - \frac{d_m}{2}, \mathbf{d}^{m-1}\right). \tag{24}$$

The last relation produces two equations in a new variable $q = s - \frac{d_m}{2}$

$$V(q,\mathbf{d}^{m-1}) = V\left(q+\frac{d_m}{2},\mathbf{d}^m\right) - V\left(q-\frac{d_m}{2},\mathbf{d}^m\right),$$
$$V(-q,\mathbf{d}^{m-1}) = V\left(-q+\frac{d_m}{2},\mathbf{d}^m\right) - V\left(-q-\frac{d_m}{2},\mathbf{d}^m\right). \tag{25}$$

Hence if $V(q,\mathbf{d}^m)$ is an even function of $q$, then $V(q,\mathbf{d}^{m-1})$ is an odd one, and vice versa. Because $V(q,\mathbf{d}^1)$ is an even function, we arrive at (23). □

**Corollary.** *If $s_1+s_2+2\xi\{\mathbf{d}^m\}=0$, then*

$$W(s_1,\mathbf{d}^m) = (-1)^{m+1}W(s_2,\mathbf{d}^m)$$

**Proof:** This follows from the parity properties and after substitution of two new variables $s_1 = s-\xi\{\mathbf{d}^m\},\ s_2 = -s-\xi\{\mathbf{d}^m\}$ into (23). □

**Lemma 4.2.** *Let m-tuple $\{\mathbf{d}^m\}$ generate the Sylvester wave $W(s,\mathbf{d}^m)$. Then for every integer p the m-tuple $\{p\cdot\mathbf{d}^m\} = \{pd_1,pd_2,\ldots,pd_m\}$ generates the following Sylvester wave*

$$W(s,p\cdot\mathbf{d}^m) = \Psi_p(s)\cdot W\left(\frac{s}{p},\mathbf{d}^m\right),\quad or \quad V(s,p\cdot\mathbf{d}^m) = \Psi_p(s-p\xi\{\mathbf{d}^m\})\cdot V\left(\frac{s}{p},\mathbf{d}^m\right), \tag{26}$$

*where the periodic function $\Psi_p(s)=\Psi_p(s+p)$ is defined in* (11).

**Proof:** According to the definition (3)

$$\sum_s W(s,p\cdot\mathbf{d}^m)\cdot t^s = \sum_s W(s,\mathbf{d}^m)\cdot t^{ps} = \sum_{s'} W\left(\frac{s'}{p},\mathbf{d}^m\right)\cdot t^{s'}$$

Equating powers of $t$ in the latter equation and taking into account that $s'/p$ must be integral we obtain (26). □

**Lemma 4.3.** *Let m-tuple $\{\mathbf{d}^m\}$ generate the Sylvester wave $W(s,\mathbf{d}^m)$. Then $W(s,\mathbf{d}^m)$ has the following zeroes*:

- *If all exponents $d_r$ are mutually prime numbers, then the zeroes $\mathfrak{s}_0(\mathbf{d}^m)$ read*

$$\mathfrak{s}_0(\mathbf{d}^m) = -1,-2,\ldots,-\sum_{r=1}^m d_r+1,\quad if\ m=2k+1,$$
$$\mathfrak{s}_0(\mathbf{d}^m) = -1,-2,\ldots,-\sum_{r=1}^m d_r+1,-\xi\{\mathbf{d}^m\},\quad if\ m=2k; \tag{27}$$

- *If all exponents $d_r$ have a maximal common factor $p$, then $W(s,\mathbf{d}^m)$ has infinite number of zeroes $\mathfrak{S}_1(\mathbf{d}^m)$ which are distributed in the following way*

$$\mathfrak{S}_1(\mathbf{d}^m) = \mathfrak{s}_1(\mathbf{d}^m) \cup \{\mathbb{Z}/p\mathbb{Z}\}, \tag{28}$$

*where $\{\mathbb{Z}/p\mathbb{Z}\}$ denotes a set of integers $\mathbb{Z}$ with deleted integers of modulo $p$*

$$\{\mathbb{Z}/p\mathbb{Z}\} = \{\ldots, -p-1, -p+1, \ldots, -1, 1, \ldots, p-1, p+1, \ldots\} \tag{29}$$

*and*

$$\begin{aligned} \mathfrak{s}_1(\mathbf{d}^m) &= -p, -2p, \ldots, -\sum_{r=1}^{m} d_r + p, \quad \text{if } m = 2k+1, \\ \mathfrak{s}_1(\mathbf{d}^m) &= -p, -2p, \ldots, -\sum_{r=1}^{m} d_r + p, -\xi\{\mathbf{d}^m\}, \quad \text{if } m = 2k. \end{aligned} \tag{30}$$

**Proof:** Consider again the relation (6) which we rewrite as follows

$$\sum_{s=0}^{\infty} W(s,\mathbf{d}^m)\cdot t^s = \frac{1}{1-t^{d_m}} \cdot \sum_{s'=0}^{\infty} W(s',\mathbf{d}^{m-1})\cdot t^{s'} \tag{31}$$

assuming that the exponents in $\mathbf{d}^m$ are sorted in the ascending order. Note that the influence of the new $d_m$ exponent appears only in terms $t^s$ with $s \ge d_m$. This enables us to deduce that the values of $W(s,\mathbf{d}^{m-1})$ and $W(s,\mathbf{d}^m)$ coincide at integer positive values $s = 0, 1, \ldots, d_m - 1$. This means that for $0 \le s \le d_m - 1$ we have $W(s,\mathbf{d}^m) = W(s,\mathbf{d}^{m-1})$. Recalling the main recursion relation (5) we conclude that

$$W(s,\mathbf{d}^m) = \quad 0\,(-d_m \le s \le -1).$$

Using the last relation for $m$ and $m-1$ in (5) we can find also

$$W(s-d_m,\mathbf{d}^m) = \quad 0\,(-d_{m-1} \le s \le -1) \Rightarrow W(s,\mathbf{d}^m) = \quad 0\,(-d_{m-1} - d_m \le s \le -1).$$

Repeating this procedure and taking into account that at the last step it leads to the zeroes of $\Psi_{d_1}$ which are located at $(1 - d_1 \le s \le -1)$, we get the set of the zeroes for $W(s,\mathbf{d}^m)$ with odd number of exponents $m = 2k+1$

$$W(s,\mathbf{d}^m) = \quad 0\,(1 - \sum_{i=1}^{m} d_i \le s \le -1). \tag{32}$$

The eveness of $m$ gives one more zero of $W(s,\mathbf{d}^m)$ which arises from the parity properties of $V(s,\mathbf{d}^m)$, namely, $V(0,\mathbf{d}^{2k}) = 0$. The last equality immediately generates a zero $-\xi\{\mathbf{d}^{2k}\}$ of $W(s,\mathbf{d}^{2k})$ that together with (32) proves the first part (27) of Lemma 3.

The second part of Lemma 3 follows from (26) and from the first part of (27) because a set of integers $\{\mathbb{Z}/p\mathbb{Z}\}$ represents the zeroes of the periodic function $\Psi_p(s)$. □

The complexity of the exponents sequence $\{\mathbf{d}^m\}$ and its large length make the calculative procedure of restoration of $Q_j^m(s)$ very cumbersome. Therefore it is important to find the inner properties of $\{\mathbf{d}^m\}$ when this procedure could be essentially reduced.

**Lemma 4.4.** *Let m-tuple* $\{\mathbf{d}^m\} = \{d_1, d_2, \ldots, d_r, d_r, \ldots, d_m\}$ *contain an exponent* $d_r$ *twice. Then the Sylvester wave* $V(s,\mathbf{d}^m)$ *is related to the Sylvester wave* $V(s,\mathbf{d}^{m_1})$ *produced by the the non-degenerated tuple* $\{\mathbf{d}^{m_1}\} = \{d_1, d_2, \ldots, d_r, \ldots, d_m, 2d_r\}$ *as follows*

$$V(s,\mathbf{d}^m) = V\left(s - \frac{d_r}{2}, \mathbf{d}^{m_1}\right) + V\left(s + \frac{d_r}{2}, \mathbf{d}^{m_1}\right). \tag{33}$$

**Proof:** According to the definition (3)

$$\left(1 + t^{d_r}\right) \cdot \sum_s W(s,\mathbf{d}^{m_1}) \cdot t^s = \sum_s W(s,\mathbf{d}^m) \cdot t^s.$$

Taking into account that $\xi\{\mathbf{d}^{m_1}\} - \xi\{\mathbf{d}^m\} = d_r/2$ and equating powers of $t$ in the latter equation we obtain the stated relation (33) according to the definition (22). □

We will make use of relation (33) during the evaluation of the expression $V(s,\mathbf{d}^m)$ for the Coxeter group $D_m$.

## 5. Recursion formulas for $V(s,\mathbf{d}^m)$

The shift (22) transforms the relation (8) into

$$V(s + \tau\{\mathbf{d}^m\}, \mathbf{d}^m) = V(s,\mathbf{d}^m) + \sum_{p=0}^{\delta_m - 1} V(s + \tau\{\mathbf{d}^m\} - \lambda_p \cdot d_m, \mathbf{d}^{m-1}), \quad \lambda_p = p + \frac{1}{2} \tag{34}$$

and the relation (17) into

$$V(s,\mathbf{d}^m) = R_m^m(s) + \sum_{j=1}^{m-1} R_j^m(s) \cdot s^{m-j}, \tag{35}$$

where

$$R_j^m(s) = \sum_{i=1}^{j} C_{m-i}^{j-i} \cdot (-\xi\{\mathbf{d}^m\})^{j-i} \cdot Q_i^m(s - \xi\{\mathbf{d}^m\}),$$

i.e., $R_1^m(s) = Q_1^m(s - \xi\{\mathbf{d}^m\})$; $R_2^m(s) = Q_2^m(s - \xi\{\mathbf{d}^m\}) - (m-1)\cdot\xi\{\mathbf{d}^m\}\cdot Q_1^m(s - \xi\{\mathbf{d}^m\})$ etc. This means that the functions $R_j^m(s)$ and $Q_j^m(s)$ have the same period $\tau\{\mathbf{d}^j\}$.

Inserting the expansion (35) into the relation (34) and equating powers of $s$ we can obtain for $k = 1, 2, \ldots, m-1$

$$\sum_{j=1}^{k} C_{m-j}^{m-1-k}\cdot R_j^m(s)\cdot\tau\{\mathbf{d}^m\}^{k+1-j}$$
$$= \sum_{p=0}^{\delta_m-1}\sum_{j=1}^{k} R_j^{m-1}(s-\lambda_p\cdot d_m)\cdot C_{m-1-j}^{m-1-k}\cdot(\tau\{\mathbf{d}^m\}-\lambda_p\cdot d_m)^{k-j}. \tag{36}$$

For the first successive values of $k$ the latter Eq. (36) gives

$$\begin{aligned}
R_1^m(s) &= \frac{1}{(m-1)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1} R_1^{m-1}(s-\lambda_p\cdot d_m),\\
R_2^m(s) &= \frac{1}{(m-2)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1} R_2^{m-1}(s-\lambda_p\cdot d_m)\\
&\quad+\sum_{p=0}^{\delta_m-1}\left(\frac{1}{2}-\frac{\lambda_p}{\delta_m}\right)\cdot R_1^{m-1}(s-\lambda_p\cdot d_m),\\
R_3^m(s) &= \frac{1}{(m-3)\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1} R_3^{m-1}(s-\lambda_p\cdot d_m)\\
&\quad+\sum_{p=0}^{\delta_m-1}\left(\frac{1}{2}-\frac{\lambda_p}{\delta_m}\right)\cdot R_2^{m-1}(s-\lambda_p\cdot d_m)\\
&\quad+\frac{m-2}{2}\cdot\tau\{\mathbf{d}^m\}\sum_{p=0}^{\delta_m-1}\left(\frac{1}{6}-\frac{\lambda_p}{\delta_m}+\frac{\lambda_p^2}{\delta_m^2}\right)\cdot R_1^{m-1}(s-\lambda_p\cdot d_m).
\end{aligned} \tag{37}$$

It is easy to see that in the summands of the latter formulas (37) there appear the Bernoulli polynomials $\mathcal{B}_i(1-\frac{\lambda_p}{\delta_m}) : \mathcal{B}_0(x)=1,\ \mathcal{B}_1(x)=x-1/2,\ \mathcal{B}_2(x)=x^2-x+1/6,\ \mathcal{B}_3(x)=x^3-3/2\,x^2+1/2\,x$, etc. [1]. Continuing the evaluation of the general expression for $R_j^m(s)$, $1<j<m$, we arrive at

**Lemma 5.1.** *$R_j^m(s)$ for $1 \le j < m$ is given by the formula*

$$R_j^m(s) = \frac{1}{m-j}\cdot\sum_{l=0}^{j-1}(\tau\{\mathbf{d}^m\})^{l-1}\cdot C_{m-1-j+l}^{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}\right)\cdot R_{j-l}^{m-1}(s-\lambda_p\cdot d_m). \tag{38}$$

**Proof:** Before going to the proof we recall two identities for the Bernoulli polynomials [1, 10],

$$\mathcal{B}_l(x+y)-\mathcal{B}_l(x)=\sum_{j=1}^{l}C_l^j\cdot y^j\cdot\mathcal{B}_{l-j}(x),\quad \mathcal{B}_l(1+x)-\mathcal{B}_l(x)=lx^{l-1}. \tag{39}$$

Using the definition (35) we check that formula (38) satisfies (34).

$$\begin{aligned}
V(s,\mathbf{d}^m) &= R_m^m(s)+\sum_{j=1}^{m-1}s^j\sum_{l=j}^{m-1}C_l^j\frac{(\tau\{\mathbf{d}^m\})^{l-j-1}}{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_{l-j}\left(1-\frac{\lambda_p}{\delta_m}\right)R_{m-l}^{m-1}(s-\lambda_p d_m)\\
&= R_m^m(s)+\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\\
&\quad\times\sum_{j=1}^{l}C_l^j\left(\frac{s}{\tau\{\mathbf{d}^m\}}\right)^j\mathcal{B}_{l-j}\left(1-\frac{\lambda_p}{\delta_m}\right)\\
&= R_m^m(s)+\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\\
&\quad\times\left[\mathcal{B}_l\left(1+\frac{s-\lambda_p d_m}{\tau\{\mathbf{d}^m\}}\right)-\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}\right)\right],
\end{aligned} \tag{40}$$

where we use the first of the identities (39). Having in mind the $\tau\{\mathbf{d}^m\}$-periodicity of functions $R_j^m(s)$ and $R_j^{m-1}(s)$ and the second identity (39) we may rewrite the difference in the l.h.s. of relation (34) in the following form:

$$\begin{aligned}
&V(s,\mathbf{d}^m)-V(s-\tau\{\mathbf{d}^m\},\mathbf{d}^m)\\
&=\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)\left[\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}+\frac{s}{\tau\{\mathbf{d}^m\}}\right)\right.\\
&\left.-\mathcal{B}_l\left(-\frac{\lambda_p}{\delta_m}+\frac{s}{\tau\{\mathbf{d}^m\}}\right)\right]\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}R_{m-l}^{m-1}(s-\lambda_p d_m)l\left(\frac{s-\lambda_p d_m}{\tau\{\mathbf{d}^m\}}\right)^{l-1}\\
&=\sum_{p=0}^{\delta_m-1}\sum_{l=0}^{m-2}(s-\lambda_p d_m)^l R_{m-1-l}^{m-1}(s-\lambda_p d_m)=\sum_{p=0}^{\delta_m-1}V(s-\lambda_p d_m,\mathbf{d}^{m-1}).
\end{aligned} \tag{41}$$

□

The formula (38) enables us to restore all terms $R_k^m(s)$ except the last $R_m^m(s)$. Actually we can learn about it from the following consideration. Let us separate $R_{m-k}^m(s)$ in the following way

$$R_{m-k}^m(s)=\mathcal{R}_{m-k}^m(s)+r_{m-k}^m(s),\quad 0\le k\le m-1, \tag{42}$$

where

$$\mathcal{R}^m_{m-k}(s)=\sum_{l=1}^{m-k-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l+k}\cdot C^k_{l+k}\sum_{p=0}^{\delta_m-1}\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}\right)\cdot R^{m-1}_{m-k-l}(s-\lambda_p\cdot d_m) \tag{43}$$

$$r^m_{m-k}(s)=\frac{1}{k\cdot\tau\{\mathbf{d}^m\}}\sum_{p=0}^{\delta_m-1}R^{m-1}_{m-k}(s-\lambda_p d_m),\quad r^m_{m-k}(s)=r^m_{m-k}(s-d_m),\quad (k\neq 0) \tag{44}$$

The representation (42) and $d_m$-periodicity of the function $r^m_{m-k}(s)$ make it possible to prove the following.

**Lemma 5.2.** *$R^m_{m-k}(s)$ for $0\le k\le m-1$ and $\mathcal{R}^m_{m-k}(s)$ for $0<k\le m-1$ satisfy the recursion relation*

$$\begin{aligned}R^m_{m-k}(s)-R^m_{m-k}(s-d_m)&=\mathcal{R}^m_{m-k}(s)-\mathcal{R}^m_{m-k}(s-d_m)\\&=\sum_{j=k+1}^{m-1}\left\{(-d_m)^{j-k}\cdot C^k_j\cdot R^m_{m-j}(s-d_m)+\left(-\frac{d_m}{2}\right)^{j-1-k}\cdot C^k_{j-1}\cdot R^{m-1}_{m-j}\left(s-\frac{d_m}{2}\right)\right\}.\end{aligned} \tag{45}$$

**Proof:** Inserting (35) into (24), expanding the powers of binomials into sums and equating the powers of $s$ in the latter equation we obtain the relation (45) for the function $R^m_{m-k}(s)$, $0\le k\le m-1$. Using the definition (42) we immediately arrive at the relation for the function $\mathcal{R}^m_{m-k}(s)$, $0<k\le m-1$. □

In the special case $k=0$ the general relation (45) produces the recursion for $R^m_m(s)$

$$\begin{aligned}&R^m_m(s)-R^m_m(s-d_m)\\&=\sum_{j=1}^{m-1}\left\{(-d_m)^j\cdot R^m_{m-j}(s-d_m)+\left(-\frac{d_m}{2}\right)^{j-1}\cdot R^{m-1}_{m-j}\left(s-\frac{d_m}{2}\right)\right\}.\end{aligned} \tag{46}$$

We cannot use (43) directly with $k=0$ since $r^m_m(s)$ can not be derived from (44). But it is a good mathematical intuition to exploit the formula (43) for $k=0$ in order to prove

**Lemma 5.3.** *$\mathcal{R}^m_m(s)$ is given by the formula*

$$\mathcal{R}^m_m(s)=\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}\right)\cdot R^{m-1}_{m-l}(s-\lambda_p\cdot d_m). \tag{47}$$

**Proof:** In order to prove that $\mathcal{R}^m_m(s)$ given by (47) satisfies the difference Eq. (46) we consider a difference $\mathcal{R}^m_m(s)-\mathcal{R}^m_m(s-d_m)=\Delta_m(s)=\Delta^1_m(s)+\Delta^2_m(s)$:

$$\Delta_m(s)=\sum_{l=1}^{m-1}\frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l}\sum_{p=0}^{\delta_m-1}\mathcal{B}_l\left(1-\frac{\lambda_p}{\delta_m}\right)\cdot\left[R^{m-1}_{m-l}(s-\lambda_p d_m)-R^{m-1}_{m-l}(s-\lambda_{p+1}d_m)\right]$$

with

$$\Delta_m^1(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \left\{ \mathcal{B}_l \left(1 - \frac{1}{2\delta_m}\right) - \mathcal{B}_l \left(-\frac{1}{2\delta_m}\right) \right\} \cdot R_{m-l}^{m-1} \left(s - \frac{d_m}{2}\right),$$

$$\Delta_m^2(s) = \sum_{l=1}^{m-1} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \sum_{p=1}^{\delta_m} \left\{ \mathcal{B}_l \left(1 - \frac{\lambda_p}{\delta_m}\right) - \mathcal{B}_l \left(1 - \frac{\lambda_p}{\delta_m} + \frac{1}{\delta_m}\right) \right\} \cdot R_{m-l}^{m-1}(s - \lambda_p d_m).$$

The first term $\Delta_m^1(s)$ is calculated with the help of one of the identities (39):

$$\Delta_m^1(s) = \sum_{l=1}^{m-1} \left(-\frac{d_m}{2}\right)^{l-1} \cdot R_{m-l}^{m-1}\left(s - \frac{d_m}{2}\right). \tag{48}$$

Using another identity from (39) we may write for $\Delta_m^2(s)$:

$$\Delta_m^2(s) = \sum_{l=1}^{m-1} \sum_{j=1}^{l} \frac{(\tau\{\mathbf{d}^m\})^{l-1}}{l} \cdot C_l^j \cdot \left(-\frac{1}{\delta_m}\right)^j \sum_{p=1}^{\delta_m} \mathcal{B}_{l-j} \left(1 - \frac{\lambda_{p-1}}{\delta_m}\right) \cdot R_{m-l}^{m-1}(s - \lambda_p d_m).$$

Interchanging the summation order $\sum_{k=l+1}^{m-1} \sum_{j=l+1}^{k} = \sum_{j=l+1}^{m-1} \sum_{k=j}^{m-1}$ and comparing the inner sum with (38) we arrive at

$$\Delta_m^2(s) = \sum_{j=1}^{m-1} (-d_m)^j \cdot R_{m-j}^m(s - d_m) \tag{49}$$

Then (48) and (49) prove the Lemma. □

From this Lemma follows the existence of the $d_m$-periodic function $r_m^m(s) = r_m^m(s - d_m)$ which could not be derived from (44). The unknown function $r_m^m(s)$ corresponds to vanishing harmonics in the r.h.s. of the Eq. (45). We are free to choose any basic system of continuous $\tau\{\mathbf{d}^m\}$-periodic functions. This arbitrariness can affect the behaviour of $W(s, \mathbf{d}^m)$ only for non-integer $s$ that does not violate the recursion relation (5). In the rest of the paper we will choose a basic system of the simplest periodic functions sin and cos.

The function $r_m^m(s)$ corresponds to the harmonics of the type

$$\left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \frac{2\pi n}{d_m} s$$

Because the parity of $R_m^m(s)$ coincides with that of $V(s, \mathbf{d}^m)$ itself we can rewrite (35) in the following form

$$V(s, \mathbf{d}^{2m}) = \sum_{j=1}^{2m-1} R_j^{2m}(s) \cdot s^{2m-j} + \mathcal{R}_{2m}^{2m}(s) + \sum_n \rho_n^{2m} \cdot \sin \frac{2\pi n}{d_{2m}} s, \tag{50}$$

$$V(s, \mathbf{d}^{2m+1}) = \sum_{j=1}^{2m} R_j^{2m+1}(s) \cdot s^{2m+1-j} + \mathcal{R}_{2m+1}^{2m+1}(s) + \sum_n \rho_n^{2m+1} \cdot \cos \frac{2\pi n}{d_{2m+1}} s. \tag{51}$$

In order to produce $r_m^m(s)$ we use some of the zeroes $\mathfrak{s}$, described in the preceding Section, constructing a system of linear equations for $[(m+1)/2]$ coefficients $\rho_n$; $n$ runs from 1 to $m/2$ in (50) and from 0 to $(m-1)/2$ in (51). We use a trivial identity $V(\xi(\mathbf{d}^m), \mathbf{d}^m) = 1$, and choose the values of $s$ out of the set $\mathfrak{s}$, adding homogeneous equations to arrive at a non-degenerate inhomogeneous system of linear equations. This system is solved further to produce the final expression for corresponding Sylvester wave. These explicit expressions are given in the next Section. Appendix A presents two instructive examples of the above procedure.

## 6. Sylvester waves $V(s, G)$

We start with the symmetric group $\mathcal{S}_m$ because of two reasons: first, of their relation with restricted partition numbers and, second, they form a natural basis to utilize the Sylvester waves $V(s, G)$ in all Coxeter groups.

### 6.1. *Symmetric groups $\mathcal{S}_m$*

Making use of the procedure developed in the previous section we present here the first twelve Sylvester waves $V(s, \mathcal{S}_m)$ , $m = 1, \ldots, 12$.[1]

$G = \mathcal{S}_m,\ d_r = 1, 2, 3, \ldots, m,\ \xi(\mathcal{S}_m) = \frac{m(m+1)}{4},$

$$V(s, \mathcal{S}_1) = 1,$$

$$V(s, \mathcal{S}_2) = \frac{s}{2} - \frac{1}{4}\sin\pi s,$$

$$V(s, \mathcal{S}_3) = \frac{s^2}{12} - \frac{7}{72} - \frac{1}{8}\cos\pi s + \frac{2}{9}\cos\frac{2\pi s}{3},$$

$$V(s, \mathcal{S}_4) = \frac{s^3}{144} - \frac{s}{96}\cdot(5 + 3\cos\pi s) + \frac{1}{8}\sin\frac{\pi s}{2} - \frac{2}{9\sqrt{3}}\sin\frac{2\pi s}{3},$$

$$\begin{aligned} V(s, \mathcal{S}_5) = {} & \frac{s^4}{2880} - \frac{11\cdot s^2}{1152} - \frac{s}{64}\cdot\sin\pi s + \frac{17083}{691200} - \frac{2}{27}\cos\frac{2\pi s}{3} \\ & + \frac{1}{8\sqrt{2}}\cos\frac{\pi s}{2} + \frac{2}{25}\left(-\cos\frac{2\pi s}{5} + \cos\frac{4\pi s}{5}\right), \end{aligned}$$

$$\begin{aligned} V(s, \mathcal{S}_6) = {} & \frac{s^5}{86400} - \frac{91\cdot s^3}{103680} + \frac{s^2}{768}\cdot\sin\pi s + \frac{s}{829440}\cdot\left(9191 - 10240\cos\frac{2\pi s}{3}\right) \\ & - \frac{161}{9216}\sin\pi s - \frac{1}{16\sqrt{2}}\sin\frac{\pi s}{2} - \frac{1}{81\sqrt{3}}\sin\frac{2\pi s}{3} - \frac{1}{18}\sin\frac{\pi s}{3} \\ & - \frac{2}{25\sqrt{5}}\left(\sin\frac{\pi}{5}\sin\frac{4\pi s}{5} + \sin\frac{2\pi}{5}\sin\frac{2\pi s}{5}\right), \end{aligned}$$

$$
\begin{aligned}
V(s,\mathcal{S}_7) = {} & \frac{s^6}{3628800} - \frac{s^4}{20736} + \frac{s^2}{38400}\cdot(71+25\cos\pi s) - \frac{s}{81\sqrt{3}}\cdot\sin\frac{2\pi s}{3} \\
& - \frac{52705}{6096384} - \frac{77}{4608}\cos\pi s - \frac{1}{32}\cos\frac{\pi s}{2} - \frac{5}{486}\cos\frac{2\pi s}{3} - \frac{1}{18}\cos\frac{\pi s}{3} \\
& + \frac{2}{25\sqrt{5}}\left(\cos\frac{2\pi s}{5} - \cos\frac{4\pi s}{5}\right) + \frac{2}{49}\left(\cos\frac{2\pi s}{7} + \cos\frac{4\pi s}{7} + \cos\frac{6\pi s}{7}\right),
\end{aligned}
$$

$$
\begin{aligned}
V(s,\mathcal{S}_8) = {} & \frac{s^7}{203212800} - \frac{17\cdot s^5}{9676800} + \frac{s^3}{8294400}\cdot(1343+225\cos\pi s) \\
& + s\cdot\left(-\frac{16133}{4976640} - \frac{1}{256}\cos\frac{\pi s}{2} + \frac{1}{243}\cos\frac{2\pi s}{3} - \frac{31}{12288}\cos\pi s\right) \\
& + \frac{1}{32}\left(\sin\frac{\pi s}{4} - \sin\frac{3\pi s}{4}\right) - \frac{1}{128}\sin\frac{\pi s}{2} + \frac{1}{162\sqrt{3}}\sin\frac{2\pi s}{3} + \frac{1}{18\sqrt{3}}\sin\frac{\pi s}{3} \\
& + \frac{4}{125}\left(\sin\frac{2\pi}{5}\sin\frac{4\pi s}{5} - \sin\frac{\pi}{5}\sin\frac{2\pi s}{5}\right) \\
& - \frac{1}{49}\left(\sin\frac{2\pi s}{7}\csc\frac{\pi}{7} - \sin\frac{4\pi s}{7}\csc\frac{2\pi}{7} + \sin\frac{6\pi s}{7}\csc\frac{3\pi}{7}\right),
\end{aligned}
$$

$$
\begin{aligned}
V(s,\mathcal{S}_9) = {} & \frac{s^8}{14631321600} - \frac{19\cdot s^6}{418037760} + \frac{145597\cdot s^4}{16721510400} + \frac{s^3}{73728}\cdot\sin\pi s \\
& - s^2\cdot\left(\frac{67293991}{140460687360} + \frac{1}{4374}\cos\frac{2\pi s}{3}\right) \\
& - s\cdot\left(\frac{1}{256\sqrt{2}}\sin\frac{\pi s}{2} + \frac{1}{1458\sqrt{3}}\sin\frac{2\pi s}{3} + \frac{205}{98304}\sin\pi s\right) \\
& + \frac{199596951167}{56184274944000} + \frac{1}{64}\left(\cos\frac{\pi s}{4}\csc\frac{\pi}{8} - \cos\frac{3\pi s}{4}\csc\frac{3\pi}{8}\right) \\
& + \frac{2}{125}\left(\cos\frac{4\pi s}{5} - \cos\frac{2\pi s}{5}\right) - \frac{5}{512\sqrt{2}}\cos\frac{\pi s}{2} + \frac{257}{17496}\cos\frac{2\pi s}{3} \\
& + \frac{1}{36\sqrt{3}}\cos\frac{\pi s}{3} + \frac{2}{81}\left(-\cos\frac{2\pi s}{9} + \cos\frac{4\pi s}{9} + \cos\frac{8\pi s}{9}\right) \\
& - \frac{1}{98}\left(\cos\frac{2\pi s}{7}\csc\frac{\pi}{7}\csc\frac{2\pi}{7} + \cos\frac{4\pi s}{7}\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}\right. \\
& \left. + \cos\frac{6\pi s}{7}\csc\frac{3\pi}{7}\csc\frac{\pi}{7}\right),
\end{aligned}
$$

$$
\begin{aligned}
V(s,\mathcal{S}_{10}) = {} & \frac{s^9}{1316818944000} - \frac{11\cdot s^7}{12541132800} + \frac{113113\cdot s^5}{358318080000} - \frac{\sin\pi s}{2949120}\cdot s^4 \\
& - \frac{18063859\cdot s^3}{468202291200} + s^2\cdot\left(\frac{1}{4374\sqrt{3}}\sin\frac{2\pi s}{3} + \frac{143}{1179648}\sin\pi s\right)
\end{aligned}
$$

$$+s\cdot\left[\frac{273512277643}{240789749760000}+\frac{1}{512\sqrt{2}}\cos\frac{\pi s}{2}+\frac{7}{13122}\cos\frac{2\pi s}{3}\right.$$
$$\left.+\frac{1}{625}\left(\cos\frac{4\pi s}{5}-\cos\frac{2\pi s}{5}\right)\right]-\frac{2877523}{707788800}\sin\pi s-\frac{1211}{52488\sqrt{3}}\sin\frac{2\pi s}{3}$$
$$-\frac{5}{1024\sqrt{2}}\sin\frac{\pi s}{2}-\frac{1}{108}\sin\frac{\pi s}{3}+\frac{1}{64\sqrt{2}}\left(\csc\frac{3\pi}{8}\sin\frac{3\pi s}{4}-\csc\frac{\pi}{8}\sin\frac{\pi s}{4}\right)$$
$$+\frac{1}{50}\left(\sin\frac{3\pi s}{5}-\sin\frac{\pi s}{5}\right)-\frac{2\sqrt{2}}{625}\left(\frac{\sqrt{5}+2}{\sqrt{5+\sqrt{5}}}\sin\frac{2\pi s}{5}+\frac{\sqrt{5}-2}{\sqrt{5-\sqrt{5}}}\sin\frac{4\pi s}{5}\right)$$
$$-\frac{1}{196}\csc\frac{\pi}{7}\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}\left(\sin\frac{6\pi s}{7}+\sin\frac{4\pi s}{7}-\sin\frac{2\pi s}{7}\right)$$
$$+\frac{1}{81}\left(\csc\frac{4\pi}{9}\sin\frac{8\pi s}{9}+\csc\frac{2\pi}{9}\sin\frac{4\pi s}{9}+\csc\frac{\pi}{9}\sin\frac{2\pi s}{9}\right),$$

$$V(s,\mathcal{S}_{11})=\frac{s^{10}}{144850083840000}-\frac{23\cdot s^8}{1755758592000}+\frac{23\cdot s^6}{2799360000}$$
$$-s^4\cdot\left(\frac{381869}{195084288000}+\frac{1}{5898240}\cos\pi s\right)$$
$$+s^2\cdot\left(\frac{31377037}{210691031040}+\frac{1}{13122}\cos\frac{2\pi s}{3}+\frac{539}{5898240}\cos\pi s\right)$$
$$+s\cdot\left[\frac{1}{1024}\sin\frac{\pi s}{2}+\frac{2}{6561\sqrt{3}}\sin\frac{2\pi s}{3}+\frac{2\sqrt{5}}{3125}\right.$$
$$\left.\times\left(\sin\frac{\pi}{5}\sin\frac{4\pi s}{5}-\sin\frac{2\pi}{5}\sin\frac{2\pi s}{5}\right)\right]-\frac{209272989329}{130069463040000}$$
$$+\frac{2}{121}\left(\cos\frac{2\pi s}{11}+\cos\frac{4\pi s}{11}+\cos\frac{6\pi s}{11}+\cos\frac{8\pi s}{11}+\cos\frac{10\pi s}{11}\right)$$
$$-\frac{1}{25}\left(\cos\frac{\pi}{5}\cos\frac{\pi s}{5}+\cos\frac{3\pi}{5}\cos\frac{3\pi s}{5}\right)-\frac{1}{108}\cos\frac{\pi s}{3}-\frac{3}{1024}\cos\frac{\pi s}{2}$$
$$-\frac{277}{26244}\cos\frac{2\pi s}{3}-\frac{1}{64}\left(\cos\frac{\pi s}{4}+\cos\frac{3\pi s}{4}\right)-\frac{821381}{176947200}\cos\pi s$$
$$-\frac{15+17\sqrt{5}}{12500}\cos\frac{2\pi s}{5}-\frac{15-17\sqrt{5}}{12500}\cos\frac{4\pi s}{5}+\frac{1}{162}\csc\frac{\pi}{9}\csc\frac{2\pi}{9}\csc\frac{4\pi}{9}$$
$$\times\left(\sin\frac{4\pi}{9}\cos\frac{2\pi s}{9}-\sin\frac{\pi}{9}\cos\frac{4\pi s}{9}-\sin\frac{2\pi}{9}\cos\frac{8\pi s}{9}\right)+\frac{1}{392}\csc\frac{\pi}{7}$$
$$\times\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}\left(\csc\frac{3\pi}{7}\cos\frac{2\pi s}{7}-\csc\frac{\pi}{7}\cos\frac{4\pi s}{7}+\csc\frac{2\pi}{7}\cos\frac{6\pi s}{7}\right),$$

$$\begin{aligned}
V(s,\mathcal{S}_{12}) &= \frac{s^{11}}{19120211066880000} - \frac{13\cdot s^9}{83433648291840} + \frac{2327\cdot s^7}{14485008384000} \\
&- s^5\cdot\left(\frac{351143}{5150225203200} + \frac{1}{353894400}\cos\pi s\right) \\
&+ s^3\cdot\left(\frac{22832915807}{2085841207296000} + \frac{611}{212336640}\cos\pi s + \frac{1}{472392}\cos\frac{2\pi s}{3}\right) \\
&+ s^2\cdot\left(\frac{1}{78732\sqrt{3}}\sin\frac{2\pi s}{3} - \frac{1}{24576}\sin\frac{\pi s}{2}\right) \\
&+ s\cdot\left(-\frac{710427757}{1589212348416} - \frac{1}{1296}\cos\frac{\pi s}{3} + \frac{1}{4096}\cos\frac{\pi s}{2} - \frac{301}{314928}\cos\frac{2\pi s}{3}\right. \\
&\left. - \frac{206713}{424673280}\cos\pi s - \frac{1}{625\sqrt{5}}\cos\frac{4\pi s}{5} + \frac{1}{625\sqrt{5}}\cos\frac{2\pi s}{5}\right) \\
&+ \frac{1}{121}\left(-\csc\frac{\pi}{11}\sin\frac{2\pi s}{11} + \csc\frac{2\pi}{11}\sin\frac{4\pi s}{11} - \csc\frac{3\pi}{11}\sin\frac{6\pi s}{11}\right. \\
&\left. + \csc\frac{4\pi}{11}\sin\frac{8\pi s}{11} - \csc\frac{5\pi}{11}\sin\frac{10\pi s}{11}\right) + \frac{1}{162\sqrt{3}}\csc\frac{\pi}{9}\csc\frac{2\pi}{9}\csc\frac{4\pi}{9} \\
&\times\left(\sin\frac{2\pi}{9}\sin\frac{8\pi s}{9} - \sin\frac{\pi}{9}\sin\frac{4\pi s}{9} - \sin\frac{4\pi}{9}\sin\frac{2\pi s}{9}\right) \\
&+ \frac{1}{784}\csc^2\frac{\pi}{7}\csc^2\frac{2\pi}{7}\csc^2\frac{3\pi}{7}\left(-\sin\frac{\pi}{7}\sin\frac{2\pi s}{7}\right. \\
&\left. + \sin\frac{2\pi}{7}\sin\frac{4\pi s}{7} - \sin\frac{3\pi}{7}\sin\frac{6\pi s}{7}\right) + \frac{1}{128}\left(\sin\frac{\pi s}{4} - \sin\frac{3\pi s}{4}\right) \\
&- \frac{7}{648\sqrt{3}}\sin\frac{\pi s}{3} - \frac{1087}{472392\sqrt{3}}\sin\frac{2\pi s}{3} + \frac{617}{73728}\sin\frac{\pi s}{2} \\
&- \frac{15+\sqrt{5}}{5000}\csc\frac{2\pi s}{5}\sin\frac{2\pi s}{5} - \frac{15-\sqrt{5}}{5000}\csc\frac{\pi s}{5}\sin\frac{4\pi s}{5} \\
&+ \frac{1}{50}\left(\csc\frac{\pi s}{5}\cos\frac{\pi s}{5}\sin\frac{\pi s}{5} - \csc\frac{2\pi s}{5}\cos\frac{2\pi s}{5}\sin\frac{3\pi s}{5}\right) \\
&+ \frac{1}{72}\left(\sin\frac{\pi s}{6} + \sin\frac{5\pi s}{6}\right).
\end{aligned} \tag{52}$$

Appendix C presents the figures of all twelve Sylvester waves $V(s,\mathcal{S}_m),\ m = 1,\ldots,12$.

### 6.2. *Coxeter groups*

Let us define two auxiliary functions

$$\begin{aligned}
U_+(s,p,G) &= V(s+p,G) + V(s-p,G), \\
U_-(s,p,G) &= V(s+p,G) - V(s-p,G)
\end{aligned} \tag{53}$$

with obvious properties

$$U_+\left(s, p, \mathbf{d}^m/d_r\right) = U_-\left(s, p + \frac{d_r}{2}, \mathbf{d}^m\right) - U_-\left(s, p - \frac{d_r}{2}, \mathbf{d}^m\right),$$

$$U_+(s, 0, G) = 2V(s, G),$$

$$U_-\left(s, p, \mathbf{d}^m/d_r\right) = U_+\left(s, p + \frac{d_r}{2}, \mathbf{d}^m\right) - U_+\left(s, p - \frac{d_r}{2}, \mathbf{d}^m\right),$$

$$U_-(s, \frac{d_r}{2}, \mathbf{d}^m) = V(s, \mathbf{d}^m/d_r),$$

where the $(m-1)$-*tuple* $\{\mathbf{d}^m/d_r\} = \{d_1, d_2, \ldots, d_{r-1}, d_{r+1}, \ldots, d_m\}$ doesn't contain the $d_r$-exponent.

Sylvester waves for the Coxeter groups are given below expressed through the relations elaborated in the previous Sections.

$G = A_m,\ d_r = 2, 3, \ldots, m+1;\ \xi(A_m) = \frac{1}{4}m(m+3)$

$$V(s, A_m) = U_-\left(s, \frac{1}{2}, \mathcal{S}_m\right). \tag{54}$$

$G = B_m,\ d_r = 2, 4, 6, \ldots, 2m;\ \xi(B_m) = \frac{1}{2}m(m+1)$

$$V(s, B_m) = \frac{1}{2}\Psi_2(s - \xi(B_m)) \cdot U_+\left(\frac{s}{2}, 0, \mathcal{S}_m\right). \tag{55}$$

In the list for $D_m$ groups the degree $m$ occurs twice when $m$ is even. This is the only case involving such a repetition.

$G = D_m,\ d_r = 2, 4, 6, \ldots, 2(m-1), m,\ m \geq 3;\ \xi(D_m) = \frac{1}{2}m^2,$

$$V(s, D_{2m}) = \Psi_2(s) \cdot U_+\left(\frac{s}{2}, \frac{m}{2}, \mathcal{S}_{2m}\right), \tag{56}$$

$$V(s, D_{2m+1}) = \sum_{s_1=0}^{s-\xi(D_{2m+1})} V\left(s + \frac{2m+1}{2} - s_1, B_{2m}\right) \cdot \Psi_{2m+1}(s_1),$$

$$V(s, D_3) = V(s, A_3),$$

$$V(s, D_5) = U_-\left(s, \frac{11}{2}, \mathcal{S}_8\right) - U_-\left(s, \frac{9}{2}, \mathcal{S}_8\right) - U_-\left(s, \frac{5}{2}, \mathcal{S}_8\right) + U_-\left(s, \frac{3}{2}, \mathcal{S}_8\right).$$

$G = G_2,\ d_r = 2, 6;\ \xi(G_2) = 4,$

$$V(s, G_2) = \Psi_2(s) \cdot U_-\left(\frac{s}{2}, 1, \mathcal{S}_3\right). \tag{57}$$

$G = F_4,\ d_r = 2, 6, 8, 12;\ \xi(F_4) = 14,$

$$V(s, F_4) = \Psi_2(s) \cdot \left[U_+\left(\frac{s}{2}, \frac{7}{2}, \mathcal{S}_6\right) - U_+\left(\frac{s}{2}, \frac{3}{2}, \mathcal{S}_6\right)\right]. \tag{58}$$

$G = E_6,\ d_r = 2, 5, 6, 8, 9, 12;\ \xi(E_6) = 21,$

$$\begin{aligned} V(s, E_6) = {} & U_+(s, 18, \mathcal{S}_{12}) - U_+(s, 17, \mathcal{S}_{12}) - U_+(s, 15, \mathcal{S}_{12}) \\ & + U_+(s, 13, \mathcal{S}_{12}) + U_+(s, 5, \mathcal{S}_{12}) - U_+(s, 2, \mathcal{S}_{12}). \end{aligned} \tag{59}$$

$G = E_7,\ d_r = 2, 6, 8, 10, 12, 14, 18;\ \xi(E_7) = 35,$

$$V(s, E_7) = \Psi_2(s-1) \cdot \left[U_+\left(\frac{s}{2}, 5, \mathcal{S}_9\right) - U_+\left(\frac{s}{2}, 3, \mathcal{S}_9\right)\right]. \tag{60}$$

$G = E_8,\ d_r = 2, 8, 12, 14, 18, 20, 24, 30;\ \xi(E_8) = 64,$

$$\begin{aligned} V(s, E_8) = \Psi_2(s) \cdot \Bigg[ & U_-\left(\frac{s}{2}, 28, \mathcal{S}_{15}\right) + U_-\left(\frac{s}{2}, 21, \mathcal{S}_{15}\right) + U_-\left(\frac{s}{2}, 12, \mathcal{S}_{15}\right) \\ & + U_-\left(\frac{s}{2}, 11, \mathcal{S}_{15}\right) - U_-\left(\frac{s}{2}, 8, \mathcal{S}_{15}\right) - U_-\left(\frac{s}{2}, 7, \mathcal{S}_{15}\right) \\ & - U_-\left(\frac{s}{2}, 6, \mathcal{S}_{15}\right) - U_-\left(\frac{s}{2}, 26, \mathcal{S}_{15}\right) - U_-\left(\frac{s}{2}, 25, \mathcal{S}_{15}\right)\Bigg]. \end{aligned} \tag{61}$$

$G = H_3,\ d_r = 2, 6, 10;\ \xi(H_3) = 9,$

$$V(s, H_3) = \Psi_2(s-1) \cdot \left[U_+\left(\frac{s}{2}, 3, \mathcal{S}_5\right) - U_+\left(\frac{s}{2}, 1, \mathcal{S}_5\right)\right]. \tag{62}$$

$G = H_4,\ d_r = 2, 12, 20, 30;\ \xi(H_3) = 32,$

$$\begin{aligned} V(s, H_4) = {} & U_+(s, 32, E_8) - U_+(s, 24, E_8) - U_+(s, 18, E_8) - U_+(s, 14, E_8) \\ & + U_+(s, 10, E_8) - U_+(s, 8, E_8) + U_+(s, 6, E_8) + U_+(s, 0, E_8). \end{aligned} \tag{63}$$

$G = I_m,\ d_r = 2, m;\ \xi(I_m) = 1 + \frac{1}{2}m$

$$\begin{aligned} & V(s, I_m) = \sum_{s_1=0}^{s-\xi(I_m)} \Psi_2(s - \xi(I_m) - s_1) \cdot \Psi_m(s_1), \\ & V(s, I_2) = V(s, B_1), \quad V(s, I_3) = V(s, A_2), \quad V(s, I_4) = V(s, B_2), \\ & V(s, I_5) = U_+\left(s, \frac{7}{2}, A_4\right) - U_+\left(s, \frac{1}{2}, A_4\right), \\ & V(s, I_6) = V(s, G_2), \quad V(s, I_8) = U_+(s, 5, B_4) - U_+(s, 1, B_4) \\ & V(s, I_{10}) = U_-(s, 3, H_3), \quad V(s, I_{12}) = U_+(s, 7, F_4) - U_+(s, 1, F_4). \end{aligned} \tag{64}$$

## Appendix A: Derivation of Sylvester waves $V(s, \mathcal{S}_4)$ and $V(s, \mathcal{S}_5)$

We will illustrate how the formulas (38)–(51) work in the case of the symmetric groups $\mathcal{S}_4$ and $\mathcal{S}_5$.

We start with Sylvester wave $V(s, \mathcal{S}_3)$ taken from (52)

$$V(s, \mathcal{S}_3) = \frac{s^2}{12} - \frac{7}{72} - \frac{1}{8}\cos\pi s + \frac{2}{9}\cos\frac{2\pi s}{3} \tag{A1}$$

and with successive usage of the formulas (38) and (47) one can obtain

$$R_1^4(s) = \frac{1}{144}, \quad R_2^4(s) = 0, \quad R_3^4(s) = -\frac{1}{96}\cdot(5 + 3\cos\pi s), \quad \mathcal{R}_4^4(s) = -\frac{2}{9\sqrt{3}}\sin\frac{2\pi s}{3}. \tag{A2}$$

Now we will use the representation (50)

$$V(s, \mathcal{S}_4) = \sum_{j=1}^{3} R_j^4(s)\cdot s^{4-j} + \mathcal{R}_4^4(s) + \rho_1^4\cdot\sin\frac{\pi}{2}s + \rho_2^4\cdot\sin\pi s. \tag{A3}$$

Since $V(s, \mathcal{S}_4) = W(s-5, \mathcal{S}_4)$ the variable $s$ takes only integer values what makes the last contribution in (A3) into the $V(s, \mathcal{S}_4)$ irrelevant. The unknown coefficient $\rho_1^4$ is determined with help of *zeroes* (27) of $W(s, \mathcal{S}_4)$

$$0 = V(1, \mathcal{S}_4) = \sum_{j=1}^{3} R_j^4(1) + \mathcal{R}_4^4(1) + \rho_1^4, \quad \text{or} \quad \rho_1^4 = \frac{1}{8} \tag{A4}$$

Thus we arrive at the Sylvester wave $V(s, \mathcal{S}_4)$ presented in (52).

Repeating the same procedure with symmetric group $\mathcal{S}_5$ we find

$$\begin{aligned} R_1^5(s) &= \frac{1}{2880}, \quad R_2^5(s) = 0, \quad R_3^5(s) = -\frac{11}{1152}, \quad R_4^5(s) = -\frac{1}{64}\sin\pi s, \\ \mathcal{R}_5^5(s) &= \frac{475}{27648} - \frac{2}{27}\cos\frac{2\pi s}{3} + \frac{1}{8\sqrt{2}}\cos\frac{\pi s}{2}. \end{aligned} \tag{A5}$$

The representation (51) produces

$$V(s, \mathcal{S}_5) = \sum_{j=1}^{4} R_j^5(s)\cdot s^{5-j} + \mathcal{R}_5^5(s) + \rho_0^5 + \rho_1^5\cdot\cos\frac{2\pi s}{5} + \rho_2^5\cdot\cos\frac{4\pi s}{5}. \tag{A6}$$

Since $V(s, \mathcal{S}_5) = W(s - \frac{15}{2}, \mathcal{S}_5)$ the variable $s$ has only half-integer values. By solving three linear equations $V(\frac{1}{2}, \mathcal{S}_5) = V(\frac{3}{2}, \mathcal{S}_5) = V(\frac{5}{2}, \mathcal{S}_5) = 0$ we find

$$\rho_0^5 = \frac{217}{28800}, \quad \rho_1^5 = -\frac{2}{25}, \quad \rho_2^5 = \frac{2}{25}, \tag{A7}$$

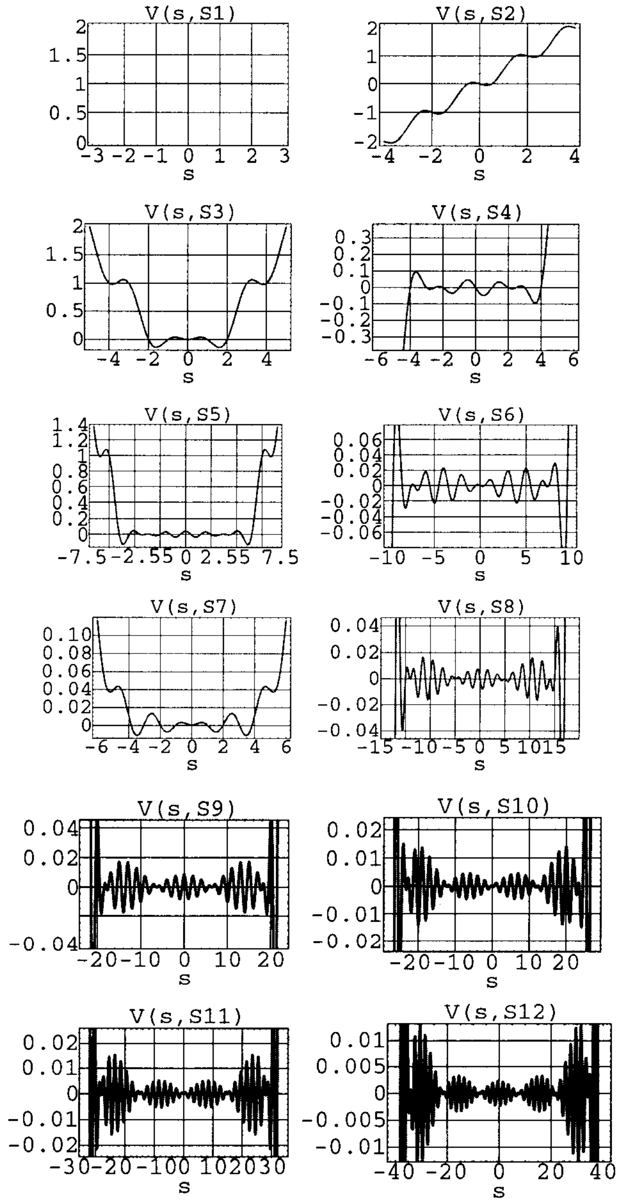
which together with (A6) produces the Sylvester wave $V(s, \mathcal{S}_5)$ from (52).

## Appendix B: Table of restricted partition numbers $W(s, \mathcal{S}_m)$

In this Appendix we give the Table of the restricted partition numbers $\mathcal{P}_m(s) = W(s, \mathcal{S}_m)$ $m \leq 10$ for $s$ running in the different ranges. One can verify that the content of this Table can be obtained with the help of the formulas (52).

| $s$ | $\mathcal{S}_1$ | $\mathcal{S}_2$ | $\mathcal{S}_3$ | $\mathcal{S}_4$ | $\mathcal{S}_5$ | $\mathcal{S}_6$ | $\mathcal{S}_7$ | $\mathcal{S}_8$ | $\mathcal{S}_9$ | $\mathcal{S}_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 1 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 1 | 3 | 5 | 6 | 7 | 7 | 7 | 7 | 7 | 7 |
| 6 | 1 | 4 | 7 | 9 | 10 | 11 | 11 | 11 | 11 | 11 |
| 7 | 1 | 4 | 8 | 11 | 13 | 14 | 15 | 15 | 15 | 15 |
| 8 | 1 | 5 | 10 | 15 | 18 | 20 | 21 | 22 | 22 | 22 |
| 9 | 1 | 5 | 12 | 18 | 23 | 26 | 28 | 29 | 30 | 30 |
| 10 | 1 | 6 | 14 | 23 | 30 | 35 | 38 | 40 | 41 | 42 |
| 51 | 1 | 26 | 243 | 1215 | 4033 | 9975 | 19928 | 33940 | 51294 | 70760 |
| 52 | 1 | 27 | 252 | 1285 | 4319 | 10829 | 21873 | 37638 | 57358 | 79725 |
| 53 | 1 | 27 | 261 | 1350 | 4616 | 11720 | 23961 | 41635 | 64015 | 89623 |
| 54 | 1 | 28 | 271 | 1425 | 4932 | 12692 | 26226 | 46031 | 71362 | 100654 |
| 55 | 1 | 28 | 280 | 1495 | 5260 | 13702 | 28652 | 50774 | 79403 | 112804 |
| 56 | 1 | 29 | 290 | 1575 | 5608 | 14800 | 31275 | 55974 | 88252 | 126299 |
| 57 | 1 | 29 | 300 | 1650 | 5969 | 15944 | 34082 | 61575 | 97922 | 141136 |
| 58 | 1 | 30 | 310 | 1735 | 6351 | 17180 | 37108 | 67696 | 108527 | 157564 |
| 59 | 1 | 30 | 320 | 1815 | 6747 | 18467 | 40340 | 74280 | 120092 | 175586 |
| 60 | 1 | 31 | 331 | 1906 | 7166 | 19858 | 43819 | 81457 | 132751 | 195491 |
| 101 | 1 | 51 | 901 | 8262 | 48006 | 198230 | 628998 | 1621248 | 3539452 | 6757864 |
| 102 | 1 | 52 | 919 | 8505 | 49806 | 207338 | 662708 | 1719877 | 3778074 | 7254388 |
| 103 | 1 | 52 | 936 | 8739 | 51649 | 216705 | 697870 | 1823402 | 4030512 | 7782608 |
| 104 | 1 | 53 | 954 | 8991 | 53550 | 226479 | 734609 | 1932418 | 4297682 | 8345084 |
| 105 | 1 | 53 | 972 | 9234 | 55496 | 236534 | 772909 | 2046761 | 4580087 | 8942920 |
| 106 | 1 | 54 | 990 | 9495 | 57501 | 247010 | 812893 | 2167057 | 4878678 | 9578879 |
| 107 | 1 | 54 | 1008 | 9747 | 59553 | 257783 | 854546 | 2293142 | 5194025 | 10254199 |
| 108 | 1 | 55 | 1027 | 10018 | 61667 | 269005 | 898003 | 2425678 | 5527168 | 10971900 |
| 109 | 1 | 55 | 1045 | 10279 | 63829 | 280534 | 943242 | 2564490 | 5878693 | 11733342 |
| 110 | 1 | 56 | 1064 | 10559 | 66055 | 292534 | 990404 | 2710281 | 6249733 | 12541802 |

**Appendix C: Figures of restricted partition numbers $V(s, \mathcal{S}_m)$**



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## Note

1. Having in mind the results of Sylvester [11, 12] and Glaisher [7] for restricted partition numbers for $m \leq 10$ and of Gupta et al. [8] for $m \leq 12$ we repeat them up to $m = 12$. The list of $V(s, \mathcal{S}_m)$ can be simply continued up to any finite $m$ with the help of the symbolic code written in *Mathematica* language [16].

## References

1. H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.
2. E.R. Canfield, *Electr. J. Combinatorics* **4**(2) (1997), R6
3. A. Cayley, *Phil. Trans. R. Soc. London* **148**(1) (1858), 47–52; F. Brioschi, *Annali di sc. mat. e fis.* **8** (1857), 5–12; S. Roberts, *Quart. J. Math* **4** (1861), 155–158. For more historical details see L.E. Dickson, *History of the Theory of Numbers*, Vol. 2, Chelsea, New York, 1952.
4. P. Erdös, *Annals of Math.* **43**(2) (1942), 437–450.
5. L.G. Fel, V. Sh. Machavariani, and D.J. Bergman, *J. Physics* A: *Math. Gen.* **33** (2000), 6669.
6. L.G. Fel, "Self-dual symmetric polynomials and conformal partitions," *Ramanujan J.* (2002), submitted, [http://arXiv.org/abs/math.NT/0111155].
7. J.W.L. Glaisher, *Quart. J. Math.* **40** (1909), 57–143; **40** (1909), 275–348; **40** (1910), 94–112.
8. H. Gupta, C.E. Gwyther, and J.C.P. Miller, *Tables of Partitions*, Cambridge University Press, Cambridge, 1958.
9. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
10. A.P. Prudnikov, Yu. A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 3, Gordon and Breach Science Publishers, New York, 1992.
11. J.J. Sylvester, *Quart. J. Math.* **1** (1857), 81–84, 141–152; Also in *Coll. Math. Papers* **2** (1973), 86–99.
12. J.J. Sylvester, Outlines of the lectures were printed privately in 1859 and republished in *Proc. London Math. Soc.* **28** (1897), 33–96; Also in *Coll. Math. Papers* **2** (1973), 119–175.
13. J.J. Sylvester, *Amer. Jour. Math.* **5** (1882), 119–136; Also in *Coll. Math. Papers* **3** (1973), 605–622.
14. G. Szekeres, *Quart. J. Math. (Oxford)* **2**(2) (1951), 85–108; G. Szekeres, **4** (1953), 96–111.
15. P. Turán, "On some connections between combinatorics and group theory," in *Colloquia Mathematica Societatis János Bolyia 4. Combinatorial Theory and its Applications*, Balaonfüred, Hungary, 1969.
16. S. Wolfram, *The Mathematica Book*, 3rd edn., Wolfram Media Inc., 1996.