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Incidence Graphs of Convex Polytopes

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ABSTRACT

This paper introduces the notion of the (r, s) incidence graph of an n-polytope P as the bipartite graph whose nodes correspond to the r-faces and the s-faces of P with an edge joining two nodes iff one of the corresponding faces contains the other. Various types of connectivity are defined for incidence graphs and bounds for these connectivities are established as functions of r, s, and n. It is shown that these bounds are also valid for a large class of cell-complexes.

1. INTRODUCTION

We define the (r, s) *incidence graph*, $G(r, s; P)$, of an n-dimensional convex polytope (*n-polytope*) P as follows: The nodes of $G(r, s; P)$ correspond to the r-dimensional faces (*r-faces*) and the s-dimensional faces of P (termed *r-nodes* and *s-nodes*, respectively). An edge joins an r-node to an s-node iff the corresponding r-face is contained in the corresponding s-face. No edge joins two r-nodes or two s-nodes. We always assume that $0 \leq r < s \leq n - 1$ for an (r, s) incidence graph. If x is a node of an incidence graph, then $\hat{x}$ denotes the corresponding face of the polytope.

A *graph* $G = (V, E)$ is a set V of vertices and a set E of edges joining pairs of vertices. We assume that graphs have no loops or multiple edges. Two vertices are said to be *adjacent* if they are joined by an edge.

The notion of an incidence graph generalizes the concept of the *edge graph* of a polytope P, which is the graph formed by the vertices and

edges of P. We will often use the natural identification between the edge graph of P and $G(0, 1; P)$. In such cases, however, we will regard the edge graph as being embedded in the polytope and $G(0, 1; P)$ as an abstract graph. In particular, we will always consider *edge paths* (paths in an edge graph) to lie on the polytope.

Balinski [1] has shown that, if P is an n-polytope, then the edge graph of P is n-connected, that is, between every pair of vertices of P there exist n paths which are disjoint except for end-points. The purpose of this paper is to define various connectivities for incidence graphs and establish bounds for them as Balinski did in the case of edge graphs. Sections 2 and 3 are devoted to these definitions, collecting relevant background results, and proving some elementary theorems. Sections 4 and 6 are concerned with one type of connectivity and Sections 7 and 8 with three other types. In Section 5 we prove a useful lemma on the number of r-faces contained in a given set of s-faces. Section 9 is devoted to extending some results of Klee [7] on separating sequences, and in Section 10 we characterize polytopes with a particular value for one connectivity. Some unsolved problems are included throughout.

2. PRELIMINARY RESULTS ON INCIDENCE GRAPHS

If u and v are two nodes of a graph, we say that a set X of nodes *separates* u and v if every path between them contains at least one member of X. Let U be a collection of nodes of a graph G. We call X a *separating set for* U in G either if X contains every member of U except possibly one, or if X separates some two members of $U \sim X$. A set U of nodes of G is said to be *k-connected* if k is the minimum cardinality of a separating set for U. If U contains all the nodes of G, we say that G is *k-connected.*

We may also restrict the type of nodes which make up X. Then by choosing various combinations of nodes for U and X a number of different connectivities may be obtained.

In the case of incidence graphs we are initially interested in six types of connectivities. Let $G = G(r, s; P)$ be an (r, s) incidence graph. We say that G is $\alpha(r, s; P)$-connected if U consists of r-nodes and X of either r- or s-nodes. More precisely, G is $\alpha(r, s; P)$-connected if the r-nodes of G are $\alpha(r, s; P)$-connected. In a similar fashion, we say that G is:

$\beta(r, s; P)$-connected if U consists of s-nodes and X of r- or s-nodes;

$\gamma(r, s; P)$-connected if U consists of r-nodes and X of r-nodes;

$\delta(r, s; P)$-connected if U consists of r-nodes and X of s-nodes;

$\varepsilon(r, s; P)$-connected if U consists of s-nodes and X of r-nodes;

$\zeta(r, s; P)$-connected if U consists of s-hodes and X of s-nodes.

We also define

$$\alpha(r, s; n) = \min\{\alpha(r, s; P)\colon P \text{ is an } n\text{-polytope}\}$$

and similar notions for the other connectivities.

Certain relationships among the connectivities are clear:

$$\alpha(r, s; n) \leq \min\{\gamma(r, s; n),\ \delta(r, s; n)\}$$
$$\beta(r, s; n) \leq \min\{\varepsilon(r, s; n),\ \zeta(r, s; n)\}. \tag{2.1}$$

We conjecture that equality always holds in (2.1), but our methods will not cover all values of n. Our best results in this direction appear in (6.4).

The two statements in (2.1) are actually equivalent because of a fundamental duality we will now describe. It arises from the existence of a *dual polytope* P^0 associated with each n-polytope P in the following way:

$$P^0 = \{x \in E^n\colon (x, y) \leq 1 \quad \text{for all } y \in P\}.$$

A general discussion of dual polytopes may be found in [10]. For our purposes, the most important results are:

If P is an n-polytope, then $Q = P^0$ is an n-polytope. Moreover, $Q^0 = P$. (2.2)

Each k-face F of an n-polytope P corresponds to a unique $(n - k - 1)$-face $F^\dagger$ of P^0. (2.3)

If $F \subset G \subset P$, then $G^\dagger \subset F^\dagger \subset P^0$. (2.4)

We say that two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* (written $G \approx G'$) if there exists a biunique mapping $\vartheta\colon V \to V'$ such that $(\vartheta(u), \vartheta(v)) \in E'$ iff $(u, v) \in E$. From the above statements on dual polytopes it is easy to prove the following useful

THEOREM.

If P is an n-polytope, $G(r, s; P) \approx G(n - 1 - s, n - 1 - r; P^0)$. (2.5)

PROOF: If F is a node of $G(r, s; P)$ let $\vartheta(F) = F^\dagger$. From (2.2) and (2.3) we see that ϑ is biunique, and (2.4) shows that edges are preserved.

COROLLARY.

If P is an n-polytope, $\alpha(r, s; P) = \beta(n - 1 - s, n - 1 - r; P^0)$. (2.6)

COROLLARY.

$\alpha(r, s; n) = \beta(n - 1 - s, n - 1 - r; n)$. Moreover, if P is an n-polytope such that $\alpha(r, s; P) = \alpha(r, s; n)$, then $\beta(n - 1 - s, n - 1 - r; P^0) = \beta(n - 1 - s, n - 1 - r; n)$. (2.7)

In the same way as above it follows:

$$\gamma(r, s; n) = \zeta(n - 1 - s,\ n - 1 - r;\ n), \tag{2.8}$$

$$\delta(r, s; n) = \varepsilon(n - 1 - s,\ n - 1 - r;\ n). \tag{2.9}$$

Some additional definitions are needed before proceeding.

A hyperplane H is said to *support* a face F of a polytope P if P lies entirely in one of the closed half-spaces determined by H and if $H \cap P = F$. Every proper face of a polytope is supported by at least one hyperplane.

A *cell complex* C is a collection of polytopes (termed *cells* of C) such that:

(1) if $P \in C$, then every face of P is a member of C;

(2) if both P and Q belong to C and $P \cap Q \neq \varphi$, then $P \cap Q$ is a face of both P and Q.

If all of the cells are simplices, we say that C is a *simplicial* cell complex. We denote by $|C|$ the set of all points which belong to some cell of C.

We can define an incidence graph $G(r, s; C)$ for a cell complex C in a way completely analogous to the way we did for polytopes. If n is the maximum dimension of a cell in C, then we assume that $0 \leq r < s \leq n$.

A *strong n-cell complex* is a cell complex such that:

(1) every cell is contained in an n-cell;

(2) every pair of n-nodes can be joined by an $(n-1, n)$ path. Our connectivity results extend to this larger class of objects as the following result shows. Since an n-polytope together with all of its faces is a strong n-cell complex, the theorem cannot be improved.

THEOREM. *Let C be a strong n-cell complex. Then*

$$\text{for } s \leq n-1,\ \mathscr{K}(r, s; C) \geq \mathscr{K}(r, s; n),$$
$$\text{for } \mathscr{K} = \alpha, \beta, \gamma, \delta, \varepsilon \text{ and } \zeta. \tag{2.10}$$

PROOF: The proof of this theorem is based on the following construction, which is due to V. L. Klee. Let P and Q be n-polytopes with a common $(n-1)$-face (or *facet*) F. Let P' be a projective image of P which leaves F fixed and which has the property that, if v is a vertex of P' but not of F, the orthogonal projection of v onto the hyperplane supporting P' at F lies in the relative interior of F. Such a projective image may be found by mapping a hyperplane which is exterior to P but which passes sufficiently near the centroid of F onto the hyperplane at infinity. Such a projective transformation will preserve not only the number of faces of each dimension but also incidences between them. Let Q' be a similar projective image of Q. If necessary, rotate Q' around an axis which leaves F fixed until both P' and Q' lie in a *flat* (a translate of a subspace) of dimension n. Then $P' \cup Q'$ is a convex polytope which contains images of all of the faces of both P and Q except F.

We will now proceed with the proof of the theorem. To be definite. we will assume that $\mathscr{K} = \beta$. All the other cases are completely analogous.

Let X be a set of $(\beta(r, s; n) - 1)$ nodes of $G(r, s; C)$, let F^s, G^s be two remaining s-nodes, and suppose $\hat{P}^n$, $\hat{Q}^n$ are n-cells of C such that $\hat{F}^s \subset \hat{P}^n$, $\hat{G}^s \subset \hat{Q}^n$. Let

$$P^n = P_0{}^n \to P_0^{n-1} \to P_1{}^n \to \cdots \to P_t{}^n = Q^n$$

be an $(n-1, n)$ path joining P^n and Q^n. Choose s-nodes $F_i{}^s$, $1 \leq i \leq t-1$, such that $\hat{F}_i{}^s \subset \hat{P}_i{}^n$ and $F_i{}^s \not\subset X$. Set $F_0{}^s = F^s$, $F_t{}^s = G^s$.

By definition of $\beta(r, s; n)$, an (r, s) path joins $F_i{}^s$ to F_{i+1}^s if both of them lie in the same n-cell of C. If they do not, construct an n-polytope Q_i from $\hat{P}_i{}^n$ and $\hat{P}_{i+1}^n$ as indicated above. Identify in the obvious way all of the faces of $\hat{P}_i{}^n$ and $\hat{P}_{i+1}^n$ (except for $\hat{P}_i^{n-1}$) with faces of $\hat{Q}_i$. Under this correspondence, an (r, s) path exists in $G(r, s; \hat{Q}_i)$ which joins $F_i{}^s$ to F_{i+1}^s and misses X. Thus, an (r, s) path joining $F_i{}^s$ to F_{i+1}^s and missing X exists in $G(r, s; C)$. Since this argument is valid for all i, X does not separate F^s and G^s in $G(r, s; C)$ and the conclusion follows.

COROLLARY.

$$\mathscr{K}(r, s; n) \geq \mathscr{K}(r, s; m) \text{ if } s < m \leq n,$$
$$\text{for } \mathscr{K} = \alpha, \beta, \gamma, \delta, \varepsilon \text{ and } \zeta. \tag{2.11}$$

PROOF: Let P be an n-polytope and $\mathscr{B}(P)$ its *boundary complex* (the cell complex formed by all proper faces of P). Then $\mathscr{B}(P)$ is a strong $(n-1)$-cell complex, and it follows from (2.10) that $\mathscr{K}(r, s; \mathscr{B}(P)) \geq \mathscr{K}(r, s; n-1)$, whenever $s \leq n-2$. But $\mathscr{K}(r, s; P) = \mathscr{K}(r, s; \mathscr{B}(P))$, and hence $\mathscr{K}(r, s; n) \geq \mathscr{K}(r, s; n-1)$. Iterating this argument, we obtain the result.

COROLLARY.

$$\mathscr{K}(r, s; n) \geq \mathscr{K}(r-k, s-k; n-k) \text{ if } k \leq r$$
$$\text{for } \mathscr{K} = \alpha, \beta, \gamma, \delta, \varepsilon \text{ and } \zeta. \tag{2.12}$$

PROOF: By (2.7) and (2.11),

$$\alpha(r, s; n) = \beta(n-1-s, n-1-r; n) \geq \beta(n-1-s, n-1-r; n-k)$$
$$= \alpha(r-k, s-k; n-k).$$

Exactly analogous proofs work for the other connectivities.

An important special type of strong n-cell complex is the pseudo-manifold. An *n-dimensional pseudo-manifold* may be defined as a finite, simplicial, strong n-cell complex in which every $(n-1)$-cell lies in exactly two n-cells. Of course, since pseudo-manifolds are topologic objects, the "simplices" which make up the cell complexes are actually homeomorphs of the standard simplex (which is a polytope). The distinction is not too important, as each simplicial cell complex is homeomorphic cell by cell to a simplicial cell complex each cell of which is a polytope.

In turn, the n-dimensional pseudo-manifold is a generalization of the basic notion of the *n-dimensional manifold* (or *n-manifold*), which is a finite simplicial cell complex in which each point has a neighborhood of the same homotopy type as the n-dimensional sphere. The proof that each n-manifold is an n-dimensional pseudo-manifold may be found in [9, p. 238].

Thus each n-manifold is homeomorphic to a strong n-cell complex. Hence, if C is the cell complex associated to an n-manifold, $G(r, s; C)$ is the (r, s) graph of some strong n-cell complex. Combining this fact with (2.10) shows that merely knowing the connectivities of an incidence graph (or even of all of the incidence graphs of a cell complex) is not sufficient to characterize those cell complexes which arise from polytopes. Other conditions are needed, and it would be of great interest to determine sufficient ones.

In view of the fact that each n-manifold is homeomorphic to a strong n-cell complex, the following result takes on special interest:

Let K be a strong n-cell complex, and let L be a finite cell complex such that $|K| = |L|$. The L is also a strong n-cell complex. (2.13)

PROOF: First suppose that K consists of a single n-cell, $\hat{S}$. By considerations of dimensionality, it is clear that each cell of L lies in an n-cell. Let $\hat{P}$, $\hat{Q}$ be two n-cells of L and choose points $x \in \hat{P}$, $y \in \hat{Q}$, such that the line segment $[x, y]$ does not intersect any cell of L of dimension less than $n - 1$. Such a line segment will clearly determine an $(n-1, n)$ path between P and Q.

Now suppose that K is a general strong n-cell complex. Once again it is clear that every cell of L lies in an n-cell. Suppose that $\hat{S}$ and $\hat{T}$ are two n-cells of K with a common $(n-1)$-cell $\hat{U}$, and let $\hat{P}$, $\hat{Q}$ be two n-cells of L such that $\operatorname{int} \hat{S} \cap \operatorname{int} \hat{P} \neq \varphi \neq \operatorname{int} \hat{T} \cap \operatorname{int} \hat{Q}$. If there exists an n-cell $\hat{R}$ of L such that $\operatorname{int} \hat{R} \cap \operatorname{rel\,int} \hat{U} \neq \varphi$, it is easy to find an $(n-1, n)$ path from P to R and one from R to Q by the results of the first paragraph. Thus a path in $G(n-1, n; L)$ joins P to Q. If no n-cell such as $\hat{R}$ exists, then choose two n-cells $\hat{R}_1$, $\hat{R}_2$ in L such that $\operatorname{rel\,int} (\hat{R}_1 \cap \hat{R}_2) \cap \operatorname{rel\,int} \hat{U} \neq \varphi$, and such that $\operatorname{int} \hat{R}_1 \cap \operatorname{int} \hat{S}$ $\hat{S} \neq \operatorname{int} \hat{R}_2 \cap \operatorname{int} \hat{T}$. Then as above it is easy to use R_1 and R_2 to construct an $(n-1, n)$ path from P to Q. It is clear how to extend this argument to construct an $(n-1, n)$ path between any two n-cells of L.

One other result which will be of use to us later is Dirac's generalization of Menger's Theorem [2, p. 151]. It might be mentioned that the

slightly weakened version given below can be proved much more simply than in the original paper by using the Max-Flow Min-Cut Theorem, just as Balinski [1, p. 434] simplified the proof of Whitney's Theorem.

THEOREM (DIRAC).

Let G be a graph and let $A = \{a_1, ..., a_k\}$, and $B = \{b_1, ..., b_m\}$ be two sets of nodes of G such that no node of A can be separated from any node of B by a set with fewer than n nodes. Let $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_m$ be non-negative integers such that $\lambda_1 + \cdots + \lambda_k = \mu_1 + \cdots + \mu_m$. Then there exist n paths in G such that:

(1) λ_i of the paths start at a_i;
(2) μ_j of the paths end at b_j; (2.14)
(3) the paths are disjoint except for end-points.

Menger's Theorem [8]. *Let G be a graph and let a, b be two nodes of G which cannot be separated by any set of $k - 1$ nodes. Then at least k paths, disjoint except for endpoints, join a and b.* (2.15)

3. PRELIMINARY RESULTS ON POLYTOPES

In this section we collect a few known results about polytopes and use them to prove some elementary theorems. Our first two results are essentially due to Balinski [1]:

Let f be an affine function defined on an n-polytope P such that $f(x) > 0$ for some $x \in P$. If u and v are two vertices of P such that $f(u) \geq 0$, $f(v) \geq 0$, then there exists an edge path

$$u = w_0 \to w_1 \to \cdots \to w_t = v \tag{3.1}$$

joining them such that $f(w_i) > 0$ for $1 \leq i \leq t - 1$.

PROOF: Let $M = \max\{f(x): x \in P\}$ and assume $f(u) < M$. Let H be a hyperplane which *strictly separates* u from the other vertices of P; that is, u lies in one of the open half-spaces determined by H and the remaining vertices of P lie in the other open half-space. Let $Q = H \cap P$. Then f is an affine function on Q which attains its maximum (on Q) at a vertex q_1. Let

$$H' = \{x \in E^n: f(x) = f(u)\}.$$

By assumption, H' intersects the interior of P and passes through u, so it intersects the interior of Q. Hence, $f(q_1) > f(u)$. Since $q_1 = H \cap (u, u_1)$ for some edge (u, u_1), it follows that $f(u_1) > f(u)$.

If $f(u_1) < M$, repeat the argument above to find a vertex u_2 adjacent to u_1 such that $f(u_2) > f(u_1)$. Continue this process to generate a path $u = u_0 \to u_1 \to \cdots \to u_k$ such that $f(u_i) > f(u_{i-1})$ for $1 \leq i \leq k$ and $f(u_k) = M$. By hypothesis $M > 0$.

In a similar way construct a path $v = v_0 \to v_1 \to \cdots \to v_m$ such that $f(v_j) > f(v_{j-1})$, and $f(v_m) = M$. Since

$$F = \{x \in P: f(x) = M\}$$

is a face of P, we can join u_k and v_m by an edge path lying entirely on F. Since $f(u_i) > 0$ for all $i > 0$, and $f(v_j) > 0$ for all $j > 0$, combining these three paths gives an edge path with the required property.

We can apply this lemma to prove two useful results:

THEOREM.

$$\alpha(0, 1; n) \geq n. \tag{3.2}$$

PROOF: Let P be an n-polytope and let X be a set of $n - 1$ nodes in $G(0, 1; P)$. Let u and v be two remaining 0-nodes. Associate to each 0-node of $G(0, 1; P)$ the corresponding vertex of P and to each 1-node the midpoint of the corresponding edge. Let X' indicate the points of P corresponding to members of X. Choose an additional vertex $\hat{p}$ of P and let H be a hyperplane passing through X' and $\hat{p}$. Let f be an affine function so that

$$H = \{x: f(x) = 0\}.$$

If $\hat{x}$ is a vertex of P such that $f(\hat{x}) \geq 0$, an edge path joining $\hat{x}$ to $\hat{p}$ exists which avoids X' by (3.1). Similarly, if $f(\hat{x}) \leq 0$, an edge path joining $\hat{x}$ to $\hat{p}$ exists which avoids X'. In particular, edge paths exist which miss X' joining both $\hat{u}$ and $\hat{v}$ to $\hat{p}$, and thus joining $\hat{u}$ and $\hat{v}$ to each other. The edge path between $\hat{u}$ and $\hat{v}$ is reflected in an obvious way in a (0, 1) path missing X in $G(0, 1; P)$ which joins u and v. Thus X does not separate any two remaining 0-nodes of $G(0, 1; P)$ and the conclusion follows.

Balinski's Theorem is an immediate corollary.

The edge graph of an n-polytope is n-connected. (3.3)

Let P be an n-polytope and let u, v be two vertices of P which do not lie in a given face F of P. Then there exists an edge path joining u to v which does not pass through F. (3.4)

PROOF: Let H be a hyperplane such that $F = P \cap H$ and let f be an affine function which vanishes on H and is positive on the interior of P. By hypothesis $f(u) > 0$ and $f(v) > 0$. The conclusion then follows from (3.1).

If F is a face of a polytope P, the *anti-star* of F, denoted ast(F), is the set of all faces of P which do not intersect F.

Let P an n-polytope and v a vertex of P. Then ast(v) is a strong $(n - 1)$-cell complex. (3.5)

PROOF: It is clear that every face of P which does not intersect v lies in a facet which does not intersect v. All that remains is to show that an $(n - 2, n - 1)$ path joins every pair of facets in ast(F).

It is easier to do this by considering the dual polytope P^0. The facets of ast(v) correspond to vertices of P^0 which do not lie in $v^\dagger$. By (3.4) any pair of such vertices can be joined by an edge path which does not pass through $v^\dagger$. Or, the corresponding 0-nodes can be joined by a (0, 1) path in $G(0, 1; \text{ast}(v^\dagger))$. Hence, by (2.6) any two $(n - 1)$-nodes in $G(n - 2, n - 1; \text{ast}(v))$ are connected, and the result follows.

The *facial lattice* (or *lattice of faces*) of a polytope is the set of all of its faces, including the empty face and the polytope itself, with a partial ordering defined by set inclusion. Proofs of the next two results may be found in [4, §3.4, exercise 9 (iii)] and [6, p. 712], respectively.

Let P be an n-polytope and F a k-face of P. Then the lattice of faces of P which contain F is isomorphic to the lattice of faces of an $(n - k - 1)$-polytope Q where each t-face of P which contains F corresponds to a $(t - k - 1)$-face of Q. (3.6)

Every n-polytope contains at least $\binom{n+1}{s+1}$ s-faces for $0 \leq s < n$. Moreover, equality is attained only for the n-simplex. (3.7)

From these two theorems we can easily derive a useful corollary:

Let P be an n-polytope and F a k-face of P. Then P contains at least $\binom{n-k}{s-k}$ s-faces which contain F for every $s > k$. (3.8)

PROOF: Let N be the number sought. (3.6) N equals the number of $(s-k-1)$-faces in some $(n-k-1)$-polytope. From (3.7) it then follows that

$$N \geq \binom{n-k}{s-k}.$$

4. BOUNDS FOR α- AND β-CONNECTIVITIES

Our first numerical bounds for α- and β-connectivities will be established in this section. In the light of (2.7), once a general bound for either one of the connectivities is established the other will follow immediately. This same duality principle also allows us to choose between two proofs of a given result and thus often simplifies our considerations.

The main result of this section is

THEOREM.

$$\alpha(r, s; n) \geq n - r, \tag{4.1}$$

and its dual formulation,

$$\beta(r, s; n) \geq s + 1. \tag{4.2}$$

After these results have been established, some examples will be given to show that the bounds are exact whenever $r = 0$, $s = n - 1$, or $s = r + 1$.

We first establish

$$\alpha(0, s; n) \geq n \quad \text{for} \quad 0 < s < n. \tag{4.3}$$

PROOF. The proof goes by induction on n for fixed s. By (3.2) the result is known for $s = 1$ for all n and in particular the proposition is true as stated for $n = 2$. We assume that $\alpha(0, s; k) \geq k$ for all $s \leq k - 1$ if $k \leq n - 1$.

Let P be an n-polytope, where $n \geq 3$. Assume $s \geq 2$.

(A) Let $\hat{p}$ and $\hat{q}$ be two adjacent vertices of P and let $\hat{E}$ be the edge they determine. Remove a set X of $n - 1$ nodes from $G(0, s; P)$ such that neither p nor q is a member of X. We wish to show that a (0, s) path still connects p and q in $G(0, s; P)$.

By (3.8) there are at least $\binom{n-1}{s-1}$ s-faces of P which contain the edge $\hat{E}$. Since $2 \leq s \leq n - 1$, there are at least $n - 1$ s-faces of P which contain $\hat{E}$. If a node corresponding to one of these faces is not in X we have an easy (0, s) path remaining between p and q. If all these nodes have been removed, then the only members of X are the s-nodes corresponding to s-faces containing $\hat{E}$. In this case, let

$$\hat{p} = \hat{p}_0 \to \hat{p}_1 \to \cdots \to \hat{p}_j = \hat{q}$$

be another edge path between p and q (this exists by (3.3) since $n \geq 2$). Since the only members of X are s-nodes corresponding to faces containing the edge $\hat{E}$, then for each i there exists an s-face $\hat{F}_i$ such that $\hat{p}_i \in \hat{F}_i$, and $\hat{p}_{i+1} \in \hat{F}_i$. Thus,

$$p_0 \to F_0 \to p_1 \to \cdots \to F_{j-1} \to p_j$$

is a (0, s) path between p and q and hence X does not separate p and q.

(B) Now let X be any set of $n - 1$ nodes in $G(0, s; P)$ and let p and q be any two remaining 0-nodes. Let

$$\hat{p} = \hat{p}_0 \to \hat{p}_1 \to \cdots \to \hat{p}_t = \hat{q}$$

be an edge path joining $\hat{p}$ to $\hat{q}$ which contains no vertex corresponding to a member of X. This is possible by (3.3). By (A) a (0, s) path missing X exists between p_i and p_{i+1} for $0 \leq i \leq t - 1$. Joining these paths gives a (0, s) path between p and q.

Thus no set of cardinality $n - 1$ can disconnect two 0-nodes of $G(0, s; P)$. That is, $\alpha(0, s; P) \geq n$. Since P was arbitrary, the result follows.

From (2.12) we see that $\alpha(r, s; n) \geq \alpha(0, s - r; n - r)$, and this inequality together with (4.3) completes the proof of (4.1).

Now that lower bounds have been established for α- and β-connectivities, we turn our attention to finding upper bounds for $\alpha(r, s; n)$ and $\beta(r, s; n)$. By considering the n-simplex, Σ^n it is easy to see that

$$\beta(r, s; n) \leq \binom{s+1}{r+1}. \tag{4.4}$$

This statement follows immediately from the fact that each s-node in $G(r, s; \Sigma^n)$ is adjacent to $\binom{s+1}{r+1}$ r-nodes since each s-simplex contains exactly $\binom{s+1}{r+1}$ r-simplices.

By duality it follows that

$$\alpha(r, s; n) \leq \binom{n-r}{n-s}. \tag{4.5}$$

Considering the *bipyramid* over the $(n-1)$-simplex (that is, the polytope formed by taking the union of two n-simplicies with a common facet) we obtain another bound.

$$\alpha(r, s; n) \leq \binom{n}{r+1}. \tag{4.6}$$

Proof: Let P be the bipyramid over the $(n-1)$-simplex Σ^{n-1}. Then removing all of the r-nodes corresponding to r-faces in Σ^{n-1} will disconnect $G(r, s; P)$. For let $\hat{F}$ and $\hat{G}$ be two r-faces of P such that $\hat{p} \in \hat{F}$ and $\hat{q} \in \hat{G}$, where $\hat{p}$ and $\hat{q}$ are the two vertices of P which do not lie in Σ^{n-1}. Observe that no face of P contains both $\hat{p}$ and $\hat{q}$. Thus, on any (r, s) path between F and G there is a last node, A, such that $\hat{p} \in \hat{A}$, but no node on the path between A and G corresponds to a face containing p. Moreover, it is clear that A is an s-node since, if any r-face contains $\hat{p}$, then every s-face which contains it also contains $\hat{p}$. Let B be the next r-node in the path beyond A. Since $\hat{B}$ lies in a face containing p but does not itself contain $\hat{p}$, $\hat{B}$ lies in Σ^{n-1}.

Thus, removing all of the r-nodes corresponding to r-faces contained in Σ^{n-1} will separate F and G. The conclusion follows.

The dual result, obtained by removing s-faces from a cylinder over Σ^{n-1}, reads:

$$\beta(r, s; n) \leq \binom{n}{s}. \tag{4.7}$$

Combining these upper bounds with our previous lower ones, we see that our bounds are exact in three cases.

$$\alpha(r, s; n) = n - r \quad \text{if} \quad r = 0, \quad s = n-1, \quad \text{or } s = r+1. \tag{4.8}$$

$$\beta(r, s; n) = s + 1 \quad \text{if} \quad r = 0, \quad s = n-1, \quad \text{or } s = r+1. \tag{4.9}$$

5. A Combinatorial Lemma

In order to extend our results we need an estimate of the number of r-faces contained in a collection of s-faces. Klee (see (3.7)) settled the problem for a single s-face. Here we generalize his result to the case of a small number of s-faces.

If $F_1, ..., F_k$ are faces of a polytope P, $\varrho_r(F_1 \cup \cdots \cup F_k)$ denotes the number of r-faces of P contained in one or more of the F_i. In a similar way we define $\varrho_r(F_k \sim (F_1 \cup \cdots \cup F_{k-1}))$ as the number of r-faces of P which are contained in F_k, but not in any of the F_i for $i < k$.

Theorem. *Let P be an n-polytope and let $F_1, ..., F_k$ be k different s-faces of P. If $k \leq s + 2$, then*

$$\varrho_r(F_1 \cup \cdots \cup F_k) \geq \binom{s+2}{r+1} - \binom{s+2-k}{r+1-k}. \tag{5.1}$$

Moreover, if equality holds:

(a) *all of the F_i are s-simplices,*
(b) *$F_i \cap F_j$ is an $(s-1)$-simplex if $i \neq j$,*
(c) *$F_i \cap F_j \neq F_i \cap F_m$ if $j \neq m$.*

The proof of this theorem is based upon the following observation:

$$\begin{aligned}\varrho_r(F_1 \cup \cdots \cup F_k) = \varrho_r(F_1) + \varrho_r(F_2 \sim (F_1 \cap F_2)) + \cdots \\ + \varrho_r(F_k \sim (\bigcup_{i<k} F_i \cap F_k))\end{aligned} \tag{5.2}$$

The remainder of the argument will be devoted to showing that each term on the right-hand side attains its minimum value if P is the n-simplex and the F_i all lie in the same $(s+1)$-face. The numerical bound in (5.1) will then follow immediately by direct calculation. We conclude the proof by showing that certain terms in (5.2) attain their minimum only if (a), (b), and (c) are satisfied.

Definition. Let P and Q be n-polytopes. A homeomorphism $\sigma: P \to Q$ is called a *refinement homeomorphism* if $\sigma^{-1}(F)$ is a cell complex of P for any face $F \subset Q$.

Let P, Q be n-polytopes and let $\sigma: P \to Q$ be a refinement homeomorphism. For $k \leq n-1$, let $F_1, ..., F_k$ be k facets of P (not necessarily distinct). Then there is a collection of k different facets of Q, $G_1, ..., G_k$, such that

$$\varrho_r(P \sim (F_1 \cup \cdots \cup F_k)) \geq \varrho_r(Q \sim (G_1 \cup \cdots \cup G_k)). \tag{5.4}$$

Proof: First observe that if K^t is a t-face of P, then $\sigma(K^t)$ is contained in a unique face K^u of smallest dimension where $u \geq t$. It is clear that

$u \leq n - 1$, so that $u = n - 1$ whenever $t = n - 1$. For each F_i let G_i' be the unique facet of Q such that $\sigma(F_i) \subset G_i'$. Note that the G_i' are not necessarily distinct.

Let L^r be an r-face in $Q \sim (G_i' \cup \cdots \cup G_k')$. Since σ is a homeomorphism, $\sigma^{-1}(L^r)$ does not lie in $\sigma^{-1}(G_i' \cup \cdots \cup G_k')$, and thus it does not lie in $F_1 \cup \cdots \cup F_k$ since the latter is contained in $\sigma^{-1}(G_i' \cup \cdots \cup G_k')$. It is clear that at least one r-face of P lies in $\sigma^{-1}(L^r)$ and that this r-face will not lie in $\sigma^{-1}(K^r)$ for any other r-face K^r of Q. Thus, for each r-face in $Q \sim (G_1' \cup \cdots \cup G_k')$, there is at least one r-face in $P \sim (F_1 \cup \cdots \cup F_k)$. That is,

$$\varrho_r(P \sim (F_1 \cup \cdots \cup F_k)) \geq \varrho_r(Q \sim (G_i' \cup \cdots \cup G_k')).$$

If the G_i' are not all distinct, then removing additional facets until k different ones have been selected will not increase that number of r-faces in their complement. The statement follows.

If P is any n-polytope, F an s-face of P, and $F_1, \ldots, F_k$ $(k \leq s + 2)$ are different faces of P contained in F, then there is a collection of k different $(s - 1)$-faces $G_1, \ldots, G_k$ of Σ^n contained in an s-face G such that

$$\varrho_r(F \sim (F_1 \cup \cdots \cup F_k)) \geq \varrho_r(G \sim (G_1 \cup \cdots \cup G_k)). \quad (5.5)$$

Proof: Let G be any s-face of P. By the proof of Grünbaum's Refinement Theorem [5] it follows that there exists a refinement homeomorphism σ mapping P onto Σ^n such that $\sigma(F) = G$. For all i let F_i' be an $(s - 1)$-face of P such that $F_i \subset F_i' \subset F$. Then

$$\begin{aligned}\varrho_r(F \sim (F_1 \cup \cdots \cup F_k)) &\geq \varrho_r(F \sim (F_1' \cup \cdots \cup F_k')) \\ &\geq \varrho_r(G \sim (G_1 \cup \cdots \cup G_k))\end{aligned}$$

where the last inequality follows from (5.4). This completes the proof.

Suppose that $F_1, \ldots, F_k$ are k different s-faces of an n-polytope P where $k \leq s + 2$. Then for $r < s$, $\varrho_r(F_1 \cup \cdots \cup F_k) \geq \varrho_r(G_1 \cup \cdots \cup G_k)$ where $G_1, \ldots, G_k$ are k different s-faces of Σ^n which all lie in an $(s + 1)$-face G. Moreover, equality holds iff all of the F_i satisfy conditions (a), (b), and (c) of (5.1). (5.6)

Proof. According to (5.2)

$$\varrho_r(F_1 \cup \cdots \cup F_k) = \sum_{j=1}^{k} \varrho_r(F_j \sim (\bigcup_{i<j} F_i \cap F_j)).$$

By (5.5), we have

$$\varrho_r(F_j \sim (\bigcup_{i<j} F_i \cap F_j)) \geq \varrho_r(G_j \sim (\bigcup_{i<j} G_i \cap G_j)) \quad (5.7)$$

since $G_i \cap G_j$ is an $(s - 1)$-face of each for all $i < j$ and since $G_i \cup G_j \neq G_i \cap G_m$ for $j \neq m$. Using (5.2) to sum both sides of (5.7) the first assertion of the proposition follows.

Now assume that $\varrho_r(F_1 \cup \cdots \cup F_k) = \varrho_r(G_1 \cup \cdots \cup G_k)$. By (3.7) it follows that $\varrho_r(F_1) = \varrho_r(G_1)$ iff F_1 is an s-simplex. Since the ordering of the F_i is arbitrary, each of the F_i is an s-simplex. We next observe that

$$\varrho_r(F_2 \sim (F_1 \cap F_2)) > \varrho_r(G_2 \sim (G_1 \cap G_2))$$

unless $\dim(F_1 \cap F_2) = \dim(G_1 \cap G_2) = s - 1$. From this we see that $F_1 \cap F_2$ is an $(s - 1)$-face of P and hence that $F_i \cap F_j$ is an $(s - 1)$-face of P for all $i \neq j$. Finally, we see that

$$\varrho_r(F_3 \sim ((F_1 \cap F_3) \cup (F_2 \cap F_3)) > \varrho_r(G_3 \sim ((G_1 \cap G_3) \cup (G_2 \cap G_3)))$$

unless $F_1 \cap F_3 \neq F_2 \cap F_3$. For an s-simplex lacking two $(s - 1)$-faces will always contain strictly fewer r-faces than an s-simplex lacking just one s-face, and we know that $G_1 \cap G_3 \neq G_2 \cap G_3$. Thus, $F_i \cap F_j \neq F_i \cap F_m$ for any $j \neq m$ and the proposition is established.

The proof of (5.1) is now complete except for computing the numerical bound. Using (5.6) the problem reduces to evaluating $\varrho_r(G_1 \cup \cdots \cup G_k)$ where the G_i and G are as in (5.6). To do this note the total number of r-faces in G is $\binom{s+2}{r+1}$, and that each r-face which does not lie in any of the G_i is the intersection of $s + 1 - r$ of the remaining $s + 2 - k$ s-faces. Thus, $\binom{s+2-k}{r+1-k}$ r-faces of G lie in none of the G_i. Hence,

$$\varrho_r(G_1 \cup \cdots \cup G_k) = \binom{s+2}{r+1} - \binom{s+2-k}{r+1-k}.$$

This concludes the proof of (5.1).

It might be conjectured that the three conditions (a), (b), and (c) of (5.1) would imply that all of the s-faces would lie in an $(s + 1)$-face. This conjecture is seen to be false by considering the bipyramid P over an

$(s+1)$-simplex, Σ. There are $s+2$ s-faces in Σ which satisfy (a), (b). and (c) but they do not lie in any $(s+1)$-face of P. In Section 10 we will return to this problem and show that for $s = n-2$ the counter-example above is essentially unique.

6. Further Connectivity Results

The theorem of the last section will applied through the following

LEMMA. *Let P be an n-polytope and let X be a collection of s-nodes in $G(0, s; P)$ such that card $X < \binom{n}{s}$. Then there exists a $(0, s)$ path missing X between any two 0-nodes of $G(0, s; P)$.* (6.1)

PROOF: Let $\hat{p}$ and $\hat{q}$ be two vertices of P. By (3.3) there exist n disjoint edge paths connecting $\hat{p}$ and $\hat{q}$. Then for at least one of these paths, say

$$\hat{p} = \hat{p}_0 \to \hat{p}_1 \to \cdots \to \hat{p}_r = \hat{q},$$

there exist s-nodes $F_1, ..., F_r$ which do not belong to X such that $\hat{E}_i \subset \hat{F}_i$ $(1 \leq i \leq r)$, where $\hat{E}_i$ is the edge containing $\hat{p}_{i-1}$ and $\hat{p}_i$.

For, otherwise, on each of the n edge paths between $\hat{p}$ and $\hat{q}$ there exists an edge $\hat{A}_i$ such that X contains every s-node corresponding to an s-face containing $\hat{A}_i$. Considering the dual polytope, this means that every $(n-s-1)$-face which lies in one of the n $(n-s)$-faces $\hat{A}_i^*$ corresponds to a member of X. But by (5.1),

$$\varrho_{n-s-1}(\hat{A}_1^{\dagger} \cup \cdots \cup \hat{A}_n^{\dagger}) \geq \binom{n}{n-s} = \binom{n}{s}.$$

Thus X must contain at least $\binom{n}{s}$ s-nodes, contrary to hypothesis. The contradiction completes the proof.

With the above proposition at our disposal, it is not difficult to prove that we need determine only the connectivity of those s-nodes which correspond to intersecting s-faces in order to evaluate $\beta(r, s; n)$.

In order to make this notion more precise, we introduce some additional notation.

We say that $G(r, s; P)$ has connectivity $\alpha^v(r, s; P)$ if $\alpha^v(r, s; P)$ is a minimal cardinality of a set needed to separate some two r-nodes of $G(r, s; P)$ which correspond to r-faces having a common vertex. We also define $\alpha^f(r, s; P)$ as the minimal cardinality of a set needed to separate some two r-nodes of $G(r, s; P)$ which correspond to r-faces lying in the same facet of P. In a similar way we may define $\beta^v(r, s; P)$, $\gamma^f(r, s; n)$, etc.

THEOREM.

$$\beta(r, s; n) = \min\left\{\beta^v(r, s; n), \binom{n}{s}\right\}. \tag{6.2}$$

PROOF: Let

$$z = \min\left\{\beta^v(r, s; n), \binom{n}{s}\right\}.$$

It is clear from (4.7) and the definition of $\beta^v(r, s; n)$ that $\beta(r, s; n) \leq z$.

In order to show the reverse inequality, let P be an n-polytope and remove a set X of $z-1$ nodes from $G(r, s; P)$. Let F and G be two remaining s-nodes of $G(r, s; P)$ and let $\hat{v}, \hat{w}$ be vertices of P such that $\hat{v} \in \hat{F}$, and $\hat{w} \in \hat{G}$. By (6.1), there exists a $(0, s)$ path which contains no member of X between v and w in $G(0, s; P)$. Let this path be

$$v \to F_0 \to v_1 \to F_1 \to \cdots \to F_t \to w.$$

Since $z \leq \beta^v(r, s; n)$, there exists an (r, s) path which misses X joining F_i to F_{i+1} for all i. There also exist (r, s) paths missing X which join F to F_0 and G to F_t. Combining all of these paths gives us an (r, s) path between F and G which avoids X. Since F and G were arbitrary, X is not a separating set. Thus $\beta(r, s; n) \geq z$ and the result follows.

With the above result in mind we now turn our attention to estimating $\beta^v(r, s; n)$.

THEOREM. *For $1 \leq r < s \leq n-2$,*

$$\beta^v(r, s; n) \geq \beta(r-1, s-1; n-1) + \min\left\{\binom{s}{r+1}, \alpha(r, s; n-1)\right\}. \tag{6.3}$$

PROOF: Let p be any n-polytope and let $\hat{F}, \hat{G}$ be two s-faces of P with a common vertex $\hat{v}$. Let X be a set of $z-1$ nodes of $G(r, s; P)$ which does not contain either F or G, where z is the value of the right-hand side of (6.3). We shall show that there exists an (r, s) path missing X and joining F and G which is of one of the two following special types:

(1) Every node of the path corresponds to a face containing $\hat{v}$.

(2) Every node of the path, except for F and G, corresponds to a face in $\operatorname{ast}(\hat{v})$.

Assume that no path of either type exists. Let H be a hyperplane which strictly separates $\hat{v}$ from the other vertices of P and let $Q = H \cap P$. Define

$$X_1 = \{K \in X\colon \hat{v} \in \hat{K}\} \text{ and } Y = \{L\colon \hat{L} = \hat{K} \cap H,\ K \in X_1\}.$$

Since no path of type (1) exists, every path from $F \cap H$ to $G \cap H$ in $G(r-1, s-1; Q)$ must contain a member of Y. But distinct members of X_1 determine distinct members of Y and so

$$\operatorname{card} X_1 \geq \alpha(r-1, s-1;\ Q) \geq \alpha(r-1, s-1;\ n-1).$$

Each path of type (2) must connect an r-node adjacent to F to an r-node adjacent to G by a path in $G(r, s; \operatorname{ast}(\hat{v}))$. The face $\hat{F}$ contains at least one $(s-1)$-face in $\operatorname{ast}(\hat{v})$ and hence, by (3.7), at least $\binom{s}{r+1}$ r-faces in $\operatorname{ast}(\hat{v})$. Similarly with $\hat{G}$. Let

$$X_2 = \{K \in X\colon \hat{K} \in \operatorname{ast}(\hat{v})\}.$$

If no path of type (2) exists, then either all nodes adjacent to F (or G) lie in X_2 or else $\alpha(r, s; \operatorname{ast}(\hat{v}))$ other nodes lie in X_2. That is,

$$\operatorname{card} X_2 \geq \min\left\{\binom{s}{r+1},\ \alpha(r, s; \operatorname{ast}(v))\right\}.$$

Since $\operatorname{ast}(\hat{v})$ is a strong $(n-1)$-cell complex by (3.5), it follows from (2.10) that

$$\alpha(r, s; \operatorname{ast}(\hat{v})) \geq \alpha(r, s; n-1).$$

Since X_1 and X_2 are disjoint,

$$\operatorname{card} X \geq \operatorname{card} X_1 + \operatorname{card} X_2 \geq \alpha(r-1, s-1; n-1) + \min\left\{\binom{s}{r+1}, \alpha(r, s; n-1)\right\}$$

or $\operatorname{card} X \geq z$, contrary to hypothesis. The contradiction completes the proof.

It should be possible to obtain a strengthening of the above result by allowing more general types of paths. A better bound might also take into account the possibility of $\hat{F}$ and $\hat{G}$ having common r-faces. However, we can still use (6.2) and (6.3) together to obtain a number of useful corollaries.

$$\beta(r, s; n) = \binom{s+1}{r+1} \quad \text{for } n \geq \binom{s}{r+1} + r + 1. \tag{6.4}$$

PROOF: By (4.7), $\beta(r, s; n) \leq \binom{s+1}{r+1}$ for all n. Here we must establish the reverse inequality. Using (2.11), we see that the assertion need be proved only for the case

$$n = \binom{s}{r+1} + r + 1$$

and it will then follow immediately for all larger n.

We use induction on r and s to establish the result. By (4.9) the assertion is true for all s and n when $r = 0$. Assume that the result is known for all triples $(r, s; n)$ when

$$r < r_0,\ s < s_0,\ r < s \text{ and } n \geq \binom{s}{r+1} + r + 1.$$

Now it follows from (4.1) that

$$\alpha(r_0, s_0;\ n-1) \geq \binom{s_0}{r_0+1}$$

whenever

$$n \geq \binom{s_0}{r_0+1} + r_0 + 1.$$

Using this fact in (6.3), we find that

$$\beta^{v}(r_0, s_0; n) \geq \beta(r_0 - 1, s_0 - 1; n-1) + \binom{s_0}{r_0+1}. \tag{6.5}$$

Applying our inductive assumption to $\beta(r_0 - 1, s_0 - 1; n-1)$ (noting that n is large enough that the inductive assumption applies to it), we obtain

$$\beta^v(r_0, s_0; n) \geq \binom{s_0}{r_0} + \binom{s_0}{r_0+1} = \binom{s_0+1}{r_0+1}. \quad (6.6)$$

The result now follows from (6.2), once it is established that

$$\binom{s_0+1}{r_0+1} \leq \binom{n}{s_0} \quad \text{for } n \geq \binom{s_0}{r_0+1} + r_0 + 1.$$

For this, let $k = s_0 - r_0$. The inequality is trivial in case k is 1 or 2. For $k \geq 3$, $s_0 \geq 4$ (since $r_0 \geq 1$) and $2 \leq r_0 + 1 \leq s_0 - 2$. Hence,

$$n \geq \binom{s_0}{2} + r_0 + 1 \geq 2(s_0 - 1) + r_0 + 1 \geq 2s_0 - r_0 = s_0 + k.$$

Then

$$\binom{n}{s_0} < \binom{n-k+1}{r_0+1} \geq \binom{s_0+1}{r_0+1}$$

since $n - k \geq s_0$. This completes the proof.

In much the same way as in the above corollary we can combine (6.2) and (6.3) to get better lower bounds for β-connectivity. For example:

If $1 \leq r \leq n-4$, then $\beta(r, n-2; n) \geq (r+1)(n-r-1)$. (6.7)

PROOF: The proof goes by induction on r. For $r = 1$, by (6.3)

$$\beta^v(1, n-2; n) \geq \beta(0, n-3; n-1) + \min\left\{\binom{n-2}{n}, \alpha(1, n-2; n-1)\right\}$$

$$\geq n - 2 + \min\left\{\binom{n-2}{2}, n-2\right\} \geq 2(n-2).$$

Since $\binom{n}{n-2} \geq 2(n-2)$ for all n, it follows from (6.2) that $\beta(1, n-2; n) \geq 2(n-2)$, and thus the result is true for $r = 1$.

Assume that the proposition is true for $r - 1$ and all n. Then, as above

$$\beta^v(r, n-2, n) \geq \beta(r-1, n-3, n-1) + \min\left\{\binom{n-2}{r+1}, \alpha(r, n-2; n-1)\right\}$$

$$\geq r(n-r-1) + \min\left\{\binom{n-2}{r+1}, n-r-1\right\} \geq (r+1)(n-r-1)$$

since $\binom{n-2}{r+1} \geq n - r - 1$. The proof will be completed by applying (6.2) once we show that $\binom{n}{n-2} \geq (r+1)(n-r-1)$. But this inequality follows easily from the fact that $n \geq r + 3$.

Combining the above result with (2.11), we obtain

If $r \neq 0, s \neq n-1$, and $s \neq r+1$, then

$$\beta(r, s; n) \geq (r+1)(s-r+1). \quad (6.8)$$

More results of this general nature could be given by utilizing a suitable mixture of (6.2) and (6.3) as we have done above. Better results will entail more restrictions on r and s, however, as further upper bounds are attained. For example, (6.8) gives an exact bound for $\beta(1, 3; n)$, so a better bound would have to exclude this case. Another approach is probably needed to make a significant improvement in these bounds.

Using the results of this section, we summarize some known values for $\alpha(r, s; n)$ and $\beta(r, s; n)$ by means of Table 1. We exclude the cases when $r = 0$ or $s = n - 1$ as these bounds are exact for all values of n.

TABLE 1

n	r	s	$\alpha(r, s; n)$	$\beta(r, s; n)$
4	1	2	3	3
5	1	2	4	3
	1	3	6	6
6	1	2	5	3
	1	3	$8 \leq \alpha \leq 10$	6
	1	4	$8 \leq \alpha \leq 10$	$8 \leq \beta \leq 10$
	2	3	4	4
	2	4	6	$8 \leq \beta \leq 10$
	3	4	3	5

We conclude by stating the dual formulation of the more important results.

$$\alpha(r, s; n) = \min\left\{\alpha^f(r, s; n), \binom{n}{r+1}\right\}. \quad (6.9)$$

For $1 \leq r < s \leq n-2$,

$$\alpha^f(r, s; n) \geq \alpha(r, s; n-1) + \min\left\{\binom{n-r-1}{n-s}, \beta(r-1, s-1; n-1)\right\}. \tag{6.10}$$

$$\alpha(r, s; n) = \binom{n-r}{n-s} \text{ whenever } s \geq \binom{n-1-r}{n-s}. \tag{6.11}$$

If $r \neq 0$, $s \neq n-1$, and $s \neq r+1$, then

$$\alpha(r, s; n) \geq (n-s)(s-r+1). \tag{6.12}$$

7. γ, δ, ε-, AND ζ-CONNECTIVITIES

We now turn our attention to investigating the connectivities of incidence graphs in which only one type of node is removed. Precise definitions of the four types we wish to consider were given in Section 2. Essentially the same methods can be used to investigate these connectivities as we have used to this point, but much better bounds can be obtained here.

As before, these connectivities are paired in a natural way by duality. We recall from Section 2 that:

$$\gamma(r, s; n) = \zeta(n-1-s, n-1-r; n), \tag{2.8}$$

$$\delta(r, s; n) = \varepsilon(n-1-s, n-1-r; n). \tag{2.9}$$

Using this duality and the examples of the n-simplex and the bipyramid over the $(n-1)$-simplex from Section 4, we find:

$$\gamma(r, s; n) \leq \binom{n}{r+1}, \tag{7.1}$$

$$\delta(r, s; n) \leq \binom{n-r}{n-s}, \tag{7.2}$$

$$\varepsilon(r, s; n) \leq \binom{s+1}{r+1}, \tag{7.3}$$

$$\xi(r, s; n) \leq \binom{n}{s}. \tag{7.4}$$

For δ-and ε-connectivities, we have the following strong result:

THEOREM. *For all n,*

$$\varepsilon(r, s; n) = \binom{s+1}{r+1}.$$

Moreover, if $s \leq n-2$, then $\varepsilon(r, s; P) = \varepsilon(r, s; n)$ iff P contains an s-face which is an s-simplex. (7.5)

PROOF: It is clear that, if P is an n-polytope containing an s-face which is an s-simplex, then

$$\varepsilon(r, s; P) \leq \binom{s+1}{r+1}.$$

To prove the opposite inequality, we let P be an n-polytope and let X be a set of $\binom{s+1}{r+1} - 1$ r-nodes in $G(r, s; P)$. Let F and G be two s-nodes of $G(r, s; P)$. By (4.2) there are $s+1$ disjoint $(s-1, s)$ paths between F and G in $G(s-1, s; P)$. Assume that for one of these paths, say

$$F = F_0^s \to F_0^{s-1} \to F_1^s \to \cdots \to F_k^s = G,$$

there exist r-nodes $F_1^r, \ldots, F_{k-1}^r$ which are not in X and such that $\hat{F}_i^r \subset \hat{F}_i^{s-1}$ for all i. Then we can easily obtain an (r, s) path between F and G which misses X, namely,

$$F_0^s \to F_0^r \to \cdots \to F_k^s.$$

If no path such as that described above exists, this means that there are at least $s+1$ $(s-1)$-faces of P which contain only r-faces corresponding to members of X. Hence by (5.1), X contains at least $\binom{s+1}{r+1}$ r-nodes, contrary to assumption.

Thus, an (r, s) path avoiding X always exist between F and G and the first statement follows.

To show that $s \leq n-2$ and $\varepsilon(r, s; P) = \varepsilon(r, s; n)$ implies that P contains an s-simplex requires two additional lemmas.

LEMMA. *Let P be an n-polytope, H a hyperplane, and H^+ one of the open halfspaces associated with H such that $H^+ \cap P \neq \varphi$. Suppose $\hat{F}$ and $\hat{G}$ are two r-faces of P such that $H^+ \cap \hat{F} \neq \varphi \neq H^+ \cap \hat{G}$. Then there exists an (r, s) path joining F and G which contains no node corresponding to a face lying in H.* (7.6)

PROOF: By (3.4) an edge path missing H joins any vertex in $H^+ \cap \hat{F}$ to any vertex in $H^+ \cap \hat{G}$. Choose s-nodes corresponding to faces containing edges along this path. If $r = 0$, we are done.

If $r > 0$, observe that none of the faces containing any vertex $v \in H^-$ lie in H and that an (r, s) path may be found joining any two s-nodes corresponding to faces which contain v such that each member of the path contains v.

Therefore the required path exists.

LEMMA. *Let $F_1, \ldots, F_{s+2}$ be $s + 2$ s-faces of an n-polytope P which satisfy the following three conditions:*

(a) *each F_i is an s-simplex;*
(b) *$F_i \cap F_j$ is an $(s-1)$-simplex if $i \neq j$;* (7.7)
(c) *$F_i \cap F_j \neq F_i \cap F_k$ if $j \neq k$.*

Then exactly $s + 2$ vertices of P are contained in one or more of the F_i.

PROOF: Let $v_1, \ldots, v_t$ be the vertices of P which lie in one or more of the F_i. Without loss of generality assume that F_1 is the convex hull of $\{v_1, \ldots, v_{s+1}\}$ (written $F_1 = \text{con}\{v_1, \ldots, v_{s+1}\}$). Using (b) we can likewise assume that

$$F_2 = \text{con}\{v_1, \ldots, v_s, v_{s+2}\}.$$

Assume that $v_t \in F_3$. Since $F_3 \cap F_1$ contains exactly s vertices, F_3 omits v_i for some $1 \leq i \leq s$; that is,

$$F_3 = \text{con}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s+1}, v_t\}.$$

It is not possible that $i = s + 1$, for otherwise we would have $F_3 \cap F_1 = F_2 \cap F_1$ contrary to (c). If $t \neq s + 2$, then

$$F_3 \cap F_2 = \text{con}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_s\},$$

an $(s-2)$-simplex, which contradicts (b). Hence, $t = s + 2$ and the proof is complete.

Using these two lemmas it is easy to complete the proof of (7.5). Suppose that P is an n-polytope such that

$$\varepsilon(r, s; P) = \binom{s+1}{r+1}.$$

Let X be a set of $\binom{s+1}{r+1}$ r-nodes which separate two r-nodes F and G of $G(r, s; P)$. Using the reasoning from the first part of the proof, we see that P contains at least $s + 1$ $(s-1)$-faces, $\hat{F}_1, \ldots, \hat{F}_{s+1}$, which contain only r-faces corresponding to members of X. Since card $X = \binom{s+1}{r+1}$, it follows from (5.1) that these $(s-1)$-faces satisfy conditions (a), (b), and (c) of (7.7). Thus, they contain only $s + 1$ vertices in all.

Now let $\hat{A}$ be an r-face not lying in any of the $\hat{F}_i$, let $z \in \text{rel int } \hat{A}$, and let H be a hyperplane containing z as well as the vertices of P which lie in one of the $\hat{F}_i$, but not containing all of $\hat{A}$. Since $s \leq n - 2$, we are specifying at most n points and such a hyperplane can be found.

By (7.6) an (r, s) path missing X joins A to any r-node which does not correspond to a face in one of the F_i. Thus if $\hat{B}$ and $\hat{C}$ are s-faces each of which contains r-faces which do not lie in one of the $\hat{F}_i$, an (r, s) path missing X joins B and C. Since F and G are separated by X, every r-face in $\hat{F}$, say, lies in one of the $\hat{F}_i$. By (3.7), since $\hat{F}$ is an s-face it contains $\binom{s+1}{r+1}$ r-faces iff $\hat{F}$ is an s-simplex. Thus $\hat{F}$ is an s-simplex and the proof is complete.

$\delta(r, s; n) = \binom{n-r}{n-s}$. *Moreover, if* $r \geq 1$, $\delta(r, s; P) = \delta(r, s; n)$ *iff P contains an r-face which lies in exactly $n - r$ facets.* (7.8)

The result for γ- and ζ-connectivities are unfortunately not as complete. We do, however, have the following strong lower bound.

THEOREM. $$\zeta(r, s; n) \geq r\binom{n-r-1}{s-r} + \binom{n-r}{s-r}. \quad (7.9)$$

In order to prove this statement we use the same type of reasoning used in Section 6. In fact, the same proof as in (6.2) may be given to show

THEOREM. $$\zeta(r, s; n) = \min\left\{\zeta^v(r, s; n), \binom{n}{s}\right\}. \quad (7.10)$$

Likewise, we can duplicate the proof of (6.3) to evaluate $\zeta^v(r, s; n)$. However, in this case, as we are considering only the removal of s-nodes, the term $\binom{s}{r+1}$ does not appear in our estimate as it arose from the possibility of removing r-nodes. Thus our new inequality may be written as:

THEOREM.

$$\zeta^{v}(r, s; n) \geq \zeta(r-1, s-1; n-1) + \delta(r, s; n-1),$$
$$\text{for } 1 \leq r < s \leq n-2. \tag{7.11}$$

These last two propositions can be used to prove (7.9) as soon as we have a lower bound for $\zeta(0, s; n)$. The needed result is:

$$\zeta(0, s; n) \geq \binom{n}{s}. \tag{7.12}$$

PROOF: Let P be an n-polytope and let X be a set of $\left(\binom{n}{s} - 1\right)$ s-nodes of $G(0, s; P)$. Let F and G be two remaining s-nodes and let p and q be two 0-nodes such that $\hat{p} \in \hat{F}$, and $\hat{q} \in \hat{G}$. By (6.1), a $(0, s)$ path which misses X joins p and q, and hence a $(0, s)$ path which misses X joins F and G.

The proof of (7.9) now follows by induction on r. If $r = 0$, the result is given by (7.12) for all s and n. For $r \geq 1$, we use (7.11) and the inductive hypothesis:

$$\begin{aligned} \zeta^{v}(r, s; n) &\geq \zeta(r-1, s-1; n-1) + \binom{n-r-1}{n-s-1} \\ &\geq (r-1)\binom{n-r-1}{s-r} + \binom{n-r}{s-r} + \binom{n-r-1}{s-r} \\ &\geq r\binom{n-r-1}{s-r} + \binom{n-r}{s-r}. \end{aligned} \tag{7.13}$$

Letting $f(r, s, n)$ denote the right-hand side of (7.13), it is easy to verify that

$$f(r, s, n) \leq f(r-1, s, n),$$

and thus that

$$f(r, s, n) \leq f(0, s, n) = \binom{n}{s}.$$

Applying (7.10) completes the proof.

In (7.10) we showed that to evaluate $\zeta(r, s; n)$, it usually suffices to consider $\zeta^{v}(r, s; n)$. In the next theorem we strengthen this result to show that we only need to evaluate ζ^{v} for cones.

THEOREM. *If $\zeta(r, s; n) = \zeta^{v}(r, s; n)$, then there exists a cone P such that $\zeta(r, s; P) = \zeta(r, s; n)$. Moreover, P contains two s-faces, $\hat{F}$ and $\hat{G}$, each of which contains the vertex of the cone, such that F and G can be separated in $G(r, s; P)$ by a set of $\zeta(r, s; n)$ s-nodes.* (7.14)

The proof requires two lemmas.

LEMMA. *Let P and Q be n-polytopes and let $\sigma: P \to Q$ be a refinement homeomorphism which is linear on each face of P. If $\hat{F}$ is an s-face of Q, and $\hat{G}$, $\hat{H}$ are s-faces of P contained in $\sigma^{-1}(\hat{F})$, then for any $r < s$ an (r, s) path between G and H exists in $G(r, s; \sigma^{-1}(F))$.* (7.15)

PROOF: It follows from the conditions on σ that $\sigma(\hat{G})$ and $\sigma(\hat{H})$ are s-polytopes contained in F. Choose points $x \in \text{int}\, \sigma(\hat{G})$ and $y \in \text{int}\, \sigma(\hat{H})$ such that the line segment $[x, y]$ does not intersect the σ-image of any face of P of dimension less than $s - 1$. Since $[x, y] \subset \hat{F}$ it determines in an obvious way an $(s-1, s)$ path between G and H in $G(s-1, s; \sigma^{-1}(F))$. Given an $(s-1, s)$ path between G and H, it is an easy matter to find an (r, s) path between G and H in $G(r, s; \sigma^{-1}(F))$.

LEMMA. *Let P and Q be n-polytopes and let $\sigma: P \to Q$ be a refinement homeomorphism which is linear on each face of P. Let $\hat{F}$, $\hat{G}$ be two s-faces of P such that $\sigma(\hat{F})$ and $\sigma(\hat{G})$ are s-faces of Q. Then at least as many s-nodes must be removed from $G(r, s; P)$ to separate F and G as must be removed from $G(r, s; Q)$ to separate the nodes corresponding to $\sigma(\hat{F})$ and $\sigma(\hat{G})$.* (7.16)

PROOF: For notational convenience, we will write $\sigma(F)$ as the node corresponding to $\sigma(\hat{F})$, etc.

Suppose that m paths disjoint in s-nodes exist in $G(r, s; Q)$ between $\sigma(F)$ and $\sigma(G)$. Let these paths be:

$$\sigma(F) \to F^{r}_{11} \to F^{s}_{11} \to \cdots \to F^{r}_{1n_1} \to \sigma(G),$$
$$\sigma(F) \to F^{r}_{m1} \to F^{s}_{m1} \to \cdots \to F^{r}_{mn_m} \to \sigma(G).$$

Consider the "paths":

$$F \to \sigma^{-1}(F_{11}^r) \to \sigma^{-1}(F_{11}^s) \to \cdots \to G,$$
$$F \to \sigma^{-1}(F_{m1}^r) \to \sigma^{-1}(F_{m1}^s) \to \cdots \to G.$$

For each i, j let $\hat{G}_{ij}^r$ be an r-face of P contained in $\sigma^{-1}(\hat{F}_{ij}^r)$. Choose $\hat{G}_{ij}^s$ in $\sigma^{-1}(\hat{F}_{ij}^s)$ such that $\hat{G}_{ij}^r \subset \hat{G}_{ij}^s$ and choose $\hat{H}_{ij}^s$ such that $\hat{G}_{ij+1}^r \subset \hat{H}_{ij}^s$ ($\hat{G}_{ij}^s$ are chosen for $1 \leq j \leq n_i - 1$ for all i, while $\hat{H}_{ij}^s$ are chosen for $2 \leq j \leq n_i$ for all i). By (7.15) an (r, s) path joining H_{ij}^s to G_{ij}^s can be found in $G(r, s; \sigma^{-1}(\hat{F}_{ij}^s))$ for all i, j.

As σ is a homeomorphism, no s-face of P lies in more than one of the $\sigma^{-1}(F_{ij}^s)$. Hence, all the (r, s) paths from H_{ij}^s to G_{ij}^s in $G(r, s; \sigma^{-1}(F_{ij}^s))$ for different i, j are all disjoint. Connecting H_{ij}^s to $G_{i,j+1}^s$ by means of $G_{i,j+1}^r$ gives us m paths between F and G which are disjoint in s-nodes. This concludes the argument.

We can now prove (7.14). Assume that Q is an n-polytope such that $\zeta^v(r, s; Q) = \zeta(r, s; n)$. Let $\hat{F}$ and $\hat{G}$ be two s-faces of Q with a common vertex, $\hat{v}$ such that F and G may be disconnected in $G(r, s; Q)$ by removing exactly $\zeta(r, s; n)$ s-nodes.

Let H be a hyperplane strictly separating $\hat{v}$ from the remaining vertices of Q, let $Q_1 = H \cap Q$, and let $P = $ convex hull $\{Q_1, v\}$. Then the map determined by rays through $\hat{v}$ is a refinement homeomorphism of ast($\hat{v}$) onto Q_1, and it can easily be extended to a refinement homeomorphism of Q onto P. Moreover, it is linear on faces of Q and $\sigma(F)$ and $\sigma(G)$ are s-faces of P. By (7.16) no more s-nodes must be removed from $G(r, s; P)$ to separate $\sigma(F)$ from $\sigma(G)$ than were needed to separate F and G in $G(r, s; P)$. Hence, $\zeta^v(r, s; P) = \zeta(r, s; n)$ and the result follows.

Note that the same arguments can be used to show that the minimum δ-connectivity is attained for two r-nodes with a common vertex. The essential part of the proof revolves around the fact that the separating set consists of s-nodes and not on the type of node which we were trying to separate.

It would be of interest to know if (7.14) could be improved still further, say to the point of being able to assert that, if $\zeta^v(r, s; n) = \zeta(r, s; n)$ we can always find two s-faces with a common $(s-1)$-face whose corresponding nodes have a separating set of cardinality $\zeta(r, s; n)$. In such a case, we could repeat the refinement argument used above on all of the vertices of the common $(s-1)$-face and it would follow that $\zeta(r, s; n) = \zeta(r, s; \Sigma^n)$.

The upper bound for $\zeta(r, s; n)$ given earlier is not the best possible.

A somewhat better one is given by:

$$\zeta(r, s; n) \leq \sum_{i=1}^{s-r} \binom{n-s}{i} \binom{s+1}{i}. \tag{7.17}$$

Note that the sum on the right is $\binom{n}{s}$ in case $2s - r \geq n$, but that we improve the previous bound for larger values of n.

PROOF: This estimate is obtained by considering the set X of all s-faces of the n-simplex which intersect a particular s-face $\hat{F}_0$ in at least an r-face. Then the set of nodes corresponding to members of X will clearly separate F_0 from the other s-nodes of $G(r, s; \Sigma^n)$. It remains to determine the cardinality of X.

Let X_t be the set of s-faces of Σ^n which intersect $\hat{F}_0$ in a face of dimension t. Then

$$X = X_r \cup \cdots \cup X_{s-1}.$$

Since all of the X_t are distinct,

$$\text{card } X = \text{card } X_r + \cdots + \text{card } X_{s-1}.$$

To find card X_t observe that if $\hat{G}$ is a t-face contained in $\hat{F}_0$, then $\binom{n-s}{s-t}$ s-faces of Σ^n will intersect $\hat{F}_0$ in exactly $\hat{G}$. Since $\hat{F}_0$ contains $\binom{s+1}{s-t}$ t-faces,

$$\text{card } X_t = \binom{n-s}{s-t} \binom{s+1}{s-t}.$$

Hence

$$\text{card } X = \sum_{t=r}^{s-1} \binom{n-s}{s-t} \binom{s+1}{s-t}.$$

Setting $i = s - t$ and reversing the order of summation establishes the result.

We conclude with the dual formulation of the more important results:

$$(n-1-s)\binom{s}{r} + \binom{s+1}{r+1} \leq \gamma(r, s; n) \leq \sum_{i=1}^{s-r} \binom{n-r}{i} \binom{r+1}{i}, \tag{7.18}$$

$$\gamma(r, s; n) = \min\left\{\gamma^f(r, s; n), \binom{n}{r+1}\right\}. \tag{7.19}$$

If $\gamma(r, s; n) = \gamma^f(r, s; n)$, *then there exists a cone P such that* $\gamma(r, s; P) = \gamma(r, s; n)$. *Moreover, P contains two r-faces,* $\hat{F}$ *and* $\hat{G}$, *in its base such that F and G can be separated in* $G(r, s; P)$ *by removing exactly* $\gamma(r, s; n)$ *r-nodes.* (7.20)

8. Another Type of Connectivity

B. Grünbaum suggested investigating the connectivities of incidence graphs obtained by removing clusters of nodes, consisting of a central node and all adjacent ones. Such a cluster would be analogous to the usual case in an edge graph when removing a vertex in effect removes all of the incident edges. In accordance with this suggestion, we define:

$\eta(r, s; P)$ $[\theta(r, s; P)]$ to be the minimal cardinality of a set of r-nodes [s-nodes] which, together with all s-nodes [r-nodes] adjacent to at least one of them, must be removed to separate some two remaining r-nodes [s-nodes] or to leave just one unremoved r-node [s-node] in $G(r, s; P)$. (8.1)

We define $\eta(r, s; n)$ and $\theta(r, s; n)$ in the usual way and observe the basic duality:

$$\eta(r, s; n) = \theta(n - 1 - s; n - 1 - r; n). \tag{8.2}$$

As with the other connectivities, our results extend to strong cell-complexes, except for one reservation (see (2.10) for proof):

Let C be a strong n-cell complex. Then

$$\begin{aligned} \eta(r, s; C) &\geq \eta(r, s; n) \quad \text{for } 0 \leq r < s \leq n - 1. \\ \theta(r, s; C) &\geq \theta(r, s; n) \quad \text{for } 0 \leq r < s \leq n - 2. \end{aligned} \tag{8.3}$$

It is the purpose of this section to establish the following dual results.

Theorem. $\eta(r, s; n) = n - s + 1$. *Moreover, if P is an n-polytope with at least one r-face which is contained in exactly* $n - r$ *facets, then* $\eta(r, s; P) = n - s + 1$. (8.4)

$\theta(r, s; n) = r + 2$. *Moreover, if Q is an n-polytope with at least one s-face which is an s-simplex, then* $\theta(r, s; Q) = r + 2$. (8.5)

We first prove that the bound stated is the best possible and that it is attained for the type of polytope described. For this purpose it is more convenient to prove the formulation in (8.5).

Let Q be as described in (8.5) and let $\hat{F}$ be an s-face of Q which is an s-simplex. Note that each r-face contained in $\hat{F}$ is the intersection of $s - r$ faces of dimension $s - 1$, and that $\hat{F}$ contains only $s + 1$ $(s - 1)$-faces. Thus, if $\hat{F}_1, \ldots, \hat{F}_{r+2}$ are any $r + 2$ of these $(s - 1)$-faces, each r-face contained in $\hat{F}$ will lie in at least one of them. For $1 \leq i \leq r + 2$, let $\hat{G}_i$ be an s-face of Q such that $\hat{F}_i = \hat{G}_i \cap \hat{F}$. Then removing the G_i and all adjacent r-nodes in $G(r, s; Q)$ will clearly separate F from any remaining s-nodes.

In order to prove the inequality in the opposite direction, several lemmas are needed.

$$\eta(0, n - 1; n) \geq 2. \tag{8.6}$$

Proof: Let P be an n-polytope and remove one 0-node, p, from $G(r, s; P)$ together with all adjacent $(n - 1)$-nodes. Let u, v, be two remaining 0-nodes which correspond to the end-points of an edge $\hat{E}$. Let $\hat{F}$ be a facet containing $\hat{E}$ but not $\hat{p}$. Then $u \to F \to v$ is a $(0, n - 1)$ path which remains between u and v.

Now assume that x and y are any two remaining 0-nodes. Let

$$x = x_0 \to (x_0, x_1) \to x_1 \to \cdots \to x_k = y$$

be a (0, 1) path joining x and y. Since a $(0, n - 1)$ path remains from x_i to x_{i+1} for every i, a $(0, n - 1)$ path joins x to y. The conclusion follows.

$$\eta(0, s; n) \geq n - s + 1. \tag{8.7}$$

Proof: We will use induction on n for s fixed. The result is given by (8.6) if $n = s + 1$. Assume that the result is known for $n - 1$.

Let P be an n-polytope and remove a set X of $n - s$ 0-nodes and adjacent s-nodes from $G(0, s; P)$. Let u, v be two remaining 0-nodes which correspond to the end-points of an edge $\hat{E}$. Suppose $p \in X$ and let $\hat{F}$ be a facet of P which contains $\hat{E}$ but not $\hat{p}$. Since $\hat{F}$ does not contain $\hat{p}$ it contains no s-face which contains $\hat{p}$. Hence, at most $n - s - 1$ 0-nodes of X lie in $G(r, s; \hat{F})$. By our induction hypothesis, there exists a $(0, s)$ path which misses X in $G(r, s; \hat{F})$, and hence in $G(r, s; P)$.

Now if x and y are any two remaining 0-nodes we can find a (0, 1) path which misses X connecting them, and use the result of the above paragraph to find a $(0, s)$ path which joins x and y and avoids X. Hence X does not separate the remaining 0-nodes and the result is proved.

$$\eta(r, s; n) \geq \eta(r-1, s-1; n-1) \text{ if } r \geq 1. \tag{8.8}$$

PROOF: If $r \geq 2$, then the result is a corollary of (8.3) in the same way as (2.12) follows from (2.10). However, here the restriction in (8.3) that $s \leq n-2$ for 0-connectivity does not allow us to conclude that $\eta(1, s; n) \geq \eta(0, s-1; n-1)$. A separate argument is needed for this final step.

Let P be an n-polytope and let X be a set of $n-s$ 1-nodes together with adjacent s-nodes in $G(1, s; P)$. Let F_0, G_0 be two remaining 1-nodes.

Assume that $\hat{F}_0$ and $\hat{G}_0$ have a common vertex $\hat{v}$. Let H be a hyperplane which strictly separates $\hat{v}$ from the other vertices of P and let $Q = H \cap P$. There is a biunique map defined by $\hat{K} \to \hat{K} \cap H$ between the t-faces of P which contain $\hat{v}$ and the $(t-1)$-faces of Q. Moreover, this map preserves incidences.

Let

$$X' = \{F \in G(0, s-1; Q)\colon \hat{F} = \hat{G} \cap H \quad \text{for some } G \in X\}.$$

Since X' contains at most $n-s$ 0-nodes together with their adjacent $(s-1)$-nodes, a $(0, s-1)$ path which misses X' exists between $F_0 \cap H$ and $G_0 \cap H$. This path is reflected in an obvious way in a $(1, s)$ path between F_0 and G_0 missing X. Thus X does not separate F_0 and G_0.

If $\hat{F}_0$ and $\hat{G}_0$ do not have a common vertex, let $\hat{p}$ be a vertex of $\hat{F}_0$ and $\hat{q}$ a vertex of $\hat{G}_0$. By (8.7), there exist n disjoint $(0, 1)$ paths in $G(0, 1; P)$. Since $n - s < n$, at least one of these paths contains no member of X. Then we can use the result from the above paragraphs to show that a $(1, s)$ path missing X joins F_0 and G_0. The result follows.

Having these last two lemmas at our disposal, it is easy to complete the proof of (8.4). For we have

$$\eta(r, s; n) \geq \eta(r-1, s-1; n-1) \geq \cdots \geq \eta(0, s-r, n-r) = n-s-1$$

Combining this inequality with the opposite one given at the beginning of the proof concludes the argument.

9. SEPARATING SEQUENCES

Let X and Y be disjoint sets of nodes in a graph G. X is said to *totally separate* Y if every path between any two members of Y passes through X.

For any (r, s) incidence graph, let $\alpha_m(r, s; P)$ denote the greatest integer z such that z r-nodes of $G(r, s; P)$ are totally separated by m other nodes of $G(r, s; P)$. To employ the same notations as above, Y consists of z r-nodes of $G(r, s; P)$ and X of m other nodes of $G(r, s; P)$. In a similar way, we define the maximal cardinality of a totally separated set Y in $G(r, s; P)$ to be:

$\beta_m(r, s; P)$ if Y consists of s-nodes and X of m other nodes;
$\gamma_m(r, s; P)$ if Y consists of r-nodes and X of m other r-nodes;
$\delta_m(r, s; P)$ if Y consists of r-nodes and X of m s-nodes;
$\varepsilon_m(r, s; P)$ if Y consists of s-nodes and X of m r-nodes;
$\zeta_m(r, s; P)$ if Y consists of s-nodes and X of m other s-nodes;
$\eta_m(r, s; P)$ if Y consists of r-nodes and X of m other r-nodes, together with all s-nodes adjacent to at least one of them;
$\theta_m(r, s; P)$ if Y consists of s-nodes and X of m other s-nodes, together with all r-nodes adjacent to at least one of them.

The usual dualities are in evidence:

$$\alpha_m(r, s; n) = \beta_m(n-1-s,\ n-1-r;\ n), \tag{9.1}$$

$$\gamma_m(r, s; n) = \zeta_m(n-1-s,\ n-1-r;\ n), \tag{9.2}$$

$$\delta_m(r, s; n) = \varepsilon_m(n-1-s,\ n-1-r;\ n), \tag{9.3}$$

$$\eta_m(r, s; n) = \theta_m(n-1-s,\ n-1-r;\ n). \tag{9.4}$$

We also remark the following inequalities:

$$\alpha_m(r, s; n) \geq \max\{\gamma_m(r, s; n),\ \delta_m(r, s; n)\}, \tag{9.5}$$

$$\eta_m(r, s; n) \geq \max\{\gamma_m(r, s; n),\ \delta_m(r, s; n)\}. \tag{9.6}$$

Let $\mu_r(m, n)$ denote the maximum number of facets on an n-polytope with m or fewer r-faces.

THEOREM. $\gamma_m(r, s; n) \geq \mu_r(m, n)$. (9.7)

PROOF: Let P be an n-polytope with m or fewer r-faces and $\mu_r(m, n)$ facets. Let Q be the polytope obtained from P by adding simplicial caps over the facets of P. Let $V = \{\hat{v}_1, \ldots, \hat{v}_t\}$ be the collection of "new" vertices. Clearly, $t = \mu_r(m, n)$.

Let

$$W = \{F \in G(r, s; Q)\colon \hat{F} \text{ is an } r\text{-face and contains no } \hat{v}_1\}.$$

We assert that W will totally disconnect $G(r, s; Q)$ into $\mu_r(m, n)$ classes where each class consists of all the r-nodes whose corresponding faces contain some member of V. This assertion follows from the observation that no facet of Q contains more than one $\hat{v}_i$. Thus, no s-face contains more than one $\hat{v}_i$. Let F, G be two r-nodes of $G(r, s; Q)$ which are not in W. Suppose $\hat{v}_j \in \hat{F}$, $\hat{v}_k \in \hat{G}$, $j \neq k$. Then any (r, s) path from F to G in $G(r, s; Q)$ eventually contains a last s-node whose corresponding face contains $\hat{v}_j$. The next r-node along the path thus corresponds to a face containing no $\hat{v}_i$ and, hence, the r-node is a member of W. Thus W totally disconnects $G(r, s; Q)$ into $\mu_r(m, n)$ classes.

Observing that each member of W corresponds to an r-face of P completes the proof.

Note that, in the above proof, it was essential that r-nodes formed the separating set, but that it was immaterial whether r- or s-nodes were separated. So we could essentially duplicate the proof of (9.7) to show:

$$\varepsilon_m(r, s; n) \geq \mu_r(m, n). \tag{9.8}$$

Combining these last two inequalities with (9.5) and (9.6) and making use of duality, we have:

$$\mathscr{K}(r, s; n) \geq \mu_r(m, n) \quad \text{for } \mathscr{K} = \alpha, \beta, \varepsilon, \eta, \text{ and } \theta. \tag{9.9}$$

$$\mathscr{K}(r, s; n) \geq \mu_{n-1-s}(m, n) \quad \text{for } \mathscr{K} = \alpha, \beta, \delta, \zeta, \eta, \text{ and } \theta. \tag{9.10}$$

For upper bounds we cannot extend our results significantly beyond the theorem of Klee [7] who proved:

Let $\sigma_m(n)$ denote the maximum cardinality of a subset of vertices of the edge graph of an n-polytope which are totally separated by m other vertices. Then

$$\sigma_m(n) = \begin{cases} 1, & \text{if } m \leq n-1, \\ 2, & \text{if } m = n, \\ \mu_0(m, n), & \text{if } m \geq n+1. \end{cases} \tag{9.11}$$

We will show that the same bounds extend to α_m, γ_m, and η_m separating sequences with the aid of one further definition and a lemma.

Let P be an n-polytope and F a face of P. Let y be a point in E^n which is not in P, but which is sufficiently near the barycenter of F that it lies below every supporting hyperplane of P which does not contain F (that is, if $H = \{x\colon h(x) = 0\}$ is a supporting hyperplane for P which does not contain F, and $h(x) \geq 0$ for all x in P, then $h(y) > 0$). Let P' be the convex hull of P and y (denoted $P' = \text{con}\ \{P, y\}$). Then we say that P' is obtained from P by a *barycentric pulling* of F. This notion generalizes the concept of *pulling* the vertex of a polytope introduced in [3].

LEMMA. *Let P be an n-polytope such that a set Y of z 0-nodes [1-nodes] in $G(0, 1; P)$ can be totally separated by a set X of m other nodes. Then there exists an n-polytope Q such that $G(0, 1; Q)$ contains a set Y' of z 0-nodes [1-nodes] which are totally separated by a set X' of m other 0-nodes.* (9.12)

PROOF: Let $E_1, \ldots, E_k$ be the 1-nodes of X, let Q be the polytope obtained from P by a barycentric pulling of the $\hat{E}_i$, and let $\hat{q}_1, \ldots, \hat{q}_k$ be the "new" vertices of Q. Make the obvious correspondence between nodes in $G(0, 1; P)$ (except for $E_1, \ldots, E_k$) and nodes in $G(0, 1; Q)$. Let Y' be the nodes of $G(0, 1; Q)$ corresponding to members of Y and let X' be the nodes of $G(0, 1; Q)$ corresponding to members of $X \sim \sim \{E_1, \ldots, E_k\}$ together with $\{q_1, \ldots, q_k\}$. Notice that, if Y consists of 0-nodes [1-nodes], then Y' consists of 0-nodes [1-nodes].

It is clear that X' consists only of 0-nodes and that it totally separates Y'. For if u, v, are two members of Y', any (0, 1) path joining them corresponds to a (0, 1) path in $G(0, 1; P)$ unless it uses a "new" 0-node or a "new" 1-node. But every "new" 0-node is a member of X' and any path passing through a "new" 1-node also passes through a "new" 0-node. Thus any (0, 1) path between members of Y' either corresponds to a (0, 1) path in $G(0, 1; P)$ missing X (contrary to hypothesis) or else includes a member of X'. The conclusion follows.

$$\begin{aligned} \delta_m(0, 1; n) &\leq \alpha_m(0, 1; n) \\ &= \gamma_m(0, 1; n) \\ &= \eta_m(0, 1; n) \\ &= \begin{cases} 1, & \text{if } m \leq n-1, \\ 2, & \text{if } m = n, \\ \mu_0(m, n), & \text{if } m \geq n+1. \end{cases} \end{aligned} \tag{9.13}$$

PROOF: It follows from (9.12) that

$$\gamma_m(0, 1; n) \geq \delta_m(0, 1; n),$$

and that

$$\gamma_m(0, 1; n) \geq \alpha_m(0, 1; n).$$

Combining this latter inequality with (9.5) shows that

$$\gamma_m(0, 1; n) = \alpha_m(0, 1; n).$$

By general considerations of the relation between an edge graph and the corresponding (0, 1) graph of a polytope, it is easy to see that

$$\gamma_m(0, 1; n) = \eta_m(0, 1; n) = \sigma_m(n).$$

The proposition is then a consequence of (9.11).

We can also apply (9.12) to show:

$$\beta_m(0, 1; n) = \varepsilon_m(0, 1; n) \geq \zeta_m(0, 1; n). \tag{9.14}$$

In general, equality does not hold on the right. For example, it is a fairly easy matter to check that $\varepsilon_3(0, 1; 3) = 3$, while $\zeta_3(0, 1; 3) = 2$.

An interesting phenomenon occurs for certain η_m and θ_m separating sequences.

$$\eta_m(0, s; n) = \begin{cases} 1, & \text{for } m \leq n - s, \\ \infty, & \text{for } m \geq n - s + 1. \end{cases} \tag{9.15}$$

PROOF: It follows from (8.4) that $\eta_m(0, s; n) = 1$ for $m \leq n - s$.

To establish the second statement, for any positive integer z let Q be an s-polytope with at least z vertices, let P_1 be a cone over Q, and for $2 \leq j \leq n - s$ let P_j be a cone over P_{j-1}. (P_j is said to be a *j-fold suspension of* Q.) Let $\hat{p}_j$ be the vertex of P_j which is not in P_{j-1}. Clearly P_{n-s} has dimension n.

Note that for any u each u-face of P_j either contains $\hat{p}_j$ or else lies in P_{j-1}. Thus, every s-face of P_{n-s} except Q, contains one of the $\hat{p}_i$. Let $\hat{q}$ be some vertex of Q. Then removing the 0-nodes $\{q, p_1, \ldots, p_{n-s}\}$ together with all adjacent s-nodes will totally separate the remaining 0-nodes of $G(0, s; P_{n-s})$. Hence,

$$\eta_{n-s+1}(0, s; n) \geq z - 1.$$

Since z was arbitrary the result follows for $m = n - s + 1$. As it is clear that η_m is non-decreasing in m, the result follows for all m.

$$\eta_m(r, n - 1; n) = \begin{cases} 1, & \text{if } m = 1, \\ \infty, & \text{if } m \geq 2. \end{cases} \tag{9.16}$$

PROOF: Once again, the first statement follows from (8.4). For the second, for any positive integer z, let Q be an $(n - r)$-polytope with at least z r-faces, let P be the r-fold suspension of Q, and let $\hat{p}_1, \ldots, \hat{p}_r$ be the "new" vertices of P. Note that every facet of P is either an r-fold suspension of a facet of Q or else an $(r - 1)$-fold suspension of Q. Hence, every facet of P save one contains the r-face, $\hat{F}$, determined by $(\hat{q}, \hat{p}_1, \ldots, \hat{p}_r)$ where $\hat{q}$ is a vertex of Q. Let $\hat{G}$ be an r-face contained in the remaining facet.

Then removing F and G, together with all adjacent $(n - 1)$-nodes, from $G(r, s; P)$ will totally separate the remaining r-nodes. Hence,

$$\eta_2(r, n - 1; n) \geq z - 1.$$

Since z was arbitrary, the conclusion follows for $m = 2$ and thus for all larger m.

The dual statements of the two preceding theorems read:

$$\theta_m(r, n - 1; n) = \begin{cases} 1, & \text{if } m \leq r + 1, \\ \infty, & \text{if } m \geq r + 2. \end{cases} \tag{9.17}$$

$$\theta_m(0, s; n) = \begin{cases} 1, & \text{if } m = 1, \\ \infty, & \text{if } m \geq 2. \end{cases} \tag{9.18}$$

10. A STRUCTURAL THEOREM

We recall:

$$\gamma(r, s; n) = \min \left\{ \gamma^f(r, s; n), \binom{n}{r + 1} \right\}. \tag{7.19}$$

In Section 7 we investigated the case in which $\gamma(r, s; n) = \gamma^f(r, s; n)$ (actually we studied the dual problem). Here we consider the remaining situation.

THEOREM. *Let P be an n-polytope such that $\gamma^f(r, s; n) > \gamma(r, s; n)$*

$= \binom{n}{r+1}$. *Then P can be decomposed into two n-polytopes P_1 and P_2 with a common facet, P^{n-1}, such that every face of P is a face of either P_1 or P_2 and such that P^{n-1} is an $(n-1)$-simplex.* (10.1)

PROOF OF THEOREM: Let F and G be two r-nodes in $\alpha(r, s; P)$ which can be separated by a set X of cardinality $\binom{n}{r+1}$. By assumption, $\hat{F}$ and $\hat{G}$ do not lie in the same facet of P. Let $\hat{A}$, $\hat{B}$ be facets of P such that $\hat{F} \subset \hat{A}$, $\hat{G} \subset \hat{B}$. By (4.2), there exist at least n disjoint $(n-2, n-1)$ paths between A and B. As in (6.1), if for one of these paths, say

$$A = A_0^{n-1} \to A_0^{n-2} \to \cdots \to A_t^{n-1} = B,$$

there exist r-nodes $C_0^r, ..., C_{t-1}^r$ which are not members of X, and such that $\hat{C}_i^r \subset \hat{A}_i^{n-2}$ for $0 \le i \le t-1$, then an (r, s) path exists between F and G.

But by assumption, no such path exists. Hence, there exist $n(n-2)$-faces of P, $\hat{D}_1, ..., \hat{D}_n$, such that every r-node corresponding to a face in one of them lies in X. By (5.1),

$$\varrho_r(\hat{D}_1 \cup \cdots \cup \hat{D}_n) \ge \binom{n}{r+1}.$$

But since X contains only $\binom{n}{r+1}$ r-nodes, every r-node in X corresponds to a face in one of the $\hat{D}_i$. Moreover,

$$\varrho_r(\hat{D}_1 \cup \cdots \cup \hat{D}_n) = \binom{n}{r+1}.$$

Hence, by (5.1) the $\hat{D}_i$ satisfy conditions (a), (b), and (c) of (7.7). Thus, only n vertices, $\hat{v}_1, ..., \hat{v}_n$, of P occur among the $\hat{D}_i$.

Let $Q = \text{con}\{\hat{v}_1, ..., \hat{v}_n\}$. Since every $n-1$ of these vertices determine a face of P, all n of them determine a unique hyperplane H. We assert that $Q = H \cap P$.

For let $P_0 = P \cap H$. Clearly $P_0 \supset Q$. If $P_0 \ne Q$, then there exists a point $p \in (\text{rel int } P_0) \cap (\text{rel bd } Q)$. But every point on the relative boundary of Q lies in one of the $\hat{D}_i$ and hence in the boundary of P, while rel int $P_0 \subset$ int P, a contradiction. Hence, $P_0 = Q$.

From this fact it is easy to deduce that H does not intersect any face of P in a relatively interior point. For if $\hat{E}$ is a facet of P different from Q, then

$$H \cap \hat{E} = Q \cap \hat{E} = \text{bd}Q \cap \hat{E} = (\hat{D}_1 \cup \cdots \cup \hat{D}_n) \cap \hat{E} = \bigcup_1^n (\hat{D}_i \cap \hat{E}).$$

But since each $\hat{D}_i$ is an $(n-2)$-face of P, $\hat{D}_i \cap \hat{E}$ contains no interior point of $\hat{E}$ for any i, and thus

$$H \cap \text{rel int } \hat{E} = \varphi.$$

We finally observe that Q is not a face of P. Otherwise, we could use the edge path constructed as in (3.4) which does not pass through Q to show that the r-nodes in X do not separate F and G.

Thus H intersects the interior of P, but not the relative interior of any facet of P. We set $P_1 = H^+ \cap P$ and $P_2 = H^- \cap P$, where H^+ and H^- are the closed half-spaces determined by H. Clearly P_1 and P_2 are the polytopes we seek, and $P_1 \cap P_2 = Q$ is an $(n-1)$-simplex, so the proof is complete.

Precisely the same argument works for α-connectivity, so we can also state the following

THEOREM. *Let P be an n-polytope such that*

$$\alpha^f(r, s; P) > \alpha(r, s; P) = \binom{n}{r+1}.$$

Then P can be decomposed into two n-polytopes P_1 and P_2 with a common facet, P^{n-1}, such that every face of P is a face of either P_1 or P_2 and such that P^{n-1} is an $(n-1)$-simplex. (10.2)

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On Comparing Connecting Networks

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Abstract

Seeing how *structural* differences between connecting networks lead to differences in their *performance* is a basic problem in telephone traffic theory. The object is to transform combinatorial information about networks into an inequality between suitable blocking probabilities. This paper stresses the relevance of routing to this problem, and takes an initial step toward answering the equestion: What kinds of relationships between two networks ensure that one is "better" than the other?

A relation $\leq$ is defined which partially orders all the possible networks ν on given inlets and outlets. With an *assignment* defined as a specification of what inlet is to be connected to what outlet, $\nu_1 \leq \nu_2$ means roughly that it is possible to map a subset of the states of ν_1 that is closed under hangups onto those of ν_2 so as to preserve assignments, and in such a way that only states comparable in the natural partial ordering can have comparable images.

With $b(\nu, R)$ the probability of blocking of network ν under routing rule R (appropriate to ν), it is proved (i) that $\min_R b(\nu, R)$ is isotone on $\leq$, and (ii) that $\nu_1 \leq \nu_2$ implies the existence of an isomorph of the states of ν_2 within ν_1. The latter result, suggested by S. Darlington, provides a different, very natural proof of the isotony (i). The intuitive meaning of these two results is that, if $\nu_1 \leq \nu_2$, then any way of operating ν_2 can be mimicked in ν_1, so that the best way of routing in ν_1 gives a loss no greater than that achieved by the best way of routing in ν_2.

1. Introduction and Summary

In the design of connecting networks it is customary to compare alternative networks by estimating their respective carried loads and loss