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A TRIANGULATION OF THE n-CUBE

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This paper is concerned with estimating $\varphi(n)$, the minimum number of n-simplices required to triangulate an n-cube.

ntroduction

n 1967, H. Scarf developed a finite algorithm for approximating a fixed point continuous mapping of a simplex into itself [8]. This algorithm was refined extended by H. Kuhn, B.C. Eaves and O. Merrill, among others. (See amardian [4] and Todd [10] for these and other references.) Several al- thms use a technique of pivoting among simplices which triangulate an ube, and there is some hope that reducing the number of simplices in a ngulation may lead to more efficient algorithms. This paper is concerned with mating $\varphi(n)$, the minimum number of n-simplices required to triangulate an ube. We show in particular that

$$L_n \leq \varphi(n) \leq P_n$$

re $P_1 = 1$ and the P_i satisfy the recursion

$$P_{n+1} = (n+1)(P_n + 2^{n-1}) + 2^{n+1} - (n-1),$$

d where

$$L_n/H_n \approx \tfrac{1}{4}\sqrt{e}\,(n+1)^{\frac{1}{2}},$$

being the number of simplices given by Hadamard's inequality. As is plain m the large gap between them, these bounds are far from sharp. In particular, r upper bound of $0.477(n!)$ should be compared with the fact that $n!$ is mmonly quoted [3] as the value $\varphi(n)$. We also show that $\varphi(4) = 16$ and (5)≤ 67, thus sharpening the results of Mara [5, 6]. Cottle [1] has also shown at $\varphi(4) = 16$. His proof is interesting in that it uses one of Mara's theorems.

2. The triangulation

We use, without specific reference, a number of basic properties of (convex) polytopes. They can all be found in the book by Grunbaum [2]. In particular, an *n-polytope* is an n-dimensional, compact, convex set with a finite number of extreme points (vertices). And an *n-simplex* is an n-polytope with $n+1$ vertices. An *affine* set is a translate of a linear set. For $A \subseteq \mathbb{R}^n$, aff A (con A) is the intersection of all affine (convex) sets which contain A.

A *triangulation* of an n-polytope P is a finite set S of n-simplices such that
(i) $P = \bigcup S$,

(ii) for all $a, b \in S$, $a \cap b$ is a face of both a and b.

An *n-complex* is a finite set C of n-polytopes such that $P_1 \cap P_2$ is a face of both P_1 and P_2 for all $P_1, P_2 \in C$.

In the rest of this paper, P will be an n-polytope, C an n-complex and S a set of n-simplices unless the context clearly indicates otherwise.

S is a *triangulation* of C if (i) for each $P \in C$ there is a subset of S which triangulates P, and (ii) no proper subset of S has this property. If v is a vertex of P, then a face F of P is *opposite* v if $v \notin F$.

If v is a vertex of P and S is a triangulation of the complex of facets opposite v, then the set $S_v = \{\text{con}(\{v\} \cup s) \colon s \in S\}$ is a triangulation of P.

The notation $A \subset B$ means $A \subseteq B$ and $A \neq B$. If $P = F_n \supset F_{n-1} \supset \cdots \supset F_0 \neq \emptyset$ is a sequence of faces of P and $v_n, v_{n-1}, \ldots, v_0$ is a sequence of vertices of P such that $v_i \in F_i$ and $v_{i+1} \notin F_i$ for $0 \leqslant i \leqslant n-1$, then the v_i's are the vertices of an n-simplex. To see this note that $v_{i+1} \notin \text{aff}\{v_i, \ldots, v_0\}$ for $0 \leqslant i \leqslant n-1$.

These ideas lead naturally to the following construction of a triangulation for a complex C. Let $v_1, v_2, \ldots, v_L$ be an ordering of the set of vertices of the polytopes of C. For each face $F \neq \emptyset$ of $P \in C$, let $i_F = \min\{k \colon v_k \in F\}$ and $V_F = v_{i_F}$. Each sequence of faces $P = F_n \supset \cdots \supset F_0 \neq \emptyset$ with $v_{F_{i+1}} \notin F_i$ for $0 \leqslant i \leqslant n-1$ has an associated simplex $s = \text{con}(\{v_{F_n}, \ldots, v_{F_0}\})$. Let $S_C(v_1, \ldots, v_L)$ be the set of all simplices generated by sequences of faces as defined above.

Lemma. *For any complex C, the set $S_C(v_1, \ldots, v_L)$ is a triangulation of C.*

Proof. By induction. For $n = 2$, the lemma is obviously true.

Assume that the result holds for n and let C be a complex of $(n+1)$-polytopes. Let C_n be the complex of facets of the polytopes of C. Then $S_{C_n}(v_1, \ldots, v_L)$ is a triangulation of C_n. For any polytope $P \in C$, the complex of facets opposite v_P is triangulated by a subset S' of $S_{C_n}(v_1, \ldots, v_L)$. Each simplex in S' has an associated sequence of faces $F_n \supset \cdots \supset F_0 \neq \emptyset$. Since $v_P \notin F_n$, this sequence can be extended to the sequence $P = F_{n+1} \supset F_n \supset \cdots \supset F_0$. Thus

$$S_P = \{\text{con}(\{v_P\} \cup s) \colon s \in S'\} \subseteq S_C(v_1, \ldots, v_L)$$

is a triangulation of P.

In order to guarantee that $S_C(v_1, \ldots, v_L)$ is a triangulati hat if $s = \mathrm{con}(\{v_P, v_{F_{n-1}}, \ldots, v_{F_0}\})$ is a simplex in $S_{c_n}(v_1, \ldots, v_L)$ containing en there is an associated sequence of faces $P = F_{n+1} \supset F_n' \supset \cdots \supset F_0'$ such that $_{n-1}, \ldots, v_{F_0}\} \subset \{v_P, v_{F_n'}, \ldots, v_{F_0'}\}$.

arly $\{v_{F_{n-1}}, \ldots, v_{F_0}\}$ is a set of vertices in an $(n-1)$-face of P which does not in v_P. Thus there is an n-face F_n' of P which contains this set and not v_P. If F_{n-1}, then $P \supset F_n' \supset F_{n-1} \supset \cdots \supset F_0$ is the required sequence of faces. If F_{n-1}, then by induction there is a sequence of faces of F_n' such that $\cdots \supset F_0'$ with $\{v_{F_{n-1}}, \ldots, v_{F_0}\} \subset \{v_{F_n'}, \ldots, v_{F_0'}\}$. Hence $\{v_p, v_{F_{n-1}}, \ldots, v_{F_0}\} \subset$ $_{F_n'}, \ldots, v_{F_0'}\}$. This proves the lemma.

rem. *Let $\varphi(n)$ be the minimum number of simplices required to triangulate an be. Then $\varphi(n) \leq P_n$, where $P_1 = 1$ and*

$$P_{n+1} = (n+1)(P_n - 2^{n-1}) + 2^{n+1} - (n+1) \quad \textit{for } n \geq 1.$$

of. For the n-cube $I^n = [0,1]^n$, we will call a vertex $v = (v_1, \ldots, v_n)$ odd if an number of the v_i's are 1 and even otherwise. For each vertex v of I^n, there is perplane which passes through the vertices adjacent to it. For example, the rplane $H = \{x: x_1 + \cdots + x_n = 1\}$ passes through the vertices adjacent to $0 =$ $\ldots, 0)$. It can be shown by a simple induction that $H \cap I^n$ is the $(n-1)$-lex whose vertices are the vertices of I^n adjacent to 0. For $n \geq 3$, let $\mathcal{O}$ be the f odd vertices of I^n and for each $v \in \mathcal{O}$, let H_v be the hyperplane which passes ugh the vertices adjacent to v. Let H_v^- be the closed halfspace which contains d H_v^+ be the closed halfspace which contains the rest of I^n. Let T^n denote the cated cube $I^n \bigcap_{v \in \mathcal{O}} H_v^+$. It can be shown that $T^n = \mathrm{con}(E)$, where E is the set en vertices of I^n. If $A^n = \{I^n \cap H_v^-: v \in \mathcal{O}\}$, then A^n is a set of simplices and $= 2^{n-1}$. Finally, let $C^n = \{T^n\} \cup A^n$. It is clear that $I^n = \bigcup C^n$ and that a gulation of C^n is a triangulation of I^n.

t S^n be the triangulation of C^n given by the lemma and $P_n = |S^n|$. Clearly, S^n, thus $S^n - A^n$ is a triangulation of T^n and $|S^n - A^n| = P_n - 2^{n-1}$. Consider riangulation S^{n+1} of C^{n+1}. To triangulate T^{n+1} we note that T^{n+1} has 2^n lex faces and $2(n+1)$ faces congruent to T^n. Of these faces, $n+1$ simplex and $n+1$ T^n-faces are adjacent to $v_{T^{n+1}}$. Each T^n-face opposite $V_{T^{n+1}}$ will rate $P_n - 2^{n-1}$ simplices and the $2^n - (n+1)$ simplex faces opposite v_{n+1} will generate one simplex for a total of $(n+1)(P_n - 2^{n-1}) + 2^n - (n+1)$ simplices $^{+1} - A^{n+1}$ which triangulate T^{n+1}. Thus

$$\begin{aligned} P_{n+1} &= |S^{n+1}| = |S^{n+1} - A^{n+1}| + |A^{n+1}| \\ &= (n+1)(P_n - 2^{n-1}) + 2^n - (n+1) + 2^n \\ &= (n+1)(P_n - 2^{n-1}) + 2^{n+1} - (n+1). \end{aligned}$$

An upper bound

The common triangulations of I^n require $n!$ simplices. The first ten values of P_n 1, 2, 5, 16, 67, 364, 2445, 19296, 173015 and 1720924.

Consider the ratio $P_n/n!$. Since

$$\frac{P_{n+1}}{(n+1)!}=\frac{P_n}{n!}-\frac{2^{n-1}}{n!}+\frac{2^{n+1}}{(n+1)!}-\frac{1}{n!}\leqslant\frac{P_n}{n!}$$

for $n\geqslant 3$, the sequence is decreasing. We also have

$$\frac{P_{n+1}}{(n+1)!}\geqslant\frac{P_n}{n!}-\frac{2^{n-1}}{n!}-\frac{1}{n!}\geqslant\frac{P_n}{n!}-\frac{2^n}{n!}$$

and by induction

$$\frac{P_{n+k}}{(n+k)!}\geqslant\frac{P_n}{n!}-\frac{2^{n+i}}{(n+i)!}\geqslant\frac{P_n}{n!}-\sum_{i=n}^{\infty}\frac{2^i}{i!}$$
$$\geqslant\frac{P_n}{n!}-\left(e^2-\sum_{i=0}^{n-1}\frac{2^i}{i!}\right).$$

For $n=10$,

$$\frac{P_n}{n!}=0.47636 \quad\text{and}\quad e^2-\sum_{i=1}^{9}\frac{2^i}{i!}=0.00034.$$

Thus $P_n/n!\in[0.47600,\ 0.47636]$ for $n\geqslant 10$.

4. A lower bound

From Hadamard's inequality, if M is a $(1,-1)$ $n\times n$ matrix, then $|\det(M)|\leqslant n^{n/2}$. Thus any $(0,1)$ $n\times n$ matrix M satisfies the inequality $|\det(m)|\leqslant n^{n/2}/2^{n-1}$. By the well-known formula [9] for the volume of a simplex, the maximum volume, V_n, of a simplex in I^n satisfies $V_n\leqslant(1/n!)\,|\det(M)|$, where M is a $(0,1)$ $(n+1)\times(n+1)$ matrix. Let

$$H_n=(2^n\cdot n!)/(n+1)^{(n+1)/2}.$$

Then $\varphi(n)\geqslant 1/V_n\geqslant H_n$ and H_n/P_n approaches 0 as n becomes large, which shows that there is a huge gap between H_n and P_n.

However, if we consider only triangulations of I^n whose vertices coincide with those of I^n we can do slightly better. For $K\subseteq R^n$, let $V(K)$ denote the volume of K. If K is a set of subsets of R^n, we will use $V(K)$ instead of $V(\cup K)$. If F is a facet of the polytope P and S is a triangulation of P, we will say that a simplex $s\in S$ *belongs* to F if a facet of s is a subset of F. If S_1 and S_2 are the sets of simplices belonging to two adjacent facets of I^n, then

$$V(S_1)=1/n \quad\text{and}\quad V(S_1\cap S_2)\leqslant 1/n(n-1).$$

Theorem. *If S is a triangulation of I^n whose vertices coincide with those of I^n and $L_n=\frac{1}{2}(H_n+n\cdot H_{n-1})$, then $|S|\geqslant L_n$ and $L_n/H_n\approx\frac{1}{2}+\frac{1}{4}\sqrt{e}(n+1)^{\frac{1}{2}}$ for large n.*

Proof. Let F_{i_1} and F_{i_2} be distinct pairs of opposite facets of I^n for $1\leqslant i\leqslant n$ and

S: s belongs to F_{i_1} or F_{i_2}}. Then for each i, $V(A_i) = 2/n$. If $B_k = \bigcup_{i=1}^{k} A_i$,

$$V(B_k) \geq \sum_{i=1}^{k} \left(\frac{2}{n} - \frac{4(i-1)}{n(n-1)}\right).$$

is last inequality, note that $B_k = B_{k-1} \cup A_k$. There are $4(k-1)$ intersec-
ets belonging to the two facets which determine A_k and the $2(k-1)$
hich determine B_{k-1}. Hence

$$V(B_k) = V(B_{k-1}) + V(A_k) - V(B_{k-1} \cap A_k)$$
$$\geq \sum_{i=1}^{k-1} \left(\frac{2}{n} - \frac{4(i-1)}{n(n-1)}\right) + \left(\frac{2}{n} - \frac{4(k-1)}{n(n-1)}\right).$$

raightforward exercise to show that $V(B_k) \geq \frac{1}{2}$ for $k = \lfloor \frac{1}{2}(n+1) \rfloor$. Thus the
lumes of the simplices in S which belong to at least one facet must be at
The volume of a simplex which belongs to a facet is less than or equal to
$_{-1}$, thus $|S| \geq \frac{1}{2}(H_n + nH_{n-1})$ and the theorem follows.

ing a lower bound for $\varphi(n)$ significantly better than H_n seems to be a
t problem.

ension $n \leq 6$

$= 3$ and $n = 4$, it can be shown that the minimum triangulations require 5
simplices respectively, thus

$$\varphi(3) = P_3 \quad \text{and} \quad \varphi(4) = P_4.$$

$= 3$, $|B_1| \geq 4$ and $V(B_1) \leq \frac{2}{3}$, showing that there must be at least one more
to fill I^n. For $n = 4$, we use the facts that the maximum volume of a
is $\frac{3}{24}$ and that if a facet has exactly five simplices, the simplex with base
rea $\frac{1}{3}$ can belong to only that facet. The fact that $\varphi(4) \geq 16$ then follows
straightforward consideration of the following cases: all facets have 5
es; there is a facet having 6 or more simplices, but no opposite pair of
ach having 6 or more simplices; there is a pair of opposite facets with 6 or
implices each. For example, in the second case, there must be at least 11
es with total volume less than or equal to $\frac{1}{2}$. Since no two opposite facets
or more simplices, there must be at least 4 simplices with bases of area 2,
s been counted and the other 3 have total volume less than or equal to $\frac{1}{4}$
he maximum altitude of a simplex belonging to a facet is 1. It is clear that
ust be at least 2 more simplices to fill the remainder of I^4.
$n = 5$, $\varphi(5) \leq P_5 = 67$. It is not known whether or not equality holds.
$n = 6$, since there is a triangulation of I^6 with only 344 simplices while
364, it is likely that $\varphi(n) < P_n$ for all $n \geq 6$.

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