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15 de Octubre de [illegible]

Hola Claudio, te mando copia de lo que le mandé a Sternberg (perdóname que no te traduzca para ti) y más un dibujo adicional en el que se aprecian los "volumitos".

También copio (por fin!) las fotos de mi estancia por allá. Para ti sólo hay tres, pero te lo pasarás bien viendo las que les mandé a David y Liz, que son de una fiesta en su casa de Syracuse.

Un abrazo para todos los matemáticos latinos (as) de Cornell! (Bueno, también para los que no sean latinos, no somos discriminatorios)

[illegible]

Santander, September 30, 1994

Dear ~~Ilia~~ Jesús,

These are the pictures corresponding to my construction of improved (and failed) counterexamples to the Ragsdale conjecture.

The basis of the construction is shown in Figure 1.a. The "diamonds" (in thick in the picture) composing the pattern are the same as your hexagons with two corners added, as in Haas'construction. In the way I dispose them you can cover 9/10 of the area with diamonds having their centers in an odd-odd point. The rest of the area is occupied by smaller diamonds, there being (assymptotically) the same number of big and small diamonds. As shown in figure 1.c, giving appropiate signs to the vertices one obtains 16 positive (and 2 negative) ovals per each primitive region of the pattern, i.e. 16 positive ovals per each 20 area units in the Newton triangle. Up to now the process gives a regular triangulation, as can be deduced from figure 1.b: the whole pattern (no matter how big you do it) can be decomposed in vertical strips as the one shown, and for any convex function that one is given in the left edge of the strip one can easily find a convex function on the whole strip producing my triangulation.

Now, in order to increase the density of positive ovals, I construct diamonds of the same height (equal to 6) but of width $6 \times n$, where n is any arbitrary integer fixed in advance (the previous construction is the case $n = 1$). This diamonds and the way thay are triangulated inside are actually the same thing as Haas' "lucarnes" except that my lucarnes are always symmetrical. Figure 2.a shows the diamonds for $n = 2$. Again, this bigger diamonds can be placed in an arbitrarily big amount in such a way that they cover 9/10 of the area. For reasons that are obvious in figure 2.b, the center of the diamonds must now be in an even-odd point, for $n = 2$. This will happen for every even n, but a universal rule for all n is that the centers of the small diamonds are in even-even points. For the same reasons as in Haas' construction, for bigger n one obtains greater density of positive ovals. Namely, in a primitive region of the pattern (composed by one big diamond of width $6n$ and height 6 and one small diamond of width $2n$ and height 2) one obtains $17 \times n - 1$ positive ovals (and $3 \times n$ negative), and the region has an area of $20 \times n$. For example, in fig. 2.b, I obtain 33 positive ovals per each 40 area units, for $n = 2$. For big n this gives the asymptotic value of 17 positive ovals per 20 area units.

Thus, once chosen an integer n no matter how big, one has a procedure to produce "Viro curves" of degree $2k$ having $\frac{17-1/n}{10}k^2 - O(k)$ positive ovals, where the $-O(k)$ part comes from the obvious reason that one has to leave unused a certain neighborhood of the boundary of the Newton triangle. The constant in the $O(k)$ depends on n (n was supposed large but fixed) and an estimate for it is $12n$ (because $12nk$ is more or less the number of points where you cannot put the center of one of my big diamonds).

Incidentally, if one wants to know the optimal n to be used in a construction for a given (large) k, with the approximate formula *positive ovals* $= \frac{17-1/n}{10}k^2 - 12nk$, the optimal turns out to be attained for $n = \sqrt{\frac{k}{240}}$. Substituting this in the formula one obtains that the maximum number of ovals that one can achieve is $\frac{17}{20}k^2 - \sqrt{\frac{12}{5}}k^{\frac{3}{2}}$. So, it is not completely true that I obtain $\frac{17}{20}k^2 - O(k)$ positive ovals. The real asymptotics is rather $\frac{17}{20}k^2 - O(k^{\frac{3}{2}})$.

Concerning regularity, the proof I used for $n = 1$ is no longer valid for $n > 1$ because now the "vertical" strips will unavoidably have some turn-in and turn-outs (as in fig 3.a). I can actually prove that the triangulation is not regular, for any $n \geq 2$.

In order to simplify things, in figure 2.a I have drawn the complete triangulation only in one of the diamonds. In the rest of them I have only drawn what we can call a skeleton of the triangulation, meaning by this that the complete triangulation can be obtained from its skeleton by "flips" of the triangulation that add a new point. It is clear that such flips do not altere regularity. Thus, my triangulation will be regular if and only if this skeleton is.

Finally, the fact that the skeleton is not regular can be proved focusing on the eight triangles of figure 3b, which are actually triangles in the skeleton. Either by algebraic or geometrical arguments one can prove that no triangulation containing those eight triangles is not regular. This will also be true for any $n > 2$ because a condition necessary for those eight triangles to be part of a regular triangulation is that at least one of the two "diagonals" that I show in grey in figures 3b, 3c and 3d do not cut the triangles.

Things can be arranged a little, but this arrangement complicates the construction and it still does not provide a better bound than Haas'. Anyway, I will explain it. The idea is to take a big fixed n and dispose diamonds of width $6n$, as before, but do not fill them completely with our triangulation. Instead, one can fill a central subdiamond with this triangulation and the rest with any triangulation that makes the whole regular (and use Harnack's rule in these parts). The results for regularity that I mentioned before seem to indicate that one can fill 5/6 of the diamonds with our triangulation without losing regularity (more precisely, my result shows that if one fills in 5/6 or more one loses regularity, but I am quite confident now of the converse). In the remaining 1/6 and in the small diamonds one would use Harnack's rule. In this way, one covers a fraction of $\frac{5}{6} \times \frac{9}{10}$ with a density of 16 ovals per each 18 area units, and the rest (i.e. a fraction of $\frac{1}{4}$) of the area with Harnack's rule, which can be seen to give a density of 2/3 for these parts. Joining this two things together I obtain exactly the same asymptotic density of positive ovals that the one obtained by Haas, i.e. 10 positive ovals per 12 area units.

Let me finish this 'report' with a somehow personal question. My result loses a big part of its importance from the fact that the triangulation is not regular, because this implies that it cannot be directly applied to algebraic curves. In the other hand, it may throw a bit of light into the question of why regularity is needed in Viro's constructon. I have always wandered whether one could construct combinatorially curves by Viro's method that are not algebraic curves (by using non regular triangulations) or, in the contrary, the condition of regularity of the triangulation is truly needed for Viro's theorem, appart from being used in the proof. My belief was that regularity is needed, but the non regular triangulations that I tried before never gave anything with more ovals or with a more, say, complicated structure than one obtains with regular ones. Even more, non-regularity -in small examples- seems to prevent instead of reinforce the formation of ovals.

Now, my result is the first time that I have a triangulation that produces curves that one doesn't know how to produce with regular triangulations (even if this does not mean that they cannot be produced, nor that no algebraic curve have that structure). My question to you is your opinion on these things, both from a purely mathematacal and from a "practical" point of view. Also, I would thank any suggestions on how should I try to prove that these curves cannot be algebraic curves (I see no way of proving the contrary). I have tried the (improved by Arnol'd) Petrovski's inequalities but they satisfy them. For example, any restriction on the topology of (M-4)-curves would be helpful because 4 is the minimal number of diamonds I need to place to obtain something non regular, and each diamond makes one oval disappear (in global terms) form Harnack's curves.

Receive my best regards and thanks in advance,

Paco.

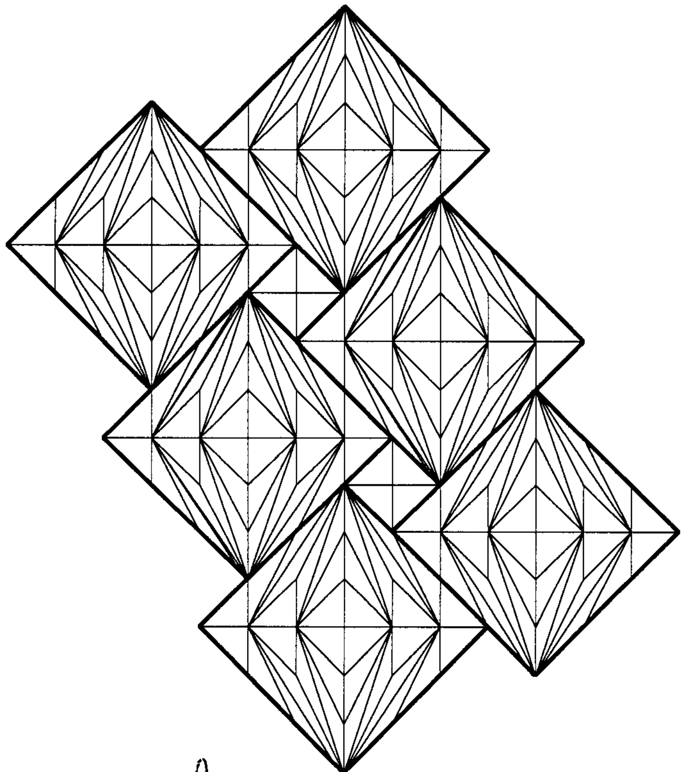


fig 1.a

n=1

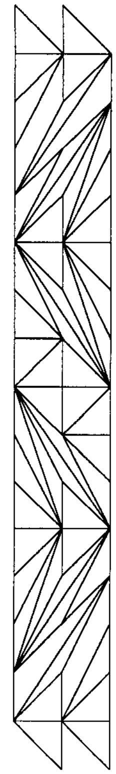


fig 1.b

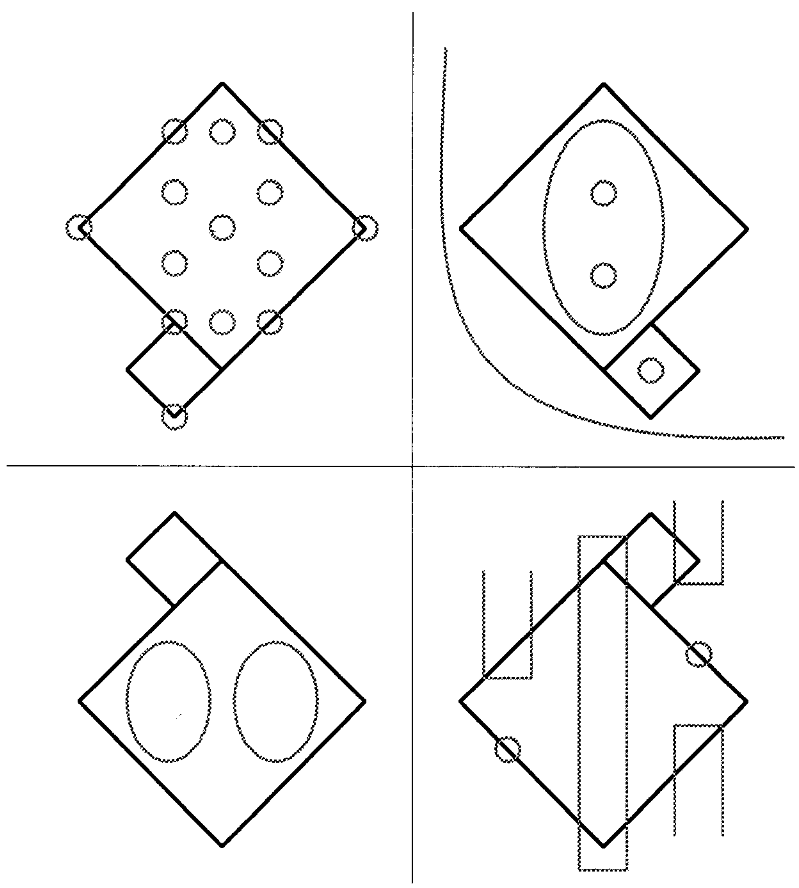


fig 1.c

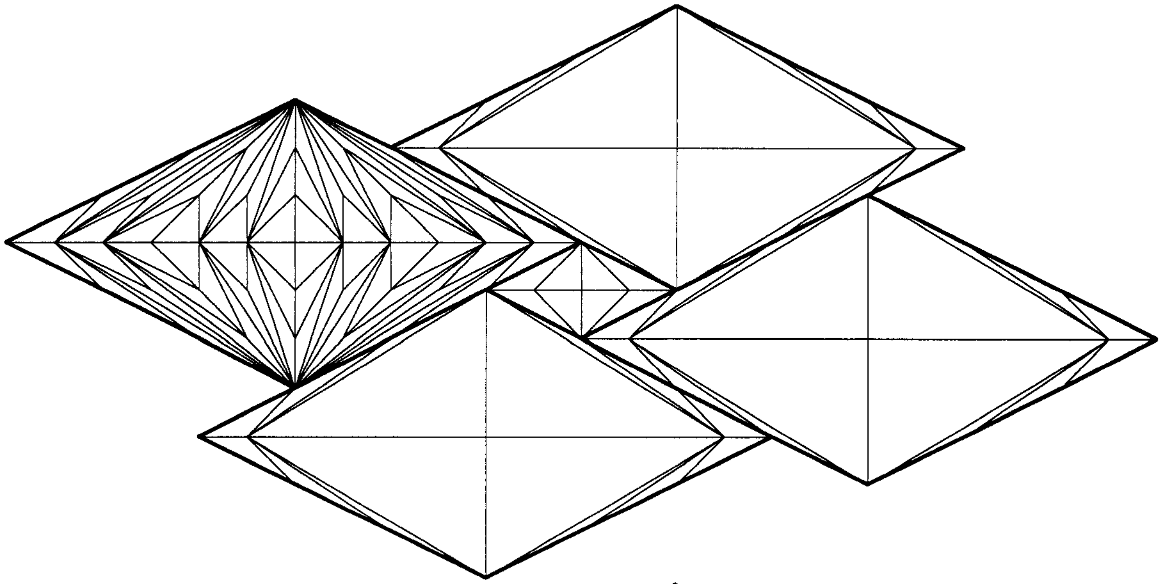
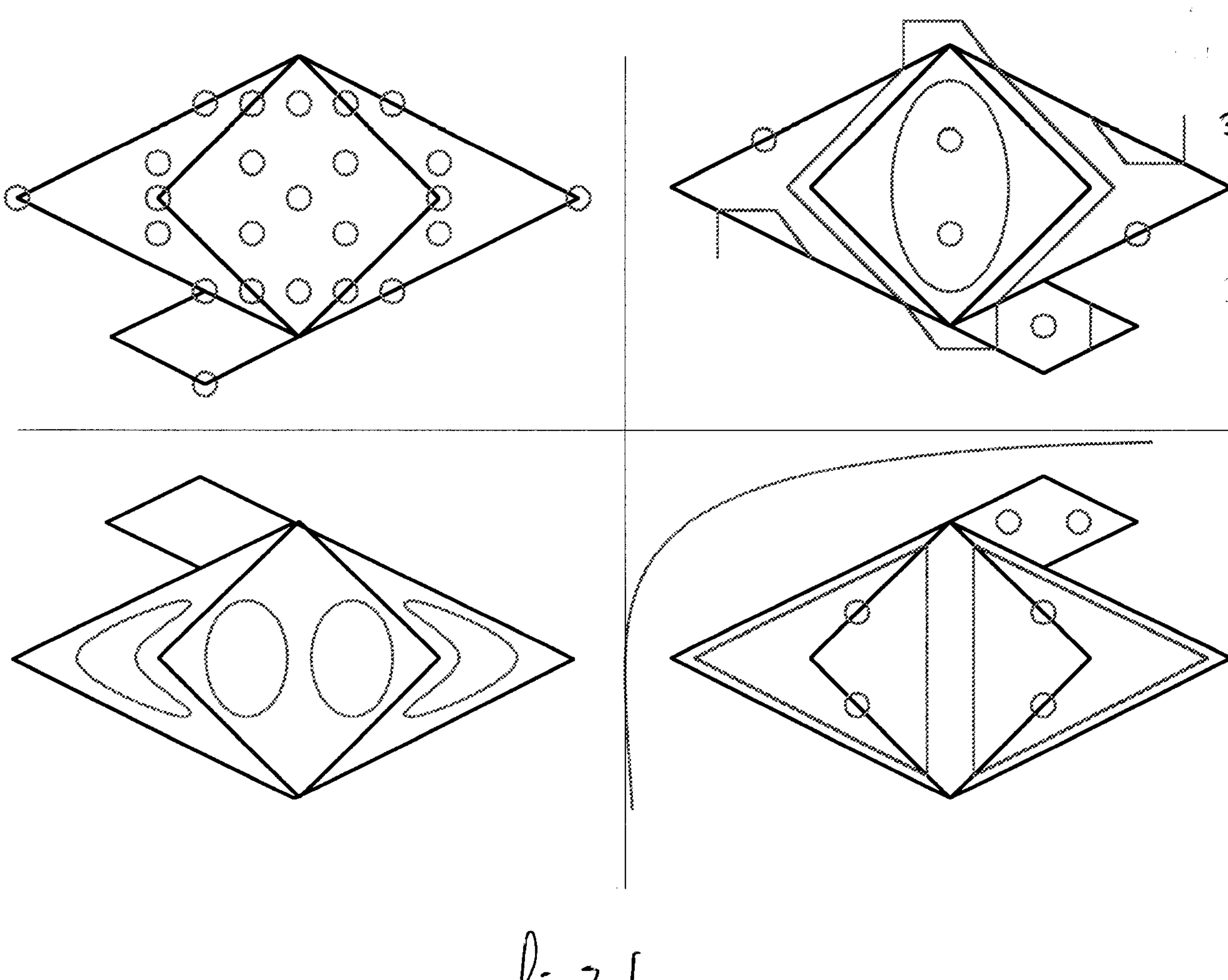


fig 2.a

n=2



3.144

16 -18+

1

8

fig 2.b

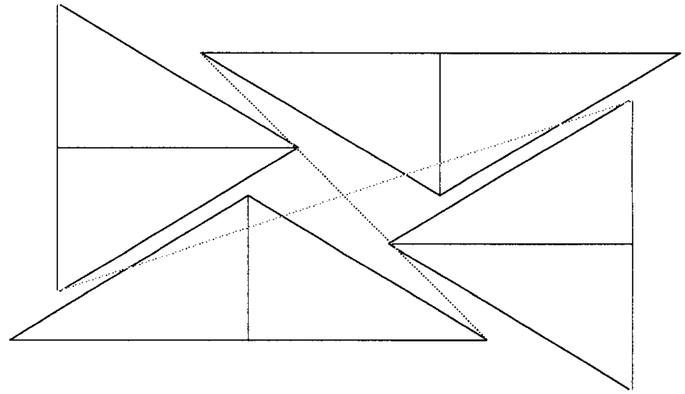


fig 3.b n=2

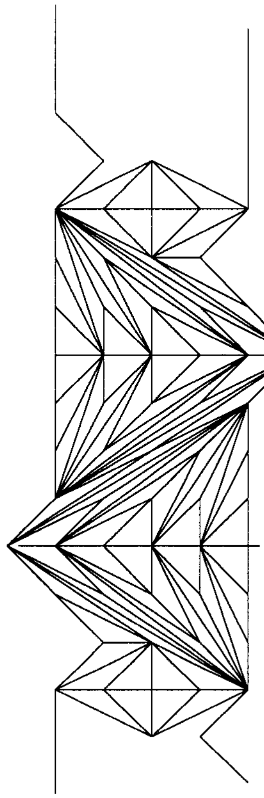


fig 3.a

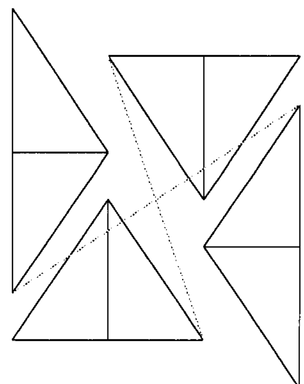
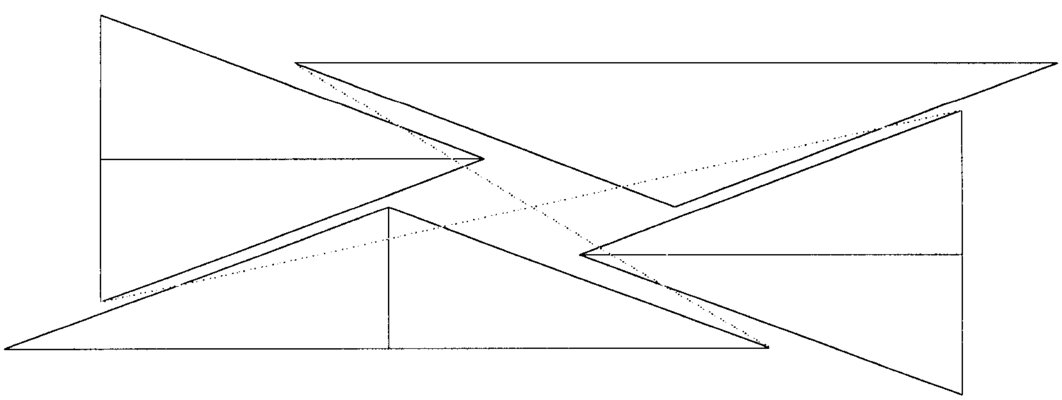


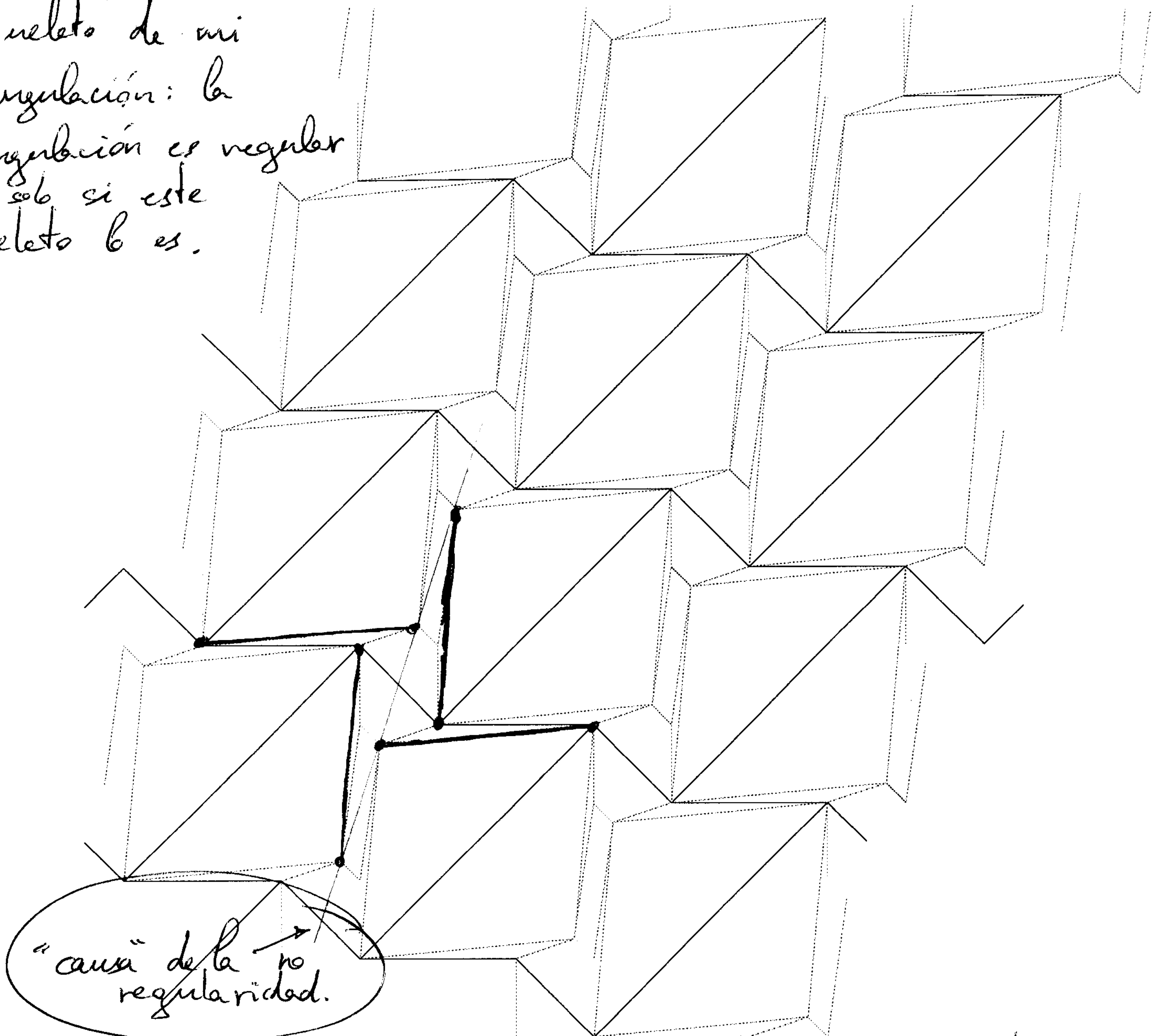
fig 3.c n=1



n=3

fig 3.d

ueleto" de mi angulación: la riangulación es regular si y solo si este esqueleto lo es.



Aquí puedes ver el "remolinillo". Este dibujo está girado 45° respecto a la realidad y corresponde a $n=2$,* así que el remolino ~~...~~ ya impide la regularidad de la triangulación. A medida que aumenta n los paralelogramos ▯ se hacen cada vez más estrechos (en relación al resto) y eso es lo que provoca la no regularidad.

(* aunque el dibujo está deformado por una contracción en la dirección diagonal ↘, de forma que los "diamantes" de los que le hablo a Itenberg