Inscribing a symmetric body in an ellipse.

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Abstract

We prove that any bounded, centrally symmetric object K in the plane can be inscribed in an ellipse E touching its boundary ∂K at at least four points. Two applications of this result in the context of Voronoi diagrams and Delaunay oriented matroids for convex distances are given.

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1 Our result

This note is devoted to prove the following result, illustrated in figure 1.

Theorem 1.1 *Let K be a compact, centrally symmetric body in the plane $\mathbf{E}^2$, not contained in a straight line. Then, there exists an ellipse E containing K and such that the boundaries ∂K and ∂E intersect in at least two pairs of opposite points.*

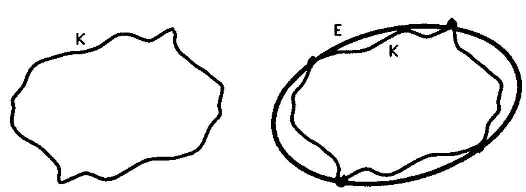

Fig. 1. Inscribing K in an ellipse through four points.

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For convenience, in Theorem 1.1 and in the rest of the paper the word *ellipse* and also the word *circle* will be used meaning not only the "curve" but also its interior region. We will say *boundary* of an ellipse/circle (or use the symbol ∂) when referring to the curve. The word *compact* means "bounded and topologically closed", as usual. The condition of K being closed is not essential (because our result applies to its boundary), but the conditions of being bounded and centrally symmetric are necessary: an unbounded object can never be inscribed in an ellipse and some non symmetric objects (such as a triangle) cannot be inscribed through more than three points.

Theorem 1.1 is proved with the following sequence of lemmas.

Lemma 1.2 *Let K be a compact, centrally symmetric, convex body in the plane $\mathbf{E}^2$, not contained in any straight line. Then, there exists an ellipse E such that ∂K and ∂E intersect in at least three pairs of opposite points.*

Proof: Let P, P', Q and Q' be any two pairs of opposite points in ∂K, not in a straight line. Then, either there exists a third pair of opposite points R, R' in ∂K such that $PQRP'Q'R'$ is a strictly convex, centrally symmetric hexagon, or K coincides with the cuadrilateral $PQP'Q'$ (recall that K is assumed to be convex). In the first case there exists an ellipse passing through the six points. In the second case, if we slightly reduce any ellipse passing through P, Q, P' and Q' we will obtain an ellipse passing through four pairs of opposite points of the boundary of the cuadrilateral. ∎

Lemma 1.3 *Let K be a compact, centrally symmetric, convex body in the plane $\mathbf{E}^2$, not contained in any straight line. Then, there exists a circle C and a linear transformation l of the plane such that the image M of K through l is contained in C and the boundaries ∂M and ∂C intersect in at least two pairs of opposite points.*

Proof: Let us apply Lemma 2.2 to K and then let us make a linear transformation l' in the plane sending the ellipse E obtained there into a circle C'. Let $M' = l'(K)$. Let $f : [0, 2\pi] \to \mathbf{R}_+$ be the map describing $\partial M'$ in polar coordinates, as a function of the angle. Then, f is periodical of period π (because of M' being centrally symmetric) and takes the same value in three different points $0 \leq x < y < z < \pi$ (the points where $\partial M'$ intersects the circle C'). In these conditions, f has at least two local maxima in a period. In fact, either at least two of the open intervals (x, y), (y, z) and $(z, x + \pi)$ contain a local maximum of f, or one of them (say (x, y)) contains a local maximum and z is another local maximum, or the three points x, y and z are local maxima.

Let α and β be two local maxima of f in the period $[0, \pi)$ and suppose without loss of generality that α is actually a global maximum. Consider the collection of linear transformations l_r $(0 < r \leq 1)$ that fix the direction in which β is and that contract its perpendicullar direction by a ratio r. Call f_r the transformed of f by l_r, i.e. $f_r = f \circ l_r$. Then, for r close to 0 β is clearly a global maximum of f_r. Call r_0 the supremum of the values of r for which this happens. Our claim is that in these conditions β is a global maximum for f_{r_0}, but it is not the only one.

To prove the claim, the fact that β is a global maximum of f_r, for r arbitrarily close to r_0 implies that it is also a global maximum of f_{r_0}. On the other hand, for any $r > r_0$ the absolute maximum of f_r is not attained on β, nor in a certain interval $[\beta - \epsilon, \beta + \epsilon]$ around β (because β is a local maximum of every f_r). Consider a sequence $r_1 > r_2 > \ldots$ with limit r_0, and for every r_i let γ_i be an absolute maximum of l_{r_i}. Then the sequence γ_i has at least one limit point γ in the compact $[0, \beta - \epsilon] \cup [\beta + \epsilon, \pi]$ and this limit point must be an absolute maximum of f_{r_0}.

The claim finishes the proof of the lemma as follows. Let $l = l_{r_0} \circ l'$, $M = l(K) = l_{r_0}(M')$ and C be the circle of radius $f_{r_0}(\beta)$. This circle contains M and the boundaries ∂M and ∂C intersect in the two pairs of opposite points in the directions of β and γ. ∎

Proof: (of Theorem 1.1). If K is convex let us apply lemma 1.3 to it, and obtain a circle C and a linear transformation l sending K to a convex M in such a way that $M \subseteq C$ and $\partial C \cap \partial M$ consists on at least two pairs of opposite points. The inverse image $E = l^{-1}(C)$ is then an ellipse in the conditions of Theorem 1.1.

If K is not convex, apply the previous remark to its convex hull $conv(K)$. We will prove that any point in $\partial conv(K) \cap \partial E$ is also in $\partial K \cap \partial E$, and that will finish the proof. Let P be one of the intersection points in $\partial conv(K) \cap \partial E$. As we have $\partial conv(K) \subseteq conv(K) = conv(\partial K)$, P is contained in a segment $[Q, R]$ with $Q, R \in \partial K \subseteq E$. Then, as $P \in \partial E$, the only possibility is $P = Q$ or $P = R$. Thus, $P \in \partial K$. ∎

2 Two Applications

In [5] and in [10] (see also the extended abstract in [9]), Theorem 1.1 has been used in the context of Voronoi diagrams and Delaunay oriented matroids for convex distance functions in the plane. In this section we will summarily describe these results, emphasizing the role of Theorem 1.1 in their proofs.

Let K be a compact, convex body in the Euclidean plane $\mathbf{E}^2$ with the origin in its interior. For any two points P and Q in the plane, K defines a *K-distance* function from P to Q (denoted $D_K(P, Q)$) as the minimum scaling factor λ that makes a scaled translation $P + \lambda K$ of K centered at P to pass through Q. The map $D_K : \mathbf{E}^2 \times \mathbf{E}^2 \to \mathbf{R}_+$ so obtained is called the *convex distance function induced by K*. Voronoi diagrams for convex distance functions were introduced in [3]. They have also been studied in [4], [5], [6], [7] and [8].

The words *smooth* and *strictly convex* applied to a convex distance function D_K mean that its unit ball K is respectively smooth (it has one only supporting line at each boundary point) and strictly convex (its boundary contains no straight line segment). These two properties have important geometrical consequences, namely that every three points are either collinear or D_K-cocircular (but not both) and if they are cocircular then there exists one only D_K-circle passing through them. Concerning Voronoi diagrams and Delaunay triangulations, a Voronoi diagram with respect to a non-strictly convex distance function may have "edges" with non empty interior and a Delaunay triangulation with respect to a non-smooth convex distance function may not be a triangulation of the whole convex hull of the sites (see [5] for details).

Here we will always assume that the convex K (which is called the *unit ball* of D_K) is centrally symmetric, in which case D_K is actually a metric. If K is an ellipse, then the distance D_K is an affine transform of the Euclidean distance and thus Voronoi and Delaunay triangulations for D_K are also affine transforms of Euclidean Voronoi and Delaunay diagrams. Theorem 3 in [5] (reproduced below and whose proof is based in Theorem 1.1) shows that for any other K, the Voronoi diagram and Delaunay triangulation of some point sets S with respect to the metric D_K will have a topological type forbidden with the Euclidean distance.

Proposition 2.1 *([5]) If D_K is a symmetric, convex distance function whose defining convex K is not an ellipse, then there exists some collection S of nine points whose Voronoi diagram with respect to distance D_K has not the topological type of any Euclidean Voronoi diagram.*

Proof: (sketch) The proof is made via Delaunay triangulations, instead of Voronoi diagrams. Delaunay triangulations are defined as being the (topological) duals of Voronoi diagrams, having the Voronoi sites as vertices and straight line segments as edges (even if in degenerate cases they are not true triangulations). They are only well-defined if the distance D_K is strictly convex, but Proposition 2.2 is trivially true for non-strictly convex distance functions.

Using our Theorem 1.1 one can find nine points as in figure 2.a, the four in each corner lying in a scaled translation of the boundary of both the unit ball K and in an ellipse E. This implies that the Delaunay triangulation of those nine points respect to the distance D_K is actually as in figure 2.a.

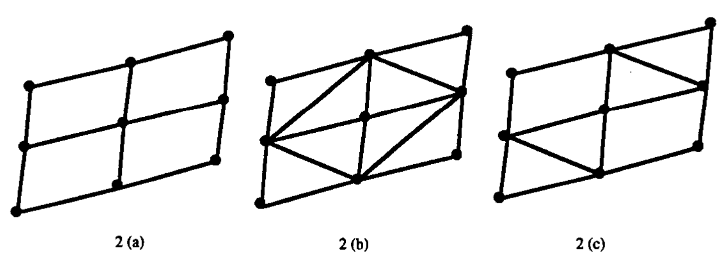

Fig. 2. An Euclidean and two non-Euclidean Delaunay triangulations.

Now, the fact that E is exterior to K implies that a certain perturbation of the four corner points along their respective ellipses gives a Delaunay triangulation as the one in figure 2.b or 2.c, which are topologically forbidden for the Euclidean distance (see [5] for details). ∎

The second result in this section concerns oriented matroids defined from circles, which in [10] are called *Delaunay oriented matroids* (see [1] for information on oriented matroids) and regularity of Delaunay triangulations.

A triangulation T of the convex hull of some point set S in the plane is called *regular* (sometimes the words *coherent* or *convex* are used) if the point set can be lifted in 3-space in such a way that the faces of the lower envelope of the lifted point set project down onto the faces of T. The fact that Delaunay triangulations for the Euclidean distance are regular is well-known and sometimes called the *lifting property* of Delaunay triangulations: the Delaunay triangulation of a point set S can be computed as the projection of the lower envelope of the lifting of the sites into the paraboloid $z = x^2 + y^2$ in 3-space (cf. [2]). For a smooth, strictly convex distance function K, the Delaunay triangulation of any point set S is still a triangulation of its convex hull $conv(S)$, but in Proposition 2.3 below we prove that it will be a non-regular triangulation for some point sets, whenever the defining convex K is not an ellipse. Thus, these distances do not have a "lifting property": Delaunay triangulations for convex distance functions cannot be computed as being the projection of the lower envelope of any lifting of the sites.

The Delaunay oriented matroid DOM(S) of a set S of sites describes how circles and lines partition S. More precisely, the covectors of DOM(S) are the signed partitions of S obtained as the interior, exterior and boundary of arbitrary circles, or the two half-planes defined by an arbitrary line and the line itself. The concept can be generalized to non-Euclidean metrics, in particular to convex distance functions, changing Euclidean circles for the D_K-circles of a convex distance function D_K (i.e. the scaled translations of the unit ball K). Nevertheless, it is not clear a priori whether the resulting covectors will satisfy the axioms for an oriented matroid.

In [10] we show that, if the convex distance function is *strictly convex* and *smooth*, then the set of covectors obtained using D_K-circles is actually an oriented matroid for any point set S, but with one important difference with respect to the Euclidean case: Euclidean Delaunay oriented matroids are *realizable*, again because of the lifting property of Euclidean Delaunay triangulations. In the contrary, for any smooth, strictly convex distance D_K with non-elliptical unit ball K we can find eight points with non-realizable Delaunay oriented matroid with respect to D_K.

Proposition 2.2 *[10] Let D_K be a smooth strictly convex distance in the plane. If D_K is not affinely equivalent to the Euclidean distance, then there exists a set S of eight points such that the combinatorial structure of the Delaunay triangulation of S (with respect to D_K) contains the eight triangles in figure 3.b and such that points P_1, P_2, P_5 and P_6 are collinear, as well as points P_3, P_4, P_7 and P_8. Thus,* *i) the Delaunay oriented matroid* $\mathrm{DOM}_{D_K}(S)$ *obtained for them with respect to* D_K *is not realizable.*

ii) the Delaunay triangulation is not regular (it is not the projection of the lower envelope of any point set in 3-space).

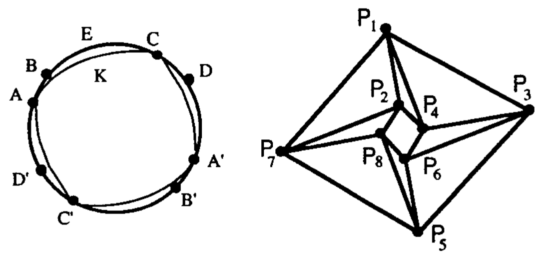

Fig. 3. Eight points with non-realizable D. O. M.

Proof: (sketch) Let us again apply Theorem 1.1 to the unit D_K-ball K and, without loss of generality, let us suppose that the ellipse E obtained in the Theorem is actually a circle (modifying K by a linear transformation, if needed). Call A, A', C and C' the four points obtained in Theorem 1.1, lying both on ∂E and on ∂K (as in figure 3.a). Consider another four points B, B', D and D' lying on ∂E and outside K, with arcs $[A, B]$, $[C, D]$, $[A', B']$ and $[C', D']$ being of equal length.

Now, let the point set S consist on the eight points in figure 3.b, obtained as

$$P_1 = A, \qquad P_2 = D', \qquad P_3 = B', \qquad P_4 = C',$$

$$P_5 = P_4 + BD', \qquad P_6 = P_4 + BA,$$

$$P_7 = P_6 + DB = P_2 + A'C', \qquad P_8 = P_6 + DC = P_2 + A'B'.$$

By construction, points $P_1P_2P_5P_6$ and $P_3P_4P_7P_8$ are collinear. Also, points $P_1P_2P_3P_4$, $P_3P_4P_5P_6$, $P_5P_6P_7P_8$ and $P_7P_8P_1P_2$ are cocircular with respect to Euclidean circles but, with respect to the distance D_K, their Delaunay triangulation contains the eight triangles in figure 3.b (for example, triangles $P_1P_2P_4$ and $P_1P_3P_4$ come from the fact that there exists a D_K-circle passing through P_1 and P_4 and having P_2 and P_3 outside). This finishes the proof of the first statement.

We will only complete the proof of part (*ii*) of the Proposition. See [10] for the proof of (*i*). If the triangulation was regular, there would be a lifting $\{Q_1, \dots Q_8\}$ of $\{P_1, \dots, P_8\}$ whose lower envelope would project down onto the triangulation. Consider the intersection point O of the lines $P_1P_2P_5P_6$ and $P_3P_4P_7P_8$ and call v the vertical line passing through it. Call A, B, C and D the intersections of v with the lines passing, respectively, through Q_1Q_2, Q_3Q_4, Q_5Q_6 and Q_7Q_8. The fact that triangles $Q_1Q_2Q_4$ and $Q_1Q_3Q_4$ are in the lower envelope of the lifting implies that A is below B in v. With the same arguments for the other triangles we conclude that B is below C, C is below D and D is below A. That is a contradiction. ∎

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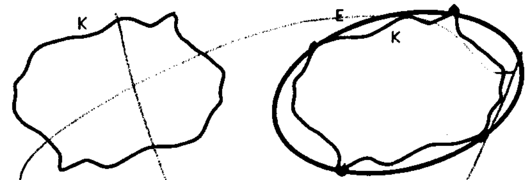
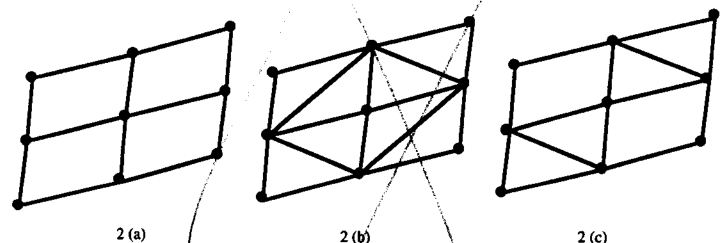


Fig. 1. Circumscribing an ellipse through four points.



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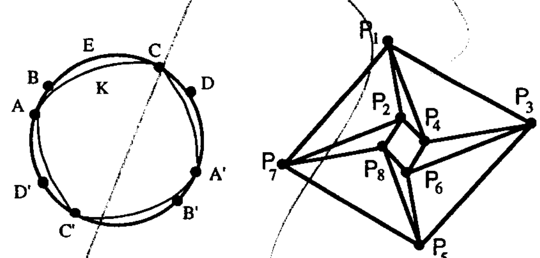

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