Construction of real algebraic plane curves with given topology

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Abstract. We study the problem of finding an algebraic curve in the real (affine or projective) having each of the topological shapes that an algebraic curve can have. Our aim is to give an effective algorithm that, with input a topological model T for the algebraic curve, gives a polynomial whose zero set is isotopic to T, with a bound for the degree needed in terms of some topological invariants of the model (number of connected components, number and order of multiple points,...). studied here.

We show a method to construct such a polynomial which works if the topological model has only double singularities and, in that case, gives a degree $4n+2K$ or $4n+2K-1$ polynomial, where n and K are the numbers of double points and connected components of the model.

The construction is based on a preliminar topological manipulation of the topological model, and then some perturbation techniques to obtain the polynomial.

0. Introduction.

If we have two subsets A and B in a topological space X, and a global isotopy which moves A to B we say that (A, X) and (B, X) are *topologically equivalent*, or that A and B have the same *topological shape* in X. In the context of real algebraic geometry an interesting question is knowing which are the possible pairs $(V, \mathbb{R}^n)$, or $(V, \mathbb{RP}^n)$ up to topological equivalence, with V an algebraic set.

The answer to this question is far from trivial in the general case (see for example [**BCR**], or [**AK**]), but simple if we restrict ourselves to the real (affine or projective) plane: any imbedded graph in $\mathbb{RP}^2$ with even order (possibly zero) in every vertex has the shape of an algebraic set, and conversely any algebraic set $V \subsetneq \mathbb{RP}^2$ has the same shape that an imbedded graph with even order. For $\mathbb{R}^2$ the characterization is the same except that there can be a certain number (finite and even) of branches going to infinity.

The proofs of this characterization normally use polynomial approximation of C^∞ functions, and thus say nothing about the degree needed to 'realize' a given topological shape by an algebraic curve. Our goal is to give a new, constructive proof of the characterization, and to bound the minimal degree of a polynomial

Partially supported by DGICyT PB 89/0379 and "POSSO", ESPRIT-BRA 6846.

whose zero set has a given shape in terms of its number of some topological invariants of the given shape. Note that every plane real algebraic set can be obtained as the zero set of a single polynomial, so we are going to speak of algebraic curves, rather than algebraic sets.

This problem is somehow related to Hilbert's 16^{th} problem, which asks for the possible topological configurations of the ovals of algebraic M-curves in the projective plane. (A nonsingular algebraic projective curve of degree d is an M-curve if it has $(d-1)(d-2)/2$ ovals, which is the greatest number of ovals it can have, according to Harnach's Theorem.) the main differences are that in our problem we deal with possibly singular curves, and that we pose the inverse question to Hilbert's: instead of asking what possible shapes an algebraic curve with fixed degree can have we ask what degree we need to construct an algebraic curve with given shape. The perturbation techniques we use in the final part of the construction (cf. Th.2.7) are similar to some also used in Hilbert's problem. (See for example [Vi] and [Gu].)

The construction we propose works (both in the projective and the affine plane) if the topological model we want to realize is compact and has only double singular points, and in this case the bound obtained is that every topological model in the projective plane can be realized with degree $d = 4N + 2K$, or $d = 4N + 2K - 1$ (depending on the parity of the topological model, cf. Definition 3.5 and Corollary 5.4), where N is the number of singular points and K the number of connected components in the topological model.

For nonsingular curves this bound gives $d = 2K$, or $d = 2K - 1$ which is trivial (for we can construct any non singular model as a product of K circles, or may be $K - 1$ circles and a line), but also optimal (if the model consists on K nested ovals it can not be "realized" by an algebraic curve of degree lower than $2K$). For singular curves we shall give examples of models with K components and N double points in them which cannot be realized algebraically with degree lower than $2N + 2K$, so our bound is reasonably close to the optimal one (cf. section 6.2).

The paper begins with to sections devoted respectively to introduce the background on algebraic curves we wil need, and to proof the main result on perturbations that will be used in the construction (Theorem 2.7). Section 3 introduces some topological procedures to be used as a previous work on the topological model, and sketchs the algorithm for the construction, which is developed in sections 4 and 5. Finally in section 6 we make some final comments, including the possible generalization to higher order singularities.

1. Algebraic curves. Singularities.

Although we have announced results both in the projective and the affine planes we will work always in the projective plane; the affine results will be a consequence of the projective ones. Thus, throughout the paper we will use the term *algebraic curve* or simply *curve* as an abbreviation for *plane projective real algebraic curve*, meaning by this a real homogeneous polynomial $f \in \mathbb{R}[X, Y, Z]$ in three variables, considered up to a constant factor. Sometimes we will call curve the zero set of f, $V_f = \{f(x, y, z) = 0\} \subset \mathbb{R}\mathbb{P}^2$, but always having in mind which is the polynomial who defines it.

We consider the real affine plane imbedded in $\mathbb{R}\mathbb{P}^2$ by the canonical map $(x, y) \longmapsto (x, y, 1)$, and so the *origin* of $\mathbb{R}\mathbb{P}^2$ is the point $(0, 0, 1)$, the *infinity line* is the projective line $\{z = 0\}$ and the affine curve associated to f is $\tilde{f}(x, y) = f(x, y, 1)$

A point $(x, y, z) \in V_f$ is a *singular point* of f if $f_X(x, y, z) = f_Y(x, y, z) = f_Z(x, y, z) = 0$. We say that a singular point $P = (x, y, z)$ is an *r-fold* singular point (or,that it has *order r*) if all the derivatives of f up to order $r - 1$ vanish at (x, y, z) and there is an order r derivative of f not vanishing at (x, y, z). If we make $P = (0, 0, 1)$ (the orign) by a change of projective coordinates, then the order of P coincides with the lowest degree of the monomials in the affine associated curve $\tilde{f}(x, y) = f(x, y, 1)$. (According to this, sometimes we will consider nonsingular points as order 1 points, and points not in the curve as order 0). In these conditions, in a small neighbourhood of P, f is the product of a certain number of *analytic branches*, each of which can be parametrized by a series in the form

$$\begin{cases} x = a_n t^n + \ldots \\ y = b_n t^n + \ldots \\ z = 1 \end{cases}$$

with $a_i, b_i \in \mathbb{C}$ for $i \geq n$. The branches are said to be in *primitive form* if not both a_n and b_n equal to 0 and the exponents of t in the non zero terms of the series have no common factor (i.e. if the exponents cannot be lowered by a change of parameter $u = t^k$ for any $k > 1$ (see [Wa] or [Se]).

The number n is called the *order* of the branch, and the line defined by $x : y = a_n : b_n$ is the *tangent line* of the branch. A branch is called *nonsingular* if it has order 1.

The sum of the orders of all the branches passing by P equals to the order r of P; in particular, there are at most r branches passing by P. Moreover, the tangent lines of the branches at P coincide with the linear factors of the part $\tilde{f}_r$ of lowest degree in the associated affine curve, and the sum of the orders of branches with a fixed tangent line, say $x : y = a : b$, equals the multiplicity of the factor $bx - ay$ in $\tilde{f}_r$.

Each analytic branch has the (local) topological shape either of an isolated point (if it is a complex branch, i.e. if it cannot be parametrized with all the a_i and b_i real numbers) or of a line passing by P (if the branch is real), so the topological (local) shape of the curve in a neighbourhood of P is that of a certain number s of lines passing by P. This number is at most the order of P, and can be 0, meaning that P is an isolated point of the curve.

We shall be concerned mainly with two types of singularities: *nondegenerate singularities* and *2-fold singularities having two real different branches*:

An r-fold point P of f is called *non-degenerate* if there are r different tangent lines at P, i.e. if all the branches at P are non singular and have different (real or complex) tangent lines. If P is the origin, an easy characterization exists: the origin is an r-fold nondegenerate point if and only if the part $\tilde{f}_r$ of lowest degree of f has degree r and is square-free.

For nondegenerate r-fold points we can say that not only the topological shape but also the differential shape of f in a neighbourhood of P is that of a certain number s of straight lines intersecting at P, where s is the number of real branches at P, i.e. the number of real linear factors of $\tilde{f}_r$, if P is the origin. s must be a number from 0 to r and congruent to r modulo 2, because each complex branch appears joint with its conjugated. If $s = r$, i.e. if the r branches at P are real, we will say that P is a *real-nondegenerate r-fold point.*

A 2-fold singularity with two real different branches is always diffeomorphic to the singularity of the curve $X^2 - Y^k$ for some positive and even integer k (see [Vi]). In the terminology of [AGV] and [Vi] it is then called an A^-_{k-1} *singularity*. The actual value of $k-1$ (which is called the Milnor number of the singularity), will not be important for us, and thus we shall call A^- *singularities* those singularities which are of type A^-_k for some even k.

Curves with A^- singularities in a point P can be obtained (and will usually be in our constructions) multiplying two curves f and g, both passing by P, non singular in it and with no common factor (or at least with no common factor passing by P).

2. Perturbation of curves.

For the construction of the curve in §5 we will need to perturb some given curve, in order to change its topology in a controlled way. Here we give the technical results which enable us to do that. The main results are Propositions 2.4, 2.5 and 2.6 which combined give Theorem 2.7. Both 2.4 and 2.6 are particular cases of the 'Lemma on the class of a point' and the 'Lemma on isotopy' from [Gu], and 2.5 can easily be deduced from [Vi]. Nevertheless, we give proofs of them, using little algebraic background.

Definitions 2.1 Let f and g be two algebraic curves of the same degree n. A *perturbation of f by g* is any curve of the form $f + \varepsilon g$, with $\varepsilon \in \mathbb{R}$, and we say that a property is true for *small perturbations of f by g* if there exists an ε_0 such that the property is true for any perturbation of f by g with $|\varepsilon| \leq \varepsilon_0$.

A *regular domain* G will be for us an open set in $\mathbb{RP}^2$ whose boundary is a finite union of disjoint rectangles, meaning by a rectangle anything which is a rectangle for some projective system of coordinates. We say that a curve f is *regularly disposed* in G if it has only a finite number of singularities in G, no singularity on its boundary and crosses this boundary transversally.

We say that two curves f and g are *transversal* to one another if they intersect only at non singular points of both of them, and they have different tangents at these intersection points.

Lemma 2.2 *Let $\{f_t\}$ be a family of univariate real polynomials of the same degree, whose coefficients vary continuously with one or more parameters $t \in \mathbb{R}^n$, and suppose that for $t = 0$ f_0 has r real roots (counted with multiplicities) in a certain closed interval I. Then, for sufficiently small values of the parameters t the polynomial f_t has at most r roots (counted with multiplicities) in I. Moreover, if the r roots of f_0 in I are different and not in the extremal points of I then f_t has exactly r different real roots in I, for small t.*

Proof. Let d be the degree of the polynomials $\{f_t\}$. Then, the d complex roots of the polynomials f_t vary continuously with t, (see for example [BR]); thus, for small t, the roots of f_0 which were outside the interval I remain outside (because of I being closed) and the number of roots in I cannot increase. Moreover, for the number of roots to decrease, some of the r roots of f_0 in I must get out from I either as a real root (and that implies f_0 having a root in a extremal point of I), or as a complex one (and that implies f_0 having a double root in I, because a complex root can only appear jointly with its conjugated). □

Lemma 2.3 *Let f and g be two curves of the same degree, and let U be an open neighbourhood of the zero set of f. Then the perturbed curve $f + \varepsilon g$ is contained in U for ε sufficiently small.*

Proof. Let us call $F(\varepsilon, X, Y, Z) = f(X, Y, Z) + \varepsilon g(X, Y, Z)$, which is a continuous function in $\mathbb{R} \times \mathbb{RP}^2$ and suppose that the lemma is not true. Then there exists a sequence $\{\varepsilon_i\}$ which tends to zero, and for each ε_i a zero z_i of $f + \varepsilon_i g$ outside U. Now, $\mathbb{RP}^2 \setminus U$ is compact, so $\{z_i\}$ contains a convergent subsequence, which we still notate $\{z_i\}$, with some limit z outside U. Then, the sequence $\{(\varepsilon_i, z_i)\} \subset \mathbb{R} \times \mathbb{RP}^2$ tends to $(0, z)$, and by continuity of F, $F(0, z) = 0$. But $F(0, z) = f(z)$, which gives the contradiction, because z is not in the zero set of f (it is not in U). □

Proposition 2.4 *Let f be a curve with a real-nondegenerate r-fold point at P, and let g be another curve of the same degree d than f and order at least r at P. Then a small perturbation of f by g still has a real-nondegenerate r-fold point at P, and moreover there exists a small rectangle U around P such that the perturbed curve $f + \varepsilon g$ is isotopic to f in U, for small ε.*

Proof. Without loss of generality we can suppose that P is the origin, and work with affine coordinates, for the problem is local.

Firstly, $f+\varepsilon g$ has an order r singular point at the origin, because f and g have no terms of degree lower than r, and the term of degree r, which is $f_r+\varepsilon g_r$, does not vanish for small ε. Moreover, the r projective roots of $f_r+\varepsilon g_r$ vary continuously with ε, and so they must be all different and real for small ε, because they are different and real for $\varepsilon=0$. We conclude then that the perturbed curve has a real-nondegenerate r-fold point, for small ε.

To find the rectangle U for the isotopy, eventually rotating the coordinate axes, we can obtain that neither the part of lower degree f_r of f, nor the part of higher degree f_d, have X as a factor (i.e. f has not a vertical tangent at the origin, and does not pass through the projective (0,1,0)). In these conditions, $f(0,Y)$ has an r-fold root at $Y=0$, and $f(x,Y)$ has degree d for every fixed x.

Call $m_1<m_2<\ldots<m_r$ the slopes of the r tangent lines of f at P, and take $r+1$ lines $l_0,\ldots l_r$ passing by P and with slopes $a_0<m_1<a_1<m_2<\ldots<m_r<a_r$ (i.e. such that $f_r(x,Y)$ has alternatively opposite signs at them, for every $x\neq 0$).

Now, if we choose an $y_0>0$, such that $f(0,Y)$ has only 0 as a root in $[-y_0,y_0]$, and x_0 sufficiently small, we have:
i) $-y_0<x_0a_1$ and $x_0a_r<y_0$.
ii) $f(x,Y)$ has at most r roots in the interval $[-y_0,y_0]$, for every $x\in[-x_0,x_0]$.
iii) for every $x\in[-x_0,x_0]$, f has the same sign than f_r at the l_i lines (i.e., alternatively opposite signs).

Statement (*ii*) comes from Lemma 2.2, and (*iii*) can be obtained because along each of the l_i's (i.e., making the substitution $Y=a_iX$ in f) we have

$$f(X,a_iX)=f_r(X,a_iX)(1+\alpha_1X+\ldots+\alpha_{d-r}X^{d-r}),$$

where $\alpha_i=f_{r+i}(X,a_iX)/(f_r(X,a_iX)X^i)$ are real constants.

Consider then the rectangle $U=[-x_0,x_0]\times[-y_0,y_0]$, and let us proof the proposition.

By (*iii*) and (*i*), for every $x\in[-x_0,x_0]$ $f(x,Y)$ has at least r roots in the interval $[-y_0,y_0]$ and because of (*ii*) it cannot have more than r roots, so $f(X,Y)$ in U has the shape of r lines crossing at P, one going between each pair of consecutive l_i lines.

Now let us call $F(\varepsilon,X,Y)=f(X,Y)+\varepsilon g(X,Y)$. Along the lines l_i we have now

$$F(\varepsilon,X,a_iX)=f_r(X,a_iX)(\beta_0(\varepsilon)+\beta_1(\varepsilon)X+\ldots\beta_{d-r}(\varepsilon)X^{d-r}),$$

with $\beta_i(0)=\alpha_i$, $\beta_0(0)=1$, and the β_i varying continuously with ε. In particular, for $x\in[-x_0,x_0]$ and small ε, F has along each line l_i the same sign that f_r had and, thus, $F(\varepsilon,x,Y)$ has at least one root between each two consecutive of the l_i lines. Again by Lemma 2.2, $F(\varepsilon,x,Y)$ cannot have more than r roots in $[-y_0,y_0]$, so it has exactly one root between each two l_i lines. So, for sufficiently small ε the shape of the curve $f+\varepsilon g$ in the rectangle U is again that of r lines crossing the rectangle, each between two consecutive l_i lines, and thus $f+\varepsilon g$ is isotopic to f in U. □

Proposition 2.5 *Let G be a regular domain and let f be an algebraic curve regularly disposed in G and nonsingular in G. Let g be any other curve of the same degree d than f. Then, a small perturbation of f by g is non singular in G, regularly disposed in G and isotopic to f in G.*

Proof. f being regularly disposed in G implies that it has no singular points in its boundary, so f is non singular in the closure $\bar{G}$ of G and transversal to its boundary. By continuity, for small ε $f+\varepsilon g$ is still nonsingular in $\bar{G}$ and transversal to its boundary, and thus regularly disposed in G.

Now, for the isotopy, Lemma 2.3 ensures that for small ε the perturbed curves stay close to the original f. Let P be any point of f in $\bar{G}$, and let us suppose that it is the origin and has horizontal tangent. If $g(P)=0$, then also $f(P)=0$, and we apply Proposition 2.4 to P (with $r=1$), to find a (local) isotopy from f to $f+\varepsilon g$ in a neighbourhood of P. If $g(P)\neq 0$, then $f+\varepsilon g$ is, in a neeighbourhood U of P, a level set of the rational function f/g, which has no critical points in U. The implicit function theorem says then that for small ε the level set $f+\varepsilon g$ is a graph of a function $Y=funct(X)$ in a neighbourhood of P, the function varying continuously with ε and that gives the (local) isotopy. $\bar{G}$ being compact, we can cover f in $\bar{G}$ by a finite number of such local isotopies, and they glue to one another to give the global isotopy on G. □

Proposition 2.6 *Let f be a curve with an A^- singularity at P, and let g be another curve of the same degree not passing by P. Then, there exists a small rectangle U around P such that any small perturbation of f by g has no singular point in U, and has locally one of the two shapes shown in figure 1.*

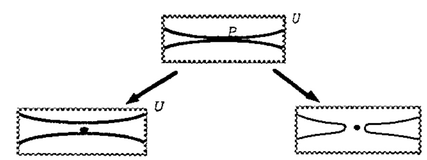

Figure 1

Moreover, the decision of which of the two perturbations occurs depends only on the sign of g at P, the sign of f at the regions adjacent to P and the sign of the parameter ε of the perturbation.

Proof. Because g is not zero at P, for any small rectangle U around P, g does not vanish in U . Then, the polynomial $f+\varepsilon g\in\mathbb{R}[X,Y,Z,\varepsilon]$ is non singular in $U\times\mathbb{R}$, for its derivative with respect to ε is g. We can apply to it Bertini's Theorem **[BCR 9.5.4]** to conclude that the level sets (i.e. the perturbed curves in U for each ε) are non singular except for a finite set of ε's, in particular they are non singular for small $\varepsilon\neq 0$. Besides, $f+\varepsilon g$ does not pass through P, for $\varepsilon\neq 0$.

For the isotopy, we again suppose that P is the origin and that the Y axis is not a tangent line of f at P, and work with affine coordinates. Under these conditions

$Y = 0$ is a root of multiplicity 2 of $f(0, Y)$, and the two nonsingular branches of f at the origin can be solved for Y in the way $Y = f_1(X)$ and $Y = f_2(X)$, where the f_i are two different analytic functions with $f_i(0) = 0$ (cf. [Wa]).

Take y_0 such that $f(0, Y)$ has no root different from 0 in $[-y_0; y_0]$. Now, for $x \neq 0$ sufficiently small we have $f_1(x) \neq f_2(x)$, $f_i(x) \in [-y_0; y_0]$ for $i = 1, 2$, and, by Lemma 2.2, the only two roots of $f(x, Y)$ in $[-y_0; y_0]$ are $f_1(x)$ and $f_2(x)$.

Our rectangle U will be $[-x_0; x_0] \times [-y_0; y_0]$ such that the above is true for $x \in [-x_0; x_0], x \neq 0$ and, in U, f has then the shape of two lines going from left to right, intersecting at the origin.

Now, for ε small the perturbed curve is non singular and has still two different roots in the left and right sides of U (by Lemma 2.2), and there must be then at least two branches joining these four points in two pairs (here and in the rest of the proof the word 'branch' has the imprecise meaning of nonsingular connected parts of the curve, and not that of 'analytic branch'). These two branches can be disposed only in the two ways of figure 1, and must approach P as ε tends to 0, because of Proposition 2.5: outside any small circle around P the perturbed curves are isotopic to f for small ε.

What we must see is that the perturbation makes not appear more branches in U (for example, new ovals) appart from these two.

If it did appear new branches, in any case for sufficiently small ε they would contain no singularities, they would not touch the boundary of U and they would collapse in P as ε tends to zero (again by Proposition 2.5) . The first two conditions imply that they must be ovals (homeomorphic to circles) and we are going to see that the third one gives a contradiction.

First note that the perturbed curves $f + \varepsilon g$ can not intersect each other for different ε's, because of g having no zeros in U. Now suppose that the perturbed curves have an oval for arbitrarily small ε's. For a given ε_0 let us call c_0 one of the ovals. c_0 cannot contain P, because $f + \varepsilon_0 g$ does not intersect f in U, and thus has P in its interior or in its exterior. Moreover, as ε tends to 0, P cannot pass from the interior to the exterior of the oval nor viceversa. Now, if P is exterior to c_0, it is impossible for the oval to collapse in P as ε tends to 0 without intersecting c_0 for some $\varepsilon < \varepsilon_0$. If P is interior, it is impossible for the two exterior branches of the curve to approach P without intersecting c_0, and that gives the contradiction in any of the cases.

The assertion relating the perturbation to the signs of f, g and ε is trivial, for the perturbed curve $f + \varepsilon g$ lies in the part of $\mathbb{RP}^2$ in which f has opposite sign to εg. □

theorem 2.7 *Let f be a real curve whose singular points divide in two groups D and A, D consisting only on real-nondegenerate points and A on A^- type points. Let g be a curve of the same degree which has a singular point of at least the same order than f in the points in D, and not passing by the points in A.*

Then, any sufficiently small perturbation $f + \varepsilon g$ has a real-nondegenerate singular point of the same order than f at each point in D, has no other singular point and its topological shape in $\mathbb{RP}^2$ can be obtained modifying each A^- singularity of f in that of the two ways in figure 1 compatible with the signs of f, g and ε.

Proof. the proof is straightforward from Propositions 2.4-2.6. It suffices to take a sufficiently small rectangle around each singular point of f to apply 2.4 or 2.6 and consider the regular domain consisting on the whole $\mathbb{RP}^2$ without these rectangles to apply 2.5. □

3. Topological models. Flips.

The problem we want to solve can be stated as "given a *topological model* T in $\mathbb{RP}^2$ (or $\mathbb{R}^2$), construct an algebraic curve which *realizes* it", according with the following definitions:

Definition 3.1 Let T be any subset of $\mathbb{RP}^2$. We say that T is a *topological model* for an algebraic curve if it is homeomorphic to a graph with an even (possibly zero) number of edges reaching to each vertex. A subset $\tilde{T}$ of $\mathbb{R}^2$ is a topological model if it is the intersection with $\mathbb{R}^2$ of a topological model in $\mathbb{RP}^2$ which cuts the infinity line in only a finitenumber of points.

We call *order* of a point in T half the number of edges reaching to it if it is a non-isolated vertex of the graph, 2 if it is an isolated vertex and 1 otherwise. We call *singular points* of T the points of order greater than 1, which are finite in number.

Definition 3.2 Let $X = \mathbb{RP}^2$ or $\mathbb{R}^2$, and let $A, B \subset X$. We say that A and B have the same *(topological) shape* if there exists a global isotopy in X which sends A to B. We say that an algebraic curve f *realizes* T if the zero set V_f of f has the same shape than T.

Remarks 3.3

(i) We recall that the even order in each vertex of a connected graph is the Euler condition for the graph being the image of a closed curve with a finite number of self-intersections: even order connected graphs are the graphs which can be continuously 'drawn' without passing twice by any edge, and finishing in the starting point.

(ii) Normally we will consider topological models up to shape equality. For example, the drawings in figure 2 represent the same topological model in $\mathbb{RP}^2$.

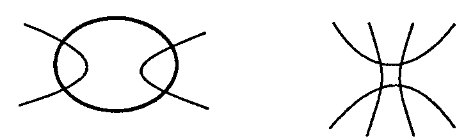

Figure 2

(iii) We will treat only topological models with order 2 in all their singular points. In section 6.3 we will refer to multiple points (points of order bigger than 2).

(iv) In §4 we will find one combinatorial characterization of the shape of a topological model. This is important if we want to our construction to be an algorithm, for the data characterizing the shape of the model will actually be the input of it. A more natural characterization (based on the so-called Gauss-codes) can be found in [Sa], [Go] and [GS]. By now, topological models can be thought as simple drawings in a sheet of paper.

Definition 3.4 Let l be a connected and non singular topological model in $\mathbb{RP}^2$. Then l is either isotopic to a line or to a circle, and is called a *pseudo-line* or an *oval*, respectively. An imbedded graph in $\mathbb{RP}^2$ is called *orientable* if it does not contain any pseudo-line or, equivalently, if there exists a pseudo-line not intersecting it.

Definition 3.5 Let T be a topological model in $\mathbb{RP}^2$ and let l be a pseudo-line transversal to T. We will say that T is *even* (resp. *odd*) if T and l have an even (resp. odd) number of intersections. The definition does not depend on the choice of l and is a shape invariant of T.

An algebraic curve is even if and only if it has even degree, and orientable models are even.

The next lemmas enable us to consider only connected models:

Lemma 3.6 *Let T be a topological model in $\mathbb{RP}^2$. Then all the connected components of T are orientable except for may be one of them.*

Proof. This comes from the fact that two pseudo-lines in $\mathbb{RP}^2$ do necessarilly intersect, so T cannot have two different non-orientable connected components. □

Lemma 3.7 *Let T be a topological model in $\mathbb{RP}^2$ and suppose that each of its connected components $T_1, \dots, T_k$ is realizable by an algebraic curve f_i of degree d_i, and such that if the component T_i is orientable then its realization f_i does not touch the infinity line.*

Then the whole model T can be realized algebraically with degree $\sum d_i$.

Proof. the condition that the realization of the orientable components is made without touching the infinity line implies that we can make these realizations as small as we want, just contracting homotetically the affine part of $\mathbb{RP}^2$. We can afterwards translate this contracted curves anywhere in $\mathbb{RP}^2$ by a projective translation, and none of these two operations will change the degree of the curve.

To realize the whole model T we realize firstly the non orientable component (if there is one) and then place the realization of each of the orientable components, sufficiently reduced, in the appropiate place to have a curve with the same shape of the topological model. This curve –the product of the curves realizing each component– will have degree $\sum d_i$. □

From now on we will suppose that our topological model $T \subset \mathbb{RP}^2$ is connected, and that it is not a single point (for a single point is trivially realized by a degree 2 curve). We will suppose also that it has only double singularities.

Definition 3.8 Let P be a singular order-2 point of a topological model T. A ***flip*** of T at P is a topological model obtained topologically dissipating the singularity of T at P, i.e. replacing a little disc around the crossing point P by a disc with two disjoint arcs. There are two possible flips at P, up to shape equality, as in figure 3 below.

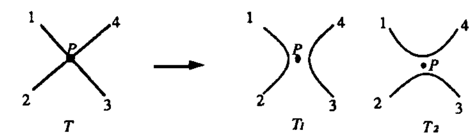

Figure 3

Lemma 3.9

i) If T is connected, at least one of the two flips at a given vertex P is connected.

ii) If T is even (odd) then any flip of T is even (odd).

Proof.

i) Let us call '1', '2', '3' and '4' the four edges reaching to P, numbered in a circular order, and let T_1 and T_2 be the two possible flips at P, as shown in figure 3.

If T_1 is not connected, the component of T_1 which contains '1' and '2' is Eulerian and thus starting by '1' we can arrive to '2' by a path outside the small neighbourhood in which the flip is made. Thus, '1' and '2' are connected in T by a path not passing by P, and T_2 is connected.

ii) It is straightforward because the line in the definition of the parity of T can be chosen not intersecting the small disc in which the flip is made. □

Proposition 3.10 *By a suitable sequence of flips in the singular points of T we can transform it to a non singular topological model T_0 which is a pseudoline if T was odd and an oval if T was even.*

Proof. It suffices to make a flip in each singular point of T until we have a non singular topological model T_0. Lemma 3.9 ensures that these flips can be chosen such that the final model is connected and has the same parity than T. Now, a nonsingular connected topological model is either a pseudoline (which is odd) or an oval (which is even) so the result is proved. □

With this we can sketch the complete construction of algebraic curves with only double points, as follows:

We begin with a topological model T that we may suppose connected by Lemma 3.7. If T is non singular, or a single point, then we can trivially realize it by a degree 1 or 2 algebraic curve. If it is not, by a sequence of flips we can get from it a nonsingular connected model T_0 (by Lemma 3.9) which will be either an oval or a pseudoline (depending on the parity of the original model T). In each flip made we join with a line (that we shall call a 'flip line', and draw in grey in the figures) the two nonsingular branches that appear, as in figure 4. We will call T_0, together with these flip lines, the *skeleton* of the topological model T and will notate it $T^{\sim}$. Up to shape equality we can suppose that T_0 is the X-axis if T was odd, or the unit circle if T was even, and the only thing we know about the flip lines in the skeleton is that they are simple arcs joining two different points of T_0, and that these arcs do not intersect each other nor T_0 (see figure 4 for an example with T even).

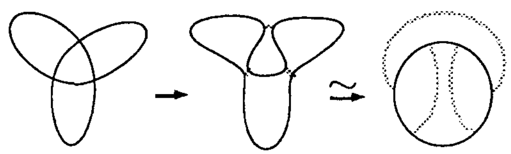

Figure 4

The line or circle T_0 can be algebraically realized with degree 1 or 2, and to recover the shape of the original model T from $T^{\sim}$ it suffices to make the inverse operation of a flip along each flip line. To make this, algebraically, we will insert along each flip line an algebraic curve with the shape of an 'eight' (as in figure 5.a) and then perturb the product of T_0 and these 'eights' (as in figure 5.b). To use Theorem 2.7 for the perturbation we need that:

- each 'eight' will be tangent to T_0 at the ends of the flip line. Apart from these tangencies (which will be A^- type singulariries in the product) the 'eights' will not intersect T_0 nor the other 'eights'.
- the only singular point in each 'eight' will be a 2-fold nondegenerate point, and the tangency points of the 'eights' with T_0 will be of type A^-.

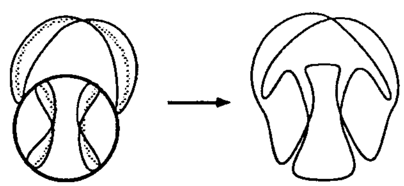

Figure 5

To ensure that the product of T_o with the 'eights' has degree lower or equal than $4n+2$ (with n the number of double points in T), we must construct each 'eight' with degree 4 and then make the perturbation without increasing the degree. In §4 we are going to study the construction of the 'eights', and in §5 how to make the perturbation.

4. Construction of the 'eight' curves.

Throughout this section T will be a connected topological model with only order 2 singular points, $T^{\sim}$ a skeleton obtained from it by a sequence of flips, and $T_0 \subset T^{\sim}$ the X-axis if T is odd and the unit circle if even.

Lemma 4.1 *Let $T^{\sim}$ be the skeleton of a topological model. Then the flip lines that represent the flips made in T can be put, by an isotopy, in such a way that they cross the infinity line transversally and no more than once each.*

Proof. Firstly, by a small isotopy of the flip lines we can make them transversal to the infinity line. Now suppose that $T^{\sim}$ has minimal number of crossings with the infinity line among the skeletons that can be obtained from $T^{\sim}$ by an isotopy. We will show that under this assumption each flip line of $T^{\sim}$ crosses the infinity line at most once.

If there is one flip line that crosses the infinity line more than once, let P and Q be two such crossing points, consecutive along the flip line. The arc from P to Q in the flip line divides the affine part of $\mathbb{RP}^2$ in two open regions, one of them containing T_0 (the X-axis or the unit circle) and the other one (we denote it by U) not intersecting it. By an isotopy in $\mathbb{RP}^2$ we can make U and the arc of flip line PQ traverse the infinity line, thus decreasing the number of crossings of the skeleton with the infinity line. That contradicts the hypothesis. □

Lemma 4.2 *Suppose that the skeleton $T^{\sim}$ has only one flip line and let P and Q be the points of T_0 joined by it. Then, an isotopy on the skeleton can put it in one of the four different dispositions shown in figure 6(i) and figure 6(ii) for the even and odd cases, respectively.*

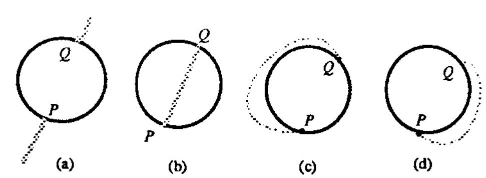

Figure 6(i)

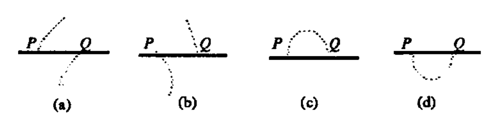

Figure 6(ii)

Proof. In the odd case, $\mathbb{RP}^2 \setminus T_0$ is simply connected, so the isotopy class of the flip line depends only in the fact that it goes upwards or downwards at P and Q, and that gives the four possibilities (a), (b), (c) and (d). In the even case, the flip line must be either interior or exterior to the circle. If it is interior the only possibility is (b), for the disc is simply connected; if it is exterior and does not cross the infinity line the two possibilities are that it goes 'clockwise' or 'anticlockwise' from P to Q, and that gives (c) and (d). Finally if it crosses the infinity line we can suppose that it does only once (because of Lemma 4.1), and it is of type (a). □

This fact can be generalized to any number of flip lines, in the following result:

Proposition 4.3 *the shape of the skeleton $T^{\sim}$ is characterized by the following data: the parity of T; the (ordered) pairs of points in the X-axis or the unit circle joined by flip lines; and a letter 'a', 'b', 'c' or 'd' for each pair of points to say which is the disposition of the corresponding flip line respect to the X-axis in the odd case or the unit circle in the even case.*

Proof. As in Lemma 4.2, once we know whether the skeleton is odd or even (that is, whether T_0 is the X-axis or the unit circle), for each pair of points joined by a flip line we will have the four choices in figure 6 . Note that 'the pair of points' means actually 'the ordered pair', because otherwise possibilities 'c' and 'd' of the even case would be the same one. □

Corollary 4.4 *the shape of a skeleton $T^{\sim}$ with n flip lines can be combinatorially characterized by the following data:*
- *its parity;*
- *a list which contains twice each number from 1 to n, and*
- *a letter 'a', 'b', 'c' or 'd' associated to each number from 1 to n.*

Proof. Let us number the flip lines in $T^{\sim}$ with the numbers from 1 to n. Then, walk along T_0 (from left to right if it is the X-axis, and in counterclockwise sense with an arbitrary starting point if it is the unit circle), and make the list of the $2n$ extremal points of the flip lines as they appear along T_0, denoting each point with the number of its flip line. That produces a list which contains each number from 1 to n twice. The letter associated to each number indicates the type of the flip line, as in Proposition 4.3, choosing as order for the pair of points joined the order in which they appear in the list. This characterizes the shape of the skeleton, because in Proposition 4.3, what is important is not which are the actual points of T_0 joined by flip lines, but its order along the X-axis (or their circular order along the circle), and this information is contained in our list. □

Remarks 4.5:

(i) The shape of the original topological model T can be recovered from the skeleton $T^{\sim}$, and so the data described characterizes its shape too. As mentioned in Remark 3.3.(iv) we can consider these data as the input for an algorithm of construction of algebraic curves (connected and with only double points).

(ii) Not all the data structures of the form described in Corollary 4.4 are the data associated to a suitable skeleton. There are some extra conditions that we will not specify and that are imposed by the fact that the flip lines can not intersect one another.

(iii) the data characterazing the shape is not unique, because there are some arbitrary choosings (the numbering of the flip lines and the starting point to make the list) and, what is more important, we can have the same shape with apparently different disposition of the flip lines in the skeleton. In fact the four drawings for the odd case in figure 6 have the same shape, as well as 'c' and 'd' in the even case.

We can use the combinatorial characterization of the shape of $T^{\sim}$ shown in Corollary 4.4 to construct a new skeleton with the same shape (we still notate this new skeleton $T^{\sim}$), and in which all the flip lines are going to be line segments or arcs of conics, and that will enable to insert the 'eights' we need along them. In fact only very special arcs of conics are going to be needed:

the even case:

We consider first only the 'a' flip lines (the ones that cross the infinity line). Because these flip lines do not cross each other, the two points joined by each of them must be opposite in the circular ordering of all of them, so we can place these points to be actually opposite in the unit circle, and the flip lines joining them to be segments of straight lines, crossing the infinity and normal to the unit circle (figure 7.a).

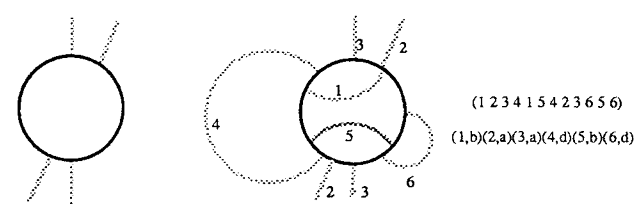

Figure 7

Then we can place the rest of the points in the circle compatibly with their ordering in the list, and take as flip lines joining each pair the (unique) arc of circle normal to the unit circle which joins them in the corresponding region (inside the circle for 'b' lines and outside for 'c' and 'd'). The flip lines so constructed do not touch each other (figure 7.b).

Just one remark: if there is no 'a' type flip line we must be aware of the distinction between 'c' and 'd' lines, and place the points in the circle such that the arc of normal circle joining them is the right 'c' or 'd' type: for example the two skeletons in figure 8 do not produce the same topological shape.

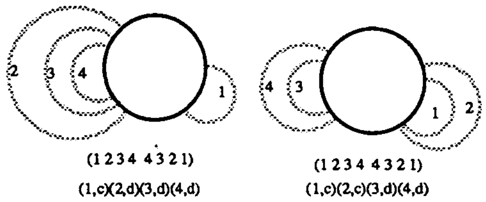

Figure 8

The odd case:

We place the points along the X-axis in the order they appear in the list, and take as flip lines of types 'c' and 'd' semicircles in the appropriate side of the X-axis (up or down), and for those of tipe 'a' and 'b', arcs of hyperbolae with axis $\{Y = 0\}$ (see figure 9). If their assymptotic lines are chosen sufficiently sloped the hyperbolae will not intersect the half-circles, and if they are chosen with the exterior ones more sloped than the inerior ones they will not intersect each other.

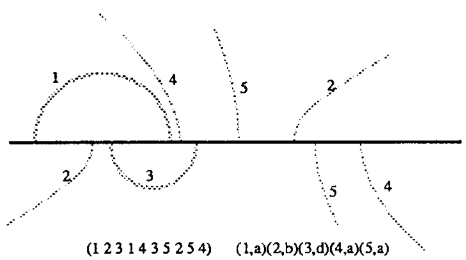

Figure 9

Now we can construct an 'eight' along each flip line with the conditions stated at the end of §3:

Proposition 4.6 *For each of the 'flip lines' described above it is possible to construct an algebraic curve of degree 4 with the shape of an 'eight' tangent to T_0, with a 2-fold nondegenerate singular point, and so close to the flip line as we want.*

Proof. We will consider each case separately:

'c' and 'd' flip lines, in the odd case.

T_0 is the X-axis and, without loss of generality, we suppose that the flip line is the upper half unit circle (for case 'c', case 'd' is analogue).

Let a be a real positive constant, and consider the curve

$$f_0(X,Y,Z) = (X^2+Y^2-2aXZ-Z^2)(X^2+Y^2+2aXZ-Z^2),$$

which consists on two circles with centers at $(a,0)$ and $(-a,0)$, both passing through the points $(0,-1)$ and $(0,1)$ (figure 10.a. Remark that we use affine coordinates for the points not in the infinity line. The third projective coordinate is supposed to be always 1. In the polynomials we maintain the variable Z, to preserve their degree). If we perturb the curve in the form

$$f_\varepsilon(X,Y,Z) = f_0(X,Y,Z)+\varepsilon(Y-Z)^2Z^2 = (X^2+Y^2-Z^2)^2-4a^2X^2Z^2+\varepsilon(Y-Z)^2Z^2,$$

with $\varepsilon > 0$, the resulting curves have the following properties, all easy to check:

i) they have a nondegenerate 2-fold singular point at $(0,1)$,

ii) they do not have any other singular point,

iii) they lie in the region where $f_0(X,Y,Z)$ is negative, i.e. the two small regions between the circles that compose f_0.

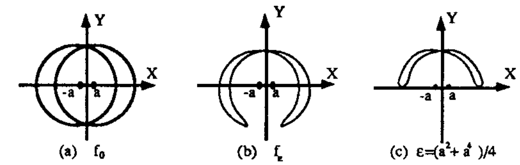

Figure 10

Moreover, they have in fact the shape of an 'eight', as we want:

Call $f = f_0$ and $g = \varepsilon(Y-Z)^2Z^2$. Point $(0,1)$ is a nondegenerate 2-fold point of f which is also a 2-fold point of g and $(0,-1)$ is an A^- singularity of f which is not a zero of g, so Theorem 2.7 describes the perturbation for small ε, which must be as in figure 10.b according to property *(iii)*. Now, if for some $\varepsilon > 0$ the shape of f_ε changes, call ε_0 the infimum of them; thus there exist $\varepsilon_1 < \varepsilon_0 < \varepsilon_2$ as close to ε_0 as we want, and such that f_{ε_1} and f_{ε_2} < have different shapes. Applying Theorem 2.7 to f_{ε_0}, which has only $(0,1)$ as singularity by properties *(i)* and *(ii)*, we obtain that for any ε sufficiently close to ε_0 the shape of f_ε is the same than that of f_{ε_0}, and that gives the contradiction.

Moreover, if the parameter a was chosen sufficiently small the curves f_ε are as close to the unit circle as we want, again because of property *(iii)*; thus, all that we need to do is to find some f_ε which is tangent at both sides to the X-axis and not intersecting it, except in the two tangencies (figure 10.c), and this is obtained by posing that $f_\varepsilon(X,0,1)$ has two double roots, which happens for

$$\varepsilon = (a^2+a^4)/4.$$

'a' and 'b' type lines, in the odd case.

Without loss of generality we suppose that the hyperbola is the standard $X^2 - Y^2 - Z^2$, and by a projective change of variables $X \leftrightarrow Z$ the arc of hyperbola becames a half unit circle and we reduce this case to the last one.

'b', 'c' and 'd' type lines, in the even case.

Now we have the flip line being an arc of circle normal to the unit circle. By affine transformations we can inverse the situation, and suppose that the flip line is an arc of the unit circle, which joins two points of another circle C normal to it, and assume moreover that this latter circle has its center on the Y-axis, and that the flip line is the upper arc of the unit circle. (figure 11.a and 11.b).

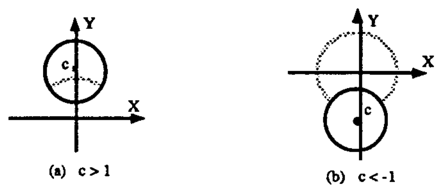

Figure 11

The general equation of such a circle is $C(X,Y) = X^2+Y^2-2YZc+Z^2$, where $(0,c)$ is its center, and $\sqrt{c^2-1}$ its radius. Now the situation is similar to the first case and we can use the f_ε 'eights' that we defined there, except that now we must find an f_ε tangent to the circle C instead of the X-axis. Note that again parameter a in the equation of f_ε is fixed in advance and as small as we need it, and we want to find ε for a and c given. This can be done making $X^2 = -Y^2+2YZc-Z^2$ in the equation of f_ε, which gives a degree two equation on Y. f_ε and C are tangent when this equation has only one, double, solution, and this is so for

$$\varepsilon = (a^2+a^4)(c-1)/(c+2+a^2+1).$$

'a' type lines, in the even case.

Now, without loss of generality, we suppose that the flip line is the part of the X-axis outside the unit circle, and by a change of coordinates $X \to Z$ the flip line becomes the segment $[-1,1]$ of the X-axis, and the unit circle the standard hyperbola. We can take then our 'eight' to be $(X^2+Y^2/a^2)^2 - X^2*Z^2+Y^2*Z^2/a^2$, which is an 'eight' along the segment of height $2a$. □

5. Perturbing the curve.

We have T a topological model connected and with n double points, T_0 the X-axis if T is odd and the unit circle if T is even, and T^{*} the skeleton of T, that is T_0, with n flip lines joining points of T_0, in such a way that T^{*} is isotopic to the result of making a flip in each double point of T. By the previous results (Proposition 4.6) we suppose that we have constructed n algebraic curves of degree 4 $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ with the shape of 'eights' along the flip lines, disjoint and tangent to T_0, and with a real nondegenerate 2-fold singularity in each. Call f the product of T_0 with the ϵ_i, which is an algebraic curve of degree $4n+1$ or $4n+2$ in the odd and even cases respectively. Our task now is to perturb f (as we did in figure 5 to recover the shape of T:

Proposition 5.1 *In these conditions, there exists a curve g of the same degree than f, and such that:*

(i)g has a singular point at the nondegenerate singular point of each 'eight'.
(ii)g does not pass by the tangency points between the 'eights' and T_0.
(iii)In a neighbourhood of each tangency point, the sign of g coincides with the sign that f takes in the region inside the 'eight'.

Proof. We do it separately for the even and odd cases.

the odd case:

Let us consider first the sign conditions for g: Let $P_1, \ldots, P_{2n}$ be the $2n$ tangency points of the n 'eights' with the X-axis, numbered in the order they appear along the axis. We will call *sign of P_i* the sign that f takes in the interior region of the 'eight' at P_i, i.e. the sign that we want g to take at P_i. We will say that there is a *sign change* between P_i and P_{i+1} if P_i and P_{i+1} have opposite sign.

Let $s \leq 2n-1$ be the number of sign changes along the X-axis. We can construct a curve g_1 of degree s which has the appropiate sign in each P_i as the product of s vertical lines, each crossing the X-axis in one of the sign changing intervals. Let us complete g_1 with $2n+1-s$ vertical lines at the right of P_{2n} (or the left of P_1) to get a curve g_1 of degree $2n+1$ with the appropiate sign at each P_i and not passing by the tangency points.

The condition of g having a singular point in the n nondegenerate double points of the 'eights' is easy to achieve multiplying g_1 by the $2n$ degree factor

$$g_2 = \prod_1^n ((c_iX - a_iZ)^2 + (b_iX - a_iY)^2 + (b_iZ - c_iY))^2.$$

where (a_i, b_i, c_i) are the coordinates of the n nondegenerate points of the 'eights'. The factor g_2 is strictly positive everywhere except in the nondegenerate points, so it does not altere the sign of g_1 outside these points, nor makes the product $g = g_1g_2$ pass by the tangency points.

the even case:

Now f has degree $4n+2$.

We can use the same notation that in the odd case except for the fact that the points $P_1, \ldots, P_{2n}$ are now in circular order along the unit circle, and that we must consider an eventual sign change between P_{2n} and P_1. The number of sign changes along the circle will be $s \leq 2n$, and even. Let us consider g_1 be the product of $s/2$ lines, each crossing the circle in two of the sign changing arcs. Again, these lines do not pass by the tangency points, and g_1 has in each P_i the sign that g must have. We complete g_1 to have degree $2n+2$ multiplying it by $2n+2-s/2$ lines not intersecting the circle, and take g_2 in the same way than in the odd case, to obtain $g = g_1g_2$, of degree $4n+2$. □

Proposition 5.2 *In the conditions of Proposition 5.1 the perturbed curve $f+\varepsilon g$ of f by g with ε sufficiently small and positive has the same topological shape than the topological model T. Moreover, if T was orientable, then $f+\varepsilon g$ does not intersect the infinity line.*

Proof. The singular points of f are only the real-nondegenerate singular points of each of the 'eights' and the tangency points of the 'eights' with T_0, which are of type A^-. Besides, g satisfies all the conditions needed to apply Theorem 2.7, so the perturbed curve for ε small has the same shape of f but with the tangency points dissipated. ε being positive, the dissipation of the tangency points is made towards the regions in which f and g have opposite signs, i.e. joining the 'eights' to T_0. That produces along each flip line the topological effect of reversing the 'flip' made to obtain the skeleton T^{*} from T, so the perturbed curve $f + \varepsilon g$ has the same shape than the original topological model T.

If the topological model was orientable, then its skeleton is also orientable, because flips do not affect orientability. It must then be even and have no 'a' type flip lines. In this case the 'eights' are all constructed along arcs of circle, so they do not touch the infinity line. For ε small the perturbed curve will still not touch it. □

This directly implies our main theorem:

theorem 5.3 *Any topological model T in $\mathbb{RP}^2$, connected and with only double points, can be realized by an algebraic curve of degree $4n+2$ if it is even or $4n+1$ if it is odd, where n is the number of (topologically) singular points of T. Moreover if T is orientable its realization can be made without touching the infinity line.* □

Corollary 5.4 *Any topological model T in $\mathbb{RP}^2$ with only double points can be realized by an algebraic curve of degree*

$$\sum_{i=1}^{K}(4N_i + 2) = 4N + 2K,$$

if it is even, or

$$\sum_{i=1}^{K}(4N_i + 2) - 1 = 4N + 2K - 1,$$

if it is odd, where the N_i are the singular points in each component of T, K the number of components and N the total number of singular points.

Proof. If T is even then all its connected components are even and can be realized with degree $4N_i + 2$ each. If T is odd then one, and only one, of its components is odd and can be realized with degree $4N_i + 1$. Lemma 3.7 directly gives the result. □

For the affine plane we can say:

Corollary 5.5 ***Any compact topological model T in $\mathbb{R}^2$ with only double points can be realized by an algebraic curve of degree $d = 4N + 2K$ where N is the number of singular points in the model and K its number of connected components.***

Proof. the compactness of T in $\mathbb{R}^2$ implies that the corresponding projective model does not touch the infinity line and thus is orientable. Its projective realization with degree $4N + 2K$ can be made without touching the infinity line, and that gives in fact an affine realization of T.□

6. Some final questions.

6.1 Algorithmic remarks.

All along the construction we have not worried about asserting things like "*...for a sufficiently small ε ...*" or "*...by a certain isotopy ...* which make the construction seem not so constructive. Here we are going to analyze in a bit detail each step made to see the difficulties that arise, for example, to implement it in a machine:

*a)*How can we obtain the skeleton T^{*} in its final form (i.e. with the flip lines being straight line segments and arcs of conic) from the topological model T?

To answer this question we would need to know exactly how the topological model is given to us as an input. Nevertheless, any combinatorial characterization of the shape of T will be easily translatable to the data given in §4 (cf. Corollary 4.4 and Remark 4.5.(i)), and from these data we can obtain the arcs of conic and segments which form the skeleton, as we did there: for the odd case we can just take the coordinates of the touching points for the flip lines along the X-axis to be the first $2n$ integers, and then find the suitable half-circles and half-hyperbolae to join them. The even case requires a bit more care with the choosing of the points.

*b)*Can we effectively find the eights along the 'flip lines', and the perturbing curve g?

The proof of 4.6 gives a method to find 'eights' "*as close to the flip line as we want, if parameter a is chosen sufficiently small*": the problem is how small needs a to be. a is, basicly, the maximal separation of the 'eight' from the flip line, and the maximal separation permitted can be taken as half the minimum distance from one flip line to another; this minimum distance is not difficult to find from the construction of the 'flip lines'.

To obtain the perturbing curve g is easy: in Proposition 5.1 g is the product of g_1 and g_2; to find g_2 we just need to know the coordinates of the nondegenerate singular point of each eight, and to find g_1, a point in each of the 'sign changing' intervals of f in the X-axis or the unit circle. Both things are easily found either by construction of the 'eights' or by standard methods.

*c)*Finally, how small does the ε in the perturbation $f + \varepsilon g$ need to be?

There are two possible ways to find a bound for ε: either one makes a deeper theoretical study of the perturbation $f + \varepsilon g$ in the particular case of the curves f and g obtained in our construction, which seems rather difficult, or proceeds as follows:

We know that any ε sufficiently small and positive will work. Let ε_0 be the infimum of the ε's that will not. Then $f+\varepsilon_0 g$ has different topological shape from $f+\varepsilon g$ for any $0<\varepsilon<\varepsilon_0$ and this would not be possible (because of Theorem 2.7 on perturbations) if $f+\varepsilon_0 g$ has only the n nondegenerate 2 fold points of the 'eights'. Thus, we must look for an ε_0 such that either some new singular points appear in $f+\varepsilon_0 g$, or some of the nondegenerate 2-fold points become degenerate, or of higher order.

The second thing is easy to check: it suffices to develop $f+\varepsilon g$ as a Taylor polynomial around each nondegenerate singular point, and study the degree two term. The first one is more difficult, but can be made as follows: a singular point of $f+\varepsilon g$ is a common zero of it and of $f_X+\varepsilon g_X$ and $f_Y+\varepsilon g_Y$, i.e. a point where

$$f/g = f_X/g_X = f_Y/g_Y = \varepsilon.$$

Such points are exactly the common zeroes of $fg_X - gf_X$ and $fg_Y - gf_Y$ (exception made of the singular points in the 'eights' where $f=g=f_X=g_X=f_Y=g_Y=0$), and can be found by standard methods in real algebraic geometry. It suffices then to evaluate f/g in all these points and take for ε_0 the lowest positive one. Note that in fact we require only a lower estimate of the value of ε_0 and thus numerical methods can be used to find the points, if we control the errors.

6.2 How good the bound obtained for the degree is?

We would like to know if the construction that we have presented here gives optimal degrees for the realization of an algebraic curve with a given topological shape. The only thing we can do is to compare it with two lower bounds, one coming from Bezout's Theorem, and the other from a particular example:

Suppose that a curve f with only 2-fold singularires has no vertical tangent at any of its singular points. Applying Bezout's Theorem to it and to its derivative f_Y we obtain $d(d-1)\geq 2n$, where d and n are the degree and the number of singular points of f. If f is irreducible the bound can be lowered to $(d-1)(d-2)\geq 2n$. Moreover, both bounds are reachable: for example, for any number n such that $n=d(d-1)/2$ for some d, the product of d different lines has n double nondegenerate points. We can conclude:

Proposition 6.1

(i) Any topological shape with only double points needs at least degree sqrt2n to be realized by an algebraic curve.

(ii) For any given number d there exists a connected topological shape with $n=d(d-1)/2$ double points that can be realized with degree $d\simeq\sqrt{2n}$. □

This seems not to be very good for us, because in these formulae the needed degree d increases with the root of n, and in ours linearly with n. Nevertheless, the possible shapes that make Bezout's inequality to be an equality are probably very few among all the possible shapes with n double points.

In fact, figure 12 shows a curve with 3 double points that can not be realized algebraically with degree lower than 8, and the example can be generalized to any given number of singular points and to non connected curves:

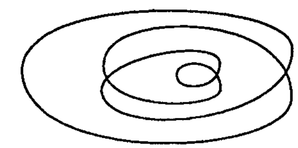

Figure 12

Proposition 6.2

(i) For any given number n there exists a connected topological shape with n double points that can not be realized by an algebraic curve f degree lower than $2n+2$.

(ii) For any given sequence of numbers $n_1,\ldots,n_K$ there exists a topological model with K connected components having $n_1,\ldots,n_K$ double points respectively that can not be algebraically realized with degree lower than $d=2\sum_1^K n_i+2K$.

Proof. *(i)*Consider a chain of $n+1$ tangent circles, one inside the next one. This cannot be realized with degree lower than $2n+2$ because any line passing by the interior of the inner circle has $2n+2$ intersections with the curve (in figure 12, $n=3$).

*(ii)*Consider for each n_K the topological model of *(i)* and then nest them one inside another.□

This later bound is reasonably closer to the bound found in the construction, and in fact shows that the actual bound for the minimal degree realization of *every* topological shape with n double points is linear in n (for a fixed number of connected components), although we do not know if the factor is 2 or 4. As a conjecture we would say that the factor is 2 and that the actual reachable bound is that of Proposition 6.2. The reason for this conjecture can be found in GS . There, another method for costructing algebraic curves with given shape is shown (in the real plane instead of the projective one), that gives exactly that reachable bound. Roughly speaking the reason for which this construction decreases the factor 4 to 2 is that it makes a deeper topological pre-processing of the model, which permits to use degree 2 ellipses instead of 'eights' to rejoin the points separated by a 'flip'. Nevertheless tat construction is not proved to work in all cases.

6.3 What about multiple points?

We have said that our construction would not work so well with singularities of order bigger than 2. Here we are going to see how we could make such a construction, and where are the bigger problems to make it work.

Suppose that we have a topological model T with multiple points. We can suppose it connected because Lemma 3.7 also works with multiple points. We can define flips for multiple points in the same way that we did for double points, but the first problem that arises is that now we are not sure that we can make flips conserve connectivity (see figure 13, and compare with Lemma 3.9).

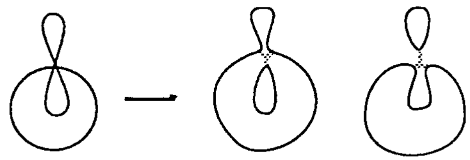

Figure 13

So, even if T is connected, the nonsingular model T_0 obtained by flips on T may have to be non connected.

This may not be a big trouble because we know how to construct any non singular model, even if it is not connected, as a product of circles (and eventually a line, in the odd case), and have the degree of this realization bounded by twice the sum of the indices of the singular points (a flip in a index i vertex produces at most i connected components).

The problem comes when we want to rejoin the curve along the flip lines obtained, which will be in fact 'stars' with as many rays as the index of the singular point was. An algebraic way to make it would consist in inserting at the vertex of each flip star a curve with the shape of an "i-petals flower" as shown in figure 14.a. Such curves can be obtained with degree $i+1$ (for odd i), or $i+2$ (for even i), but we do not know whether we can place the skeleton in an adecuate form to make them be tangent to T_0 in the i appropriate branches.

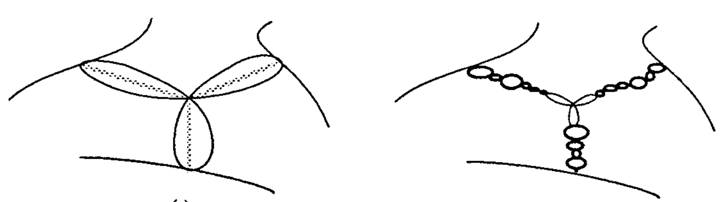

Figure 14

One solution exists, but we must forget our pretension of having the degree of the final curve somehow bounded: we can draw each 'flower' sufficiently small to not touch T_0 and then join each petal to the corresponding branch of T_0 by a chain of tangent circles, as shown in figure 14.b. It is possible to prove the existence of such a chain of circles, but we do not know how to bound the number of circles needed.

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