Very rough draft.

# Improved counterexamples to the Ragsdale conjecture.

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**Abstract**

A longstanding conjecture by Ragsdale ([5]) said that a real algebraic plane curve of degree $d$ could not have more than $\frac{3}{8}d^2 \pm O(d)$ positive ovals. The conjecture was disproved by Itenberg ([3]), who constructed curves with $\frac{13}{32}d^2 \pm O(d)$ positive ovals. B. Haas has recently improved the result to $\frac{10}{24}d^2 \pm O(d)$ and an upper bound of $\frac{7}{16}d^2 \pm O(d)$ can be derived from Rokhlin's theorem. In this note we improve their construction to obtain $\frac{17}{40}d^2 \pm O(d)$.

KEYWORDS :

## 1 Introduction

Throughout this work we will use the words *algebraic curve* as an abbreviation for real non-singular algebraic plane curve i.e. for a real homogeneous polynomial in three variables with no critical points and considered up to a constant factor. It is well-known that the topology of such a curve is that of a certain number of topological circles embedded in the projective plane. Moreover, if the degree $d$ of the curve is even, all the circles are embedded two-sidedly (they are called *ovals*) and if $d$ is odd there is exactly one which is embedded one-sidedly. The first part of Hilbert's $16^{\text{th}}$ problem, regarded in a wide sense, asks to study the possible arrangements of the ovals of real algebraic plane curves, for each degree.

The first general result on this subject is Harnack's Theorem [2]: for any degree $d$, an algebraic curve with $\frac{(d-1)(d-2)}{2} + 1$ ovals exists and no algebraic

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curve can have more than $\frac{(d-1)(d-2)}{2}+1$ ovals. Also interesting for us will be the Petrovski inequalities [4] (later improved by Arnol'd). Let $F$ be an algebraic curve of even degree $d=2k$. Assume that the sign of the curve is chosen so that it is negative in the non-orientable connected component of $\mathbf{RP}^2 \setminus F$ and call positive (resp. negative) ovals of $F$ those which bound a positive (resp. negative) connected component of $\mathbf{RP}^2 \setminus F$ (so that the most exterior ones are positive). Denote by $p$ and $n$ the numbers of positive and negative ovals of $F$, respectively. Then

$$p-n \leq \frac{3k^2-3k+2)}{2} \qquad n-p \leq \frac{3k^2-3k}{2}.$$

Joining together Harnack's and Petrovski's inequalities one obtains the following inequalities for the numbers of positive and negative ovals:

$$p \leq \frac{7k^2-9k+6}{4}, \qquad n \leq \frac{7k^2-9k+4}{4}.$$

On the other hand, a longstanding conjecture by Ragsdale [5], recently disproved by Itenberg [3], proposed the inequalities

$$p \leq \frac{3k^2-3k+2)}{2} \qquad n \leq \frac{3k^2-3k}{2}.$$

In this paper we will be interested in the gap between Ragsdale conjecture (and the counterexamples to it given by Itenberg [3] and Haas [1]) and the Petrovski-Harnack inequality, for the number of positive ovals (similar things can be said for negative ovals, as done in [3], but we will skip this). Clearly, the most interesting part in the inequalities is the term in $k^2$. Thus, we can more compactly write Petrovski-Harnack inequality as $p \leq \frac{7k^2}{4} \pm O(k)$ and Ragsdale conjecture as $p \leq \frac{3k^2}{2} \pm O(k)$. In this setting, Itenberg counterexamples to the Ragsdale conjucture give curves with $p = \frac{13k^2}{8} \pm O(k)$ and the improvement by Haas produces curves with $p=\frac{10k^2}{6} \pm O(k)$. In this paper we construct curves with $p=\frac{17k^2}{10} \pm O(k)$.

**Theorem 1.1** *For any integer $k$, there exists a non-singular real projective plane algebraic curve with .......... positive ovals.*

# 2 Viro's Theorem and previous counterexamples to Ragsdale

The most important step forward in the construction of algebraic curves with controlled topology was done by Viro in the eighties. Viro's theorem is much more general than we are going to state it, but our statement is sufficient for our purposes.

Regular triangulations will play an important role. Given a convex polygon $P$ in the plane, a triangulation of it with set of vertices $S$ is said to be *regular* (sometimes called *convex* or *coherent*), if there exists a lifting $S'$ of the vertices into 3-space such that the triangles in the triangulation of $P$ coincide with the projections of the faces of the lower envelope of $S'$.

Let $T$ be the square with vertices $(0,d)$, $(d,0)$, $(0,-d)$ and $(-d,0)$ and let $t_1,\dots,t_n$ be a triangulation of $T$, obtained from a regular triangulation of the triangle $T\cap(\mathbf{R}_{\geq 0})^2$ by symmetries respect to the $X$ and the $Y$ axis. Give signs to the vertices of the triangulation in a way compatible with the reflections, meaning by this that if $(a,b)$ is one of the vertices and $\epsilon$ the sign given to it, then the sign of $(-a,b)$ is $(-1)^a\epsilon$ and the sign of $(a,-b)$ is $(-1)^b\epsilon$. For each triangle $t_i$ with its three vertices not having the same sign consider a line segment $l_i$ joining the midpoints of the two sides of the triangle in which the sign changes. Call $L$ the union of these line segments.

**Theorem 2.1** *In the conditions above, if we identify opposite points in the boundary of $T$ to obtain a topological space $T^*$ homeomorphic to the real projective plane, the image $L^*$ of the union of segments $L$ is a curve embedded in $T^*$ and it is isotopic to some real, nonsingular, algebraic plane curve of degree $d$.*

From the statement it is clear that if we want to construct a curve using the Theorem, it will be sufficient to triangulate and choose signs for vertices in the first quadrant, and then extend the triangulation and signs to the other quadrants following the rules. We will normally suppose that the triangulation of $T$ is *primitive* in the terminology of Itenberg, meaning by this that every integer point in $T$ is a vertex of the triangulation or, equivalently, that all the triangles are of area $1/2$.

In 1993, Itenberg [3] used this same version of Viro's Theorem to find counterexamples to the Ragsdale conjecture. Itenberg first defines a certain distribution of signs on the integer points of the square $T$ that produces the maximal number $(d-1)(d-2)/2+1$ of ovals permitted and $3k^2-3k+2/2$ positive ovals for a curve of degree $d=2k$, independently of the primitive triangulation of $T$ chosen. In the first quadrant this distribution of signs (that he calls *Harnack distribution of signs*) consists on giving minus sign to a point $(i,j)$ if both $i$ and $j$ are even, and plus sign elsewhere.

After that he considers the hexagon in figure 1, which placed with its center having both coordinates odd makes the number of positive ovals increase by one. He proves that $\lfloor\frac{(k-3)^2+4}{8}\rfloor$ such hexagons can be placed in the Newton triangle of a curve of degree $2k$ and that the triangulation obtained is regular, which finishes its proof. So, its proof is based in the hexagon in figure 1, which has the property that produces a "density of positive ovals" higher than permitted by Ragsdale conjecture (namely, Ragsdale conjacture allows to have $3/4$ of positive oval per area unit in the average, while Itenberg's hexagon produces 13 positive ovals in an area of 16 units).

Very recently, B.Haas has found a way to increase the "density of positive ovals" produced by Itenberg's hexagon. His idea consists in completing the hexagon into a square and then repeating the configuration in the square periodically towards the right and the left to obtain what he calls a "multicarne". This multicarnes have the property of producing 16 positive ovals in an area of 18. Nevertheless, due to the constraint that the center point of the initial square must be placed in an odd-odd oval it is impossible to fill the Newton triangle completely with these multicarnes.

Incidentally, note that a density of 16 positive ovals per 18 area units would violate not only Ragsdale's conjecture but also Harnack-Petrovski inequality. In Haas "packing" of multicarnes only 3/4 of the area can be covered with multicarnes and in the rest he uses Harnack's rule for signs. The combined effect of this two things gives 10 positive ovals per each 12 area units.

Our main new idea here is placing Haas' multicarnes in a way that they fill in 9/10 of the area of the Newton triangle. In the rest of the area we will use Harnack's distribution of signs, which will give us 1 positive oval per each 2 area units, in the average. Joining this two things together we will obtain $9/10 \times 8/9 + 1/10 \times 1/2 = 17/20$, i.e., 17 positive ovals per each 20 area units. Of course we need to prove that our "packing" is a reguar triangulation of the Newton triangle.

# References


[1] B. Haas, Les multicarnes: Nouveaux contre-examples à la conjecture de Ragsdale, preprint.

[2] A. Harnack, Uber Vieltheiligkeit der ebenen algebraischen Curven, *Math. Ann.* **10** (1876), 189–199.

[3] I. Itenberg, Contre-examples à la conjecture de Ragsdale, *C. R. Acad. Sci. Paris*, t. 317, Série I (1993), 277–282.

[4] I. Petrovski, On the topology of real plane algebraic curves, *Ann. Math.* **39**, No.1 (1938), 187–209, Série I (1993), 277–282.

[5] V. Ragsdale, On the arrangement of the real branches of plane algebraic curves, *Amer. J. Math.* **28** (1906), 377–404.

[6] J.-J. Risler, Construction d'hypersurfaces réelles (d'après Viro), *Séminaire N. Bourbaki* n. 763 (1992).

[7] O. Ya. Viro, Gluing of plane real algebraic curves and construction of curves of degrees 6 and 7, in Lecture Notes in Mathematics Vol. 1060, Springer-Verlag, 1984, 185–200.