Representation of Curves in the Real Plane, and Construction of Curves with Given Topology.

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Abstract.- *We are interested in the following problem: if we are given a topological model for an algebraic curve in the real plane (i.e. something which is isotopic to a certain algebraic curve), what is the minimum degree of a polynomial which 'realizes' it?*

In the particular case of the model being compact and with only double points, a superior bound for the needed degree is $4N+2K$, *where* N *and* K *represent the numbers of double points and connected components respectively ([Sa2]), and in the other hand for any* N *and* K *we show examples not realizable with degree lower than* $2N+2K$.

Here we claim that this later is actually the worst-case optimal superior bound, and we show a method to construct the polynomial with this degree from the topological model, although the proof is not complete.

We introduce the notion of 'prime factors' of a curve (which are the essential components in which the curve can be decomposed) and show that these prime factors have good geometrical properties, which we enclose under the name of 'quasiconvexity'. We also study the problem of combinatorially characterizing the topology of a plane curve, and show a data structure appropiate for this characterization, based on the so-called 'Gauss codes'.

1. Introduction.

If we have two subsets A and B in a topological space X, and a global homeomorphism which sends A to B, we say that (A,X) and (B,X) are *topologically equivalent* or that A and B have the same *topological shape* in X. In the context of real algebraic geometry an interesting question is knowing which are the possible pairs $(V,\mathbb{R}^n)$, or $(V,\mathbb{RP}^n)$ up to topological equivalence, with V an algebraic set.

The answer to this question is far from trivial in the general case (see for example [BCR], or [AK]), but simple if we restrict ourselves to the real (affine or projective) plane: any imbedded graph in $\mathbb{RP}^2$ or $\mathbb{R}^2$ with even order (possibly zero) in every vertex has the shape of an algebraic set, and conversely any algebraic set $V \subsetneq \mathbb{RP}^2$ has the same shape that an imbedded graph with even order. For $\mathbb{R}^2$ the characterization is the same except that there can be a certain number (finite and even) of branches going to infinity, and thus the algebraic set can be noncompact.

Nevertheless, the classical proofs of this characterization normally use polynomial approximation of C^∞ functions ([AK]), and thus say nothing about the degree needed to 'realize' a given topological shape by an algebraic curve.

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In [Sa2] we show a construction which works well (both in the projective and the affine plane) if the topological model we want to realize is compact and has only double singular points, and in this case the bound obtained is that every topological model can be realized with degree

$$d \leq 4N + 2K,$$

where N is the number of singular points and K the number of connected components in the topological model. For nonsingular curves this bound gives $d \leq 2K$, which is trivial (for we can construct any non singular model as a product of circles, plus may be a line), but also optimal (if the model consists on K nested ovals it can not be 'realized' by an algebraic curve of degree lower than $2K$, because any line crossing the most inner oval intersects the model $2K$ times).

In the other hand, for any N and K there exist examples of singular curves with N double points which cannot be realized with lower degree then $2N + 2K$, due to topological obstructions:

Let us see first the case of a connected curve, and let N be an arbitrary number of double points. If we construct $N + 1$ circles one inside the next one, two consecutive ones being tangent, the resulting topological model has N double points, and cannot be realized with degree lower than $2N + 2$, because in any realization of it a line passing by the most inner region necessarily cuts the curve in at least $2N + 2$ points. The example generalizes to non connected curves with N double points and K connected components just considering $K - 1$ additional circles inside the inner region and one inside another. We could say even more: for any sequence of numbers $N_1, N_2, \ldots, N_K$, with $\sum N_i = N$, a curve can be constructed with K components each having N_i double points, and not realizable with degree lower than $2N + 2K$ (see figure 1 for an example with 2 connected components and $2 + 3$ double points).

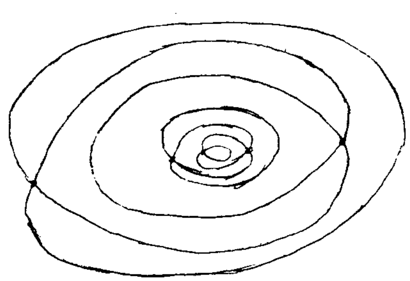


Figure 1

The question is whether these examples are the worst case for each pair of numbers N, K or not. The method discribed in this paper makes us think that they are, i.e. that any compact topological model in the plane with N double points and K connected components can be realized with degree at most $2N + 2K$ (this will be our corollary 7.8); the construction we show would give such a realization, except for some detail that we will remark in section 7 (see conjecture 7.5).

Moreover our results indicate why the examples we have mentioned as worst-cases are indeed worse than others. Proposition 7.7 says that the only connected topological models that possibly need degree $2N + 2$ to be realized are those in which every vertex disconnects the model if we delete it (as it happens in the examples). The rest of connected, double points models can always be realized with degree at most $2N$. (The converse is not true, some models in which every vertex disconnect them can be also realized with low degrees).

In the first part of the work (sections 2 and 3) we abord, as a previous question, the probl of how we can combinatorially characterize the topological shape of a diagram by means of a fin data structure. Such a characterization is necessary if we want to have an algorithm of constructi of algebraic curves with given topology, because the data characterizing the shape would actua be the input for the algorithm.

What we show there is a brief summary of some parts of the coauthors respective works [G and [Sa]:

Section 2 is devoted to introduce our topological representations of algebraic curves, (wl we call a *diagram* is in fact the topological model we will use to make the constructions), a section 3 introduces the data structure we propose to represent their topological shape, based the so-called Gauss-codes.

Other authors have given different solutions to this question: [Roy], [AM], [GT], work ir context closely related to ours: they are given a polynomial (or more) and they give algorith which compute the topological shape of its real zero set, by means of a Cylindrical Algebr Decomposition (the two formers), and seminumerical root finding methods (the later). Nev theless they do not have a good representation of topological shapes. Both Gianni-Traverso a Arnon-McCallum represent non singular curves by some data containing the number and mutu disposition of the connected components of the curve (which are in this case either ovals or line but say few or nothing for the singular case, while M. F. Roy gives for the singular case a da structure which permits to recover the topological shape of the curves, but which is not an inva ant of the shape (in fact it depends even on the cartesian coordinates chosen). This makes ve difficult to know if two such structures correspond to the same topological shape or not.

Guibas and Stolfi ([G-S]) propose, in the context of Voronoi diagrams, a representation means of what they call an algebra of edges, representation which could be applied to algebr curves but seems less appropiate than ours.

The data structure we propose here has the following three good features:

i) it characterizes the topological shape of a diagram (two diagrams with the same code ha the same topological shape).

ii) it is a topological shape invariant, up to certain basic operations roughly consisting permutations of the symbols that compose the code. We can easily compute whether two su codes come from the same topological shape.

iii) it has a good relation with the topology of the curve, in the sense that topological man ulations are well translated to codes.

Sections 4 and 5 deal with the topological manipulations we will need in the algebraic constru tions, and give a self-interesting topological result (proposition 5.4) which is that every connect diagram with only double points that cannot be disconnected by cutting only two (different) edg can be put in quasiconvex form (quasiconvexity is defined in 5.1).

Section 6 shows the algebraic realization of diagrams in the general case (in which we do n know how to bound the degrees), and section 7 in the particular case of diagrams with only doul points.

2. Curves and diagrams.

In this section we use the word 'curve' in its topological meaning, a (closed) curve being th a continuous map from the standard circle into the real plane.

Definition 2.1 A *diagram* is a finite set of topological curves, i.e., a continuous map f from a topological space X into $\mathbb{R}^2$, where X is a finite, disjoint union of circles. We call *vertices* of the diagram the points of $\mathbb{R}^2$ which have more than one inverse image in X, and *order* of a vertex its number of inverse images. We pose to diagrams the following finiteness conditions: they must have a finite number of vertices, each having finite order.

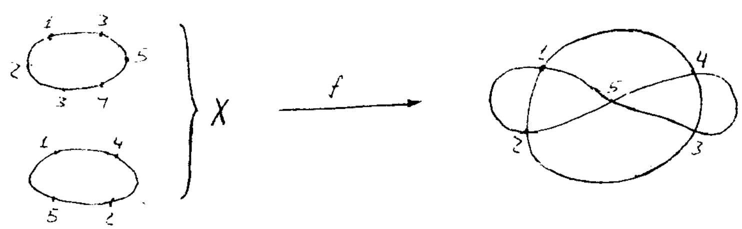


Figure 2: A diagram in the plane.

Sometimes we are going to call diagram not the continuous map but only its image in the plane. With this language flexibility a *diagram* is always a compact subset of the plane and is homeomorphic to a *graph* with even order in all its vertices. Thus every diagram is isotopic to an algebraic curve, and we can consider diagrams as being the topological models of compact algebraic curves: any compact algebraic curve will consist on a (topologically) 1-dimensional part, which is represented by a diagram, and a finite number of isolated points. Isolated points are not important for us, because any isolated point can be algebraically realized by a degree-2 polynomial, and glued into the rest of the curve without affecting the bound $2N + 2K$: each isolated point increases by 2 the degree of the curve, but it also increases by 1 the number K of connected components.

For diagrams we could give a stronger definition of shape than we gave for subsets of the plane, because the $2i$ edges that reach to a given vertex of order i are associated in pairs by the map which defines the diagram, forming what we may call the i (local) branches of the diagram at the vertex (note the analogy with the local branches of an algebraic curve at a singular point). This branches make possible to distinguish between, for example, tangencial and transversal double points, and so we can consider two diagrams whose images are isotopic, not having the same shape as diagrams if their branches do not coincide. A strong definition of shape for diagrams is:

Definition 2.2 Let $f(X)$ and $g(Y)$ be two diagrams in the real plane, f and g being their defining maps. Then we say that f and g have the same *strong topological shape* if there exists an homeomorphism h from the plane into itself such that $h(f(X)) = g(Y)$ and a new homeomorphism t from X into Y such that $h \circ f = g \circ t$.

The condition $h(f(X)) = g(Y)$ is superfluous in the definition, but we include it to make explicit that this new definition of shape is stronger than the old one. Another concept related with the local branches just mentioned is that of transversality:

Definition 2.3 Let $f(X)$ be a diagram in the plane and V be one of its vertices, of order i. We will say that the diagram is *transversal* at V if all the branches of the diagram at V have equal number of edges at each side (this number being necessarily $i - 1$). We say that a diagram is transversal if it is transversal in all its vertices.

If we consider diagrams just as imbedded graphs, thus forgetting that some edges prolong each other, we can not distinguish between transversal and non transversal diagrams, neither between weak and strong topological shape. In fact:

i) Every diagram has the same (weak) shape than one transversal diagram.

ii) Two transversal diagrams have the same weak shape if and only if they have the sar strong one.

Coming back to the relation between diagrams and algebraic curves, the above consideratio give us two canonical ways to associate a 'diagram structure' to a given compact plane algebra curve without isolated points: in the first one we follow the algebraic branches of the curve build the map f, giving a diagram which can be non transversal, and in the second one we cross the vertices transversally, in the sense of our definition. The second procedure gives a transvers diagram which contains only the topological information of the algebraic curve as a subset of t plane (its weak shape), while the first one contains a part of the algebraic information of the cur it says which pairs of topological half-branches form the analytical branches of the curve at ea singular point (its strong topological shape).

Although the first procedure seems more natural to deal with algebraic curves in this wo we are going to adopt the transversal method which has two advantages for our purposes: first it is simpler to deal only with transversal diagrams, and secondly in the algebraic constructio we are going to make we obtain always nondegenerate singular points (which are topologica transversal).

3.Gauss Codes.

In this section we are going to describe the announced characterization of plane curves a diagrams, and see its properties.

The starting point is a coding method for curves described by Gauss ([Ga]): Gauss associat to any *normal* curve in the plane (normal means here having only double transversal vertice the list of the double points of the curve, given in their cyclic order (and thus each double poi appearing twice). If we name vertices with the numbers from 1 to N, where N is the number vertices of the curve, the so obtained Gauss code of the curve is a list containing twice each of t symbols $1, \ldots, N$. Nevertheless it is easily seen that not every list having twice each number fr 1 to N is the Gauss code of a curve in the plane (for example the list $(1, 2, 1, 2)$ is not), so Ga asked what were the necessary and sufficient conditions for such a list being a Gauss code. (He ga the necessary condition of every symbol from 1 to N having exactly an even number of symb between its two appearances, but this condition proved not to be sufficient. Recent authors ha given the complete solutions [RT], [Ros], [LM], [Go]. See also [KMPS] for a recent survey Gauss codes).

We can easily generalize Gauss codes to our diagrams considering, instead of one list, as ma lists as curves form the diagram, a list consisting on the vertices one crosses when moving alo a curve (see figure 3). The set of these lists is the Gauss code of the diagram. Note that one several of the lists in the diagram can be the empty list, if the associated curve is an oval with vertices.

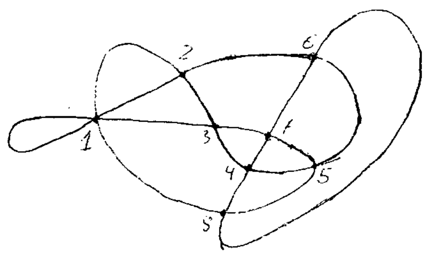


Figure 3: A diagram in the plane with Gauss code (1,2,3,4,5,6,2,1,1,3,7,5,8) (8,4,7,6).

If we want Gauss codes to be an invariant of the shape of a diagram, we must introduce some equivalence relations between codes, because of the arbitrary choices made in the construction process. We say that two codes are equivalent if we can obtain one of them from the other by a finite sequence of operations of the following types:

-Renumbering of the vertices (which corresponds with the choice of the 'names' for the double points).

-Cyclic permutation of the symbols in one of the lists (which corresponds to the choice of an initial point to start the list in each curve).

-Inversion of one of the lists (which corresponds to changing the direction to move along the curve), and

-Reordering of the lists in the code.

Note that if two codes are equivalent, then the maximum number of operations required to obtain one from the other is one of the first and forth types and one of the second and third for each list forming the code, because of the conmutativity of operations of different kinds. This is important because ensures that we can algorithmically construct all the codes which are equivalent to a given one, for example to test whether two diagrams. given by their codes, have the same shape or not.

Gauss codes are now a shape invariant of the diagram up to this equivalence relation. (A strong shape invariant, properly speaking, because diagrams with the same weak shape can have different codes depending on the transversality relations between the branches.) Nevertheless, they do not, in general, characterize the topological shape of a diagram, i.e. the same code has different-shape realizations as a diagram. We need to add some extra information to obtain a shape characterizing code.

We do it as follows: firstly, we choose one of the two possible global orientations of the real plane, and for each vertex of the diagram we number cyclically its $2i$ edges (where i is the order of the vertex), starting by an arbitrary one and following the chosen orientation. Then, we construct the Gauss code of the diagram as we did before, but we include in the code not only the vertex number, but also the edges by which we come in and out of the vertex when moving along the curve. We write the numbers corresponding to these two edges as a subscript and a superscript in the number which represents the vertex (see figure 4).

We call the so constructed code the *extended Gauss code* of the diagram. Again the extended Gauss code is a strong shape invariant up to equivalence of codes if we define two new equivalence operations with codes:

- A cyclic renumbering of the edges in one given vertex, and
- A global orientation change, i.e. an inversion on the cyclic ordering of the edges of all t
vertices.

In figure 4 we have chosen clockwise orientation of the plane and we start numbering the edg from the horizontal-right position (as showed).

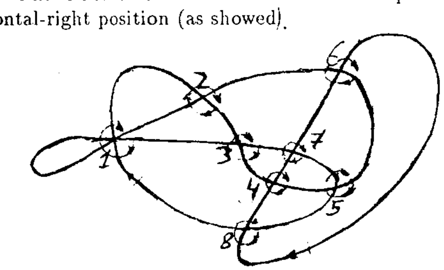


Figure 4. Extended Gauss code: (${}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_45^2\ {}_18^3$) (${}_28^4\ {}_24^4\ {}_27^4$

Obviously, the extended Gauss code contains much more topological information of the di gram than the non extended one, for it includes the local disposition of the edges of a vertex. L us see that it permits for example to recover the cycles that form the boundary of the faces (whi is not true for the usual Gauss codes). We see this in our example: firstly, we can obtain the edg of the diagram by simply breaking the code into pieces in the following way:

$$[1^5\ {}_32]\ [2^1\ {}_43]\ [3^2\ {}_34]\ [4^1\ {}_35]\ [5^1\ {}_16]\ [6^3\ {}_42]\ [2^2\ {}_61]$$

$$[1^3\ {}_41]\ [1^1\ {}_33]\ [3^1\ {}_37]\ [7^1\ {}_45]\ [5^2\ {}_18]\ [8^3\ {}_21]$$

$$[8^4\ {}_24]\ [4^4\ {}_27]\ [7^4\ {}_26]\ [6^4\ {}_28]$$

Now we can recover the cycle of the edges in the boundary of a face (in the anticlockwi sense) starting by an arbitrary edge, say $[1^5\ {}_32]$ and looking for the edge which has its secor vertex (2) with the sub- or superscript which immediately follows in the clockwise sense. In o example, we must look for a 2^4 or a ${}_42$, and that gives us the edge $[6^3\ {}_42]$. We revert this edge $[2^4\ {}_36]$, and glue it to the first one to give $[1^5\ {}_32][2^4\ {}_36]$. We look then for a 6_4, and we find $[6^4\ {}_2$ The cycle follows with $[8^3\ {}_21]$ and ends with $[1^3\ {}_41]$ (because the next edge to this one would l the original $[1^5\ {}_32]$. The obtained cycle is then $[1^5\ {}_32][2^4\ {}_36][6^4\ {}_28][8^3\ {}_21][1^3\ {}_41]$, which is the cyc of the exterior face. We can obtain all the cycles in the same way, the process finishing when eve edge has been taken twice. The complete face cycles obtained for our example are listed below:

$$[1^5\ {}_32][2^4\ {}_36][6^4\ {}_28][8^3\ {}_21][1^3\ {}_41] \qquad [2^1\ {}_43][3^1\ {}_37][7^4\ {}_26][6^3\ {}_42]$$

$$[3^2\ {}_34][4^4\ {}_27][7^3\ {}_13] \qquad [4^1\ {}_35][5^4\ {}_17][7^2\ {}_44] \qquad [7^1\ {}_45][5^1\ {}_16][6^2\ {}_47]$$

$$[2^2\ {}_61][1^1\ {}_33][3^4\ {}_12] \qquad [5^2\ {}_18][8^2\ {}_46][6^1\ {}_15] \qquad [8^1\ {}_25][5^3\ {}_14][4^2\ {}_48]$$

$$[8^4\ {}_24][4^3\ {}_23][3^3\ {}_11][1^2\ {}_38] \qquad [1^4\ {}_31] \qquad [1^6\ {}_22][2^3\ {}_51]$$

Yet the extended Gauss code of a diagram does not characterize completely its shape, and th is for two reasons: firstly, from the extended Gauss code we can recover which are the connect components of a diagram (because that is a part of its graph structure), but not how they are mutually disposed in the plane; secondly, even for connected diagrams, the extended Gauss code does not say which are the exterior and interior parts of the diagram.

We can see this second fact more clearly if we consider the one point compactified of the real plane, which is a sphere. Every plane diagram can then be viewed at as a diagram in the sphere, the sphere having one special point which represents the infinity. The Gauss code of the diagram in the sphere can be obtained in the same way as we did in the plane, but the code does not tell us in which region of the diagram is the infinity point placed. The topological shape of the diagram in the plane depends on this isposition of the infinity point respect to it, so the extended Gauss code cannot characterize its shape. Nevertheless, we can say the following (proof can be found in [Sa):]

Proposition 3.1 *The extended Gauss code of a connected diagram characterizes its strong topological shape as a diagram in the sphere, i.e. two given connected diagrams have the same strong topological shape in the sphere if and only if they have the same extended Gauss code (up to the equivalence relation for codes).* ■

What we do then to make codes characterize the strong shape for plane diagrams? First of all, we build the codes asociated to the connected components of the diagram, including in the code something which tells us which is the exterior face of each component (for example the cycle of edges of the exterior face). Then we can build a rooted tree to represent the disposition of the different components, in the same way as [GT] do for non singular curves (with each component represented by a node in the tree, and the components which are included in others represented below them), and add to the tree some information saying in which face of the inmediately upper component we must place a given one. This finishes the problem of characterizing the shape of diagrams in the plane .

We are going to mention finally the solution to the original Gauss problem applied to our extended Gauss codes, i.e. the decision of whether a given code is realizable as a diagram in the plane. The solution is very simple and generalizes easily to other surfaces.

Definition 3.2 We call an *(extended) gauss-like code* a sequence of lists globally containing all the symbols $1, \ldots, N$ at least twice, each of the symbols having a subscript and a superscript, and with the sub/superscripts of each symbol $k = 1, \ldots, N$ going from 1 to $2i_k$, where i_k is the number of appearences of k (its order).

First of all a gauss-like code is realizable in the plane if and only if each of its connected components is, so we can restrict ourselves to the connected case. Secondly we recall that we know how to get from an extended Gauss code the edges that form the cycle of a face in the corresponding diagram. In particular, we can find the number of faces, because each face of a connected diagram in the plane has only one cycle of edges. We claim that:

Proposition 3.3 *A connected gauss-like code is realizable in the plane if and only if it satisfies the Euler formula $F - E + V = 2$, where F is the number of faces (cycles of edges) that result from the code, V is the number of vertices, and E is the number of edges, which coincides with the total number of vertex symbols composing in the code (the 'code length').* ■

The proof can again be found in [Sa]: necessity of the condition is trivial once we know that the faces of a connected plane diagram are simply connected with the exception of the exterior one which is a ring, while the sufficiency is due to a more general result saying that every gauss-like code can be realized in some compact orientable surface, and the Euler characteristic of the

minimal one that realizes a code is given by the stated Euler formula. Thus a code that satisf the formula can be realized in a sphere, and deleting a point to the sphere, in the plane. Note t the formula is satisfied by our example: $11 - 17 + 8 = 2$.

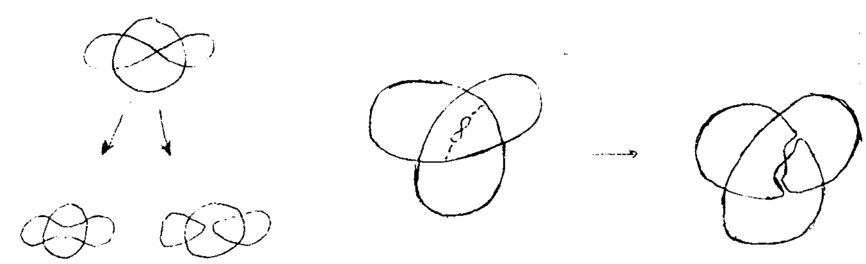
For more detailed descriptions and proofs, and for a generalization of everything concerni extended Gauss codes to compact surfaces, even in the non-empty boundary and in the no orientable cases see [Sa].

4 Flips and Flops. Prime Diagrams.

We are going to begin now the study of the geometrical manipulations on diagrams that w led to the algebraic constructions (and to some interesting topological results also). From now we are going to work only with transversal diagrams, and in some places we will demand th to be connected or to have only double vertices. Being transversal means that we can think diagrams as being just drawings in the plane, and forget the continuous map from which they the image (because of the equivalence between shape and strong shape for these diagrams).

Our aim is to give a method to construct any diagram from a collection of simpler diagra and a sequence of well defined topological operations that transform this simpler diagrams into one we had. The two basic operations we need are one to delete singular points from a diagr and other to add them. We call this operations *flips* and *flops* respectively:

Definition 4.1 To make a *flip* in a vertex of the diagram we take the $2i$ edges of the ver and join them two by two in consecutive pairs, thus making the singular point dissappear. T can be made in two possible ways up to isotopy, shown in figure 5.a. Flips can be easily trea with extended Gauss codes: if we have the code for the original diagram, a flip is characterized the name of the vertex in which we make the flip and some additional information distinguish the two possible flips.



(a) (b)

Figure 5

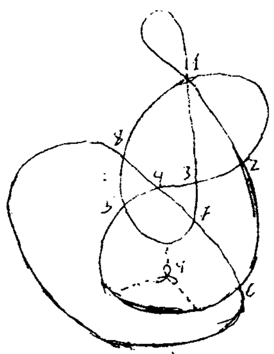
Definition 4.2 *Flops* are the inverse operation of flips. To make a flop we must choose of the faces of the diagram and a list of some (at least two) of the edges which bound this fa An edge may appear more than once in the list, and the total number i of edges in the list will called the *order* of the flop. The geometrical flop is made inserting in the chosen face an i-pet flower (as shown in figure 5.b), and then joining each petal to one of the edges.

If the face is simply connected this can be made in only one way up to shape equality; in ot case we will need some extra information about the 'paths' along which we must place the 'pet of the flower. Nevertheless we are only going to be concerned with simply connected faces; note that in a connected diagram in the plane all the faces are simply connected except for the infinity one.

Both flips and flops can be easily made in the Gauss code that represents the diagram. We show with an example the way to find the code of the resulting diagram of a flop from the old one's code. Consider the diagram of section 3, whose extended Gauss code was

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_45^2\ {}_18^3)\ \ ({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

and suppose that we want to make a flop in the face $[7^1\ {}_45][5^1\ {}_16][6^2\ {}_47]$, joining the edges $[7^1\ {}_45]$, $[5^1\ {}_16]$, and again $[5^1\ {}_16]$ (the geometrical flop is showed in figure 6).


(a)

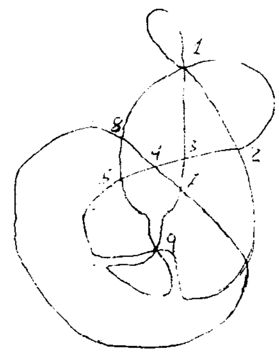

(b)

Figure 6

We give the name '9' to the new vertex, and add as many symbols 9 in each edge of the cycle as its number of appearings in the list; this gives the cycle $[7^1\ \mathbf{9}\ {}_45][5^1\ \mathbf{9}\ \mathbf{9}\ {}_16][6^2\ {}_47]$.
We then add the to the new '9' symbols the subscripts $1,\ldots,6$ in decreasing order:

$$[7^1\ {}_6\mathbf{9}^5\ {}_45][5^1\ {}_4\mathbf{9}^3\ {}_2\mathbf{9}^1\ {}_16][6^2\ {}_47],$$

and put these new 'edges' instead of the old ones in the original code:

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_4\mathbf{9}^3\ {}_2\mathbf{9}^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_69^5\ {}_45^2\ {}_18^3)$$

$$({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

this is the extended Gauss code for a diagram having the required (weak) shape, but which is not transversal. To make it transversal we just break the code in all the appearings of the symbol '9' and reglue the pieces in such a way that each symbol '9' has as subscripts two opposite edges (i.e. two numbers whose difference is 3):

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_49]\ [9^3\ {}_29]\ [9^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_69]\ [9^5\ {}_45^2\ {}_18^3),$$

$$({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

and regluing:

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_49]\ [9^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_69]\ [9^3\ {}_29]\ [9^5\ {}_45^2\ {}_18^3),$$

$$({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

i.e.:

$$({}_21^5\ {}_32^1\ {}_43^2\ {}_34^1\ {}_35^1\ {}_49^1\ {}_16^3\ {}_42^2\ {}_61^3\ {}_41^1\ {}_33^1\ {}_37^1\ {}_69^3\ {}_29^5\ {}_45^2\ {}_18^3),\ ({}_28^4\ {}_24^4\ {}_27^4\ {}_26^4)$$

which is the code for the new diagram.

The way in which flips and flops are used to build up diagrams is the following: we m flips to a given diagram D_0 until we arrive to a simpler one D_k, and in each step $i = 1,\ldots,k$ compute the code of the new diagram obtained by the i^{th} flip D_i, as well as the information recover the code of D_{i-1} from D_i (i.e. the information concerning the inverse flop of the flip). T shape of the final diagram D_k joint to the inverted sequence of flops determines the shape of The choice of the 'simpler' diagram D_k to stop the process depends on our purposes, but clea it is always possible to arrive to a diagram without any vertices (i.e. a collection of ovals), if want to.

For connected diagrams with only double points this flip/flop decomposition of diagram specially useful, because of the following result:

Proposition 4.3 *Let D be a connected diagram in the plane and let V be one of its verti of order 2. Then one of the two possible flips in vertex V leaves the diagram connected.*

Proof: Let '1', '2', '3' and '4' represent the four edges in vertex V in a cyclic order, and l what happens to D when we delete the point V.

If $D\setminus\{V\}$ is connected, then both flips on V are connected. If it is not, each of the four ed at V must be connected in $D\setminus\{V\}$ to another one, because the arc beginning in an 'open edge $D\setminus\{V\}$ must end in an open edge.

Now suppose that one of the flips in V gives a diagram which is not connected, for exam the flip which joins '1' to '2' and '3' to '4'. Then '1' can only be connected in $D\setminus\{V\}$ to (for otherwise the four edges would be connected to each other in the 'flipped' diagram), and t the other flip gives a connected diagram (because it connects '1' to '4' and '2' to '3'). A cour example for higher order points is an *'eight figure'* with an oval crossing its double point (see fig 7).

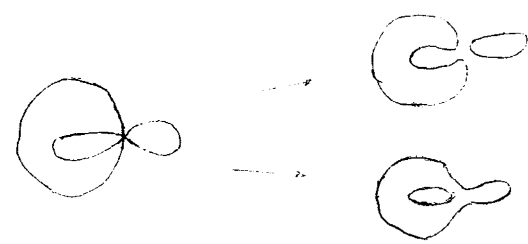


Figure 7

The proposition implies that the final diagram in the flip decomposition of a connected, double points diagram can be always chosen to be connected andnonsingular, and thus a single oval. This fact is used in the construction of algebraic curves with given topology shown in [Sa2] (in fact, there the final diagram can be either an oval or a pseudo-line, because the context in which diagrams are defined is more general).

Nevertheless, here we will prefer to use other diagrams instead of ovals to start the construction, and for that we need to introduce a type of connected, double points, diagrams with good decomposition properties, which we call *prime diagrams*. (Prime diagrams can be defined with vertices of higher order, but the good properties we mention are obtained only for double points).

Definition 4.4 Let D be a connected diagram with only double points. We say that D is *prime* if it cannot be disconnected by 'cutting' only two edges, or equivalently if there do not exist two adjacent faces in the diagram which have two different common edges on their boundaries.

The three main features about prime diagrams are:

Proposition 4.5 *Let Z be a (connected, transversal, double points) diagram. Then:*

i) The non-extended Gauss code of Z characterizes its shape in the sphere (compactified plane). Therefore the information added in the extended codes is irrelevant for this diagrams and their plane shape is determined by the Gauss code and the 'infinity face' additional information.

ii) Let V be an arbitrary vertex of Z. Then at least one of the two flips at V gives a new prime diagram.

iii) If a flop gives as final diagram Z, then the flop is made joining two **different** *edges of the initial diagram.*

In (ii) and (iii) the initial diagrams of both the flip and the flop are assumed to have at least one vertex.

Proof: *i)* The proof of this can be found in [Go]. It is too long to put it here, and in fact this property of prime diagrams, although it may be the main one to express the meaning of being prime, is not relevant to our purposes. We indicate just that the reason why non prime diagrams with the same non extended code can have different shapes (in the sphere) is that one of the parts of the diagram can be turned 'inside-out' as in figure 8, and that does not happen for prime diagrams (a prime diagram can be turned inside out as a whole, but that does not change its shape in the sphere).

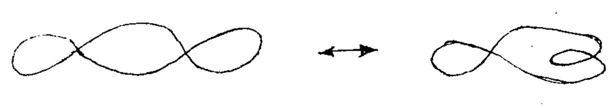


Figure 8

ii) Consider the following sketch of the two possible flips at V (figure 9). Suppose that (a) is prime and both (b) and (c) are not prime, and we are going to arrive to a contradiction.

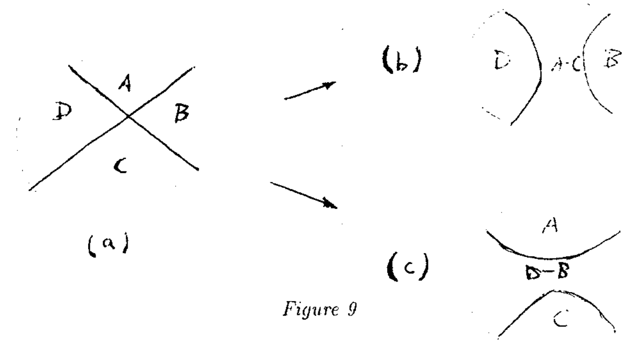


Figure 9

Diagram (b) being not prime, one of the two faces that share two edges must be the midd face $(A \cup C)$, for in other case (a) would not be prime. For the same reason the other face cann be B nor D, so we call it E, E being then a face of the initial diagram which is adjacent to bo A and C. With the same considerations for the horizontal flip we obtain another face F adjace to both D and B, and the following sketch:

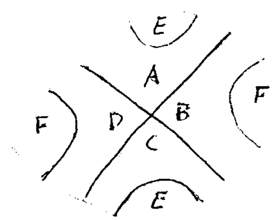


Figure 10

Now the contradiction arrives if we study whether E and F are the same face or differe ones: they can not be the same, for in figure 10 we can find a line that goes from E to F crossi the diagram in exactly three points, and if they are different we can find two lines going from to E and from F to F respectively, and crossing each other in exactly one point. Both thin are impossible because transversaly crossing curves in the plane must have an even number intersections (we recall that our diagram is a finite union of curves).

iii) It is easy to prove in its reciprocal form: an order 2 flop in the same edge of a diagra which has at least one vertex gives a diagram which is not prime. The following picture sho this. The final faces A and B share the two edges a and b, and the existence of at least one init vertex ensures that a and b are not the same edge, so the final diagram is not prime.

Figure 11

■

When we have a diagram which is not prime we can 'factorize' it by cutting the two edges that disconnect it and then regluing the pairs of open edges that lay in the same connected component. (This process can be described as a flop in the two edges followed by a flip in the new vertex obtained, as shown in figure 12, and gives as a result two connected diagrams each having at least one of the initial vertices.)

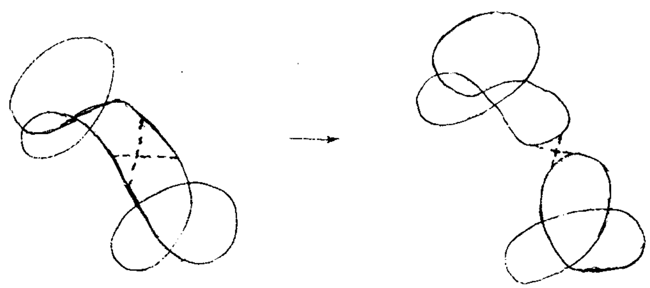


Figure 12: Decomposition of a diagram

The process can be continued with these two new diagrams if they are not prime until we have a finite collection of prime diagrams in the plane which we call the prime factors of the initial diagram.

The factorization of a diagram is not unique, because when we 'reglue' the open edges in pairs we can do it in two different ways (or, equivalently, if we make a flop in the two disconnecting edges followed by a flip on the new vertex, there are two ways to make the flop, one in each of the two faces which share the edges). Nevertheless, these different ways give diagrams which are equivalent as independent diagrams in the sphere, i.e. diagrams with the same extended Gauss codes, but possibly with different dispossition respect to one another and to the infinity point. It can be also shown that the factorization does not depend on the order we choose to make the decompositions.

The important point concerning this prime-factors decomposition is that if we know how to realize by an algebraic curve each of the prime factors of the decomposition it is easy to 'reglue'

the algebraic prime factors to have a realization of the whole diagram. We will come back to th point in section 7.

In the case of only double points we can refine a little the construction, and section 5 is devote to prepare this refinement.

5. Quasiconvexity of Prime Diagrams.

Definition 5.1 Let D be a (connected, double points) diagram in the plane. We will say tha D is *quasiconvex* if we can choose a point P_e in every edge e of the diagram in such a way that th two following conditions are satisfied:

i) for every face F of D different from the infinity one, the polygon whose vertices are th points P_e, with e the edges in the boundary of F is a strictly convex polygon contained in F an touching its boundary only in $\{P_e\}$.

ii) if an edge e is in the boundary of the infinity face, then a straight line exists passing by I and not touching the diagram in any other point (a 'tangency line' on P_e).

In figure 13 we show an example of a quasiconvex diagram. This section is devoted to proc that every prime diagram has the same shape of a quasiconvex one; for non-prime diagrams th result is not true in general, but nevertheless the diagram in figure 13 is not prime.

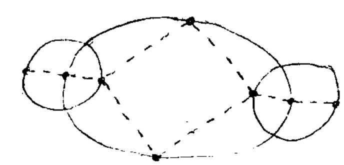


Figure 13: Quasiconvexity.

Lemma 5.2 *Let D be a quasiconvex diagram. Then every flop on D joining two different edge in a face different from the infinity one, can be made in such a way that the resulting diagram quasiconvex.*

Proof: Let a and b the edges to make the flop, and F the face. We We make the following sma peturbations in a (and b):

- if a is an edge not touching the infinity face (an interior edge), we make it to be a straight lin segment in a sufficiently small neighbourhood of P_a, without altering the quasiconvexity condition The quasiconvexity condition remains then true if we change the point P_a by sufficiently near one Q_a or R_a in a (figure 14-a).

- if a is one of the edges of the infinity face, we make it to be an 'angle' in P_a, without alterin the quasiconvexity conditions. There exist then Q_a and R_a such that the line passing by them i parallel to the tangency line in P_a and does not cross the diagram in any other point (fig 14- b).

Figure 14

Now, we make the flop joining Q_a to R_b and Q_b to R_a by straight lines, and deletintg the parts of edges a and b between these points (figure 15). The diagram so obtained is quasiconvex: the quasiconvexity condition is automatically verified in the interior faces, and in the exterior one it suffices to modify a little the line passing by Q_a and R_a (or Q_b and R_b) to two lines each passing by one of them and not crossing the diagram in any other point (as in figure 15).

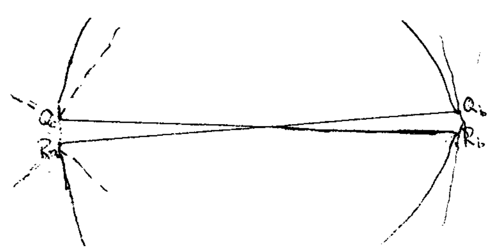


Figure 15

■

Lemma 5.3 *Every (connected, double points) prime diagram with no interior vertices have one of the following shapes (by an interior vertex we mean a vertex not adjacent to the infinity face):*

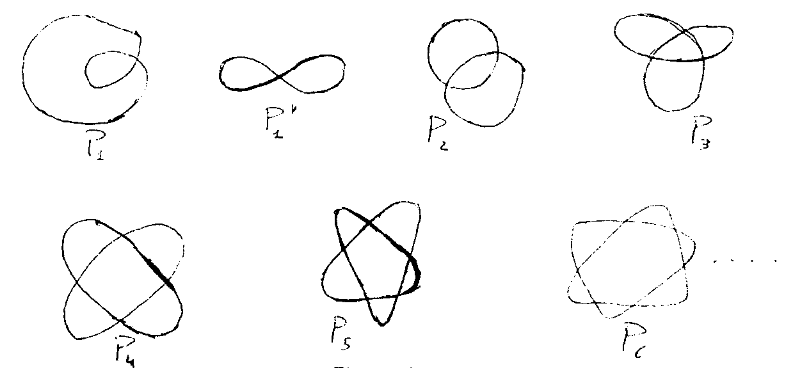


Figure 16

Proposition 5.4 *Every (connected, double points) prime diagram in the plane has the sa topological shape than a quasiconvex one.*

Proof: Let D be our diagram, and let us prove the result by induction on the number of interi vertices. By lemma 5.3 all diagrams with no interior vertices have the shape of a quasiconvex or

If D has $N+1$ interior vertices we choose one of them and make a prime flip on it (rec that one of the two flips in a vertex of a prime diagram gives a prime diagram). By inducti hypothesis this new diagram has the shape of a quasiconvex one, and by lemma 1 the flop th recovers D from it can be made to give a quasiconvex diagram. We have used here the fact that flop which produces a prime diagram must be made joining two different edges.

The lemmas and proposition 5.4 prove that if D is a prime diagram with only double poin a sequence of flops on it can lead to one of the prime diagrams in lemma 5.3 and that the flo that recover the shape of D from this final diagram can be made preserving quasiconvexity. Th is going to be the procedure we will use to construct an algebraic curve with the topological sha of a given prime diagram. and moreover let us see that making flops in the interior vertices we a never going to arrive to diagrams P_1 and P_1' of lemma 5.3:

Lemma 5.5 *Let D be a prime diagram (connected, with only double points) with at least o interior vertex. Then, D has at least 2 exterior vertex. i.e. a sequence of flops in its interi vertices cannot lead to the diagrams P_1 nor P_1'.*

Proof: The lemma reduces to proof that there are no prime diagrams with only one exteri vertex and at least one interior vertex.

This is true, because if there is only one exterior vertex, say V, then the cycle of edges of t exterior face has either one only edge $[V,V]$, or two edges, $[V,V][V,V]$. In this second case t diagram can only be P_1', and in the first case it is either P_1 or not prime (the other two edges V apart from $[V,V]$ are different and disconnect the diagram).

6. Algebraic Construction of Curves with Given Shape. The General Case.

In this final section we are going to show how we can use the 'flip-flop' techniques on diagrams to construct an algebraic curve with a given in advance topological shape, and how we can profit of the quasiconvexity properties of prime diagrams to obtain the optimal degree $2N+2$ for the realization of any compact curve in the plane with only double points. We introduce first some well known concepts in algebrain geometry:

Definitions 6.1

-By an ***algebraic plane curve*** **in** $\mathbb{R}^2$ we mean a polynomial $f \in \mathbb{R}[X,Y]$, and also its *zero set* $V_f = \{f(X,Y)=0\} \subset \mathbb{R}^2$ when there is no ambiguity in the polynomial we consider to define V_f. It is necessary to remark this because different curves (polynomials), can have the same zero set. We say that a curve f *realizes* a diagram D if the zero set of f is isotopic to D.

-A point $P=(a,b) \in \mathbb{R}^2$ is called a *singular point* of the curve $f(X,Y)$ if $f(a,b) = f_X(a,b) = f_Y(a,b) = 0$ (where f_X and f_Y are the derivatives of f). We consider only curves with a finite number of singular points. (If a plane curve has an infinite number of singular points it means that it has a repeated factor).

-The *order* of a singular point is the least order of a derivative which is not zero in (a,b). If $P=(0,0)$, then the multiplicity of P is the least degree of the monomials of f. For P arbitrary the same thing holds if we develop f around the point P.

An important class of singular points are *nondegenerate* singular points. The general definition for complex curves is well known (see, for example, [Wa]), but we give here a slightly different one for the case of real curves:

Definitions 6.2 Let $f(X,Y)$ be a real curve of degree n, and let $P=(a,b)$ be a singular point of f of order m. We can then write $f(X,Y) = f_m(X-a,Y-b) + f_{m+1}(X-a,Y-b) + \ldots + f_n(X-a,Y-b)$, where f_k are homogeneous polynomials of degree k. We will say that P is *real-nondegenerate* (or real-ordinary) if f_m decomposes in m real different linear factors. (Remark: a bivariate homogeneous polynomial always decomposes totally in complex linear factors, here we demand this factors to be different and real. The usual definition of ordinary points demands them only to be different).

We are going to be specially interested in singular points of order 2. The local structre of a real algebraic curve in a neighborhood of such an order 2 point is either that of one order 2 analytic branch (this is the case of a 'cusp') or that of two nonsingular branches crossing at the point, and in this later case these branches can be either both complex or both real. We are going to call *singularities of type* A^- those order 2 singularities which consist on two real analytic branches (the name comes from the terminology used in [AGV] to clasify singularities). An example of these A^- singularities (in fact the only one we are going to be concerned with in the constructions) is the product of two curves both passing by a point P which is regular for both of them.

Finally we say that a curve f of degree n *has no points at infinity* if the monomial of highest degree f_n of f has no real zeroes different from the origin (i.e. if the projective curve associated to f has no points on the infinity line of the projective plane $\mathbb{R}\mathbb{P}^2$). Note that if two curves f and g have no points at infinity, then neither the product fg has.

Our construction of algebraic curves is based on perturbation techniques: a perturbation on a polynomial is a small, continuous change in its coefficients. A particular case of a perturbation of a polynomial f is the family of polynomials $f+\epsilon g$, where ϵ is supposed to be a small parameter which varies continuously and g is supposed to be of degree lower or equal to g (due to technical

reasons). We call it a *perturbation of f by g* and say that a property is true for *sufficiently sm[illegible]* perturbations of f by g if it is true for every curve $f+\epsilon g$ with $|\epsilon|$ smaller than a certain ϵ_0.

The following result says how a small perturbation of this type affects the topological shape [illegible] the polynomial f, in a particular case that will suffice to our purposes. We give it without proo[f] for it is the affine version of theorem 2.7 in [Sa2], and it can also be deduced from two lemmas [illegible] [Gu] (the 'lemma on the class of a point' and 'the lemma on isotopy').

Proposition 6.3 *Let f be a real curve with no points at infinity and of degree n, and suppo[se] that the singular points of f are $P_1, P_2, \ldots, P_k$ and $Q_1, Q_2, \ldots, Q_l$, from which the P_i are rea[l-] nondegenerate and the Q_i are of A^- type. Let g be a curve of the degree $\deg(g) \le \deg(f)$ whic[h] has a singular point of at least the same order in the points P_i, and which does not pass by t[he] points Q_i.*

Then, any sufficiently small perturbation of f by g of the form $f+\epsilon g$:

- has a real-nondegenerate singular point of the same order in each of the P_i,

-has no other singular point, and no points at infinity, and

-its topological shape in $\mathbb{R}^2$ can be obtained modifying each A^- singularity of f in that of t[he] two ways in figure 17 which is compatible with the signs of f, g and ϵ.

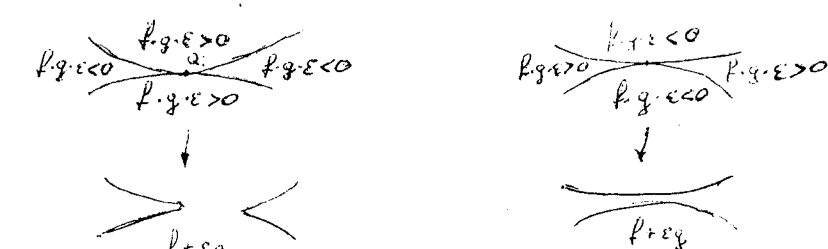


Figure 17

Proposition 6.3 is enough to describe the construction of algebraic 'flops', as we do in th[e] proof of the following lemma:

Theorem 6.4 *Let D be a diagram which is realized by an algebraic curve $f(X,Y)$, f havi[ng] only real-nondegenerate points and no points at infinity. Let D' be a diagram obtained by a fl[op] on D. Then D' can also be realized by an algebraic curve f' with only real-nondegenerate singul[ar] points and no point at infinity.*

Proof: Let $P_1, P_2, \ldots, P_{k-1}$ be the singular points of f, which are all real-nondegenerate ar[e] with orders $m_1, m_2, \ldots, m_{k-1}$. We can identify D with the zero set of f.

The flop of order m in D is given by one of its faces F, an m-petals flower in the face, and [illegible] paths joining the petals with points in the boundary of F.

An m-petals flower can be constructed algebraic by the formula: $R = \cos(mt)$ if m is od[d] and $R^2 = \cos(mt)$ if m is even (it is easy to check that these equations define algebraic curv[es] of degrees $m+1$ and $m+2$ respectively, and that they have the shape of an m-petals flower, [n]o points at infinity, and their only singular point is real-nondegenerate of order m).

Now we can place the m-petals flower in the face F by translations and homoteties, and ca[ll] $f*$ the product of f with the polynomial defining the flower.

To make the flop we have to join each petal of the flower to the corresponding points in f, along some given paths. To do this algebraically, we first cover each path with a 'chain of circles' satisfying:

-The first circle is tangent to the point in the petal, the last one to the point in f, and each circle is tangent to the next one.

-The circles in the chains do not intersect each other nor $f*$ in other points than the mentioned tangencies (see figure 18).

(To construct the chains we first put a tangent circle in each of the two extremal points of the path, sufficiently small not to touch $f*$ in other points than the tangency one, and then cover the part of the path not covered by these two circles with a finite number of circles not touching $f*$. If we delete the superfluous circles and reduce the remaining ones to be each tangent to the next one the circles will satisfy the conditions).

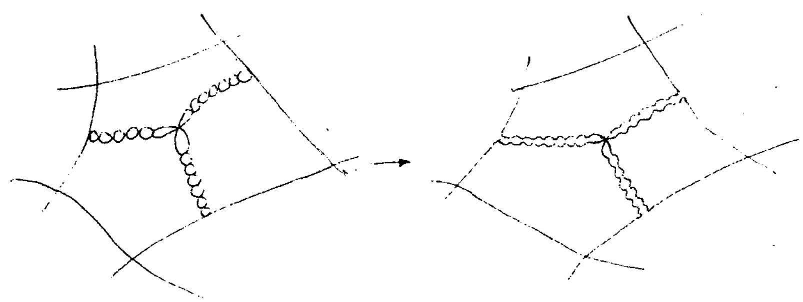


Figure 18

We still call $f*$ The product of $f*$ with all the circles in the chains.

$f*$ satisfies the hypothesis in proposition 6.3, if we call $P_k = P$ (the singular point in the flower), and $Q_1, Q_2, \ldots, Q_l$ the tangency points in the chains of circles. We can moreover suppose that $f*$ has positive sign outside the circles and the flower, and negative inside. We are going to perturb the curve $f*$ to have the same shape than the diagram obtained from the flop:

For each nondegenerate point $P_i = (a_i, b_i), i = 1, \ldots, k$ of f we consider the polynomial $g_i = ((X - a_i)^2 + (Y - b_i)^2)^{m*_i}$, where $m*_i$ is an exponent to make P_i be of order at least m_i in g_i. (It suffices $m*_i \geq m_i/2$), and call g the product of the g_i's. g is everywhere positive (except in the points P_i which are its zeroes), and we can suppose that its degree is smaller than the degree of $f*$ (if it is not we multiply $f*$ by a factor not affecting its zero set, such as $(X^2 + Y^2 + 1)^p$).

In these conditions, proposition 6.3 ensures that for a small positive ϵ the curve $f' = f * -\epsilon g$ realizes the wanted diagram D' of the flop: At each tangency point the deformation compatible with the signs of f, g and ϵ is that which joins the petals of the flower with the original curve f along the chains of circles. ∎

Corollary 6.5 *Every diagram in the plane can be realized algebraically, with only real-nondegenerate points and no points at infinity.*

Proof: By induction on the number of singular points. A diagram with no singular points is a finite collection of ovals which can always be realized by some product of circles. To realize a diagram D_N with N singular points, we make a flip to it, obtaining a diagram D_{N-1} with $N - 1$ singular points, and by induction suppose this new diagram realized algebraically by a curve f_{N-1}

with only real-nondegenerate points and no points at infinity, and apply the theorem to the inve flop to the flip made. That gives a realization of D_N

7. The Case of only Double Points.

Corollary 6.5 gives a constructive proof of the characterization of the possible shapes of co pact algebraic sets in the plane: every diagram is realizable as an algebraic set implies tha sufficient condition for something to have the shape of an algebraic set is to be an imbedded gra with even order in all the vertices, and as we mentioned in the introduction this is also an easy proof necessary condition for an algebraic compact set in the plane. But more interesting that t is that this kind of construction permits us to think in controlling the degree of the curves we to realize a diagram. In theorem 6.4 and corollary 6.5 this is not possible because we do not kn a priori how to bound the number of circles needed in the chains of circles for the flop. Nevertl less if we restrict ourselves to diagrams having only double points we can refine the constructi thanks the 'prime factors decomposition' of diagrams and the 'quasiconvexity' properties of pri diagrams.

For algebraic curves we need a slightly different definition of quasiconvexity than for diagran

Definition 7.1 We say that an algebraic curve f (connected, with only double points) *quasiconvex* if its zero set is quasiconvex (in the sense of definition 5.1 for diagrams), and moreo the points P_e in the esterior edges of the curve are not flexes.

Note that the exterior condition of quasiconvexity, in the case of algebraic curves, implies th the tangent line to the curve at points P_e in the exterior edges does not have any other intersectio with the curve. The additional assumption of the points P_e not being flexes (i.e. having fin curvature) implies that a sufficiently big circle tangent to the curve at P_e has the same proper it does not intersect the curve in any other point. This will be used in the next proposition 'glue' the quasiconvex algebraic realizations of the prime factors of a diagram. The proposition true for any number of prime factors, but we proof it for 2 factors, for the sake of simplicity.

Proposition 7.2 *Let D be a connected diagram with only double points which decompo in two factors D_1 and D_2 such that D_1 and D_2 are one outside another, or D_2 inside D_1, (l not the converse). Suppose that D_1 and D_2 are realized by two algebraic curves f_1 and f_2 w only real-nondegenerate singular points (of order 2), and no points at infinity. Suppose also tl $d = \deg(f_1) + \deg(f_2) \geq 2N$, where N is the number of double points in D, and that f_2 quasiconvex. Then D can be realized by an algebraic curve f of degree d.*

Proof: In any of the two dispossitions of D_1 and D_2 (D_2 inside D_1 or one outside another), recover D from D_1 and D_2 we need only to place a copy of D_2 in the appropiate face of D_1 (whi would be the exterior face if D_1 and D_2 lie one outside another), and join them by the appropia edges.

Let us then do that with the algebraic curves f_1 and f_2 which realize D_1 and D_2. We c put the curve f_2 in the appropiate face of f_1 by translations and homotecies, which do not affe its degree. Let e_1 and e_2 be the edges by which we must join f_2 to f_1, and let P_1 and P_2 be t points of the quasiconvexity conditions in these edges. e_2 is an exterior edge, and thus we c construct a big circle tangent to the curve f_2 at P_2 and containing the whole curve f_2. We can al construct a small circle tangent to f_1 at P_1 and contained in the appropiate face of f_1 (because is regular), and by some rotations, translations and homotecies in f_2 make the two circles coincid and identify the points P_1 and P_2, which become a tangency point between f_1 and f_2 (see figure 19).

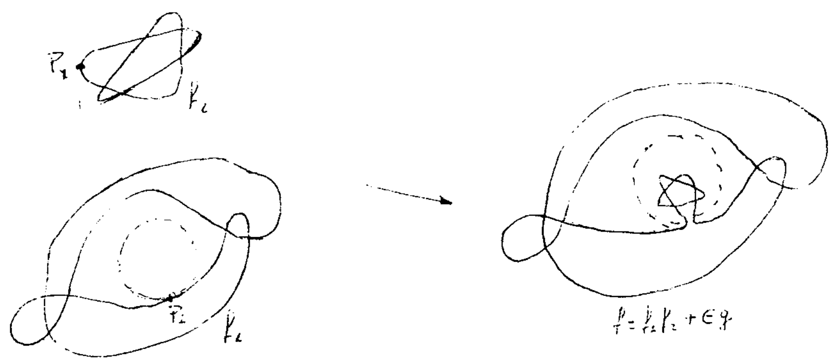


Figure 19

Consider the product f_0 of f_1 and f_2 under these conditions. It has degree $d \geq 2N$, and N order 2, real-nondegenerate singular points, which correspond to the double points of D, plus another singular point which is also of order 2 and type A^-: the tangency point. Moreover, it has the same shape of the diagram D except for the tangency point. Thus, we want to perturb it to make the tangency disappear, and it is easy to do this thanks to proposition 6.3:

f_0 satisfies the conditions of the proposition, and we can take as perturbing curve the product g of the factors $g_i = (X - a_i)^2 + (Y - a_i^2)$, (where (a_i, b_i), with $i = 1, \ldots, N$ are the coordinates of the singular points). g has degree $2N \leq d = \deg(f_0)$ and has a singular point of order 2 at each of the real-nondegenerate singular points of f_0 (which are also of order 2). Then, the perturbed curve $f = f_0 + \epsilon g$, with ϵ sufficiently small has the same shape of f_0 except for the tangency which disappears and thus, with the adequate sign for ϵ, the shape of D. ∎

Proposition 7.2 (generalized to any number of prime factors) will permit to realize any connected diagram in the plane by an algebraic curve of controlled degree if we know how to realize its prime factors by a quasiconvex curve. To realize the prime factors we are going to make use of their quasiconvexity properties, but we must make note that in the induction process we describe, we have not a proof that quasiconvexity can be preserved by the perturbations made (see conjecture 7.5). Thus the induction hypothesis is not ensured, and thus 7.6, 7.7 and 7.8 are true only if the conjecture is.

We start realizing the 'basic' prime diagrams, which are prime diagrams with no interior points:

Lemma 7.3 *Every prime diagram with only double points and no interior vertices can be realized by a quasiconvex algebraic curve of degree $2N$ (where N is the number of exterior vertices), with no points at infinity and only real-nondegenerate singular points, except for P_1 and P_1', which can be realized with degree 4 (and the same properties).*

Proof: : We recall lemma 5.3 which said that the only possible prime diagrams without interior vertices where the P_1, P_1' and the P_i, for $i = 2, \ldots$; we will show the algebraic construction for each of them:

P_1 is realized by the lemniscata $(X^2 + Y^2)^2 = X^2 - Y^2$, and P_1' can be constructed perturb the product of two circles which intersect transversally (we consider one of the intersection poi as real-nondegenerate and the other one as of type A^- to apply proposition 6.3), as shown in fig 20, and that gives degree 4. The quasiconvexity properties needed are easily verified.

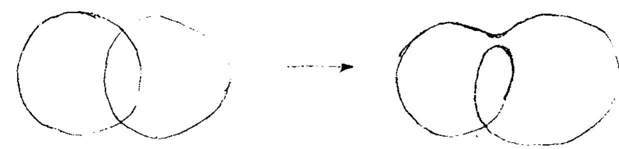
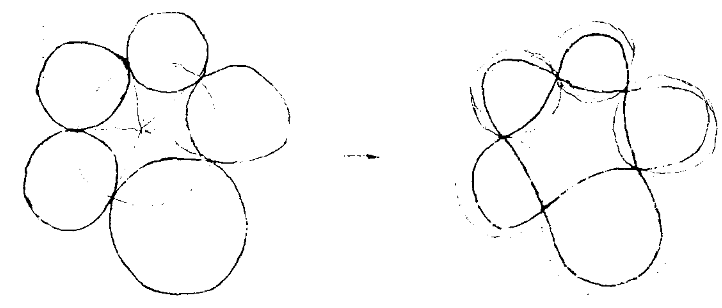


Figure 20

To realize the rest of the P_i we use the following procedure, which we describe only for P_5: consider 5 different ratii of the unit circle from the origin, and find the 5 circles which are tang to two consecutive ones in the points where the ratii touch the circle (two consecutive such cir are tangent to one another, as in figure 21.a). We call f the product of the 5 circles, with posi sign at the origin and at infinity. f has clearly degree 10 and no points at infinity, and we going to perturb it by the curve $g = (X^2 + Y^2 - 1)^2$, which is positive everywhere except in unit circle. This perturbation is not included in proposition 6.3, because g passes by the singu points of f which are degenearte, but it is easy to describe its effect on the curve:



(a) *Figure 21* (b)

First, the perturbed curve $f + \epsilon g$, for sufficiently small, positive ϵ, must be included in interior of the 5 circles (because these are the regions in which f and g have opposite sign), an locally isotopic to f in its non singular points. It rests only to see what happens at the tange points of the circles. If we translate one of this points to be the origin, and rotate the figure u the tangency is horizontal, then the terms of lower degree of f and g are $f = -Y^2 + \ldots$, a $g = X^2 + \ldots$ and thus $f + \epsilon g = -Y^2 + \epsilon X^2 + \ldots$ which corresponds to two real branches w tangents $Y = \sqrt{(\epsilon)}X$, i.e. to a real-nondegenerate order 2 point, which gives the shape of figure 21.b. The quasiconvexity properties are automatically satisfied, as shown in the figure. ∎

Now let us see how two add the interior double points to the realized diagrams:

Lemma 7.4 *Let D be a prime diagram with N vertices, which are of order 2, and with at least one interior vertex V. Let D' be a prime diagram obtained by a flip on D at V, and suppose that D' is realized by a quasiconvex algebraic curve f' with only real-nondegenerate double points and no points at infinity. Then, D can be realized by an algebraic curve of degree $2N$ with only real-nondegenerate singular points and no points at infinity.*

Proof: All we have to do is an algebraic flop on f' to recover the initial D, and we are going to do this by a similar process as made in proposition 6.4. The difference now is that we can profite the quasiconvexity properties of f'.

Let a and b be the edges of f' in which we must make a flop to recover the shape of D, and let P_a and P_b be the points of the quasiconvexity definition in the edges a and b. Then, the face for the flop (the only face which has a and b in its boundary, for f' is prime) must be an interior face of f', because the vertex of D in which we made the flip was interior. Then, by quasiconvexity, there exists a convex polygon with vertex at P_a and P_b inscribed in the face, and in particular the segment P_aP_b is contained in the face. Moreover P_a and P_b are regular points in f' (they are not vertices), and this implies that an ellipse can be constructed being tangent to f' at P_a and P_b, and sufficiently close to the segment P_aP_b to be contained in the face (see figure 22.a). What we want to do is to perturb the product of f' with this ellipse in the way shown in figure 22.b.

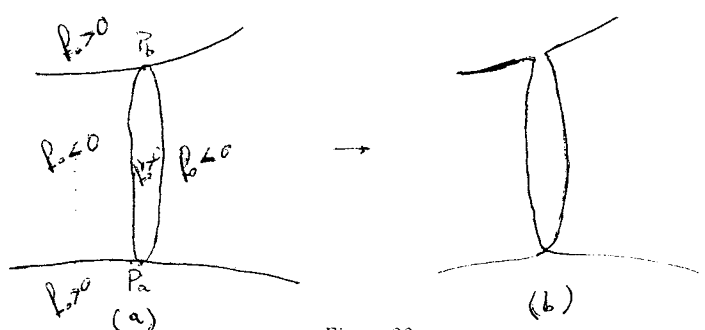


Figure 22

Call f_0 this product, $P_1, \ldots, P_{N-1}$ the singular points of f', and suppose that P_a is placed at the origin, with horizontal tagent, and with the signs dispossition for f_0 shown in figure 22.a.

For a perturbation of type $f_0 + \epsilon g$ (with ϵ small and positive) to have the shape of figure 22.b (and not to change anything elsewhere), we need a polynomial g with degree at most $2N$ (the degree of f_0), with a singular point at each of the $P_1, \ldots, P_{N-1}$ (this ensures that the real-nondegenarate singular points of f' are preserved by the perturbation), with positive sign at P_b (to break this tangency in the appropiate way) and with a singular point at P_a, such that the singularity of $f + \epsilon g$ at P_a be real-nondegenerate.

The last condition is achieved if g has no terms of degree lower than 2 and its degree 2 term is $-X^2$, and this is achieved, for example, if $g = g_1 g_2$, with g_1 positive at P_a and $g_2 = (X^2 + Y^2 + rX)(X^2 + Y^2 - rX)$, i.e. the product of two circles vertically tangent at P_a, where the

radius r is chosen sufficiently small to not interfere with the rest of the figure (i.e. such that
positive at P_b and at every P_i).

Now the conditions for g_1 are only to be positive at P_a and P_b, to have a singular poin
each $P_i, i = 1, \ldots, N-1$, and to be of degree at most $2N - 4$ (because g_2 is of degree 4.

f' has at least two vertices (in fact, at least two exterior vertices), and we can suppose wit
loss of generality that P_a and P_b are not in the line passing by P_1 and P_2, because the quasiconve
properties are also satisfied if we move a little P_a and P_b along the edges a and b of f'. Thus, we
take as g_1 the square of this line, times a factor $(X - x_i)^2 + (Y - y_i)^2$ for each $i = 3, \ldots, N-1$, w
(x_i, y_i) are the coordinates of the points P_i. This g_1 has degree exactly $2N - 4$, and is everyw
positive except in the points P_i and in the line passing by P_1 and P_2.

That ends the construction of the algebraic flop. By a perturbation theorem similar to pro
sition 6.3, the curve $f = f_0 + \epsilon g$ has the same shape than the diagram D, has no points at infi
and its N singular points which correspond to the N double points of D are real-nondegenerat

We would like to use lemma 7.4 inductively to construct every prime diagram, but we do
know how to preserve the qusiconvexity conditions in the flop. Thus we state a conjecture,
give only a partial proof:

Conjecture 7.5 *In the conditions of lemma 7.4 the final curve f can be constructed qu
convex.*

Proof: In fact, the only quasiconvexity conditions that we cannot ensure to be true are t
concerning only the four new edges that appear from a and b by the flop. The rest are preser
because the quasiconvexity conditions are 'open' in the sense that they remain true if we
perturb a little the points in the edges or the edges themselves, and this perturbation is m
'smoothly' (i.e. varying continuously not only the points but also the slopes).

In the special case that the edges a and b have their curvature towards the outside at points
and P_b, the quasiconvexity conditions of the new four edges are also preserved, if the ellipse joi
P_a and P_b is chosen sufficiently narrow: this is so because, in this case, there exists a rectangle v
vertices in the edges a and b, close to P_a and P_b (as in figure 22.a), and this rectangle construc
is preserved by the perturbaton, if the ellipse is contained in the rectangle and the perturbatio
small (see figure 22.b).

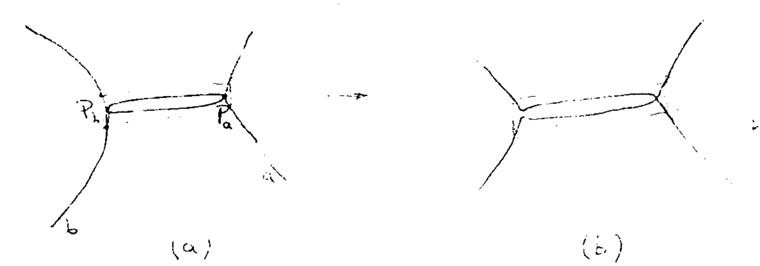


Figure 23

If the curvatures at P_a and P_b are towards the inside this construction is not possible,
possibly with a sufficiently narrow ellipse quasiconvexity is still preserved.

Corollary 7.6 *If conjecture 7.5 is true, then every prime diagram with only double points can be realized by a quasiconvex algebraic curve of degree $2N$ (where N is the number of vertices), without points at infinity and only real-nondegenerate singular points, except for P_1 and P_1', which can be realized with degree 4 (and the same properties).*

Proof: The proof is made by induction on the number of interior vertices. Lemma 7.3 gives the proof for 0 interior vertices, and for a diagram D with at least one interior vertex V, we make to D a flip at V, obtaining a new diagram D', which can be supposed prime, by proposition 4.5(ii). Besides, lemma 5.5 ensures that D' is not P_1 nor P_1', so by induction hypothesis we can suppose D' realized by a quasiconvex algebraic curve f' of degree $2N-2$, with no points at infinity and $N-1$ real-nondegenerate order-2 singular points.

Lemma 7.4 enables us to construct the curve f with degree $2N$, only real-nondegenerate points and no point at infinity, and by conjecture 7.5 we can suppose that f is also quasiconvex. ∎

Finally we state the general theorem about the construction of real algebraic compact curves in the real plane:

Theorem 7.7 *We suppose that conjecture 7.5 is true. Let D be a connected diagram with N vertices, all of order two. If at least one of the prime factors of D is not P_1 nor P_1' then D can be realized by an algebraic curve of degree $2N$, with only real-nondegenerate singular points and no points at infinity. If not, D can be realized in the same conditions with degree $2N+2$.*

Proof: For prime diagrams the theorem is already proved (corollary 7.6), and for non prime diagrams we use the same techniques of proposition 7.2: we decompose D in its prime factors D_i, and realize each by a quasiconvex algebraic curve of degree $2N_i$, where N_i is the number of double points in D_i; this can be done by proposition 7.6, except if the factor is a P_1 or a P_1' (we will treat this case separately).

Now, proposition 7.2 gives a procedure to reglue all the prime factors one by one and gives as final degree the sum of the degrees needed to realize the factors, that is $2N$, where N is the total number of double points in D, the only thing to take care of is that to use proposition 7.2 we must first realize the most exterior prime factor of D (or one of the most exterior ones, if there are more than one), and then glue the others from the exterior to the interior.

When there is some P_1 or P_1' factor this procedure would not give degree $2N$, because these prime factors can only be realized with degree 4, and they add just one singular point. Nevertheless we can 'glue' them in another, equivalent way: insert a tangent circle in the appropiate face of the curve, and then perturb the tangency (which is a singular degenerate point) to be real-nondegenerate (this can be done in the same way we did in the proof of lemma 7.4). With this procedure each P_1 or P_1' factor increases the degree only by 2, and than the final degree $2N$ is mantained.

The only case in which this cannot be done is if all the prime factors of D are P_1 or P_1', for in these case we need degree 4 to realize the first prime factor, and thus the final degree becomes $2N+2$ instead of $2N$. ∎

Corollary 7.8 *(If conjecture 7.5 is true) every diagram in the plane with only double points can be realized with degree lower or equal to $2N+2K$, where N and K are the numbers of double points and connected components, respectively.*

Proof: Let $D_1, \ldots, D_K$ be the connectec components of D. Theorem 7.7 permits to realize each D_i with degree at most $2N_i + 2$, where N_i is the number of double points in D_i. Realizing all of them and then placing them in the appropiate place from one another we will have the desired curve realizing D (the product of the curves realizing the connected components), whose degree will be at most $\sum(2N_i+2) = 2N+2K$. ∎

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