TRIANGULATIONS OF ORIENTED MATROIDS *

Francisco Santos
Mathematical Institute
24–29 St. Giles'
Oxford OX1 3LB, UK

January 23, 1997

Contents

---

*Research Supported by an HCM postdoctoral fellowship of the European Community.
Current address: Depto. de Matemáticas, Estadística y Computación, Universidad de Cantabria, E-39071 Santander SPAIN. E_mail: santos@matesco.unican.es

Abstract

We consider the concept of triangulation of an oriented matroid,. We provide a definition which generalizes the previous ones by Billera–Munson and by Anderson and which specializes to the usual notion of triangulation (or simplicial fan) in the realizable case.

Then we study the relation existing between triangulations of an oriented matroid $\mathcal{M}$ and extensions of its dual $\mathcal{M}^*$, via the so-called *lifting triangulations*. We show that this duality behaves particularly well in the class of *Lawrence matroid polytopes*. In particular, that the *extension space conjecture* for realizable oriented matroids posed by Sturmfels and Ziegler is equivalent to the restriction to Lawrence polytopes of the Generalized Baues problem for subdivisions of polytopes.

We finish showing examples and a combinatorial characterization of lifting triangulations.

Introduction

Matroids (see [22]) and oriented matroids (see [7]) are axiomatic abstract models for combinatorial geometry over general fields and ordered fields, respectively. Oriented matroids have some extra structure over usual matroids, one of whose features is the existence of a notion of *convexity* (see Chapter 9 of [7]). In particular, it seems quite natural to consider the concept of a *triangulation* of an oriented matroid. This concept is the object of this paper. Triangulations of oriented matroids were first defined by Billera and Munson [5] for the class of matroid polytopes and an account of them for the class of acyclic oriented matroids (which is a more general case) appears in Section 9.6 of [7]. We assume familiarity with the basics of oriented matroids. For the more advanced topics we will use [7] as a standard reference.

Our interest in oriented matroid triangulations comes from two sources. In one hand, it is well-known (see for example [11]) that the collection of triangulations of a convex polytope (more generally, of a point configuration) only depends on the oriented matroid structure of the configuration and not in other geometric features. This suggests that an oriented matroid approach to triangulations provides additional insight into their structure and can help to solve some fundamental questions. The problems we have in mind concern the homotopy type of the order poset of subdivisions of a polytope and also the notion of *bistellar flip*–a sort of "elementary move"– between triangulations. Both objects have received attention in recent literature, partially as a result of the theory of *secondary polytopes* developed by Gelfand, Kapranov and Zelevinski (see Chapter 7 of [15], and pages 231–233] for the original definition of bistellar flip). The following two questions are open: Has the order complex of polytopal subdivisions of any polytope with n points in dimension d the homotopy type of a $(n-d-2)$-sphere? Is the set of triangulations of a polytope connected by bistellar flips?

The first question is a particular case of the *Generalized Baues problem* posed by Billera et al. [4]. The second is a weak version of it. Both of them have been answered in very few cases, always in the affirmative. The case of "few points" ($n \leq d+3$) is answered by the results of Lee [19] and the theory of secondary polytopes by Gelfand et al. (see [7, Chapter 7] and [3]). In the case of "low dimension" ($d \leq 2$) the connectivity question is known since some time ago [18] while the homotopy one has been recently solved by Edelman and Reiner [12]. A particular case interesting because of the amount of extra combinatorial structure available is that of *cyclic polytopes*, solved by Rambau [25].

The second source of interest in oriented matroid triangulations is their connection with the theory of *combinatorial differential manifolds* introduced by MacPherson [21]. These manifolds provide a very promising interplay between differential and combinatorial geometry, whose first striking outcome has been the combinatorial formula for the Pontrjagin classes of triangulated differential manifolds obtained by Gelfand and MacPherson in [16].

The connection between differential combinatorial manifolds and triangulations of oriented matroids was exhibited by Anderson in [1]. Roughly speaking, combinatorial differential manifolds are defined as simplicial topological pseudo-manifolds whose local differential structure is combinatorially defined by means of oriented matroids. Anderson has proved that the link of every cell of a combinatorial differential manifold is an oriented matroid triangulation. She uses a different definition of triangulation than Billera and Munson and, in fact, she works with totally cyclic oriented matroids, which is the "opposite" case to acyclic ones. However, she proves that a version of her definition for the polytopal case is equivalent to the definition by Billera and Munson.

The first problem one encounters when dealing with oriented matroid triangulations is how to define them. We have mentioned the existence of two equivalent definitions, but Billera and Munson [5] consider also another possible, and apparently stronger, definition. Actually, they recognize this "stronger" definition to be 'a more direct translation of the usual definition of triangulation of a polytope', but discard it essentially for practical reasons (see our Remarks 2.5 (a) and (b)). One of our first results is that this "strong" definition was in fact equivalent to the "weak" one. Actually, we give seven different characterizations of oriented matroid triangulations, which include (more precisely, generalize) the two definitions by Billera and Munson and the one by Anderson (see our Theorem 2.4).

Our setting is more general than the ones in [1], [5] and [7] in the sense that we do not assume the oriented matroids to be neither acyclic nor totally cyclic. Of course, a condition that any reasonable definition of oriented matroid triangulations has to satisfy is that it agrees with the usual notion of triangulation if the oriented matroid is realizable. This makes sense since the triangulations of a point (or vector) configuration depend only on the underlying oriented matroid, as we have mentioned. From the equivalent characterizations in Theorem 2.4 it is easy to conclude that our definition satisfies the following properties:

- If $\mathcal{M}$ is a matroid polytope realized as a polytope P, the triangulations of $\mathcal{M}$ are exactly the triangulations of the polytope P which only use the vertices of P as vertices. There is a recent survey by Lee [20] on this topic.

- If $\mathcal{M}$ is acyclic (but perhaps non-polytopal) and realized as a point configuration $\mathcal{A}$ in a real affine space, the triangulations of $\mathcal{M}$ are exactly the triangulations of the convex hull $conv(\mathcal{A})$ which only use (perhaps not all) the points of $\mathcal{A}$ as vertices. This is a generalization of the previous case which has often been considered in recent literature (see [3, 4, 9, 10, 11, 12, 24] and Chapter 7 of [15]).

- In general, if $\mathcal{M}$ is realized as a vector configuration $\mathcal{A}$ in a real vector space, the triangulations of $\mathcal{M}$ are exactly the *simplicial fans* (see [15, Definition 4.1]) covering the positive span $pos(\mathcal{A})$ of $\mathcal{A}$ and which only use (perhaps not all) the vectors of $\mathcal{A}$ as generators of rank-1 cones.

After the definition problem is solved, our task is two-fold: in one hand we will generalize to the oriented matroid case results which are known for triangulations of point or vector configurations. In particular, most of the results in [11], in which our Sections 3.1 and 3.2 are based. In the other hand, we use oriented matroid methods in order to obtain results which are new even in the realized case. An example of this is that we show that the *extension space conjecture* for extensions of a realizable oriented matroid, posed by Sturmfels and Ziegler in [29], is equivalent to the afore-mentioned Generalized Baues problem for subdivisions of a polytope, restricted to the case of Lawrence polytopes. See Section 4.2 and in particular Remark 4.15 for details.

Summarizing, the main results of the paper are as follows:

(a) We settle down the problem of defining triangulations of oriented matroids by providing a definition which suits any oriented matroid and giving several different characterizations of it (Theorem 2.4).

(b) We generalize to this setting the results in [11] concerning the duality between triangulations and extensions and on the affine span of the universal polytope (Sections 3.1 and 3.2). In particular, we prove that all the triangulations of a uniform oriented matroid are connected by *abstract* bistellar flips (this is a rephrasing of part 3 of Corollary 3.7).

(c) We prove that under the mentioned duality, bistellar flips of triangulations correspond in a good way to mutations of extensions (Theorem 3.14).

(d) We consider the case of Lawrence polytopes and prove that in this case the duality behaves specially well, because all the triangulations are lifting triangulations (Theorem 4.14).

(e) In particular, this implies that the *extension space conjecture* (see [29]) is equivalent to the Generalized Baues problem for Lawrence polytopes (Corollary 4.17). Also, that there are (non-realizable) oriented matroids whose triangulations are not connected by bistellar flips (Corollary 4.16) since there are (non-realizable) oriented matroids whose extensions are not connected by mutations (an example was provided by Richter-Gebert in [23]).

(f) We introduce a reoriented version of the Lawrence construction (Section 4.3, Theorem 4.18), which permits to translate results on triangulations of acyclic non-polytopal oriented matroids to triangulations of matroid polytopes, and vice-versa. This shows that the "polytopal case" cannot be considered simpler than the "acyclic case" when dealing with triangulations.

(g) We give necessary conditions (Proposition 5.4) and characterizations (Proposition 5.3 and Theorem 5.13) for an oriented matroid triangulation to be a lifting triangulation. Although the concept of lifting triangulation is heavily based in oriented matroid theory, the characterizations are purely combinatorial.

(h) We construct non-lifting triangulations of the 4-cube and of a unimodular polytope (examples 5.11 and 5.8). We also show bad behavior of triangulations of non-Euclidean oriented matroids, with an example in the Edmonds-Fukuda-Mandel oriented matroid EFM(8) (Example 5.6).

The following is a more detailed description of some of these points, and of the structure of the paper:

The technical tools from oriented matroid theory that we will need concern mainly single-element extensions and, of course, convexity. That is, the first part of Chapter 7 and Chapter 9 in [7]. In Section 1 we recall these concepts and prove some results which will be important later on. All of them either are relatively easy to prove or can be found in [7]. Thus, the reader familiar with oriented matroids can skip this section and come to it only for reference. Other readers may find this section clarifying for understanding the convex geometry of oriented matroids.

In Section 2 we give our definition of oriented matroid triangulation, based in the "weak" one by Billera and Munson, and provide several equivalent characterizations of it (Theorem 2.4). In particular, we show the equivalence with the "strong" definition by Billera and Munson and with the recursive definition proposed by Anderson [1]. In Section 2.3 we prove some properties of triangulations which will be of use in the rest of the paper.

In Section 3.1 we introduce the duality existing between *lifting* triangulations of an oriented matroid and extensions of its dual in general position. Lifting triangulations were defined in [7, Section 9.6] (a particular case was mentioned in [5]). We will give two definitions of them, dual to one another (definitions 3.4 and 4.1).

The duality between triangulations of an oriented matroid and extensions of its dual is a generalization of the duality between regular triangulations of a point configuration $\mathcal{A}$ and chambers of its Gale transform $\mathcal{A}^*$, exhibited by Billera et al. in [3]. De Loera et al. [11] have already given a generalized version of this duality, still in the realizable case, with the introduction of what they call *virtual chambers*. Section 3.2 is the translation into the general oriented matroid setting of Sections 2 and 5 of [11], and most of the proofs required little changes. In particular, it is shown that for any triangulations T of $\mathcal{M}$ and T' of the dual $\mathcal{M}'$ there are unique maximal simplices (i.e., bases) of T and T' which are complements (Theorem 3.8). Also, that all the triangulations of a uniform oriented matroid can be joined by a sequence of "abstract" bistellar flips (this is a rephrasing of part (iii) of Corollary 3.7), a result first proved in the realizable case in [3]. In Section 3.3 we show the exact relation between the natural notions of elementary change on triangulations (the notion of *bistellar flip*) and on extensions (the notion of *mutation*) under the duality. Namely, that whenever two extensions differ by a mutation the corresponding triangulations either coincide or differ by a bistellar flip (Theorem 3.14).

Going further on this duality, it is easy to establish a relation between the order complexes of extensions ordered by weak maps and of subdivisions of an oriented matroid ordered by refinement (subdivisions of an oriented matroid are already defined in [7, Section 9.6] as a generalization of polytopal subdivisions of a polytope, and the relation between the order complexes is stated in Exercises 9.30 and 9.31). In Section 4.2 we are going to see that this relation is specially interesting in the case of *Lawrence polytopes*, because a Lawrence polytope only has lifting triangulations and each of them corresponds to a unique extension of its dual (Theorem 4.14). This provides the mentioned relation between the *extension space conjecture* and the homotopy of the order complex of polytopal subdivisions of Lawrence polytopes (Remark 4.15 and Corollary 4.17). For the sake of self-completeness we devote Section 4.1 to the introduction of general subdivisions and to prove some results that we will need. In Section 4.3 we introduce a reoriented version of the Lawrence construction, which is useful in order to translate results on triangulations of polytopes to non-polytopal point configurations and vice-versa.

Since lifting triangulations have played an important role in the results so far, we have devoted to them the last part of the paper. We will see some good properties and their relation with regular triangulations of point configurations in Section 5.1 and we will construct interesting examples of non-lifting triangulations in Section 5.2. In Section 5.3 we prove the following surprising fact: although the definition of lifting triangulations (even for a realized oriented matroid) relies strongly in the notion of oriented matroid, there is a purely combinatorial characterization of them which can be stated with no mention to oriented matroids (Theorem 5.13). In our opinion this implies that the concept of lifting triangulation is a natural one even outside oriented matroid theory.

We finish this introduction with some open problems on triangulations of oriented matroids:

- Of course, we can mention here the General Baues problem in the two particular cases of zonotopal and arbitrary subdivisions, although this problem applies to realizable oriented matroids only. See the details in Remark 4.15.

- The fundamental problem concerning triangulations of oriented matroids is that of their topology: Is every oriented matroid triangulation homeomorphic (or at least homotopic) to a sphere or a ball of the appropriate dimension? Anderson [1] has shown that this question is equivalent to whether every combinatorial differential manifold is a topological manifold. She proves the answer to be yes for *Euclidean* oriented matroids, and it is easily seen to be yes also for lifting triangulations. See Remark 2.15 for more information.

- Apart from their topology, there are other properties of triangulations which can only be proved using some sort of euclideanness condition (see Example 5.6, in particular the introductory comments). It would be good to either prove them in general or show counterexamples. In particular, we consider the following questions:
  - Is there an example in which the graph defined in the proofs of lemmas 2.7 and 4.6 has cycles? This seems to be connected to the topology problem; see Remark 2.8.
  - Is there a triangulation with two simplices containing respectively the positive and negative parts of a circuit? This question was pointed out to us by Jörg Rambau. A weaker property than this appears in characterization (f) of Theorem 2.4.
  - Are there two simplices σ and τ of an oriented matroid such that they are separated by a covector (they lie in different parts of the covector) but they do not simultaneously belong to any triangulation? In Example 5.6 we see a case in which two separated simplices do not belong to any lifting triangulation, which is already impossible in Euclidean oriented matroids.
- All the non-lifting triangulations which appear in this paper fail to satisfy the *necessary* condition stated in Proposition 5.4, which leads us to think that this necessary condition might also be *sufficient.* If this is the case, every triangulation of a realizable rank 3 oriented matroid will be a lifting triangulation. Even if it is not the case,is there a simpler combinatorial characterization of lifting triangulations than our Theorem 5.13?

I am grateful to Bernd Sturmfels who proposed me the study of oriented matroid triangulations as a natural continuation of the results in [11]. To Jörg Rambau with whom I had several interesting discussions specially on the topic of euclideanness. And to Jesus de Loera who carefully read different versions of the manuscript and helped with suggestions and comments.

1 Some preliminaries on oriented matroids

In this section we sum up the main oriented-matroid concepts and properties that we will need. We will follow the book by Björner et al. [7] for notation and reference, unless otherwise indicated. We assume a familiarity with the basics of oriented matroid theory.

Since we will be very seldom concerned with (non-oriented) matroids, we will use the terms *circuits*, *cocircuits*, *vectors* and *covectors* always referring to *signed* ones. We will indistinctly note them as being signed subsets $C = (C^+, C^-)$ of E and as functions $C : E \longrightarrow \{-1, 0, +1\}$, where E is the ground set of an oriented matroid. Using the second point of view we can say that a circuit "is positive" or that it "vanishes" at some given points of E, and will write $C(p) = +1$ with the same meaning as $p \in C^+$.

1.1 Convexity

Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. In order to stress the geometrical meaning of oriented matroid concepts we will call *simplices* of $\mathcal{M}$ the independent subsets of E. A k-simplex is a simplex with k-elements. Thus, r-simplices are the same thing as bases.

Observe that if $\mathcal{M}$ is a realizable oriented matroid and $\mathcal{V} \subset \mathbf{R}^r$ is a vector realization of $\mathcal{M}$ then the geometric counterpart of the k-simplices of $\mathcal{M}$ are simplicial cones of dimension k positively spanned by independent subsets of $\mathcal{M}$. If $\mathcal{M}$ is acyclic and realized by a point configuration $\mathcal{A} \subset \mathbf{R}^{r-1}$ then the k-simplices of $\mathcal{M}$ correspond to simplices of $\mathcal{A}$ of dimension $k-1$, with vertex set contained in $\mathcal{A}$ (this is the standard situation in the literature on triangulations of a point configuration).

Following [7, Chapter 9], we call *facets* of $\mathcal{M}$ the complements of supports of non-negative cocircuits of $\mathcal{M}$ and *faces* the complements of supports of non-negative covectors. Facets are the maximal faces strictly contained in E. We are not assuming $\mathcal{M}$ to be acyclic. In contrast with [7], we do not assume $\mathcal{M}$ to be acyclic. If $\mathcal{M}$ is totally cyclic then it has no proper faces (faces different from E itself).

For any $A \subset E$ we denote by $\mathcal{M}(A)$ the restriction of $\mathcal{M}$ to A. We call faces (resp. facets) of A the faces (resp. facets) of $\mathcal{M}(A)$. In particular, every subset of a k-simplex σ is a face of σ, and it is a facet if and only if it has $k-1$ elements. Of course, all the faces of a simplex are simplices.

The *convex hull* of a subset $A \subset E$ is the union of A and those elements p of $E \setminus A$ for which there is a signed circuit C of $\mathcal{M}$ with $C^+ = \{p\}$ and $C^- \subset A$. We denote this set by $conv_{\mathcal{M}}(A)$. The *relative interior* of A is the set obtained removing the convex hulls of facets of A from the convex hull of A. We denote it by $relint_{\mathcal{M}}(A)$.

In the following two lemmas we prove some properties of the convex hull and relative interior in oriented matroids.

Lemma 1.1 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. Let $p \in E$ and $A \subset E$. Then:*

(i) *$p \in conv_{\mathcal{M}}(A)$ if and only if $p \in conv_{\mathcal{M}(A \cup p)}(A)$, where $\mathcal{M}(A \cup p)$ is the restriction of $\mathcal{M}$ to $A \cup p$.*

(ii) *if $rank_{\mathcal{M}}(A) = k$, then $p \in conv_{\mathcal{M}}(A)$ if and only if there is a k-simplex $\tau \subset A$ with $p \in conv_{\mathcal{M}}(\tau)$.*

(iii) *$p \in conv_{\mathcal{M}}(A)$ if and only if every cocircuit of $\mathcal{M}$ which is nonnegative on A is nonnegative on p.*

(iv) *if $p \in conv_{\mathcal{M}}(A)$ and $A \subset conv_{\mathcal{M}}(B)$ ($B \subset E$) then $p \in conv_{\mathcal{M}}(B)$.*

(v) *if A is an r-simplex, then $p \in conv_{\mathcal{M}}(A)$ if and only if for every $a \in A$ the unique cocircuit of $\mathcal{M}$ vanishing on $A \setminus a$ and positive on a is non-negative on p.*

Proof: Part (i) follows from the fact that circuits of $\mathcal{M}(A \cup p)$ correspond exactly with circuits of $\mathcal{M}$ with support contained in $A \cup p$. Part (ii), considered on the oriented matroid $\mathcal{M}(A \cup p)$, is "Carathéodory's Theorem" [7, Theorem 9.2.1(1)]. Part (iii) is "Weyl's Theorem" [7, Theorem 9.2.1(2)]. Part (iv) follows from (iii).

The "only-if" part in (v) is a consequence of (iii). For the "if" part consider a circuit C with support contained in the spanning but not independent subset $A \cup \{p\}$. Since A is independent, p is in the support of C and without loss of generality we assume $C(p) = +1$. If C was positive at some point $a \in A$, then the orthogonality of C and the cocircuit vanishing on $A \setminus a$ would be violated in the restricted oriented matroid $\mathcal{M}(A \cup \{p\})$. □

Lemma 1.2 *Let $\mathcal{M}$ be an oriented matroid on a set E. Let $a \in E$ and $A, B \subset E$. Then:*

(i) *$p \in relint_{\mathcal{M}}(A)$ if and only if $p \in conv_{\mathcal{M}}(A)$ and for every covector $C = (C^+, C^-)$ vanishing on p, either C vanishes on A or has both negative and positive points in A.*

(ii) *if $p \in relint_{\mathcal{M}}(A)$ and $A \subset conv_{\mathcal{M}}(B)$, but A is not contained in the convex hull of any facet of B, then $p \in relint_{\mathcal{M}}(B)$.*

(iii) *if A is an independent set, then $p \in relint_{\mathcal{M}}(A)$ if and only if $(\{p\}, A)$ is a circuit of $\mathcal{M}$.*

Proof: If $p \in relint_{\mathcal{M}}(A)$, then any cocircuit which is nonnegative on A either vanishes on A or does not vanish on p; otherwise p will be in the convex hull of a facet of $\mathcal{A}$, or not in the convex hull of A. Reciprocally, if $p \in conv_{\mathcal{M}}(A)$, but $p \notin relint_{\mathcal{M}}(A)$, then there is a cocircuit which is nonnegative on A, vanishes on p and does not vanish on A. This proves (i).

For (ii), consider a cocircuit C vanishing on p but not on B. If C vanishes on A, as A is not in a facet of B, C takes both signs on B. If C does not vanish on A, it takes both signs on A and, hence, also on B (because $A \subset conv_{\mathcal{M}}(B)$, and using part (iii) of Lemma 1.1). We conclude that $p \in relint_{\mathcal{M}}(B)$, by the characterization in part (i).

In (iii), A being independent implies that there is at most one circuit with support contained in $A \cup p$. By definition of convex hull, $p \in conv_{\mathcal{M}}(A)$ if and only if the circuit is of the form $(\{p\}, B)$ for some $B \subset A$. If this is the case we have two possibilities: if $B \neq A$, then p is in the convex hull of a proper face of A, and thus not in $relint_{\mathcal{M}}(A)$. If $B = A$, then the orthogonality between circuits and covectors implies, with part (i), that $p \in relint_{\mathcal{M}}(A)$. □

1.2 Extensions. Lexicographic extensions

Let $\mathcal{M}$ and $\mathcal{M}'$ be two oriented matroids on sets E and E'. If $E \subset E'$, and every circuit of $\mathcal{M}$ is a circuit in $\mathcal{M}'$ we say that $\mathcal{M}'$ is an extension of $\mathcal{M}$. Equivalently, $\mathcal{M}'$ is an extension of $\mathcal{M}$ if $\mathcal{M}$ is obtained from $\mathcal{M}'$ by deleting some elements. We will only consider extensions which do not increase the rank, i.e., for which $rank(\mathcal{M}) = rank(\mathcal{M}')$. If $E' \setminus E = \{p\}$ is a single point we say that $\mathcal{M}'$ is a *one-element* extension, and normally use the notation $\mathcal{M} \cup p$ for $\mathcal{M}'$.

Let $\mathcal{M} \cup p$ be a one-element extension of $\mathcal{M}$. For every cocircuit $C = (C^+, C^-)$ of $\mathcal{M}$, exactly one of $(C^+ \cup \{p\}, C^-)$, $(C^+, C^- \cup \{p\})$ and (C^+, C^-) is a cocircuit of $\mathcal{M}$. In other words, there is a unique way to extend each cocircuit of $\mathcal{M}$ into a cocircuit of $\mathcal{M} \cup p$. This means that there is no ambiguity in considering C as a cocircuit in $\mathcal{M} \cup p$, and we can write $C(p) = +1, -1$ and 0, respectively. The function assigning to each cocircuit of $\mathcal{M}$ its value $C(p) \in \{-1, 0, +1\}$ on the new element p is called the *signature* of the extension $\mathcal{M} \cup p$.

Not every map from the set of cocircuits of $\mathcal{M}$ to $\{-1, 0, +1\}$ is the signature function of an extension. Also, not every cocircuit of an extension $\mathcal{M} \cup p$ is the extension of a cocircuit of $\mathcal{M}$. However, it is true that a valid signature function on the cocircuits of $\mathcal{M}$ uniquely determines the extension. More information on these points can be found in [7, Section 7.1]. In particular, the conditions that a signature function has to satisfy to be valid are in Theorem 7.1.8 and the way to obtain all the cocircuits of $\mathcal{M} \cup p$ from the cocircuits of $\mathcal{M}$ and the signature function of $\mathcal{M} \cup p$ is in Proposition 7.1.4 (both results coming from a paper by Michel Las Vergnas).

We will be particularly interested in the so-called *lexicographic extensions*, which were also introduced by Las Vergnas. We take as a definition the following characterization of them which appears in [7, Proposition 7.2.4].

Definition 1.3 Let $\mathcal{M}$ be an oriented matroid on a set E. Let $\{a_1, \dots, a_k\} \subset E$ and choose a sign $\epsilon_i \in \{+, -\}$ for each $i = 1, \dots, k$. The lexicographic extension $\mathcal{M} \cup p$ of $\mathcal{M}$ by the point $p := [a_1^{\epsilon_1}, \dots, a_k^{\epsilon_k}]$ is defined to be the one whose cocircuit signature is given by:

$$C(p) = \begin{cases} \epsilon_i C(a_i) & \text{if } i \text{ is minimal with } C(a_i) \neq 0 \\ 0 & \text{if } C(a_i) = 0, \quad \forall i = 1, \dots, k \end{cases}$$

In particular, with $p := [a^+]$ we obtain the extension by a point p parallel to a, and with $p := [a^-]$ the extension by a point antiparallel to a. In the definition, there is no loss of generality if we assume $a_1, \dots, a_k$ to be independent. In fact, if l is the first index for which $a_1, \dots, a_l$ is dependent, then the point a_l can be removed from the definition without affecting the extension obtained.

Definition 1.4 Let $\mathcal{M} \cup p$ be a one-element extension of an oriented matroid $\mathcal{M}$ of rank r on a set E. We say that the extension is *interior* if $p \in conv_{\mathcal{M} \cup p}(E)$. We say that the extension is in *general position* if $C(p) \neq 0$, for every cocircuit

C of $\mathcal{M}$; equivalently, if the support of every circuit of $\mathcal{M} \cup p$ containing p has exactly $r + 1$ elements.

Lemma 1.5 *Let $\mathcal{M}$ be an oriented matroid of rank r. Let $\mathcal{M} \cup p$ be a lexicographic extension of $\mathcal{M}$ by the point $p := [a_1^{\epsilon_1}, \dots, a_k^{\epsilon_k}]$. Then,*

(i) *the extension is in general position if and only if $rank(\{a_1, \dots, a_k\}) = r$.*

(ii) *if $\epsilon_i = +$ for all i, then $p \in relint_{\mathcal{M} \cup p}(\{a_1, \dots, a_k\})$. In particular, it is an interior extension.*

Proof: Let $A := \{a_1, \dots, a_k\}$. The rank of A in $\mathcal{M}$ equals r if and only if no cocircuit of $\mathcal{M}$ vanishes on A; with this (i) follows easily.

In part (ii), suppose first that A is an independent set in $\mathcal{M}$ (and thus in $\mathcal{M} \cup p$). By definition of lexicographic extension, every cocircuit vanishing on A vanishes also on p. Thus, there is a circuit of $\mathcal{M} \cup p$ with support contained in $A \cup p$. Since A is independent, there is exactly one such circuit C, and p is in its support. Without loss of generality we assume that $C(p) = +1$ and we will proof that, in fact, $C = (\{p\}, A)$. Suppose that the circuit is non-negative in some point a_l of A. Consider a cocircuit D obtained by extension of a cocircuit vanishing on $A \setminus a_l$ but not on a_l (such cocircuit exists, since A is independent). By definition, we have $D(p) = D(a_l)$; but then C and D do not satisfy the orthogonality axiom of circuits and cocircuits. This finishes the proof if A is independent, by part (iii) of Lemma 1.2.

If A is not independent, then we can remove the points a_l which depend on the previous ones from the lexicographic expression of p without altering the expression. The arguments above imply that (p, A') is a circuit, where A' is the independent subset of A obtained by the removal. Thus $p \in conv_{\mathcal{M} \cup p}(A') \subset conv_{\mathcal{M} \cup p}(A)$. Let us prove that p cannot be in a facet of A: if it is in a facet, then there is a cocircuit non-vanishing and non-negative on A which vanishes on p. By construction, $rank(A') = rank(A)$ and, therefore, this cocircuit does not vanish on A'. This violates orthogonality of this cocircuit with the circuit (p, A'). □

Definition 1.6 Let $\mathcal{M} \cup p$ and $\mathcal{M} \cup p'$ be two single-element extensions of an oriented matroid $\mathcal{M}$. We say that p' is a perturbation of p if for every cocircuit C of $\mathcal{M}$

$$C(p) \neq 0 \qquad \Longrightarrow \qquad C(p) = C(p')$$

For example, let $\mathcal{M} \cup p$ be any extension of $\mathcal{M}$ and consider the lexicographic extension of $\mathcal{M} \cup p$ by a point $p' := [p^+, a_1^{\epsilon_1}, \dots, a_k^{\epsilon_k}]$. Then, the extension $\mathcal{M} \cup p' := (\mathcal{M} \cup p) \cup p' \setminus p$ is a perturbation of $\mathcal{M} \cup p$. In case that p itself is given by a lexicographic extension $p := [b_1^{\delta_1}, \dots, b_l^{\delta_l}]$, then p' is the lexicographic extension given by $p' := [b_1^{\delta_1}, \dots, b_l^{\delta_l}, a_1^{\epsilon_1}, \dots, a_k^{\epsilon_k}]$.

Lemma 1.7 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. Let $A \subset E$ and let $\mathcal{M} \cup p$ and $\mathcal{M} \cup p'$ be two extensions of $\mathcal{M}$, with p' being a perturbation of p. Then:*

(i) *if* p'' *is a perturbation of* p'*, then* p'' *is a perturbation of* p *as well.*

(ii) *if* $p' \in conv_{\mathcal{M}\cup p'}(A)$*, then* $p \in conv_{\mathcal{M}\cup p}(A)$.

(iii) *if* A *has rank* r *and* $p \in relint_{\mathcal{M}\cup p}(A)$*, then* $p' \in relint_{\mathcal{M}\cup p'}(A)$.

(iv) *if* $p \in conv_{\mathcal{M}\cup p}(A)$ *and* $p' := [p^+, a_1^+, \ldots, a_k^+]$*, then* $p' \in conv_{\mathcal{M}\cup p'}(A \cup \{a_1, \ldots, a_k\})$.

Proof: Part (i) is obvious from the definition. Part (ii) follows from the characterization of the convex hull by cocircuits (part (iii) of Lemma 1.1). Part (iii) follows from the characterization of the relative interior by cocircuits (part (i) of Lemma 1.2) and the fact that every cocircuit of $\mathcal{M}$ which vanishes on p' also vanishes on p. Part (iv) follows from the fact that, in the oriented matroid $\mathcal{M} \cup \{p, p'\}$, p' is in the convex hull of $\{p, a_1, \ldots .a_k\}$ (part (ii) of Lemma 1.5) and p is in the convex hull of A. □

An important problem concerning extensions is the following: given an oriented matroid $\mathcal{M}$ on a set E and a subset $A \subset E$, suppose that we have an extension $\mathcal{M}(A) \cup p$ or $(\mathcal{M}/A) \cup p$ of a restriction or a contraction of $\mathcal{M}$. Does there exist an extension $\mathcal{M} \cup p$ of $\mathcal{M}$ which extends the given one? By this we mean that $(\mathcal{M} \cup p)/A = (\mathcal{M}/A) \cup p$ in the contraction case and $\mathcal{M} \cup p(\mathcal{A} \cup \{p\}) = \mathcal{M}(A) \cup p$ in the restriction case.

This is not true in general; for example, the oriented matroid $\mathcal{M}$ of the six vertices of a convex hexagon does not depend on the hexagon being regular or not, and has two extensions which are incompatible: the extension $\mathcal{M} \cup p_1$ by a point lying in the intersection of the three main diagonals and an extension $\mathcal{M} \cup p_2$ by a point lying in the intersection of two of them, but not on the third one. This means that the extension p_2 of $\mathcal{M}$ can not be extended to $\mathcal{M} \cup p_1$. However, extensions can be extended in the following cases:

Lemma 1.8 *Let* $\mathcal{M}$ *be an oriented matroid on a set* E *and let* $A \subset E$*. Let* $\mathcal{M}(A) \cup p$ *be a lexicographic extension of the restriction* $\mathcal{M}(A)$*. Let* $\mathcal{M} \cup p'$ *be the lexicographic extension of* $\mathcal{M}$ *with the same lexicographic expression. Then:*

- $\mathcal{M} \cup p'(A \cup \{p'\}) = \mathcal{M}(A) \cup p$.
- *if* p *is interior in* $\mathcal{M}(A)$*, then* p' *is interior in* $\mathcal{M}$.
- *if* p *is in general position and* A *spans* $\mathcal{M}$*, then* p' *is in general position.*

Proof: Every cocircuit of $\mathcal{M}(A)$ extends to a cocircuit of $\mathcal{M}$. The fact that p and p' have the same lexicographic expressions implies that their cocircuit signatures agree on that cocircuits, which in turn implies that the extension of $\mathcal{M} \cup p'(A \cup \{p'\})$ of $\mathcal{M}(A)$ has the same cocircuit signature as $\mathcal{M}(A) \cup p$. The fact that p' is interior whenever p is interior is a consequence of parts (i) and (iv) of Lemma 1.1. The fact that p' is in general position if p is in general position and $\mathcal{A}$ is spanning follows from part (i) of Lemma 1.5. □

Lemma 1.9 *Let $\mathcal{M}$ be an oriented matroid on a set E and let $a \in E$. Let $(\mathcal{M}/a) \cup p$ be an extension of the contraction $\mathcal{M}/a$. Every cocircuit of $\mathcal{M}$ which vanishes on a is a cocircuit of $\mathcal{M}/a$; thus, the following cocircuit signature is well defined: $C(p') = C(a)$ if $C(a) \neq 0$ and $C(p' = C(p)$ otherwise. Then,*

- *The cocircuit signature defines an extension $\mathcal{M} \cup p'$ of $\mathcal{M}$ which satisfies $\mathcal{M} \cup p'/a = (\mathcal{M}/a) \cup p$.*
- *if p is interior in $\mathcal{M}/a$, then p' is interior in $\mathcal{M}$.*
- *if p is in general position, then p' is in general position.*
- *if p is lexicographic, then p' is lexicographic.*

Proof: If p is the lexicographic extension defined by an expression $[a_1^{\epsilon_1}, \dots a_k^{\epsilon_k}]$, then p' is the lexicographic extension defined by the expression $[a^+, a_1^{\epsilon_1}, \dots a_k^{\epsilon_k}]$. This proves the last statement

Now, to show that $C(p')$ defines an extension an arbitrary extension p it suffices (Theorem 7.1.8 in [7]) to show that it defines an extension on every rank 2 contraction of $\mathcal{M}$. So, let $\mathcal{M}/A$ be a rank-2 contraction of $\mathcal{M}$. $\mathcal{M}/(A \cup \{a\})$ is a contraction of rank at most 2 and, thus, p induces a lexicographic extension p_0 on it (every extension of a rank 2 oriented matroid is lexicographic, as is easy to show). Applying the lexicographic case to this, we obtain that p_0 extends to a lexicographic extension of $\mathcal{M}/A$ by the procedure of the statement. But it is easy to verify that this procedure defines precisely the same cocircuit signature as the one induced by $C(p')$ on $\mathcal{M}/A$. This proves that $C(p')$ defines an extension of $\mathcal{M}$. The formula $\mathcal{M} \cup p'/a = (\mathcal{M}/a) \cup p$ follows from the construction.

Finally, part (v) of Lemma 1.1 and part (iii) of Lemma 1.2 show that $p \in relint_{(\mathcal{M}/a)\cup p}(B)$ for some simplex $B \in E \setminus \{a\}$ of $\mathcal{M}/a$ if and only if $p \in relint_{\mathcal{M}\cup p,}(B \cup \{a\})$, where $B \cup \{a\}$ is a simplex of $\mathcal{M}$. Thus, if p is interior and/or in general position, p' is interior and/or in general position. □

2 Triangulations of oriented matroids

2.1 Definition, characterizations and remarks

Remember that we are using the term *simplex* meaning *independent set* and thus *r-simplex* means the same thing as *basis*. We will define a triangulation of an oriented matroid $\mathcal{M}$ of rank r as a collection of r-simplices satisfying some properties. These properties should be the natural translation to oriented matroid terminology of properties characterizing triangulations of point configurations (for the acyclic case) or simplicial fans of vector configurations (for the general case). Candidate properties fall mainly in the following three categories: "covering properties" telling us that the union of the (convex hulls of) simplices of the triangulation covers the convex hull of the configuration; "pseudo-manifold properties" telling us that co-dimension 1 simplices which are not in a facet of $\mathcal{M}$ belong either to 0 or 2 full-dimensional simplices of the triangulation; and "good intersection properties" of the simplices of a triangulation. Also, "good intersection properties" are related to "circuit properties" of the simplices, such as "no circuit has its positive and negative parts being faces of simplices of a triangulation". The following are translations of these properties to oriented matroid terminology, in several degrees. All of them are satisfied for triangulations of point configurations:

Definition 2.1 Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. Let T be a non-empty collection T of r-simplices of $\mathcal{M}$.

- We say that T satisfies the *pseudo-manifold property* if for every $\sigma \in T$, each facet τ of σ is either contained in a facet of $\mathcal{M}$ or there exists another simplex $\sigma' \neq \sigma$ in T with $\tau \subset \sigma'$.

  If T satisfies the pseudo-manifold property, we say that it satisfies the *oriented pseudo-manifold property* if for every $(r-1)$-simplex τ contained in at least two simplices $\tau \cup a_1$ and $\tau \cup a_2$ of T, the unique cocircuit vanishing on τ has opposite signs at a_1 and a_2. In particular, this implies that τ is not contained in any other simplex of T.

- We say that T *covers all* (resp. *some of*) *the interior extensions of* $\mathcal{M}$ if for every (resp. for some) one-element extension $\mathcal{M} \cup p$ of $\mathcal{M}$ with $p \in conv_{\mathcal{M}\cup p}(\mathcal{M})$ there is a $\sigma \in T$ with $p \in conv_{\mathcal{M}\cup p}(\sigma)$. We say that T covers the extension(-s) *once* if the simplex σ is unique in T.

- We say that the simplices of T *intersect properly* if for every one-element extension $\mathcal{M} \cup p$ of $\mathcal{M}$ and every $\sigma_1, \sigma_2 \in T$,

$$p \in conv_{\mathcal{M}\cup p}(\sigma_1) \cap conv_{\mathcal{M}\cup p}(\sigma_2) \quad \Longrightarrow \quad p \in conv_{\mathcal{M}\cup p}(\sigma_1 \cap \sigma_2).$$

- We say that two simplices σ_1 and σ_2 *overlap on a circuit* $C = (C^+, C^-)$ if $C^+ \subset \sigma_1$ and there is an element $a \in C^+$ such that $\underline{C} \setminus \{a\} \subset \sigma_2$.

We take as a starting definition of oriented matroid triangulations a slightly modified version of the definition by Billera and Munson in [5].

Definition 2.2 Let $\mathcal{M}$ be an oriented matroid of rank r. Let T be a collection of r-simplices of $\mathcal{M}$. We say that T is a *triangulation* if it satisfies the pseudo-manifold property and its simplices intersect properly.

A different approach to triangulations of oriented matroids is taken by Anderson in [1]. Anderson is interested in studying combinatorial manifolds as introduced by MacPherson [21] and, thus, the property of being a triangulation is defined "locally", using the contraction operation. Roughly speaking, a collection of r-simplices is a triangulation if and only if for every k-simplex τ contained in some simplex of T the link of τ in T is a triangulation of $\mathcal{M}/\tau$. The *link* is a standard concept in piecewise linear topology, but our definition slightly differs from the usual one (compare [26]) because we deal with the maximal simplices of a simplicial complex, and not with the whole complex:

Definition 2.3 Let $\mathcal{M}$ be an oriented matroid of rank r. Let T be a collection of r-simplices of $\mathcal{M}$. Let τ be a k-simplex, for some $0 < k \leq r$. We call *link* of τ in T the collection of $(r-k)$-simplices $\{\sigma \setminus \tau | \tau \subset \sigma, \sigma \in T\}$. We denote it $link_T(\tau)$.

We can now state the main result of this section, which is the equivalence between the following properties, all characterizing triangulations of oriented matroids.

Theorem 2.4 *Let T be a non-empty collection of r-simplices of an oriented matroid $\mathcal{M}$ of rank r. Then, the following properties are equivalent:*

(a) *The simplices of T intersect properly and T satisfies the pseudo-manifold property (i.e., T is a triangulation of $\mathcal{M}$).*

(b) *The simplices of T intersect properly and T covers every interior extension of $\mathcal{M}$.*

(c) *T satisfies the oriented pseudo-manifold property and covers some interior extension of $\mathcal{M}$ in general position exactly once.*

(d) *T satisfies the oriented pseudo-manifold property and covers all interior extensions of $\mathcal{M}$ in general position exactly once.*

(e) *If $rank(\mathcal{M}) = 1$ then T consists of one simplex if $\mathcal{M}$ is acyclic and two simplices with opposite elements if $\mathcal{M}$ is totally cyclic. If $rank(\mathcal{M}) > 1$ then there is an element $a \in E$ such that*

$$\forall \sigma \in T \qquad a \in \sigma \Longleftrightarrow a \in conv_{\mathcal{M}}(\sigma)$$

and for every element $a \in E$ which appears as a vertex in a simplex of T, the collection $T_a := link_T(a)$ of $(r-1)$-simplices of $\mathcal{M}/a$ is a triangulation of $\mathcal{M}/a$.

(f) *No two simplices of T overlap on a circuit and T satisfies the pseudo-manifold property.*

(g) *T satisfies the oriented pseudo-manifold property and for every triangulation T^* of the dual oriented matroid $\mathcal{M}^*$ there is a unique simplex in T whose complement is in T^*.*

Remarks 2.5 (i) Statement (a) of our theorem is essentially the definition of an oriented matroid triangulation by Billera and Munson [5] (see also [7, Section 9.6]). The differences are that there the oriented matroid $\mathcal{M}$ was assumed to be acyclic and polytopal, and that the original definition included the extra condition that every element appears in some simplex of the triangulation. Our equivalences imply that for acyclic and polytopal oriented matroids this is redundant. For non-polytopal oriented matroids we prefer to allow triangulations not to use all the elements (as is already done in Section 9.6 of [7]), because this gives a richer structure to the collection of triangulations of $\mathcal{M}$.

(ii) In the same paper Billera and Munson mention our statement (b) to be a "more direct translation of the usual definition of a triangulation". They discarded this (apparently stronger) definition because it was not clear to them whether it was satisfied for every *lifting triangulation*, while statement (a) was. We will introduce lifting triangulations in Definition 3.4 and will devote to them Section 5.

(iii) Observe that from (e) it follows by induction that for every face τ of a simplex of T the link $link_T(\tau)$ is a triangulation of $\mathcal{M}/\tau$. This was essentially Anderson's definition of a (perhaps partial) triangulation of an oriented matroid (she calls triangulations "partial" if they do not use all the vertices). Although she is primarily interested in the totally cyclic case, she proves that a version of her definition for the acyclic polytopal case is equivalent to the one by Billera and Munson.

In her definition she poses the pseudo-manifold property, which is redundant because it is equivalent to the recursive condition for $rank(\tau) = r-1$.

We also want to mention that one of the least evident steps in our proof of Theorem 2.4 (namely, Lemma 2.7 which gives the equivalence of (c) and (d)) is inspired by Proposition 3.5 in [1].

(iv) Statements (c) and (d) only differ by the word "some" which changes to "all". This makes these conditions particularly easy to check: because of their equivalence, checking that something is a triangulation reduces to check the oriented pseudo-manifold property and count the number of simplices of T which have a certain interior extension in general position in their convex hull. If the extension is chosen lexicographic this counting is rather easy.

In the other hand, checking properties (a), (b) or (g) implies constructing either all the interior extensions of $\mathcal{M}$ in general position or all the triangulations of $\mathcal{M}^*$, which is extremely hard.

Conditions (f) and (e) might be considered at the same level as (c) and (d) for algorithmic purposes, but (f) is specially suitable for a branch-and-cut algorithm for the construction of all triangulations of a fixed oriented matroid: one can iteratively construct all the collections of simplices in which no pair overlaps and, for the maximal ones, check whether they satisfy the pseudo-manifold property.

(v) In [25, Proposition 2.2] and in [8] a characterization of triangulations of point configurations very similar to part (f) appears. The property that no two simplices overlap on a circuit is substituted for the following (stronger) one: there is no circuit with its positive and negative parts respectively contained in two simplices of the triangulation. For oriented matroids satisfying the *Generalized Euclidean intersection property* $\mathtt{IP}_2$ (cf. Definition 7.5.2 of [7]) this strong property is equivalent to the property that the simplices intersect properly, but not in general; a counterexample to this in the oriented matroid $R(12)$ (see [23]) was shown to me by Jörg Rambau. In Example 5.6 we will show a counterexample in the

Edmonds-Fukuda-Mandel non-euclidean oriented matroid EMF(8) (see [7, Example 10.4.1]).

However, the additional properties of a triangulation (e.g., the pseudo-manifold property) may imply that even in this case no circuit can have its positive and negative parts respectively contained in simplices of a triangulation. This is the case in our example (see Proposition 5.7).

2.2 Equivalence of the different characterizations

In this section we prove the equivalence of properties (a) to (f) in Theorem 2.4. The equivalence of properties (a) and (g) is postponed until Theorem 3.8.

Lemma 2.6 *Let T be a collection of simplices which intersect properly in an oriented matroid $\mathcal{M}$ of rank r.*

(i) *If an $(r-1)$-simplex τ of $\mathcal{M}$ is contained in two different r-simplices $\tau \cup b_1$ and $\tau \cup b_2$ of T, then the unique cocircuit vanishing on τ has opposite signs at b_1 and b_2. In particular, τ is not in a facet of $\mathcal{M}$ and no other simplex of T contains τ.*

(ii) *Let $\mathcal{M} \cup p$ be an extension of $\mathcal{M}$ in general position (we recall from Section 2 that by this we mean that $p \in conv_{\mathcal{M}\cup p}(A)$ only for spanning subsets $A \subset E$). Then there is at most one simplex $\sigma \in T$ with $p \in conv_{\mathcal{M}\cup p}(\sigma)$.*

(iii) *No two simplices of T overlap on a circuit.*

Proof: (i) Let $\tau = \{a_1, \dots, a_{r-1}\}$ be an $(r-1)$-simplex and suppose that it is contained in two different r-simplices $\tau \cup b_1$, and $\tau \cup b_2$. Consider the lexicographic extensions of $\mathcal{M}$ by $p_1 = [a_1^+, \dots, a_{r-1}^+, b_1^+]$ and $p_2 = [a_1^+, \dots, a_{r-1}^+, b_2^+]$. By definition, the signatures of the two extensions can only differ in the unique (up to sign reversal) cocircuit C which vanishes in τ. Moreover, the two extensions cannot agree on that cocircuit: if they did, then $p_1 \in conv(\tau \cup b_1) \cap conv(\tau \cup b_2)$ and hence $p_1 \in conv(\tau)$, by the proper intersection property. This would imply that C is zero at b_1 (also at b_2) which contradicts the fact that $\tau \cup b_1$ is an independent set. Thus, b_1 and b_2 lie on different sides of the cocircuit C. We conclude that τ does not lie in a facet of $\mathcal{M}$ and that no other simplex of T contains τ. This finishes the proof of this part.

(ii) If $\sigma \neq \sigma'$ are two different simplices of T having p in their convex hull, then $p \in conv_{\mathcal{M}\cup p}\sigma \cap \sigma'$ violates the general position assumption on p.

(iii) Suppose that two simplices σ_1 and σ_2 overlap on a circuit; that is, there is a circuit $C = (C^+, C^-)$ with $C^+ \subset \sigma_1$ and an element $a_1 \in C^+$ such that $\underline{C} \setminus \{a_1\} \subset \sigma_2$. Observe that this implies that both C^+ and C^- are non-empty (the latter because $C^+ \subset \sigma_1$ is independent) and in turn that they are both independent sets (since they are proper subsets of $\underline{C}$). Also, we have that $a_1 \notin \sigma_2$, since otherwise the vertices of σ_2 would not be independent.

Let $C^+ := \{a_1, \dots, a_k\}$ and consider the lexicographic extension of $\mathcal{M}$ by the point $p := [a_k^+, \dots, a_1^+]$. This point lies on the flat spanned by $\{a_1, \dots, a_k\}$, but not on the flat spanned by $\{a_2, \ldots, a_k\}$. This implies that it does not lie in $conv_{\mathcal{M}\cup p}(\sigma_1 \cap \sigma_2)$, because the intersection of $\sigma_1 \cap \sigma_2$ with the flat spanned by $\{a_1, \ldots, a_k\}$ is $\{a_2, \ldots, a_k\}$. Also, it is obvious that $p \in conv_{\mathcal{M}\cup p}(\sigma_1)$. We will prove that $p \in conv_{\mathcal{M}\cup p}(\sigma_2)$, which implies that σ_1 and σ_2 intersect improperly and finishes the proof.

Since σ_2 is a full-rank simplex, we can use part (v) of Lemma 1.1. That is, we have to prove that for every $a \in \sigma_2$ the unique cocircuit D which is zero on $\sigma_2 \setminus \{a\}$ and positive on a is non-negative on p. If a is not one of the elements of C^+ or C^-, then the unique cocircuit vanishing on $\sigma_2 \setminus \{a\}$ also vanishes on p. If $a \in C^+ \cap \sigma_2 = \{a_2, \ldots, a_k\}$ the property follows from the lexicographic definition of p. If $a \in C^-$ we have $D(a) = +1$ and $D(p) = D(a_1)$. Thus, $D(p) = -1$ would violate the orthogonality of the circuit C and the cocircuit D. □

Let T be a collection of maximal simplices of an oriented matroid $\mathcal{M}$. For any co-rank 1 simplex τ, let $C_\tau := (C_\tau^+, C_\tau^-)$ be the unique (up to sign reversal) cocircuit vanishing on τ. Let A_τ be the collection of elements a of $\mathcal{M}$ such that $\tau \cup \{a\} \in T$. With this notation, the oriented pseudo-manifold property can be restated as follows: for any co-rank 1 simplex τ which is not contained in a facet of $\mathcal{M}$, the cardinalities of $A_\tau \cap C_\tau^+$ and $A_\tau \cap C_\tau^-$ coincide and equal 0 or 1. If we impose the two cardinalities to coincide but admit them to be greater than 1, we get a property weaker than the oriented pseudo-manifold and which is expressed in [11] by saying that T "satisfies the interior cocircuit equations". The meaning of this will become apparent in Section 3.1. As a preparation for that section we prove the next lemma with this weak oriented pseudo-manifold property as hypothesis. Incidentally, from the lemma it follows that the oriented pseudo-manifold property in parts (c), (d) and (f) of Theorem 2.4 can be changed to this weaker one.

Lemma 2.7 *Let T be a non-empty collection of simplices of an oriented matroid $\mathcal{M}$ of rank r. For any facet τ of a simplex of T which is not contained in a facet of $\mathcal{M}$, let C_τ denote a cocircuit vanishing on τ. Suppose that for every such facet τ the following two sets have the same cardinality:*

$$\{\tau \cup a \in T | a \in C_\tau^+\}, \qquad \{\tau \cup a \in T | a \in C_\tau^-\}.$$

Let $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$ be two different one-element extensions of $\mathcal{M}$, both interior and in general position. Let n_i $(i = 1, 2)$ be the number of simplices σ of T for which $p_i \in conv_{\mathcal{M}\cup p_i}(\sigma)$. Then,

(i) $n_1 = n_2 \geq 1$.

(ii) *There is a chain of simplices $\sigma_0, \ldots, \sigma_k$ in T such that $p \in relint(\sigma_0)$, $p' \in relint(\sigma_k)$ and every two consecutive simplices in the chain share a facet (that is, T is strongly connected, in the sense of simplicial complexes).*

Proof: Let $\{a_1, \ldots, a_r\} \in T$ be one of the r-simplices and consider the lexicographic extension by the point $p := [a_1^+, \ldots, a_r^+]$. If we can prove the lemma for

p and p_1 and also for p and p_2 we will have it for p_1 and p_2. Thus, we consider without loss of generality that $p_1 = p$.

This has the advantage that if we consider the lexicographic extension of $\mathcal{M} \cup p_2$ with the same lexicographic expression $[a_1^+, \ldots, a_r^+]$, the restriction of the resulting oriented matroid $\mathcal{M}' := (\mathcal{M} \cup p_1) \cup p_2$ to $E \cup p_1$ and $E \cup p_2$ coincides with $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$; that is, the two extensions are compatible (which is not true in general, even for realizable extensions of a realizable oriented matroid). By part (i) of Lemma 1.1, there will be no difference in considering the convex hull of simplices in $\mathcal{M}'$ or in the restrictions $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$.

A second advantage of p_1 being a lexicographic extension is that then we have the following "joint general position" for p_1 and p_2 in $\mathcal{M}'$: that all the circuits of $\mathcal{M}'$ having either p_1 or p_2 (or both) in their support have rank r. This follows from the fact that p_1 is an extension of $\mathcal{M} \cup p_2$ in general position and does not follow from the fact that p_1 and p_2 separately are in general position. Finally, the choice of p_1 clearly implies that $n_1 \geq 1$. Thus, in part (i) we only need to prove $n_1 = n_2$.

Let us consider the following directed graph G whose nodes are a subset of the simplices of T:

- a simplex $\sigma \in T$ is a node in the graph if and only if $(\{p_1, p_2\}, \sigma)$ is a vector of $\mathcal{M}'$

- let τ be a certain $(r-1)$-simplex of $\mathcal{M}$ for which $(\{p_1, p_2\}, \tau)$ is a vector (actually a circuit) of $\mathcal{M}'$. In particular, τ is not in a facet of $\mathcal{M}$. Let $C = (C^+, C^-)$ be the cocircuit of $\mathcal{M}'$ vanishing on τ, with sign given so that $p_1 \in C^-$ and $p_2 \in C^+$. Let $\{\sigma_1^+, \ldots, \sigma_k^+\}$ be the simplices of T containing τ and with $\sigma \setminus \tau \in C^+$, and let $\{\sigma_1^-, \ldots, \sigma_l^-\}$ be the simplices of T containing τ and with $\sigma \setminus \tau \in C^-$. The hypothesis of the statement implies that $k = l$. Also, all the simplices σ_i and σ_i^ϵ satisfy that $(\sigma_i^\epsilon, \{p_1, p_2\})$ is a covector of $\mathcal{M}'$. Then, introduce a directed edge going from σ_i^- to σ_i^+ for each $i = 1, \ldots, k$.

We claim that the connected components of the graph G obtained are either isolated points, or linear paths coherently oriented, or oriented cycles (in other words, that G is an oriented 1-manifold except for the isolated points). We also claim that the isolated points correspond to r-simplices containing both p_1 and p_2 in the convex hull and that the starting and end points of the linear paths correspond, respectively, to r-simplices of T having p_1 or p_2 (but not both) in the convex hull. These claims imply $n_1 = n_2$. The claims in turn follow from the following facts:

(1) if a simplex σ has $p_1 \in conv_{\mathcal{M}'}(\sigma)$ and $p_2 \notin conv_{\mathcal{M}'}(\sigma)$, then it is a node of the graph and there is a unique edge incident to it, which is outgoing.

(2) if a simplex σ has $p_1 \notin conv_{\mathcal{M}'}(\sigma)$ and $p_2 \in conv_{\mathcal{M}'}(\sigma)$, then it is a node of the graph and there is a unique edge incident to it, which is in-going.

(3) if a simplex σ has $p_1 \in conv_{\mathcal{M}'}(\sigma)$ and $p_2 \in conv_{\mathcal{M}'}(\sigma)$, then it is an isolated node of the graph.

(4) if a simplex σ has $p_1 \notin conv_{\mathcal{M}'}(\sigma)$ and $p_2 \notin conv_{\mathcal{M}'}(\sigma)$, then either it is not a vertex of the graph, or it is a vertex of the graph with two edges incident to it, one in-going and one outgoing.

All the four facts can be easily proved considering the restriction of $\mathcal{M}'$ to $\sigma \cup \{p_1, p_2\}$, which is realizable and uniform (the latter because of the general position and "joint general position" assumptions). Note that in the realized setting, the signed subset $(\{p_1, p_2\}, A)$ is a vector if and only if the relative interiors of A and the segment going from p_1 to p_2 intersect.

Part (ii) of the lemma follows also from the properties of the graph G. Actually, any of the linear paths or isolated points of the graph G provides a chain in the required conditions.

□

Remark 2.8 Suppose that $\mathcal{M}' := \mathcal{M} \cup \{p_1, p_2\}$ is realized by a point configuration $\mathcal{A}$. If T is a triangulation of $\mathcal{A}$ (in particular, $n_1 = n_2 = 1$ in the statement) then the graph G is the dual (in the sense of cell complexes) to the refinement induced by T in the open segment (p_1, p_2). That is, it is homeomorphic to either a point or a segment. If T is a collection of simplices satisfying the oriented pseudo-manifold property but it is not a triangulation (that is, if $n_i > 1$ in the lemma), then the graph G can be obtained in a similar way starting with n_i copies of the segment. In particular, the graph has no cycles. This property is still true for not realizable oriented matroids if they are *Euclidean*. Actually, showing that the graph G of the previous proof is homeomorphic to either a point or a segment (and a generalization of this fact to simplices of higher dimension) is the key-step in the proof by Anderson of the fact that a triangulation of any Euclidean totally cyclic oriented matroid is PL-homeomorphic to a sphere. For non-Euclidean oriented matroids we do not know of neither a proof that the graph is acyclic nor an example in which it is not.

Lemma 2.9 *Let T be a collection of simplices of an oriented matroid $\mathcal{M}$ satisfying the oriented pseudo-manifold property. Let τ be a subset of one of the simplices of T and let $\mathcal{M} \cup p$ be an extension with $p \in conv_{\mathcal{M}\cup p}(\tau)$. Then, for every perturbation p' of p interior and in general position, there is an r-simplex σ of T containing τ and with $p' \in conv_{\mathcal{M}\cup p'}(\sigma)$.*

Proof: Consider the contraction $\mathcal{M}'_\tau := (\mathcal{M} \cup p')/\tau$ and the collection of simplices $T_\tau := link_T(\tau)$ in $\mathcal{M}'$. T_τ is easily seen to satisfy the oriented pseudo-manifold property. Also, the point p'_τ corresponding to p' in the contraction is still interior and in general position. By Lemma 2.7, there is a simplex σ in T containing τ with $p'_\tau \in conv_{\mathcal{M}'_\tau}(\sigma \setminus \tau)$. We only need to prove that $p' \in conv_{\mathcal{M}\cup p'}(\sigma)$.

We prove this using part (v) of Lemma 1.1: let a be a point in σ, and consider the unique cocircuit C vanishing on $\sigma \setminus a$, with the sign given so that $C(a) = +1$. Then, if $a \in \tau$ we cannot have $C(p') = -1$, because p' is a perturbation of p and $C(p) \neq -1$ (recall that $p \in conv_{\mathcal{M}\cup p}(\tau)$). If $a \notin \tau$, then C vanishes on τ, which implies that C contracts to a cocircuit C_τ of $\mathcal{M}'$, with $C(p') = C_\tau(p'_\tau)$. As p'_τ is in the convex hull of $\sigma \setminus \tau$, this cocircuit is nonnegative on p'_τ. □

Proof of equivalences (a) to (f) in Theorem 2.4:

(b) ⇒(a): Let $\tau := \{a_1, \dots, a_{r-1}\}$ be an $(r-1)$-simplex not contained in a facet of $\mathcal{M}$, and contained in a simplex $\sigma = \tau \cup b$ of T. Since τ is not in a facet, there is a point a_r such that the cocircuit C vanishing on τ has opposite signs at b and a_r. Consider the lexicographic extensions of $\mathcal{M}$ by points $p := [a_1^+, \ldots, a_{r-1}^+]$ and $p' := [a_1^+, \ldots, a_r^+]$. The extension by p' is in general position and has $p' \in conv_{\mathcal{M}}(E)$. As T covers $\mathcal{M}$, there is a simplex σ' in T with $p' \in conv_{\mathcal{M} \cup p'}(\sigma')$. Since p' is a perturbation of p, also $p \in conv_{\mathcal{M} \cup p}(\sigma')$ (part (ii) of Lemma 1.7). Since $p \in conv_{\mathcal{M} \cup p}(\sigma)$ as well, we conclude that $p \in conv_{\mathcal{M} \cup p}(\sigma \cap \sigma')$. But p is in the relative interior of τ, which implies $\tau \subset \sigma'$. This proves the pseudo-manifold property for T.

(a)$\Rightarrow$(c): That the oriented pseudo-manifold property is satisfied is part (i) of Lemma 2.6. For proving that T covers some extension exactly once consider the lexicographic extension of $\mathcal{M}$ by $p := [a_1^+, \ldots, a_r^+]$, where $\sigma = \{a_1, \ldots, a_r\}$ is an r-simplex in T. Clearly $p \in conv_{\mathcal{M} \cup p}(\sigma)$ and p is in general position. By part (ii) of Lemma 2.6, no other convex hull of a simplex of T contains p.

(d)$\Rightarrow$(b): Since T covers all interior extensions in general position, it also covers interior extensions not in general position, by part (ii) of Lemma 1.7.

For the good intersection property, consider an extension $\mathcal{M} \cup p$ and two simplices σ_1 and σ_2 of T with $p \in conv_{\mathcal{M} \cup p}(\sigma_1) \cap conv_{\mathcal{M} \cup p}(\sigma_2)$. Let τ be the minimal subset of σ_1 such that $p \in conv_{\mathcal{M} \cup p}(\tau)$. We will prove that $\tau \subset \sigma_2$, which implies that the simplices intersect properly. Consider the lexicographic perturbation p' of p using the points of σ_2 with positive signs, so that $p' \in conv_{\mathcal{M} \cup p'}(\sigma_2)$, by part (iv) of Lemma 1.7 and the fact that $p \in conv_{\mathcal{M} \cup p}(\sigma_2)$. Also, since p' is a perturbation of p into general position, Lemma 2.9 implies that there is a simplex σ' of T containing τ and with $p' \in conv_{\mathcal{M} \cup p'}(\sigma')$. If $\sigma' \neq \sigma_2$ we would have a contradiction with Lemma 2.7, because we know that there is an extension of $\mathcal{M}$ in general position contained in only one simplex of T.

(c) $\Rightarrow$ (d): This is straightforward from Lemma 2.7, and finishes the equivalences (a) $\Leftrightarrow$ (b)$\Leftrightarrow$ (c)$\Leftrightarrow$ (d).

(a) $\Rightarrow$ (e): Let a be as in the statement. As we already have proved the equivalence of (a) and (c), we will prove property (c) for T_a, instead of (a), and assume properties (a) and (c) for T. The oriented pseudo-manifold property for T_a follows from the fact that for any simplex ω containing a the simplex $\omega \setminus \{a\}$ is in a facet of $\mathcal{M}/a$ if and only if ω is in a facet of $\mathcal{M}$.

Let $\sigma_a := \{a_2, \ldots, a_r\}$ be an $(r-1)$-simplex of T_a and consider the lexicographic extension by the point $p_a := [a_2^+, \ldots, a_r^+]$ of $\mathcal{M}/a$, which is in general position. By part (ii) of Lemma 2.6, $\sigma := a \cup \sigma_a$ is the only simplex of T with $p := [a^+, a_2^+, \ldots, a_r^+] \in conv_{\mathcal{M} \cup p}(\sigma)$. Thus, σ_a is the only simplex of T_a with $p_a \in conv_{(\mathcal{M}/a) \cup p_a}(\sigma_a)$.

(e)$\Rightarrow$(c): The fact that the link of every point is a triangulation, together with the implication (c)$\Rightarrow$(e) implies that the link of every 2-simplex is a triangulation. Recursively, we conclude that the link of every k simplex τ of the triangulation is a triangulation of $\mathcal{M}/\tau$. This property applied to the simplices τ of rank $rank(\mathcal{M}) - 1$ is precisely the oriented pseudo-manifold property.

In order to find an extension which is covered exactly once, let a_1 be the element appearing in the statement of part (e). Let $\sigma := \{a_1, \ldots, a_r\}$ be an r-simplex of T. Consider the lexicographic extension by the point $p := [a_1^+, \ldots, a_r^+]$. We have that $p \in conv_{\mathcal{M} \cup p}(\sigma)$ and will show that σ is the only simplex of T with this property.

In fact, simplices of T not containing a_1 do not have a_1 in their convex hull and, thus, do not have p, which is a perturbation of the extension parallel to a_1; if a simplex σ' containing a_1 has p in its convex hull then, in $(\mathcal{M} \cup p)/a_1$, the contracted simplex $\sigma' \setminus a_1$ has the contracted point p' corresponding to p in its convex hull. But, since this contracted point p' is an interior extension of $\mathcal{M}/a_1$ in general position, only one contracted simplex $\sigma' \setminus a_1$ can have p_{a_1} in its convex hull, namely $\sigma \setminus a_1$.

(a)$\Rightarrow$(f) Follows from part (iii) of Lemma 2.6.

(f)$\Rightarrow$(e) The case of rank 1 is trivial. For the general case, let a be an arbitrary vertex of a simplex of T. If $a \in conv_{\mathcal{M}}(\sigma)$ for a simplex $\sigma \in T$ with $a \notin \sigma$, then there is a circuit of $\mathcal{M}$ of the form $(\{a\}, B)$ with $B \subset A$. That is, σ and any simplex of T having a as a vertex overlap on the circuit $(\{a\}, B)$. This proves the first part of (e).

To finish the proof we only need to prove that the link of every vertex of T satisfies (f). Then, inductively we assume that (f) and (a) are equivalent in rank lower than $rank(\mathcal{M})$, which implies that the link of every vertex in T is a triangulation.

Thus, $a \in E$ be a vertex of T. The pseudo-manifold property for $link_T(a)$ follows from the pseudo-manifold property of T in the same way as we proved the oriented pseudo-manifold property in (a)$\Rightarrow$(e). Now suppose that two simplices τ_1 and τ_2 of $link_T(a)$ overlap on a circuit $C = (C^+, C^-)$ of $\mathcal{M}/a$. That is, $C^+ \subset \tau_1$ and there is an element a_1 in C^+ such that $\underline{C} \setminus \{a_1\} \subset \tau_2$. If (C^+, C^-) is a circuit of $\mathcal{M}/a$, then one of (C^+, C^-), $(C^+ \cup \{a\}, C^-)$ and $(C^+, C^- \cup \{a\})$ is a circuit in $\mathcal{M}$. In the three cases we have that $\tau_1 \cup \{a\}$ and $\tau_2 \cup \{a\}$ overlap in that circuit. □

2.3 Some properties of triangulations

Here we prove several properties of triangulations which either are interesting by themselves or will be used later on. Given two collections A and B of subsets of two disjoint sets E and F respectively, the *join* $A \cdot B$ denotes the following collection of subsets of $E \cup F$:

$$A \cdot B := \{\tau \cup \sigma | \tau \in A, \sigma \in B\}.$$

We will use $A \cdot b$ as an abbreviation for $A \cdot \{\{b\}\}$, for $b \in F$.

Proposition 2.10 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E and suppose that $a \in E$ is an exterior point; i.e., that $a \notin conv_{\mathcal{M}}(E \setminus a)$. Let T be a triangulation of the restricted oriented matroid $\mathcal{M}(E \setminus a)$.*

Let T_a be the collection of facets of simplices of T which "are visible" from a. More precisely, an $(r-1)$-simplex τ of $\mathcal{M}$ is in T_a if and only if it is contained in a simplex of T and there is a cocircuit of $\mathcal{M}$ which is zero on τ, positive at a and non-positive at the rest of the points. Then:

(i) *$T \cup (T_a \cdot a)$ is a triangulation of $\mathcal{M} \cup a$. Here, $T_a \cdot a$ represents the collection of r-simplices obtained joining a to each $(r-1)$-simplex of T_a.*

(ii) T_a *is a triangulation of* $\mathcal{M}/a$.

Proof: (i) We will prove that $T' := T \cup (T_a \cdot a)$ satisfies characterization (c) of Theorem 2.4. We first prove that T' covers some interior extension exactly once. Consider an interior extension $\mathcal{M} \cup p$ with $p \in conv_{\mathcal{M}\cup p}(E \setminus a)$. Since T is a triangulation of $\mathcal{M}(E \setminus a)$ the simplices of T cover p exactly once. In the other hand, the simplices of the form $\tau \cup a$ with $\tau \in T_a$ do not cover p, since there is a cocircuit vanishing on τ and with opposite signs on p and a (compare Lemma 1.1(v)).

Secondly we prove that T' satisfies the oriented pseudo-manifold property: for those $(r-1)$-simplices which are interior to $\mathcal{M}(E \setminus a)$ this is clear, from the oriented pseudo-manifold property of T. For those which are in T_a the property follows from the addition of $T_a \cdot a$. Observe that the $(r-1)$-simplices in T_a are precisely those which are interior to $\mathcal{M}(E \setminus a)$ but not to $\mathcal{M}$, and do not use the point a. We finally have to deal with the $(r-1)$-simplices of T' which are interior and use the point a. These simplices are of the form $\rho \cup a$, where ρ is an $(r-2)$-simplex in a facet of $\mathcal{M}(E \setminus a)$ and the cocircuit C vanishing on $\rho \cup a$ has both positive and negative points.

Consider the link $L := link_T(\rho)$. By characterization (e) of Theorem 2.4, L is a triangulation of the oriented matroid $\mathcal{M}(E \setminus a)/\rho$, which has rank 2 and is not totally cyclic. Such a triangulation always has exactly two boundary 1-simplices, that is, ρ is contained in exactly two boundary $(r-1)$-simplices τ_1 and τ_2 of the triangulation T. The fact that $\rho \cup a$ is interior in $\mathcal{M}$ implies that the cocircuit of $\mathcal{M}$ vanishing on $\rho \cup a$ (which restricts to covectors in $\mathcal{M}(E \setminus a)$) and $\mathcal{M}(E \setminus a)/\rho$ with the same supports) has both positive and negative points. This in turn implies that the points $\tau_1 \setminus \rho$ and $\tau_2 \setminus \rho$ have opposite signs at the cocircuit; that is, T has the oriented pseudo-manifold property.

(ii) Since $T_a = link_{T \cup (T_a \cdot a)}(a)$, this follows from characterization (e) in Theorem 2.4. □

Corollary 2.11 *Let* T *be a triangulation of an oriented matroid* $\mathcal{M}$ *on a set* E *and let* $\mathcal{M} \cup p$ *be an extension of* $\mathcal{M}$. *Then,*

- *If* p *is interior, then* T *is a triangulation of* $\mathcal{M} \cup p$.
- *If* p *is not interior, then there is a triangulation* T' *of* $\mathcal{M} \cup p$ *which extends* T *(that is, with* $T \subset T'$*).*

Proof: The interior case follows trivially from characterization (c) of oriented matroid triangulations. For the non-interior case consider the triangulation exhibited in part (i) of the previous proposition. □

We now consider the following notion of *restriction of a triangulation to a face*. Let T be a triangulation of an oriented matroid $\mathcal{M}$ and let F be a face of $\mathcal{M}$ of rank k. That is, $\mathcal{M}(F)$ is an oriented matroid of rank k. We will call restriction of T to F the following collection of full-rank-simplices of $\mathcal{M}(F)$:

$$\{\tau \mid rank(\tau) = k, \quad \tau \subset F, \quad \exists \sigma \in T \text{ with } \tau \subset \sigma\}.$$

Corollary 2.12 *Let T be a triangulation of an oriented matroid $\mathcal{M}$ and F be a face of $\mathcal{M}$ of rank k. Then, the restriction of T to F is a triangulation of $\mathcal{M}(F)$.*

Proof: Using recursion we only need to prove the case of F being a facet, i.e., $rank(\mathcal{M}) = k+1$. In this case let $\{a_1, \ldots, a_k\}$ be a k-simplex in F and let $b \notin F$ be an element of $\mathcal{M}$. Then the lexicographic extension by $p := [a_1^+, \ldots, a_k^+, b^-]$ is exterior to $\mathcal{M}$. Moreover, the triangulation T_p of $(\mathcal{M} \cup p)/p$ of part (ii) of the previous proposition is precisely the collection of simplices T_F. Observe that although $(\mathcal{M} \cup p)/p$ and $\mathcal{M}(F)$ are different oriented matroids, the first one is an extension of the second one by some interior points. This implies that $T_p = T_F$ is a triangulation of $\mathcal{M}(F)$ as well (this is clear, for example, from characterization (c) in Theorem 2.4). □

Proposition 2.13 *Let $\mathcal{M}$ be an oriented matroid on a set E with only one pair of opposite circuits. Let $C = (C^+, C^-)$ be one of them. Then,*

- *If both parts of C are non-empty (i.e., if $\mathcal{M}$ is acyclic) the only triangulations of $\mathcal{M}$ are*
$$T^+ := \{E \setminus \{e\} | e \in C^+\}$$
and
$$T^- := \{E \setminus \{e\} | e \in C^-\}.$$
- *Otherwise, the only triangulation of $\mathcal{M}$ is*
$$T := \{E \setminus \{e\} | e \in \underline{C}\}.$$

Proof: The oriented matroid $\mathcal{M}$ has one more point than its rank. Actually, its maximal simplices (bases) are the subsets $\underline{C} \setminus e$, for $e \in \underline{C}$. In case (ii) this implies that any triangulation of $\mathcal{M}$ is contained in T. In case (i), the fact that no two simplices of a triangulation overlap on a circuit (characterization (f) in Theorem 2.4) implies that every triangulation is contained in either T^+ or T^-.

Since no two triangulations of an oriented matroid can be contained in one another, we will have finished if we prove that T, T^+ and T^- are in fact triangulations. This is easy to verify and left to the reader (for example, it can be proved recursively, using characterization (e) in Theorem 2.4. It will also be a trivial consequence of Corollary 3.3 in the next section, since the dual $\mathcal{M}^*$ has rank 1). □

Proposition 2.14 *Let H be a flat of an oriented matroid such that the restriction $\mathcal{M}(H)$ is totally cyclic. Let T_1 be a triangulation of the restriction $\mathcal{M}(H)$ and let T_2 be a triangulation of the contraction $\mathcal{M}/H$. Then, the join $T := T_1 \cdot T_2$ is a triangulation of $\mathcal{M}$.*

Proof: We will use characterization (e) of Theorem 2.4, and also induction on $rank(\mathcal{M})$. Since we have implicitly assumed that all the triangulations we consider are in oriented matroids of rank at least 1 (e.g., in characterization (e) of Theorem 2.4), the first case in our induction is $rank(\mathcal{M}) = 2$, $rank(H) = 1$

and $rank(\mathcal{M}/H) = 1$. It is easy to verify the lemma in this case. It is important to observe that at this point we need the assumption of $\mathcal{M}(H)$ being totally cyclic.

Let a be a vertex of T_1 and let us verify that for every simplex $\sigma \in T_1 \cdot T_2$, if $a \in conv_{\mathcal{M}}(\sigma)$ then $a \in \sigma$. Any such simplex σ is a union $\sigma = \tau_1 \cup \tau_2$, with $\tau_1 \in T_1$ and $\tau_2 \in T_2$. Suppose that $a \notin \tau_1$. Then, $a \notin conv_{\mathcal{M}(H)}(\tau_1)$. Thus, there is an element b such that the unique cocircuit of $\mathcal{M}(H)$ which vanishes on $\tau_1 \setminus \{b\}$ and is positive on b is negative on a. That cocircuit extends to the unique cocircuit which vanishes on $\sigma \setminus \{b\}$ and is positive on b. Thus, that cocircuit is negative on a, which implies that $a \notin conv_{\mathcal{M}}(\sigma)$.

To finish the proof of the lemma we show that for every vertex a in T, $link_T(a)$ is a triangulation of $\mathcal{M}/a$. Inductively, we assume the lemma to be true for all oriented matroids of rank $rank(\mathcal{M}) - 1$. If $a \in H$, then $H \setminus \{a\}$ is a totally cyclic flat in $\mathcal{M}/a$ of rank $rank(H) - 1$. Actually, $link_{T_1}(a)$ is a triangulation of $(\mathcal{M}/a)(H \setminus \{a\})$. Then, $link_T(a) = link_{T_1}(a) \cdot T_2$ is in the conditions of the statement and thus is a triangulation of $\mathcal{M}/a$.

If $a \notin H$, then H spans a flat H' in $\mathcal{M}/a$ of rank $rank(H)$. Actually, $(\mathcal{M}/a)(H)$ and $\mathcal{M}(H)$ are isomorphic oriented matroids and, in particular, T_1 is a triangulation of $(\mathcal{M}/a)(H)$. The fact that $(\mathcal{M}/a)(H)$ is totally cyclic implies that T_1 as also a triangulation of $(\mathcal{M}/a)(H')$ (by the first part of Corollary 2.11). In the other hand $(\mathcal{M}/a)/H' = (\mathcal{M}/a)/H = (\mathcal{M}/H)/a$ and thus, $link_{T_2}(a)$ is a triangulation of $(\mathcal{M}/a)/H'$. This implies that $link_T(a) = T_1 \cdot link_{T_2}(a)$ is in the conditions of the lemma. $\square$

Remark 2.15 (Topology of triangulations)

For any triangulation T of an oriented matroid $\mathcal{M}$ we consider the simplicial complex $\mathcal{P}(T)$ induced. This is defined as having T as its collection of maximal simplices. If $\mathcal{M}$ is realizable of rank r, then $\mathcal{P}(T)$ is PL-homeomorphic to a $(r-1)$-sphere if $\mathcal{M}$ is totally cyclic and to a $(r-1)$-ball otherwise. One of the central open problems in the theory of oriented matroid triangulations is deciding whether this is always the case also for non-realizable oriented matroids. Here we will give a partial answer, based on the work of Laura Anderson [1].

The *oriented pseudo-manifold* property implies that the topology of the simplicial complex $\mathcal{P}(T)$ is that of a pseudo-manifold whose boundary is homeomorphic to the boundary of $\mathcal{M}$. That is, the boundary of $\mathcal{P}(T)$ is empty if $\mathcal{M}$ is totally cyclic and a ball of dimension $rank(\mathcal{M}) - 2$ otherwise. It also implies that the chirotope of $\mathcal{M}$ defines an orientation on $\mathcal{P}(T)$. Finally, Lemma 2.7 has as the consequence that $\mathcal{P}(T)$ is strongly connected. All this seems to indicate that the following properties are true, but we are only able to prove that they are equivalent:

Proposition 2.16 *Let r be a natural number. The following statements are equivalent:*

(a) *For every triangulation T of every rank $r+1$ oriented matroid, $\mathcal{P}(T)$ is an r-manifold (possibly with boundary).*

(b) *For every triangulation T of every rank r oriented matroid, $\mathcal{P}(T)$ is an $(r-1)$-sphere if $\mathcal{M}$ is totally cyclic and an $(r-1)$-ball otherwise.*

(c) *For every triangulation T of every totally cyclic rank r oriented matroid, $\mathcal{P}(T)$ is an $(r-1)$-sphere.*

Proof: The equivalence of (a) and (b) is obvious, as well as the implication from (b) to (c). For the converse implication, observe that any triangulation T of a non-totally cyclic oriented matroid $\mathcal{M}$ has always a totally cyclic extension $\mathcal{M} \cup p$. A triangulation T of $\mathcal{M}$ extends to a triangulation T' of $\mathcal{M} \cup p$ by Corollary 2.11. $\mathcal{P}(T)$ is the antistar of p in the simplicial complex $\mathcal{P}(T')$ and, thus, is an $(r-1)$-sphere. □

Anderson [1] has proved condition (c) for triangulations of *Euclidean oriented matroids*, that is, those satisfying the *Euclidean intersection property* IP_3 of [7, Definition 7.5.2]. Also, any of the conditions in the statement is easy to verify for the class of *lifting triangulations* defined in [7, Section 9.6] (we will define them in definitions 3.4 and 4.1).

3 Duality of triangulations and extensions

3.1 Circuit, cocircuit, extension and triangulation vectors

The goal of this section is that every interior extension in general position $\mathcal{M} \cup p$ of a certain oriented matroid $\mathcal{M}$ has associated a triangulation of the dual oriented matroid $\mathcal{M}^*$. The triangulations which can be obtained in this way will be called *lifting triangulations* and in a certain sense are the analogue in oriented matroid terms of the *regular triangulations* of a point configuration. A different, more geometric, definition of lifting triangulations appears in [7, Section 9.6]. Our definition has the advantage of making more explicit the importance of duality in the context of triangulations. In Section 4.1 we will prove the equivalence of the two definitions.

The use of lifting triangulations will allow us to extend to the oriented matroid case most of the results in [11]. In particular, the following notations come from Section 5 in that paper.

Let $\Delta(\mathcal{M})$ denote the collection of all r-simplices of $\mathcal{M}$. Let e_σ denote the standard basis vector of $\mathbf{R}^{\Delta(\mathcal{M})}$ corresponding to an r-simplex σ of $\mathcal{M}$. For any triangulation $T \subset \Delta(\mathcal{M})$ we consider its characteristic vector v_T, which has coordinates $(v_T)_\sigma = 1$ if $\sigma \in T$ and $(v_T)_\sigma = 0$ if $\sigma \notin T$.

Let τ be an $(r-1)$-simplex of $\mathcal{M}$. Let $C = (C^+, C^-)$ be the unique (up to sign reversal) cocircuit vanishing on τ. We define the *cocircuit vector* $Co_\tau \in \{-1, 0, 1\}^{\Delta(\mathcal{M})}$ by

$$Co_\tau \quad := \quad \sum_{i \in C^+} e_{\tau \cup i} \quad - \quad \sum_{j \in C^-} e_{\tau \cup j}.$$

We say that a cocircuit vector is *interior* if both $+1$ and -1 appear among the coordinates of Co_τ (i.e., if τ is not in a facet of $\mathcal{M}$). Let $Co(\mathcal{M})$ denote the collection of all *cocircuit vectors* Co_τ, where τ runs over all $(r-1)$-simplices of $\mathcal{M}$. We denote by $Co_{int}(\mathcal{M})$ the set of interior cocircuit vectors. $\mathcal{M}$ is totally cyclic if and only if $Co(\mathcal{M}) = Co_{int}(\mathcal{M})$.

Dually, let ρ be a spanning $(r+1)$-subset of $\mathcal{M}$. Then ρ contains a unique signed circuit $C = (C^+, C^-)$ of $\mathcal{M}$. We define the *circuit vector* $Ci_\rho \in \{-1,0,1\}^{\Delta(\mathcal{M})}$ by

$$Ci_\rho \quad := \quad \sum_{a \in C^-} e_{\rho \setminus a} \quad - \quad \sum_{a \in C^+} e_{\rho \setminus a}.$$

We say that Ci_ρ is an *acyclic* circuit vector if both $+1$ and -1 appear among the coordinates of Ci_ρ (i.e., if the restriction $\mathcal{M}(\rho)$ is acyclic). Let $Ci(\mathcal{M})$ denote the set of all circuit vectors and $Ci_{ac}(\mathcal{M})$ the subset of acyclic circuit vectors. $\mathcal{M}$ is acyclic if and only if $Ci(\mathcal{M}) = Ci_{ac}(\mathcal{M})$.

Finally, let $\mathcal{M} \cup p$ be an extension of $\mathcal{M}$ with p in general position. Remember that the extension is *interior* if $p \in conv_{\mathcal{M} \cup p}(E)$. We define the *extension vector* $Ext_p \in \mathbf{R}^{\Delta(\mathcal{M})}$ of p by

$$Ext_p := \sum_{\substack{\sigma \in \Delta(\mathcal{M}) \\ p \in conv_{\mathcal{M} \cup p}(\sigma)}} e_\sigma .$$

Observe that different extensions can produce the same extension vector. An extension is interior if and only if its extension vector is non-zero.

We consider circuit and triangulation vectors as proper vectors in $\mathbf{R}^{\Delta(\mathcal{M})}$, while cocircuit and extension vectors are considered linear forms from its dual vector space. We fix the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^{\Delta(\mathcal{M})}$. If $\mathcal{M}^*$ denotes the dual oriented matroid of $\mathcal{M}$, which has rank $(n-r)$, an r-subset σ of E is a basis of $\mathcal{M}$ if and only if its complementary $E \setminus \sigma$ is a basis in $\mathcal{M}^*$. Thus, we can identify $\Delta(\mathcal{M})$ and $\Delta(\mathcal{M}^*)$ by complementarity, which induces an identification of (the dual of) $\mathbf{R}^{\Delta(\mathcal{M})}$ with $\mathbf{R}^{\Delta(\mathcal{M}^*)}$. From duality between circuits and cocircuits it follows that, under this identification, $Ci(\mathcal{M}) = Co(\mathcal{M}^*)$ and $Ci_{ac}(\mathcal{M}) = Co_{int}(\mathcal{M}^*)$.

The utility of the above notation becomes clear from the following result:

Proposition 3.1 *Let T be a collection of r-simplices of an oriented matroid $\mathcal{M}$ of rank r. Let $v_T \in \mathbf{R}^{\Delta(\mathcal{M})}$ be its characteristic vector. Then, the following conditions are equivalent:*

(a) *T is a triangulation of $\mathcal{M}$.*

(b) *$\langle Co_\tau, v_T \rangle = 0$ for every interior cocircuit vector Co_τ and $\langle Ext_p, v_T \rangle = 1$ for some interior extension $\mathcal{M} \cup p$ of $\mathcal{M}$ in general position.*

(c) *$\langle Co_\tau, v_T \rangle = 0$ for every interior cocircuit vector Co_τ and $\langle Ext_p, v_T \rangle = 1$ for every interior extension $\mathcal{M} \cup p$ of $\mathcal{M}$ in general position.*

Proof: The equivalence between (b) and (c) follows from Lemma 2.7.

If T is a triangulation, the equations $\langle Co_\tau, v_T \rangle = 0$ follow from the oriented pseudo-manifold property and the equations $\langle Ext_p, v_T \rangle = 1$ from the fact that T covers every interior extension in general position exactly once. This proves (a)$\Rightarrow$(c).

Suppose now that T is in the conditions of (c). By Theorem 2.4(c) we only need to prove that T satisfies the oriented pseudo-manifold property. Let $\tau = \{a_1, \ldots, a_{r-1}\}$ be a codimension-one simplex of $\mathcal{M}$ not contained in a facet and let $C = (C^+, C^-)$ be a cocircuit vanishing on τ. From the "interior cocircuit equations" in (c) it follows that the number of simplices of T of the form $\tau \cup a$ with $a \in C^+$ equals the number of those with $a \in C^-$. The oriented pseudo-manifold property will be established if we prove that this number is at most one. For this consider the lexicographic extension by $[a_1^+, \ldots, a_{r-1}^+, a^+]$, for any $a \in C^+ \cup C^-$. □

Proposition 3.2 *Let $\mathcal{M}$ be a rank r oriented matroid on E. Let $\mathcal{M}^*$ be the dual oriented matroid. Let $\mathcal{M} \cup p$ and $\mathcal{M}^* \cup p^*$ be extensions of $\mathcal{M}$ and $\mathcal{M}^*$ respectively, both in general position. Then,*

(i) *There is at most one r-simplex σ of $\mathcal{M}$ such that $p \in conv_{\mathcal{M}}(\sigma)$ and $p^* \in conv_{\mathcal{M}^*}(E \setminus \sigma)$. That is, $\langle Ext_p, Ext_{p^*} \rangle \leq 1$, under the identification of $\Delta(\mathcal{M})$ and $\Delta(\mathcal{M}^*)$.*

(ii) *If p is interior, then there exists a p^* for which there is at least one such σ (equality holds in the equation).*

(iii) *$\langle Ci_\rho, Ext_p \rangle = 0$, for every acyclic circuit vector Ci_ρ of $\mathcal{M}$.*

Proof: Suppose that there were two r-simplices $\sigma_1 \neq \sigma_2$ in $\mathcal{M}$ with $p \in conv_{\mathcal{M} \cup p}(\sigma_i)$ and $p^* \in conv_{\mathcal{M}^* \cup p^*}(E \setminus \sigma_i)$, for $i = 1, 2$. This implies that there are two circuits (p, τ_i) in $\mathcal{M}$ with $\tau_i \subset \sigma_1$ and that (τ_1, τ_2) is a vector of $\mathcal{M}$. Similar arguments (in the dual) imply that $\mathcal{M}$ has a covector (ω_1, ω_2) with $\omega_i \subset E \setminus \sigma_i$. This violates the orthogonality of vectors and covectors in $\mathcal{M}$, which proves (i).

If p is interior then there is an r-simplex σ of $\mathcal{M}$ for which $p \in conv_{\mathcal{M} \cup p}(\sigma)$. Let $\sigma^* := \{a_1, \ldots, a_{|E|-r}\}$ be the complement of σ in E. Then, we can take p^* to be the lexicographic extension by the point $p^* := [a_1^+, \ldots, a_{|E|-r}^+]$, which proves (ii).

For (iii), we can assume that $p \in conv_{\mathcal{M} \cup p}(\rho)$, since otherwise the inner product is clearly zero. If this is the case, the restricted oriented matroid $\mathcal{M}' := \mathcal{M} \cup p(\rho \cup p)$ is acyclic. Also, the value of the inner product will be the same in $\mathcal{M} \cup p$ and in $\mathcal{M}'$, by part (i) of Lemma 1.1. Now, the acyclic realizable oriented matroid $\mathcal{M}'$ can be realized by a point configuration in $\mathbf{R}^{r-1}$. The subconfiguration ρ has exactly two triangulations whose characteristic vectors are the positive and negative parts of Ci_ρ. Since p is in the interior of the convex hull of the configuration and in general position, the "extension equations" in Proposition 3.1 imply the equation $\langle Ext_p, Ci_\rho \rangle = 0$

□

Corollary 3.3 *Let $\mathcal{M}$ be an oriented matroid and let $\mathcal{M}^*$ be its dual oriented matroid. Let $\mathcal{M}^* \cup p^*$ be an interior extension of $\mathcal{M}^*$ in general position and let Ext_{p^*} be the corresponding extension vector in $\mathbf{R}^{\Delta(\mathcal{M}^*)}$. Then, Ext_{p^*} is the characteristic vector of a triangulation of $\mathcal{M}$, under the identification between $\Delta(\mathcal{M}^*)$ and $\Delta(\mathcal{M})$.*

Proof: Straightforward from Propositions 3.1 and 3.2, taking into account that the cocircuit vectors of $\mathcal{M}$ correspond to the circuit vectors of $\mathcal{M}^*$ in the identification of $\Delta(\mathcal{M})$ and $\Delta(\mathcal{M}^*)$. □

Definition 3.4 The triangulations of $\mathcal{M}$ obtained by interior extensions in general position of the dual oriented matroid $\mathcal{M}^*$, as in Corollary 3.3, are called *lifting triangulations.* Those obtained by lexicographic extensions are called *lexicographic triangulations.*

Section 5 is devoted to the study of lifting triangulations.

3.2 The affine span of characteristic vectors of triangulations

The following two results are dual to one another and inspired by Theorem 2.2 in [11]; actually our proof is taken almost word by word from that paper, with the obvious changes in notation and dualization. The role of regular triangulations is played here by lexicographic extensions and triangulations.

Theorem 3.5 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. Let $v = \sum_{\sigma\in\Delta(\mathcal{M})} c_\sigma x_\sigma$ ($c_\sigma \in \mathbf{R}$) be any vector in $\mathbf{R}^{\Delta(\mathcal{M})}$. The following properties are equivalent:*

(i) *v is a linear combination of the acyclic circuit vectors Ci_ρ of $\mathcal{M}$ and if v is integer the combination has integer coefficients.*

(ii) *$\langle v_{T^*}, v\rangle = 0$ for the characteristic vector v_{T^*} of every triangulation of the dual oriented matroid $\mathcal{M}^*$.*

(iii) *$\langle Ext_p, v\rangle = 0$ for every extension in general position $\mathcal{M}\cup p$ of $\mathcal{M}$.*

(iv) *$\langle Ext_p, v\rangle = 0$ for every lexicographic extension of $\mathcal{M}$ of the form $p := [b_1^+,\dots,b_r^+]$ where $\{b_1,\dots,b_r\}$ is an r-simplex of $\mathcal{M}$.*

Proof: The implication (i)⇒(ii) follows from Proposition 3.1. (ii)⇒(iii) from Corollary 3.3 and (iii)⇒(iv) is obvious, since any extension in the conditions of (iv) is in general position (Lemma 1.5(i)).

(iv)⇒(i): Let v be in the conditions of (iv). We shall prove that v is a linear combination of the acyclic circuit vectors using a double induction on $n = |E|$ and $r = rank(\mathcal{M})$. In particular, we assume the statement to be true for the deletion $\mathcal{M}\setminus a_1$ and the contraction $\mathcal{M}/a_1$, where a_1 is an element of E.

We suppose that a_1 is not a loop, since otherwise the inductive step is trivial. For any r-simplex σ not containing a_1, the set $\rho = \sigma\cup\{a_1\}$ is clearly the only spanning $(r+1)$-set containing σ and a_1; also, σ is the only r-simplex contained in ρ and not containing a_1. Subtracting from v appropriate multiples of $Ci_{\sigma\cup\{a_1\}}$ for those σ for which $\sigma\cup\{a_1\}$ is acyclic, we get another vector v' in the conditions of (iv) but in which the variables x_σ corresponding to these simplices do not appear. That is,

$$v' \quad = \sum_{\substack{\sigma:a_1\notin\sigma\\ \sigma\cup\{a_1\}\text{is acyclic}}} c_\sigma x_\sigma \quad + \sum_{\sigma:a_1\in\sigma} c'_\sigma x_\sigma .$$

Let us call v_1 and v_2 the two sums in the above expression, respectively.

We claim that under the natural identification of the simplices of $\mathcal{M}/a_1$ with the simplices of $\mathcal{M}$ containing a_1 the second sum v_2 is in the conditions of (iv) for the contracted oriented matroid $\mathcal{M}/a_1$. Indeed, take a lexicographic extension by $p' := [b_1^+, \dots, b_{r-1}^+]$ of $\mathcal{M}/a_1$ with the b_i's in increasing order, and consider the lexicographic extension by $p := [a_1^+, b_1^+, \dots, b_{r-1}^+]$ of $\mathcal{M}$. All the simplices σ with non-zero entry in the extension vector Ext_p satisfy that $\sigma \cup \{p\}$ is acyclic. Thus, $\langle Ext_p, v_1 \rangle = 0$. Since also $\langle Ext_p, v' \rangle = 0$ we conclude that $\langle Ext_p, v_2 \rangle = 0$. But the simplices of Ext_p containing a_1 are the same as the simplices of $Ext_{p'}$, and thus $\langle Ext_{p'}, v_2 \rangle = 0$.

By inductive hypothesis, v_2 is a linear combination of the acyclic circuit vectors of $\mathcal{M}/a_1$. Now, every such acyclic circuit vector $Ci_{\rho'}$ extends to an acyclic circuit vector of $\mathcal{M}$ by putting a non-zero entry in the coordinate of ρ', in case ρ' is a basis of $\mathcal{M}$. Thus, subtracting from v' the extended version of the expression of v_2 as linear combination of acyclic circuit vectors, we get another vector

$$v'' = \sum_{\sigma : a_1 \notin \sigma} c''_\sigma x_\sigma$$

in the conditions of (iv). If we prove that v'' is a linear combination of acyclic circuit vectors we will have finished, because v'' was obtained from v by subtracting linear combinations of acyclic circuit vectors. We can assume that a_1 is not a coloop, since otherwise $v'' = 0$.

Now, v'' can be considered a vector in $\mathbf{R}^{\Delta(\mathcal{M} \setminus a_1)}$. Moreover, every lexicographic extension $(\mathcal{M} \setminus a_1) \cup p'$ of $\mathcal{M} \setminus a_1$ in the conditions of the statement induces a lexicographic extension $\mathcal{M} \cup p$ of $\mathcal{M}$ in the same conditions, by just picking up the same sequence of defining points for the extension. The restriction of Ext_p to the simplices not containing a_1 coincides with $Ext_{p'}$. Thus, v'' is in the conditions of (iv), when considered in $\mathcal{M} \setminus a_1$. By inductive hypothesis, v'' is a linear combination of acyclic circuit vectors of $\mathcal{M} \setminus a_1$, which in turn are acyclic circuit vectors of $\mathcal{M}$. □

Corollary 3.6 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. Let $h = \sum_{\sigma \in \Delta(\mathcal{M})} c_\sigma x_\sigma$ ($c_\sigma \in \mathbf{R}$) be any vector in $\mathbf{R}^{\Delta(\mathcal{M})}$. The following properties are equivalent:*

(i) *h is a linear combination of the interior cocircuit vectors Co_τ of $\mathcal{M}$ and if h is integer the combination has integer coefficients.*

(ii) *$\langle h, v_T \rangle = 0$ for the characteristic vector v_T of every triangulation of $\mathcal{M}^*$.*

(iii) *$\langle h, v_T \rangle = 0$ for every lexicographic triangulation T of $\mathcal{M}$ produced by a lexicographic extension of the dual $\mathcal{M}^*$ in the conditions of part (iv) of Theorem 3.5.*

Proof: Follows from Theorem 3.5, by duality. □

Corollary 3.7 *Let $\mathcal{M}$ be an oriented matroid. Then,*

(i) *The characteristic vector of any triangulation is an affine combination of characteristic vectors of lexicographic triangulations Moreover, the lexicographic triangulations can be taken from those in the statement of part (iv) of Theorem 3.5.*

(ii) *The affine span of all the characteristic vectors of triangulations of $\mathcal{M}$ is defined by the interior cocircuit equations $\langle Co_T, \cdot \rangle = 0$ and any non-homogeneous affine equation satisfied on every characteristic vector (e.g., the equation $\langle Ext_p, \cdot \rangle = 1$, for any interior extension $\mathcal{M} \cup p$ of $\mathcal{M}$ in general position).*

(iii) *The difference $v_T - v_{T'}$ of the characteristic vectors of two triangulations of $\mathcal{M}$ is an integer combination of acyclic circuit vectors of $\mathcal{M}$.*

Proof: Part (i) follows from the equivalence of parts (ii) and (iii) of Corollary 3.6. Part (ii) follows from the equivalence of (i) and (ii) in the same corollary. Part (iii) follows from the equivalence of (i) and (iii) in Theorem 3.5 and the "extension equations" in part (iii) of Proposition 3.1. □

This leads to a stronger version of Proposition 3.1.

Theorem 3.8 *Let T be a collection of r-simplices of an oriented matroid $\mathcal{M}$ of rank r. Let $v_T \in \mathbf{R}^{\Delta(\mathcal{M})}$ be its characteristic vector. Then, the following conditions are equivalent:*

(a) *T is a triangulation of $\mathcal{M}$.*

(b) *$\langle Ci_\rho, v_T \rangle = 0$ for every acyclic circuit form Ci_ρ of the dual oriented matroid $\mathcal{M}^*$ and $\langle v_{T^*}, v_T \rangle = 1$ for some triangulation v_{T^*} of $\mathcal{M}^*$.*

(c) *$\langle Ci_\rho, v_T \rangle = 0$ for every acyclic circuit form Ci_ρ of the dual oriented matroid $\mathcal{M}^*$ and $\langle v_{T^*}, v_T \rangle = 1$ for every triangulation v_{T^*} of $\mathcal{M}^*$.*

Proof: The equivalence of the circuit equations in (b) and (c) and the cocircuit equations in Proposition 3.1 is trivial from duality. Also, the fact that every vector extension Ext_p of $\mathcal{M}$ is the characteristic vector of a triangulation gives the implications from (a) to (b) and from (c) to (a). We only need to prove the implication from (b) to (c), but this is a consequence of part (ii) of Corollary 3.7. □

Corollary 3.7 implies the relation $D = N - R - 1$, between the number N of r-simplices (bases) of $\mathcal{M}$, the rank R of the linear span of its interior cocircuit vectors and the dimension D of the affine span of incidence vectors of triangulations. If the oriented matroid is uniform we clearly have $N = \binom{n}{r}$; we also can give explicit formulas for the other quantities, as is done in [11].

Lemma 3.9 *Let $\mathcal{M}$ be a uniform oriented matroid. Then, every interior cocircuit vector of $\mathcal{M}$ is the difference of two interior extension vectors and every non-interior cocircuit vector is (up to sign reversal) an interior extension vector.*

Proof: Let C be a cocircuit. Let $\tau = \{a_1, \ldots, a_{r-1}\}$ be the unique $(r-1)$-simplex in which the cocircuit vanishes. Let a_r be a point not in τ. Consider the two lexicographic extensions $p^\epsilon := [a_1^+, \ldots, a_{r-1}^+, a_r^\epsilon]$, which are in the conditions of the previous lemma. Thus, we have that $Co_\tau = Ext_{p^+} - Ext_{p^-}$. If C is an interior cocircuit then the two extensions are interior, while if C is non-interior only one of the two extensions is interior and the other one has zero extension vector. □

Theorem 3.10 *Let $\mathcal{M}$ be a uniform oriented matroid of rank r on n elements. Let $H_\mathcal{M}$ be the affine span of the characteristic vectors of triangulations of $\mathcal{M}$. Then,*

(i) *the linear space parallel to $H_\mathcal{M}$ equals the linear span of the acyclic circuit vectors of $\mathcal{M}$.*

(ii) *The linear spans $lin(Co(\mathcal{M}))$ and $lin(Ci(\mathcal{M}))$ of the cocircuit and circuit vectors of $\mathcal{M}$ are orthogonal complements in $\mathbf{R}^{\Delta(\mathcal{M})}$.*

(iii) *$dim(lin(Co(\mathcal{M}))) = \binom{n-1}{r-1}$ and $dim(lin(Ci(\mathcal{M}))) = \binom{n-1}{r}$.*

(iv) *$H_\mathcal{M}$ has dimension $\binom{n-1}{r}$ if $\mathcal{M}$ is acyclic and $\binom{n-1}{r} - 1$ if it is not.*

Proof: We recall that $\mathcal{M}$ is uniform if and only if the dual $\mathcal{M}^*$ is uniform and that exactly one of $\mathcal{M}$ and $\mathcal{M}^*$ is acyclic and the other one is totally cyclic. Also, that every r-set is a basis and thus $dim(\mathbf{R}^{\Delta(\mathcal{M})}) = \binom{n}{r}$.

Let a_1 be an arbitrary point of $\mathcal{M}$. The cocircuit vectors Co_τ such that $a_1 \notin \tau$ are linearly independent, because each r-simplex containing a_1 appears in exactly one of them. Thus, $dim(lin(Co_\tau)) \geq \binom{n-1}{r-1}$. In the same way, the circuit vectors Ci_ρ for the circuits ρ containing a_1 are independent, because each r-simplex not containing a_1 appears in exactly one of them. Thus, $dim(lin(Ci_\tau)) \geq \binom{n-1}{r}$.

The dualized version of Lemma 3.9 tells us that every acyclic circuit vector of $\mathcal{M}$ is a difference of two (lifting) triangulations and every non-acyclic circuit vector is a triangulation vector itself. This proves that circuit vectors and interior cocircuit vectors are orthogonal. If $\mathcal{M}$ is totally cyclic (every cocircuit vector is interior, because $\mathcal{M}$ has no facets), this finishes parts (ii) and (iii). If $\mathcal{M}$ is not totally cyclic, then $\mathcal{M}^*$ is totally cyclic. We have proved (ii) and (iii) for $\mathcal{M}^*$, which imply the statements for $\mathcal{M}$. Also (i) follows from the above statements.

Finally, the dimension formulae in (iv) follow from part (ii) of Corollary 3.7: if $\mathcal{M}$ is totally cyclic we have to subtract one dimension to the orthogonal complement of $Co(\mathcal{M})$, while if $\mathcal{M}$ is acyclic we do not need to because there is some non-interior cocircuit vector Co_τ, which produces the non-homogeneous equation $\langle Co_\tau, v_T\rangle = 1$. □

The previous Theorem implies that the interior cocircuit equations (the first ones in condition (c) of Proposition 3.1) follow from the extension equations (the second ones) in the uniform case. This is not true if $\mathcal{M}$ is not uniform, as shown in [11].

3.3 Mutations versus bistellar flips

For both extensions of an oriented matroid and for triangulations there are notions of a "local" or "elementary" change between two of them. These are, respectively, the so-called *mutations* and *geometric bistellar flips*. It is not surprising that these two concepts be dual to one another under the duality of triangulations and extensions depicted in the previous sections. Here we will explore this duality. We take the following as a definition:

Definition 3.11 Let $\mathcal{M}$ be an oriented matroid. Let $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$ be two extensions of $\mathcal{M}$ in general position and let T_1 and T_2 two triangulations of $\mathcal{M}$.

(i) We say that p_1 and p_2 differ by a *mutation* if their cocircuit signatures differ only in one pair of opposite cocircuits. We say that the mutation is *supported* on those cocircuits.

(ii) We say that T_1 and T_2 differ by a *bistellar flip* if the difference of their characteristic vectors is a sum of acyclic circuit vectors which are supported on the same circuit C. We say that the bistellar flip is *supported* on C.

If $\mathcal{M}$ is uniform, our definition of mutation of the extensions is equivalent to the one in [7, Definition 7.3.8], except for a slight difference in point of view. We are interested in mutating between one-element extensions of a fixed oriented matroid $\mathcal{M}$, while there the main interest is mutating between arbitrary uniform oriented matroids of the same rank and cardinality. In the non-uniform case our mutations correspond to moving the general position point p_1 to an "almost-general" position (the circuits containing it have at least r points) and then perturbing it back to general position. Let us see this in more detail:

Proposition 3.12 *Let $\mathcal{M}$ be an oriented matroid. Let $C = (C^+, C^-)$ be a cocircuit of $\mathcal{M}$. Let C_0 denote the complement of the support of C and let $\mathcal{M}_0$ denote the restriction of $\mathcal{M}$ to C_0. Let $\mathcal{M} \cup p$ be an extension of $\mathcal{M}$ whose cocircuit signature is zero only on the cocircuit C (and its opposite). Let a be a point in the support of C.*

(i) *Consider the two lexicographic extensions of $\mathcal{M} \cup p$ obtained as $p_{a^+} := [p^+, a^+]$ and $p_{a^-} := [p^+, a^-]$. Then, $\mathcal{M} \cup p_{a^+}$ and $\mathcal{M} \cup p_{a^-}$ are the only extensions of $\mathcal{M}$ which are perturbations of $\mathcal{M} \cup p$. Moreover, for points $a \in C^+$ and $b \in C^-$ we have $\mathcal{M} \cup p_{a^+} = \mathcal{M} \cup p_{b^-}$, and vice-versa, and for points a and b in the same part of C we have that $\mathcal{M} \cup p_{a^\epsilon} = \mathcal{M} \cup p_{b^\epsilon}$, $\epsilon \in \{+, -\}$.*

(ii) *Let $\{\tau_1, \dots, \tau_l\}$ be the bases of $\mathcal{M}_0$ (i.e., the $(r-1)$-simplices contained in C_0) satisfying $p \in conv_{\mathcal{M} \cup p}(\tau_i)$. Let $a \in C^+$. Consider the cocircuit vectors $Co_{\tau_1}, \dots, Co_{\tau_l}$ with signs given so that the coefficient of the simplex $\tau_i \cup \{a\}$ is positive. Then,*

$$\sum_{i=1}^{l} Co_{\tau_i} = Ext_{p_{a^+}} - Ext_{p_{a^-}}.$$

(iii) $\mathcal{M} \cup p_{a^+}$ *and* $\mathcal{M} \cup p_{a^-}$ *differ by a mutation supported on* C. *Moreover, every pair of extensions which differ by a mutation can be obtained in this way.*

Proof: The proof of (i) is straightforward: clearly, p_{a^+} and p_{a^-} are well-defined extensions which are perturbations of p. In the other hand, any perturbation of p is determined by the value of its cocircuit signature on the cocircuit C. The relations between perturbations which use different points are trivial.

For proving (ii), let σ be an arbitrary r-simplex of $\mathcal{M}$ and let us see that its coefficient in the right hand side equals the one in the left hand side. Suppose that the left hand side is non-zero. This implies that σ contains one of the simplices τ_i; because of part (i) we can assume that $\sigma = \tau_i \cup a$ without loss of generality, and then it becomes clear that the coefficient of σ in both the right hand side and the left hand side is 1.

Reciprocally, suppose that the coefficient of σ in the right hand side is non-zero. This means that precisely one of the two extension elements p_{a^+} and p_{a^-} is contained in the convex hull of σ and, in particular, that $p \in conv_{\mathcal{M}\cup p}(\sigma)$, because of part (ii) of Lemma 1.7. Also, part (v) of Lemma 1.1 tells us that the signatures of the two extensions must differ in a cocircuit vanishing in a facet of σ, that is, that σ contains an $(r-1)$-simplex contained in C^0. This two facts together imply that σ contains one of the simplices τ_i. But then, we can assume without loss of generality that $\sigma = \tau_i \cup a$ and, as before, conclude that the coefficient of σ in both sides of the equation equals one.

For (iii), it is clear that $\mathcal{M} \cup p_{a^+}$ and $\mathcal{M} \cup p_{a^-}$ differ by a mutation supported on C. The converse follows from Lemma 7.3.3 of [7]. □

We will now look at bistellar flips. Our Definition 3.11 of them is rather abstract, while for the case of triangulations of a point configuration a more geometric definition exists. This is for example what Gelfand et al. [15, pages 231–233] call a *modification* of a triangulation, and is a concept which appears quite often in recent literature on triangulations of polytopes (see [3, 8, 10, 11, 25]). For more complete information the reader should consult the recent survey by Lee [20].

Parts (i) and (ii) of the following statement show what the more geometric definition of a bistellar flip is, and part (iii) says that this more geometric definition is equivalent to our abstract one. Examples of bistellar flips appear in Figure 1. Parts (a), (b) and (c) of the Figure show the three possible types of flips in dimension 2 (rank 3). The one in part (a) is degenerate in the sense that it is supported in a non-full-rank circuit. Parts (d) and (e) show the two possible non-degenerate flips in dimension 3 (rank 4).

Remember that we use the notation $A \cdot B$ for the *join* of two collections of simplices, defined as

$$A \cdot B := \{\sigma \cup \tau \mid \sigma \in A, \tau \in B\}.$$

Proposition 3.13 *Let* $\mathcal{M}$ *be an oriented matroid and let* $C := (C^+, C^-)$ *be an acyclic circuit (that is, a circuit with non-empty positive and negative parts) of* $\mathcal{M}$. *Let* T *be a triangulation of* $\mathcal{M}$. *Then:*

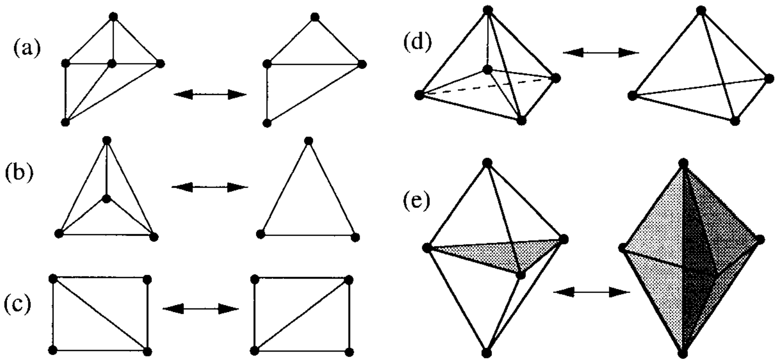

Figure 1: Some examples of geometric bistellar flips.

(i) *The restricted oriented matroid* $\mathcal{M}(\underline{C})$ *has exactly two triangulations, namely*

$$T_C^+ := \{\underline{C} \setminus \{e\} | e \in C^+\}$$

and

$$T_C^- := \{\underline{C} \setminus \{e\} | e \in C^-\}.$$

(ii) *Suppose that* T_C^- *is a subcomplex of* T *(that is, every simplex of* T_C^- *is contained in a simplex of* T*) and that the links of all the simplices of* T_C^- *in* T *coincide. In these conditions, let* L *be the link in* T *of the simplices of* T_C^-*. Then,* T *contains the join* $T_C^- \cdot L$ *and*

$$T' := T \setminus (T_C^- \cdot L) \cup T_C^+ \cdot L$$

is a triangulation of $\mathcal{A}$.

(iii) T *and* T' *differ by a bistellar flip supported on the circuit* C*. Moreover, every pair of triangulations which differ by a bistellar flip arise in this way.*

Proof: (i) This follows from Proposition 2.13.

(ii) Since L is the link of a face τ of some simplex of T, L is a triangulation of $\mathcal{M}/\tau$. Now, τ is one of the maximal simplices of T_C^-, which implies that τ spans $\underline{C}$. Thus, L is also a triangulation of $\mathcal{M}/\underline{C}$. The fact that T contains $T_C^- \cdot L$ is clear, since L is the link of every simplex of T_C^-. We will prove that T' satisfies characterization (c) of Theorem 2.4.

We first reformulate the oriented pseudo-manifold property; if instead of a collection T of rank-r simplices we consider the simplicial complex whose maximal simplices are those of T, the oriented pseudo-manifold property is equivalent to the fact that T is a pseudo-manifold whose boundary is contained in the boundary (proper faces) of $\mathcal{M}$, and that T is consistently oriented by the chirotope of $\mathcal{M}$. Thus, once we know the oriented pseudo-manifold for T, the oriented pseudo-manifold for T' will follow if we proof that $T_C^+ \cdot L$ and $T_C^- \cdot L$

are pseudo-manifolds with the same boundary. The fact that they are pseudo-manifolds follows from the fact that L, T_C^+ and T_C^- are pseudo-manifolds. The fact that they have the same boundary follows from the fact that T_C^+ and T_C^- have the same boundary (namely, the simplicial complex whose maximal simplices are $\{\underline{C} \setminus \{a,b\} | a \in C^+, b \in C^-\}$.

We finally have to find a vertex a in T' such that $a \in conv_{\mathcal{M}}(\sigma)$ and $\sigma \in T'$ implies that $a \in \sigma$. If there is no element $a \in \underline{C}$ which is a vertex in both T and T', then the positive and negative parts of C consist on one element each, the two being parallel in $\mathcal{M}$. Condition (f) in Theorem 2.4 implies that only one of them (the one in the positive part of C) is used in T; the transformation from T to T' is just the substitution of this vertex for the other and it is clear that T' will be a triangulation. Otherwise let $a \in \underline{C}$ be an element which is a vertex of both T and T'. This implies that no simplex $\sigma \in T' \setminus (T_C^+ \cdot L) \subset T$ has $a \in conv_{\mathcal{M}}(\sigma)$ unless $a \in \sigma$).

For the rest of the simplices of T', i.e., those of the form $\tau_1 \cup \tau_2$ with $\tau_1 \in T_C^+$ and $\tau_2 \in L$, if $a \notin \tau_1$, then $a \notin conv_{\mathcal{M}}(\underline{C})(\tau_1)$, because a is a vertex in the triangulation T_C^+ of $\mathcal{M}(\underline{C})$. Thus, there is an element $b \in \tau_1$ such that the unique cocircuit of $\mathcal{M}(\underline{C})$ which vanishes on $\tau_1 \setminus \{b\}$ and is positive on b is negative on a. This cocircuit extends to the unique cocircuit of $\mathcal{M}$ which vanishes on $\tau_1 \cup \tau_2 \setminus \{b\}$ and is positive in b, which thus is negative on a. This implies $a \notin conv_{\mathcal{M}}(\tau_1 \cup \tau_2)$.

(iii) For each simplex $\tau \in L$, $\underline{C} \cup L$ is a spanning $(r+1)$-subset whose circuit vector is precisely the difference of the incidence vectors of $T_C^+ \cdot \tau$ and $T_C^- \cdot \tau$. This proves that $v'_T - v_T$ is a sum of acyclic circuit vectors supported on the circuit C.

For the converse, let T and T' be two triangulations which differ by a bistellar flip supported on the acyclic circuit $C = (C^+, C^-)$. Let r and k be the ranks of $\mathcal{M}$ and C. Each circuit vector supported on C is the difference of the incidence vectors of $T_C^+ \cdot \tau$ and $T_C^- \cdot \tau$, for some $(r-k)$-simplex. Thus, the fact that T and T' are triangulations and their incidence vectors differ by a sum of circuit vectors supported on C implies that T_C^+ is a subcomplex of one of them and T_C^- a subcomplex of the other. Suppose that T_C^- is a subcomplex of T and T_C^+ of T'. Characterization (f) of Theorem 2.4 implies that T does not contain any simplex of T_C^+ as a face and T' does not contain one of T_C^-. Thus, the fact that we pass from T to T' by a sum of acyclic vectors supported on C implies that all the simplices of T_C^- have the same link L in T, which becomes the link of all the simplices of T_C^+ in T'. This finishes the proof. □

We finally show the relation between bistellar flips on lifting triangulations and mutations of the associated extensions of the dual:

Theorem 3.14 *Let $\mathcal{M}$ be an oriented matroid of rank r in n points, with dual $\mathcal{M}^*$. Let $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$ be two interior extensions of $\mathcal{M}$ in general position which differ by a mutation and let T_1 and T_2 be the corresponding lifting triangulations of $\mathcal{M}^*$. Then, either $T_1 = T_2$ or T_1 and T_2 differ by a bistellar flip.*

Proof: If the two extensions p_1 and p_2 do not coincide, then Proposition 3.12 implies that the extension vectors Ext_{p_1} and Ext_{p_2} differ by a sum of cocircuit vectors of $\mathcal{M}$ supported on the same cocircuit; it is easy to check that if p_1 and p_2 are both interior, then the cocircuit vectors are interior. Thus, the incidence vectors of T_1 and T_2 differ by a sum of acyclic circuit vectors supported on the same circuit of $\mathcal{M}^*$. □

It is natural to ask the converse of this; if T_1 and T_2 are lifting triangulations which differ by a bistellar flip, is it always true that they have associated extensions which differ by a mutation? It seems difficult to answer this question in general, because the extension associated to a lifting triangulation is not unique. In Section 4.2 we will see that the answer is positive in the case of Lawrence polytopes, for which each lifting triangulation is associated to a unique extension of the dual.

4 Subdivisions of Lawrence polytopes

4.1 Lifting subdivisions. Subdivisions

We have defined lifting triangulations of an oriented matroid $\mathcal{M}$ by means of extensions of the dual oriented matroid $\mathcal{M}^*$. In [7, pag. 410] lifting triangulations are defined in a more geometric (but equivalent) way that explains the name "lifting". The idea comes from a paper by Billera and Munson [5] although there only a particular (lexicographic) lift is considered. We introduce now that definition and show its equivalence with Definition 3.4. The following definition of a *lift* of an oriented matroid is also taken from [5] and the same concept appears in [7] under the name *one-element lifting.* It is the dual concept to a one-element extension.

Definition 4.1 Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. A *lift* of $\mathcal{M}$ is an oriented matroid $\widehat{\mathcal{M}}$ of rank $r+1$ on a set $E \cup \hat{p}$ such that $\widehat{\mathcal{M}}/\hat{p} = \mathcal{M}$. We say that the lift is *acyclic* if $\widehat{\mathcal{M}}$ is acyclic.

Given an acyclic lift $\widehat{\mathcal{M}}$ of an oriented matroid $\mathcal{M}$, the *lifting polytopal subdivision* (or *lifting subdivision* for short) T of $\mathcal{M}$ associated to the lift is the following collection of subsets of E, to be called *cells* of the subdivision:

$$T := \{A \subset E \quad | \quad A \text{ is a facet of } \widehat{\mathcal{M}}\}$$

We say that the subdivision is *simplicial* if all the cells are simplices.

Observe that Definition 4.1 is not really so far from our definition of lifting triangulations. Indeed, the dual $\widehat{\mathcal{M}}^*$ of an acyclic lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$ is a totally cyclic extension $\mathcal{M}^* \cup p$ of $\mathcal{M}^*$. The fact that $\mathcal{M}^* \cup p$ is totally cyclic is equivalent to the reorientation $\mathcal{M}^* \cup \overline{p}$ of p in it being an interior extension of $\mathcal{M}^*$.

Proposition 4.2 *Simplicial lifting subdivisions and lifting triangulations are the same thing. More precisely:*

(i) *Let T be a simplicial lifting subdivision of $\mathcal{M}$ associated with the acyclic lift $\widehat{\mathcal{M}}$. The dual $(\widehat{\mathcal{M}})^*$ is an extension of $\mathcal{M}^*$ that we denote by $\mathcal{M}^* \cup p$. Denote by $\mathcal{M}^* \cup \overline{p}$ the reorientation of the element p in $\mathcal{M}^* \cup p$. Then, any perturbation of $\mathcal{M}^* \cup \overline{p}$ into general position is an interior extension with associated lifting triangulation of $\mathcal{M}$ equal to T.*

(ii) *Let T be a lifting triangulation of $\mathcal{M}$ associated with an extension $\mathcal{M}^* \cup \overline{p}$ of the dual oriented matroid $\mathcal{M}^*$. Denote by $\mathcal{M}^* \cup p$ the reorientation at the element $\overline{p}$ of $\mathcal{M}^* \cup \overline{p}$. Then, the dual oriented matroid $(\mathcal{M}^* \cup p)^*$ is a lift of $\mathcal{M}$ whose lifting subdivision is T.*

Proof: Let us first give a different characterization of the cells of a lifting subdivision. A subset $A \subset E$ is a cell in the lifting subdivision of the lift $\widehat{\mathcal{M}}$ if and only if $((E \setminus A) \cup \widehat{p}, \emptyset)$ is a cocircuit in $\widehat{\mathcal{M}}$. This in particular implies that $A \cup \widehat{p}$ is a spanning subset of $\widehat{\mathcal{M}}$; thus, A is a spanning subset of $\mathcal{M}$ and $E \setminus A$ is an independent subset of $\mathcal{M}^*$.

The dual of an acyclic lift $\widehat{\mathcal{M}}$ of $\mathcal{M}$ is a totally cyclic extension $\mathcal{M}^* \cup p$ of the dual $\mathcal{M}^*$. Thus, the reorientation $\mathcal{M}^* \cup \overline{p}$ is an interior extension of $\mathcal{M}^*$. Even more, $\overline{p} \in relint_{\mathcal{M}^* \cup \overline{p}}(E)$. The reciprocal is also true; that is, there is a 1-to-1 correspondence between acyclic lifts of $\mathcal{M}$ and relative interior extensions of $\mathcal{M}^*$, by reorientation of the dual. Under this correspondence, $((E \setminus A) \cup \widehat{p}, \emptyset)$ is a cocircuit of the lift $\widehat{\mathcal{M}}$ if and only if $(E \setminus A, \{\overline{p}\})$ is a circuit of the extension $\mathcal{M}^* \cup \overline{p}$. This implies (ii), since the lifting triangulation T contains by definition precisely those independent sets A of $\mathcal{M}^* \cup \overline{p}$ for which $(A, \{\overline{p}\})$ is a circuit of $\mathcal{M}^* \cup \overline{p}$.

For proving (i) we have the extra difficulty that $\mathcal{M}^* \cup \overline{p}$ may not be an extension in general position. However, we have the following property, which follows from the fact that T is simplicial: any subset $A \in E$ with $\overline{p} \in conv_{\mathcal{M}^* \cup \overline{p}}(A)$ is spanning; in particular we have that $\overline{p} \in relint_{\mathcal{M}^* \cup \overline{p}}(A)$ and thus that $p' \in relint_{\mathcal{M}^* \cup p'}(A)$ for any perturbation $\mathcal{M}^* \cup p'$ of $\mathcal{M}^* \cup \overline{p}$. That is, all the simplices of T are in the lifting triangulation corresponding to p'. The reciprocal follows with the same kind of arguments. □

Example 4.3 (Lifting triangulations via lifts)

An example of the equivalence between the two definitions is shown in Figure 2. Parts (a) and (b) show an oriented matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$, both of rank 2 and realized as vector configurations. Parts (c) and (e) show two different acyclic lifts of $\mathcal{M}$, which have rank 3, realized as point configurations in the plane. Recall that if an oriented matroid is realized by a point configuration A and p is a point of A (i.e., an element of $\mathcal{M}$), the contraction $\mathcal{M}/p$ is realized by the vector configuration $\{a - p \mid a \in A \setminus \{p\}\}$.

The segments drawn in parts (c) and (e) are the facets of the oriented matroid which do not contain p; that is, the maximal simplices of the induced lifting triangulations of (a). Parts (d) and (f) of the figure show two extensions of $\mathcal{M}^*$ (noted p) and their opposites (noted $\overline{p}$). It is easily checked that the simplices of part (d) (resp. part(f)) which contain the extension $\overline{p}$ in their convex hulls are the complements of the simplices in the lifting subdivision in part (c) (resp. part (e)).

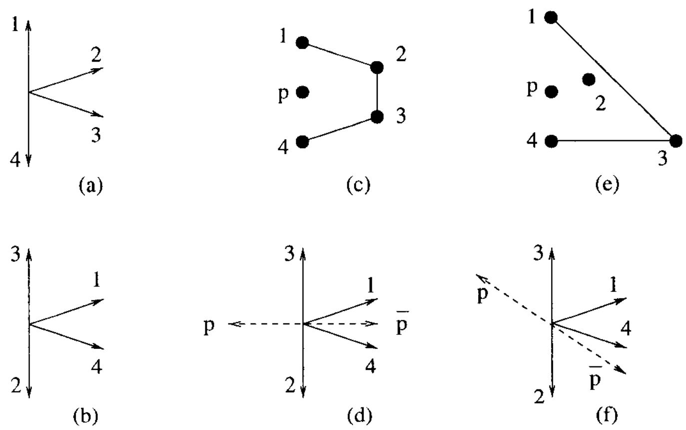

Figure 2: Lifting triangulations defined by lifts and by extensions of the dual.

Remark 4.4 (Lexicographic triangulations by "pushings and pullings")
In the realizable acyclic case, lexicographic triangulations were characterized by Carl Lee [19] as the ones that can be obtained from the trivial subdivision of a polytope by a sequence of *pushings* and *pullings* of points. This description is generalized to oriented matroid triangulations in [7, p. 410] (for the acyclic case; the description also works in the non-acyclic case, except that the starting "trivial subdivision" has no geometric meaning). Although we will not give the details, Figure 3 will help to understand the process.

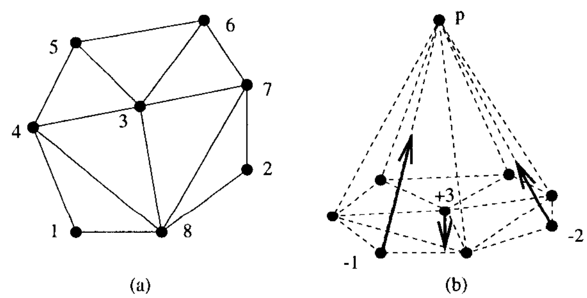

Figure 3: A lexicographic triangulation and the associated lift.

Part (a) shows a certain triangulation of a planar point configuration and part (b) shows how to obtain it by a "lexicographic lift" with the expression $[1^-, 2^-, 3^+]$. Recall that this corresponds to the triangulation associated to the opposite lexicographic extension of the dual, that is, to $[1^+, 2^+, 3^-]$. The lexicographic lift is constructed by adding a coloop p to $\mathcal{M}$ (the apex of the pyramid in part (b) of the figure) and then perturbing the points in the order they appear in the lexicographic expression, *pulling* them towards the apex if they have negative sign and *pushing* them away from the apex if they have positive sign. At the end of the process we obtain a lift of $\mathcal{M}$ such that the anti-star of p in the boundary of this lift is the lexicographic triangulation of $\mathcal{M}$.

The definition of *lifting subdivisions* suggests the concept of a general *subdivision* of an oriented matroid, which has to agree with the concept of polytopal subdivision of a polytope if the oriented matroid is polytopal, and with the concept of triangulation in the simplicial case. The following definition is taken from [7, page 408]. As it happened with triangulations, we will not assume our oriented matroids to be acyclic.

Definition 4.5 Let $\mathcal{M}$ be an oriented matroid of rank r on a set E. A non-empty collection Δ of subsets of E (called *cells*) is a *subdivision* of $\mathcal{M}$ if it satisfies:

(a) For every cell $\sigma \in \Delta$ the restriction $\mathcal{M}(\sigma)$ is acyclic and has rank r.

(b) for every one-element extension $\mathcal{M} \cup p$ of $\mathcal{M}$ and every $\sigma_1, \sigma_2 \in \Delta$,

$$p \in conv_{\mathcal{M}\cup p}(\sigma_1) \cap conv_{\mathcal{M}\cup p}(\sigma_2) \qquad \Longrightarrow \qquad p \in conv_{\mathcal{M}\cup p}(\sigma_1 \cap \sigma_2)$$

(c) If $\sigma_1, \sigma_2 \in \Delta$, then $\sigma_1 \cap \sigma_2$ is a common face of the two restrictions $\mathcal{M}(\sigma_1)$ and $\mathcal{M}(\sigma_2)$.

(d) If $\sigma \in \Delta$, then each facet of $\mathcal{M}(\sigma)$ is either contained in a facet of $\mathcal{M}$ or contained in precisely two cells of Δ.

If all the cells are full-rank simplices, Definition 4.5 specializes to Definition 2.2 of an oriented matroid triangulation. Indeed, conditions (i) and (iii) are then redundant and the other two are respectively our *pseudo-manifold* and *proper intersection* properties. The following results are proved in [7] for the acyclic case, and generalize to the perhaps-non-acyclic case with exactly the same proofs.

- Lifting subdivisions are a particular case of a subdivision.
- If $\mathcal{M}$ is acyclic and realized by a point configuration $X = \{x_1, \ldots, x_n\}$ in $\mathbf{R}^{r-1}$, then a collection of cells in $\mathcal{M}$ is a subdivision of $\mathcal{M}$ if and only if $\Delta' := \{conv\{x_i \mid i \in \tau\} \mid \tau \in \Delta\}$ is a subdivision of the polytope $P = conv(X)$.
- If $\mathcal{M}$ is realized by a vector configuration $X = \{x_1, \ldots, x_n\}$ in $\mathbf{R}^r$, then a collection of cells in $\mathcal{M}$ is a subdivision of $\mathcal{M}$ if and only if $\Delta' := \{pos\{x_i \mid i \in \tau\} \mid \tau \in \Delta\}$ is a polyhedral fan with support $P = pos(X)$.

It is reasonable to think that suitable translations of the characterizations of triangulations in Theorem 2.4 yield characterizations of oriented matroid subdivisions. Since our main interest is only in triangulations we will not get into showing this. However, the following result is probably a key step in doing it, and will be of use to us. It is a version of Lemma 2.7.

Lemma 4.6 *Let Δ be a subdivision of an oriented matroid $\mathcal{M}$. Let $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$ be two different one-element extensions of $\mathcal{M}$, both interior and in general position. Then,*

(i) *There is exactly one cell σ_{p_1} in Δ with $p_1 \in relint_{\mathcal{M}\cup p_1}(\sigma_{p_1})$ and another σ_{p_2} with $p_2 \in relint_{\mathcal{M}\cup p_2}(\sigma_{p_2})$.*

(ii) *There is a chain of cells $\sigma_{p_1} = \sigma_0, \ldots, \sigma_k = \sigma_{p_2}$ in Δ such that every two consecutive cells in the chain share a facet (in particular, Δ is a strongly connected cell complex).*

Proof: Let r be the rank of $\mathcal{M}$.

Let $\{a_1, \ldots, a_k\} \in \Delta$ be one of the cells of the subdivision and consider the lexicographic extension by the point $p := [a_1^+, \ldots, a_k^+]$. If we can prove the lemma for p and p_1 and also for p and p_2 we will have it for p_1 and p_2. Thus, we consider without loss of generality that $p_1 = p$. In these conditions, Lemma 1.8 tells us that there is a two-element extension $\mathcal{M}' = \mathcal{M} \cup \{p_1, p_2\}$ which restricts to $\mathcal{M} \cup p_1$ and $\mathcal{M} \cup p_2$ by deletion of one of the two elements. It is easy to check that Δ is also a subdivision of $\mathcal{M}'$: properties (a) and (c) are trivial; property (b) follows from the fact that any extension of $\mathcal{M}'$ restricts to an extension of $\mathcal{M}$ and property (d) follows from the fact that p_1 and p_2 are interior and in general position and thus the proper faces of $\mathcal{M}$ and of $\mathcal{M} \cup \{p_1, p_2\}$ are the same.

A second advantage of p_1 being a lexicographic extension, already mentioned in the proof of Lemma 2.7, is that then we have the following "joint general position" for p_1 and p_2 in $\mathcal{M}'$: that all the circuits of $\mathcal{M}'$ having either p_1 or p_2 (or both) in their support have rank r. Also, property (b) of Definition 4.5 for the subdivision Δ implies that $\sigma_{p_1} = [a_1, \ldots, a_k]$ is the only cell of Δ with $p_1 \in conv_{\mathcal{M}'}(\sigma_{p_1})$.

As we did in the proof of Lemma 2.7, we consider the following directed graph G whose nodes are some of the cells of Δ:

- a cell $\sigma \in \Delta$ is a node in the graph if and only if $(\{p_1, p_2\}, \sigma)$ is a vector of $\mathcal{M}'$.

- let τ be a certain $(r-1)$-face of a simplex of Δ for which $(\{p_1, p_2\}, \tau)$ is a vector of $\mathcal{M}'$. In particular, τ is not in a facet of $\mathcal{M}$ and there are exactly two cells σ^+ and σ^- in Δ having τ as a facet. Let $C = (C^+, C^-)$ be the cocircuit of $\mathcal{M}'$ vanishing on τ, and assume without loss of generality that $p_1 \cup (\sigma^+ \setminus \tau) \subset C^+$ and $p_2 \cup (\sigma^- \setminus \tau) \subset C^-$. Then, introduce a directed edge going from σ^+ to σ^-.

We claim that the connected components of the graph G obtained in this way are either isolated points, or linear paths coherently oriented, or oriented cycles (in other words, that G is an oriented 1-manifold except for the isolated points). We also claim that the isolated points correspond to cells containing both p_1 and p_2 in the convex hull and that the starting and end points of the linear paths correspond, respectively, to cells of Δ having p_1 and p_2 (but not both) in the convex hull. These claims, together with the fact that only one cell contains p_1 in the convex hull, imply the statement. The claims in turn follow from the following facts:

(1) if a cell σ has $p_1 \in conv_{\mathcal{M}'}(\sigma)$ and $p_2 \notin conv_{\mathcal{M}'}(\sigma)$, then it is a node of the graph and there is a unique edge incident to it, which is out-going.

(2) if a cell σ has $p_1 \notin conv_{\mathcal{M}'}(\sigma)$ and $p_2 \in conv_{\mathcal{M}'}(\sigma)$, then it is a node of the graph and there is a unique edge incident to it, which is in-going.

(3) any other cell σ which is a node in the graph G is either an isolated node or has two edges incident to it, one in-going and one-outgoing.

In Lemma 2.7 we proved these facts using realizability of every co-rank 2 oriented matroid. Here we need to use a different proof. Observe that since the two extensions are in general position, they being in the convex hull of a cell is the same as being in the relative interior.

Let us prove part (1). Thus, we assume p to be in the relative interior of σ and p_2 not to be. Consider the acyclic oriented matroid $\widehat{\mathcal{M}_0} := \mathcal{M}'(\sigma \cup \{p_1\}$, which is a lift of $\mathcal{M}_0 := \mathcal{M}'(\sigma \cup \{p_1\})/p_1$. Let Δ_0 be the lifting subdivision of $\mathcal{M}_0$ induced. By inductive hypothesis, the interior extension in general position $\mathcal{M}_0 \cup p_2 := \mathcal{M}'(\sigma \cup \{p_1, p_2\})/p_1$ has the extension point p_2 contained in precisely one cell of Δ_0, which means that there is exactly one facet τ of σ such that $(\tau, \{p_2\})$ is a vector of $\mathcal{M}_0 \cup p_2$. The fact that p_1 is in the convex hull of σ and p_2 is outside implies that this vector extends to the vector $(\tau, \{p_1, p_2\})$, and that the corresponding edge of G is the unique edge incident to the node of σ and is oriented as desired. Part (2) is completely analogue.

We finally deal with part (3). We assume σ to be a cell of Δ which gives a node in the graph with at least one edge, but not in the conditions of (1) or (2). Thus, $(\sigma, \{p_1, p_2\})$ is a vector. Also, there is a facet τ of σ such that $(\tau, \{p_1, p_2\})$ is a vector. The latter implies that one of p_1 and p_2 is not in the convex hull of σ. In order not to be in cases (1) or (2), the other one is also not in the convex hull of σ.

As before, consider the lift $\widehat{\mathcal{M}_0} := \mathcal{M}'(\sigma \cup p_1)$ of $\mathcal{M}_0 := \mathcal{M}'(\sigma \cup \{p_1\})/p_1$, and the extension $\mathcal{M}_0 \cup p_2 = \mathcal{M}'(\sigma \cup \{p_1, p_2\})/p_1$. $\widehat{\mathcal{M}_0}$ is acyclic. If it was not, the reorientation $\overline{p}_1$ of p_1 in $\widehat{\mathcal{M}_0}$ would be an interior extension of $\mathcal{M}(\sigma)$. Case (1) applied to $\overline{p}_1$ and p_2 would imply the existence of a facet τ of σ with p_1 and p_2 lying in one side of τ and $\sigma \setminus \tau$ lying in the other. Then, the cocircuit vanishing on τ would not be orthogonal with the vector $(\sigma, \{p_1, p_2\})$.

Thus, let Δ_0 be the lifting triangulation of $\mathcal{M}_0$ associated to the lift $\widehat{\mathcal{M}}_0$. The inductive argument shows the existence of a unique facet τ_1 of σ such that the cocircuit $C_1 = (C_1^+, C_1^-)$ vanishing on τ_1 has $p_1 \in C_1^+$, $p_2 \in C_1^-$ and $\sigma \cap C_1^- = \emptyset$. With the same arguments applied to contracting p_2 instead of p_1, we obtain a unique facet τ_2 such that the cocircuit $C_2 = (C_2^+, C_2^-)$ vanishing on it has $p_1 \in C_2^+$, $p_2 \in C_2^-$ and $\sigma \cap C_2^+ = \emptyset$. These two facets cannot coincide, (otherwise they would span σ) and thus provide the unique two edges incident to the vertex of G corresponding to σ. The edge corresponding to τ_1 is out-going and the one corresponding to τ_2 is in-going. □

The question of how to recover the lift/extension associated to a lifting subdivision of an oriented matroid is partially answered in the following lemma.

Lemma 4.7 *Let Δ be a lifting subdivision of an oriented matroid $\mathcal{M}$, defined by the acyclic lift $\widehat{\mathcal{M}}$. Let $\mathcal{M}^* \cup p$ be the (relative interior) extension of $\mathcal{M}^*$*

obtained by reorientation of $\overline{p}$ in the totally cyclic extension $\mathcal{M}^* \cup \overline{p}$ dual to $\widehat{\mathcal{M}}$. Then, the following properties hold for every cocircuit $C = (C^+, C^-)$ of $\mathcal{M}^*$:

- *If some cell of Δ contains the support $C^+ \cup C^-$ of C then the cocircuit signature of the extension $\mathcal{M}^* \cup p$ at the cocircuit C of $\mathcal{M}^*$ is $C(p) = 0$.*
- *Otherwise, if some cell of Δ contains C^+ (resp. C^-), then $C(p) = -1$ (resp. $C(p) = +1$).*

Proof: Observe that for any circuit $C = (C^+, C^-)$ of $\mathcal{M}$ exactly one of $(C^+ \cup \{\widehat{p}\}, C^-)$ $(C^+, C^- \cup \{\widehat{p}\})$ and (C^+, C^-) is a circuit of the lift $\widehat{\mathcal{M}}$, where $\widehat{p}$ denotes the extra element in the lift. If a cell of Δ contains $C^+ \cup C^-$, then $C^+ \cup C^-$ is in a facet of $\widehat{\mathcal{M}}$ not containing $\widehat{p}$. This implies that $rank(\underline{C} \cup \{p\}) = rank(\underline{C}) + 1$ and the the only possible circuit of the three above is (C^+, C^-) itself. Thus, $C = (C^+, C^-)$ is a cocircuit of $\mathcal{M}^* \cup p$; that is, $C(p) = 0$. If no cell contains $C^+ \cup C^-$, assume that a cell of Δ contains C^+ but not C^- (the other case is analogue). Then there is a positive cocircuit D of $\widehat{\mathcal{M}}$ which has empty intersection with C^+ but non-empty with C^-. Orthogonality of circuits and cocircuits implies that the only possible extension of the circuit C to $\widehat{\mathcal{M}}$ is $(C^+ \cup \widehat{p}, C^-)$, which is as required in order that the cocircuit signature of $\mathcal{M}^* \cup p$ at C be $C(p) = -1$. □

Corollary 4.8 (i) *Let Δ be a subdivision of an oriented matroid $\mathcal{M}$ and let $\mathcal{M} \cup p$ be an interior extension of $\mathcal{M}$. Then, there is at least one cell $\sigma \in \Delta$ with $p \in conv_{\mathcal{M} \cup p}(\sigma)$.*

(ii) *Let Δ and Δ' be two subdivisions of an oriented matroid $\mathcal{M}$. If one is contained in the other, then they coincide.*

Proof: If p is in general position, (i) is weaker than the statement of Lemma 4.6. If p is not in general position, then apply the same thing and part (ii) of Lemma 1.7 to any perturbation p' of p interior and in general position (which always exist; e.g., a lexicographic perturbation can do the job).

For (ii), suppose that $\Delta' \subset \Delta$ and that there is a cell $\sigma \in \Delta \setminus \Delta'$. We consider any extension $\mathcal{M} \cup p$ of $\mathcal{M}$ in general position and in the convex hull of σ (such as a lexicographic extension by the elements in σ with positive super-index). By Lemma 4.6 there is a cell $\sigma' \in \Delta'$ with p in the convex hull of σ', but this is a contradiction with the same lemma, since both σ and σ' are cells in Δ. □

4.2 Lawrence polytopes only have lifting subdivisions

We recall that an oriented matroid $\mathcal{M}$ is *polytopal* (or a *matroid polytope*) if every one-element subset is a face. In particular, it has to be acyclic. An oriented matroid $\mathcal{M}$ of rank r on n points has an associated oriented matroid $\Lambda(\mathcal{M})$ of rank $n + r$ on $2n$ points which is (essentially) polytopal and has the property that the whole oriented matroid structure of $\mathcal{M}$ is contained in the face lattice of $\Lambda(\mathcal{M})$. This construction was invented by Jim Lawrence (unpublished). References for the construction are [7, Section 9.3], [6], [2] and [30, p. 180]; the latter two deal mainly with the realizable case.

In this section we will see that every subdivision of the *Lawrence polytope* $\Lambda(\mathcal{M})$ is a lifting subdivision and that, in fact, there is a 1-to-1 correspondence between the subdivisions of $\Lambda(\mathcal{M})$ and the extensions (interior or not) of the dual oriented matroid $\mathcal{M}^*$ of $\mathcal{M}$. Under this correspondence bistellar flips correspond exactly to mutations, and the correspondence will have the ionteresting consequence of relating the *extension space conjecture* [29] to a conjecture regarding subdivisions of polytopes.

Let us fix some notation. Let $\mathcal{M}$ be an oriented matroid of rank r on a set E of n points and let $\mathcal{M}^*$ be its dual. We construct an oriented matroid $\Lambda(\mathcal{M})^*$ on the set $E \times \{1, -1\}$, which can be geometrically interpreted (e.g., in a realized setting) as the union of $\mathcal{M}^*$ and its image by the central inversion. More precisely, we identify E with $E \times \{1\}$ and will write $\overline{A}$ to denote $A \times \{-1\}$, for every subset A of E. Then, $\Lambda(\mathcal{M})^*$ is the extension of $\mathcal{M}$ by n points $\overline{e} \in \overline{E}$ antiparallel to the corresponding $e \in E$. In other words, $\Lambda(\mathcal{M})^*$ is the only oriented matroid which satisfies:

- $\Lambda(\mathcal{M})^*$ is an oriented matroid of rank $n - r$ on $E \cup \overline{E}$ whose restriction to E is $\mathcal{M}^*$.

- For any $e \in E$, the element $\overline{e} \in \overline{E}$ is a loop in $\Lambda(\mathcal{M})^*$ if and only if $e \in E$ is a loop and, if it is not a loop, then $(\{e, \overline{e}\}, \emptyset)$ is a positive circuit of $\Lambda(\mathcal{M})^*$.

Let $\Lambda(\mathcal{M})$ be the dual of $\Lambda(\mathcal{M})^*$. From the construction it follows that $(\{e, \overline{e}\}, \emptyset)$ is a covector of $\Lambda(\mathcal{M})$ for every $e \in E$ and, in particular, that $\{e, \overline{e}\}$ is a face (and the complement of a face as well). If $\{e, \overline{e}\}$ has rank 2, then both e and $\overline{e}$ are vertices. If the rank is 1, then e and $\overline{e}$ are parallel elements and they form a "double" vertex of $\Lambda(\mathcal{M})$. This happens if and only if e is a loop in $\mathcal{M}$, in which case $(\{e\}, \emptyset)$ is a cocircuit in $\mathcal{M}^*$ and $(\{e\}, \{\overline{e}\})$ is a cocircuit in $\Lambda(\mathcal{M})^*$.

Therefore, $\Lambda(\mathcal{M})$ is polytopal if $\mathcal{M}$ is loop-less, and is "almost" polytopal otherwise. If $\mathcal{M}$ is a realizable oriented matroid, then $\Lambda(\mathcal{M})$ is realizable and admits a direct geometrical construction from $\mathcal{A}$ by means of a sequence of *Lawrence extensions* (see [30, Theorem and Definition 6.26]).

Definition 4.9 In the above conditions we say that $\Lambda(\mathcal{M})$ is the *Lawrence matroid polytope* (or Lawrence polytope, for short) associated to $\mathcal{M}$.

A basic property relating $\mathcal{M}^*$ and $\Lambda(\mathcal{M})^*$ is that their extensions are in 1-to-1 correspondence and that every extension of $\mathcal{M}^*$ is interior when regarded in $\Lambda(\mathcal{M})^*$ (the latter follows from the fact that $\Lambda(\mathcal{M})^*$ is totally cyclic). This implies that a triangulation of $\Lambda(\mathcal{M})$ is lifting if it is defined by an extension of $\mathcal{M}^*$ in general position, interior or not.

Let us now characterize the circuits, cocircuits and bases of a Lawrence polytope. We introduce the following notation.

Definition 4.10 Let B be a subset of E, and $A \subset B$. We denote by

$$_AB := (B \setminus A) \cup \overline{(B \cap A)},$$

and call it the *reorientation of B by A*.

Let $C = (C^+, C^-)$ be a signed subset of E and let $A \subset C^+ \cup C^-$. We denote by

$$ {}_A C = (\; (C^+ \setminus A) \cup \overline{(C^- \cap A)} \quad , \quad (C^- \setminus A) \cup \overline{(C^+ \cap A)} \quad), $$

and call it the *reorientation of* C *by* A.

Lemma 4.11 *Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated with an oriented matroid $\mathcal{M}$. Let $\mathcal{C}i$, $\mathcal{C}o$ and $\mathcal{B}$ denote respectively the sets of circuits, cocircuits and bases of $\mathcal{M}$. Then,*

(i) *The set of circuits of $\Lambda(\mathcal{M})$ is*

$$ \mathcal{C}i_{\Lambda(\mathcal{M})} := \{ (C^+ \cup \overline{C^-}, C^- \cup \overline{C^+}) \mid (C^+, C^-) \in \mathcal{C}i \}. $$

(ii) *The set of cocircuits of $\Lambda(\mathcal{M})$ is*

$$ \mathcal{C}o_{\Lambda(\mathcal{M})} := \{ {}_A C \mid C \in \mathcal{C}o, A \subset C \} \quad \cup $$

$$ \{ \; (\{e, \overline{e}\}, \emptyset) \; , \; (\emptyset, \{e, \overline{e}\}) \mid e \in E \text{ is not a coloop of } \mathcal{M} \}. $$

(iii) *The set of bases of $\Lambda(\mathcal{M})$ is*

$$ \mathcal{B}_{\Lambda(\mathcal{M})} := \{ {}_A(E \setminus B) \cup B \cup \overline{B} \mid B \in \mathcal{B}, A \subset E \setminus B \}. $$

Proof: The proof is easy via the duals $\mathcal{M}^*$ and $\Lambda(\mathcal{M})^*$ of $\mathcal{M}$ and $\Lambda(\mathcal{M})$. See also Lemma 9.3.1 and Proposition 9.3.3 in [7]. □

The following is a first interesting consequence of this lemma:

Proposition 4.12 *Let T be a triangulation of a Lawrence polytope $\Lambda(\mathcal{M})$. Then, for every basis B of $\mathcal{M}$ there is a unique subset $A \subset (E \setminus B)$ such that ${}_A(E \setminus B) \cup B \cup \overline{B} \in T$. In particular, all the triangulations of $\Lambda(\mathcal{M})$ have the same number of simplices, equal to the number of bases of $\mathcal{M}$.*

Proof: Let B be a basis of $\mathcal{M}$. Then, $E \setminus B$ is a basis of $\mathcal{M}^*$ and the collection of reorientations ${}_A(E \setminus B)$ of $E \setminus B$ is a triangulation of $\Lambda(\mathcal{M})^*$. From part (c) of Theorem 3.8 we conclude that T has exactly one simplex which is the complement of a reorientation of $E \setminus B$. □

Now we prove that Lawrence polytopes only have lifting subdivisions. The following statement actually gives much more information.

Lemma 4.13 *Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated to an oriented matroid $\mathcal{M}$. Let Δ be a subdivision of $\Lambda(\mathcal{M})$. Then:*

(i) *The support of every circuit C of $\Lambda(\mathcal{M})$ is a face of $\Lambda(\mathcal{M})$.*

(ii) *Let F be a face of $\Lambda(\mathcal{M})$ which is the support of a circuit C. Let k be the rank of F. Consider the "restriction" of Δ to F defined as:*

$$\Delta_F := \{\sigma \cap F \mid \sigma \in \Delta,\ rank(\sigma \cap F) = k\}.$$

Then, Δ_F is either the trivial subdivision $\{F\}$ of the restricted oriented matroid $\Lambda(\mathcal{M})(F)$ or one of the triangulations T_C^+ or T_C^- of a circuit introduced in Proposition 2.13.

(iii) *The cocircuit signature of $\Lambda(\mathcal{M})^*$ defined by $C(p) = 0$ if $\Delta_{\underline{C}} = \{\underline{C}\}$ and $C(p) = +1$ (resp. $C(p) = -1$) if $\Delta_{\underline{C}} = T_C^+$ (resp. $\Delta_{\underline{C}} = T_C^-$) is the cocircuit signature of an extension $\Lambda(\mathcal{M})^* \cup p$ of $\Lambda(\mathcal{M})^*$.*

(iv) *Δ is the lifting subdivision corresponding to the acyclic lift $\widehat{\mathcal{M}}$ which is dual to the reorientation at p of the extension defined in (iii).*

Proof: (i) By part (i) of Lemma 4.11, every circuit of $\Lambda(\mathcal{M})$ is of the form $(C^+ \cup \overline{C^-}, C^- \cup \overline{C^+})$, where (C^+, C^-) is a circuit of $\mathcal{M}$. In the other hand, by part (ii) of the same lemma, every set of the form $A \cup \overline{A}$ is a face of $\Lambda(\mathcal{M})$.

(ii) Let $C = (C^+ \cup \overline{C^-}, C^- \cup \overline{C^+})$ be the circuit whose support equals F. Let $a_1 \in C^+ \cup \overline{C^-}$, $a_2 \in C^- \cup \overline{C^+}$. We first prove that at most one of F, $F \setminus \{a_1\}$ or $F \setminus \{a_2\}$ lies in Δ_F. Let $F \setminus \{a_1, a_2\} = \{b_1, \dots, b_{k-1}\}$. Consider the lexicographic extension $\Lambda(\mathcal{M}) \cup p_F$ of $\Lambda(M)$ defined by the expression $p_F := [b_1^+, \dots, b_{k-1}^+, a_2^+]$. It lies both in the relative interiors of $F \setminus \{a_1\}$ and $F \setminus \{a_2\}$ (the first thing is trivial, the second follows from the fact that a_1 and a_2 lie in opposite parts of the circuit). Then, condition (b) in Definition 4.5 applied to Δ implies that only one of $F \setminus \{a_1\}$ or $F \setminus \{a_2\}$ can be the intersection with F of a cell of Δ. If one of them is, then condition (c) in the same definition implies that F cannot be contained in a cell of Δ.

Thus, for any pair of elements $a_1 \in C^+ \cup \overline{C^-}$ and $a_2 \in C^- \cup \overline{C^+}$ at most one of F, $F \setminus \{a_1\}$ or $F \setminus \{a_2\}$ is in Δ_F. Since any spanning subset of F is either F or of the form $F \setminus \{a\}$, this implies that Δ_F is contained in one of the three subdivisions $\{F\}$, T_C^+ or T_C^- of F. If $\Delta_F \subset \{F\}$ then clearly $\{F\} = \Delta_F$. Otherwise, suppose without loss of generality that Δ_F is contained in T_C^-. We have to prove that then $\Delta_F = T_C^-$. If this is not the case, suppose that $F \setminus \{a_2\} \in T_C^-$ is one of the missing simplices. This is impossible, because then the lexicographic extension p_F defined above would not lie in the convex hull of any cell of Δ_F and, thus, would not lie in the convex hull of any cell of Δ, which contradicts part (i) of Corollary 4.8.

(iii) To prove that the cocircuit signature defines an extension it suffices to show that it defines an extension on every rank 2 contraction of $\Lambda(\mathcal{M})^*$. This is a general fact on extensions of oriented matroids, proved by Las Vergnas [17] (see also, [7, Theorem 7.1.8]).

If e is a non-loop of such a contraction, then $\overline{e}$ is also a non-loop, and vice versa. Thus, we can assume the contraction to be $\Lambda(\mathcal{M})^*/(A \cup \overline{A})$. The dual of the contraction is the restriction of $\Lambda(\mathcal{M})$ to a corank 2 face $(E \setminus A) \cup (\overline{E} \setminus \overline{A})$ of $\Lambda(\mathcal{M})$.

Observe that the cocircuit signature restricted to $\Lambda(\mathcal{M})^*/(A \cup \overline{A})$ can be obtained from the subdivision Δ restricted to $\Lambda(\mathcal{M}) \setminus (A \cup \overline{A})$ in the same way as we obtained the cocircuit signature for p from Δ. Since a corank 2 oriented matroid is always realizable and every subdivision of an acyclic realized oriented matroid of corank 2 is regular (in particular, lifting) by [19] (see also Proposition 5.8 in [11]), we conclude that the restriction of the cocircuit signature to every rank 2 contraction is the cocircuit signature of a lifting triangulation. Thus, the cocircuit signature for p in $\Lambda(\mathcal{M})^*$ defines an extension $\Lambda(\mathcal{M})^* \cup p$.

(iv) Let Δ' be the subdivision of $\Lambda(\mathcal{M})$ defined by that extension. We have to prove that $\Delta = \Delta'$; by part (ii) of Corollary 4.8 it is enough to prove that any cell of Δ is a cell of Δ' as well.

Let σ be a cell of Δ and let σ^c denote its complement in the set of elements of $\Lambda(\mathcal{M})$. We check the following two properties, the first of which is trivial:

(a) If a cocircuit C of $\Lambda(\mathcal{M})^*$ has support contained in σ, then $C(p) = 0$.

(b) For any element $a \in \sigma^c$, there is a cocircuit $C_a = (C_a^+, C_a^-)$ of $\Lambda(\mathcal{M})^*$ with $C_a(p) = +1$, $C_a^+ \cap \sigma^c = \{a\}$ and $C_a^- \cap \sigma^c = \emptyset$. Indeed, since σ is spanning in $\Lambda(\mathcal{M})$, there is a circuit C of $\Lambda(\mathcal{M})$ with support containing a and contained in $\sigma \cup \{a\}$, which we assume to be positive at a. Let $\underline{C}$ be the support of C. Since $\sigma \cap \underline{C} = \underline{C} \setminus \{a\}$, part (ii) of the Lemma implies that T_C^+ is a subcomplex of Δ, that is, $C(p) = +1$.

We consider the restriction of $\Lambda(\mathcal{M})^* \cup p$ to $\sigma^c \cup \{p\}$. Statement (b) has as a consequence that σ^c is independent. Indeed, for any $a \in \sigma^c$ there is a cocircuit vanishing on $\sigma^c \setminus \{a\}$ and not vanishing on a. Statement (a) implies that p is in the flat spanned by σ^c; that is, that σ^c is full-dimensional in the restriction. Then, part (v) of Lemma 1.1 implies that p is in the convex hull of σ^c and part (ii) of Lemma 1.2, applied with $A = \{p\}$ and $B = \{\sigma^c\}$, implies that $p \in relint_{\Lambda(\mathcal{M})^*}(\sigma^c)$.

Finally, part (iii) of Lemma 1.2 implies that $(\{p\}, \sigma^c)$ is a circuit of $\Lambda(\mathcal{M})^*$; that is, that $(\emptyset, \sigma^c \cup \{\widehat{p}\})$ is a cocircuit of the lift $\widehat{\Lambda(\mathcal{M})}$ defining the lifting subdivision Δ'. This implies that σ is a cell of Δ'. □

Theorem 4.14 *Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated to an oriented matroid $\mathcal{M}$. Then:*

(i) *There is a natural bijection between the extensions of $\mathcal{M}^*$ and the subdivisions of $\Lambda(\mathcal{M})$.*

(ii) *Under this bijection, two triangulations differ by a bistellar flip if and only if the corresponding extensions differ by a mutation.*

Proof: (i) Definition 4.1 provides a natural surjective map from the collection of interior extensions of $\Lambda(\mathcal{M})^*$ to the lifting subdivisions of $\Lambda(\mathcal{M})$. Since $\Lambda(\mathcal{M})^*$ is totally cyclic, all its extensions are interior and by construction of $\Lambda(\mathcal{M})^*$ they correspond bijectively to the extensions (interior or not) of $\mathcal{M}^*$. In the other hand, all the subdivisions of $\Lambda(\mathcal{M})$ are lifting subdivisions, by part (iv) of the previous lemma. Thus, we have a natural surjective map from the extensions of $\mathcal{M}^*$ to the subdivisions of $\Lambda(\mathcal{M})^*$. The fact that the complete cocircuit signature of an extension can be recovered from the corresponding lifting subdivision (parts (iii) and (iv) of the previous lemma), implies that the map is injective.

(ii) If two extensions of $\mathcal{M}^*$ in general position differ by a mutation, Proposition 3.13 implies that the associated triangulations of $\Lambda(\mathcal{M})$ differ by a bistellar flip (since they cannot be equal).

Reciprocally, suppose that T_1 and T_2 are two triangulations of $\Lambda(\mathcal{M})$ differing by a bistellar flip. Let C be the circuit of $\Lambda(\mathcal{M})$ in which the flip is supported, which is a cocircuit of $\Lambda(\mathcal{M})^*$. Let $\Lambda(\mathcal{M})^* \cup p_1$ and $\Lambda(\mathcal{M})^* \cup p_2$ be the extensions of $\Lambda(\mathcal{M})^*$ corresponding to the triangulations T_1 and T_2. We have that $Ext_{p_1} - Ext_{p_2}$ is a sum of cocircuit vectors supported on C, and want to derive from this that the two extensions differ by a mutation, that is, that the cocircuit signatures of p_1 and p_2 differ only on the cocircuit C (and its opposite).

Thus, let C' be a cocircuit not equal to C or its opposite. In particular, the hyperplanes H_C and $H_{C'}$ in which C and C' respectively vanish do not coincide. Let a be an element with $C'(a) = C'(p_1) \in \{+1, -1\}$. Since the restriction of $\Lambda(\mathcal{M})^*$ to the hyperplane $H_{C'}$ in which C' vanishes is totally cyclic, there is an $(r-1)$-simplex τ in $H_{C'}$ such that $p_1 \in conv_{\Lambda(\mathcal{M})^* \cup p_1}(\tau \cup a)$, that is, $\tau \cup \{a\}$ is in the support of the extension vector of p_1.

If $\tau \cup \{a\}$ is also in the support of the extension vector of p_2, we conclude that $C'(p_1) = C'(p_2)$, as we wished. If $\tau \cup \{a\}$ is not in the support of the extension vector of p_2 and since the difference of the two extension vectors is a sum of cocircuit vectors supported on C, the only way for the simplex to be in the difference of the extension vectors is that it has a facet in the hyperplane H_C in which C vanishes. In particular, $a \in H_C$ and there is an element $b \in \tau$ with $\rho := \tau \setminus \{b\} \in H_C \cap H_{C'}$. Also, the subtraction of the cocircuit vector of $\rho \cup \{a\}$ from the extension vector of p_1 implies that $\rho \cup \{a, \bar{b}\}$ is a simplex in the support of the extension vector of p_2. But $\rho \cup \{\bar{b}\} \subset H_{C'}$, which implies that $C'(p_2) = C'(a) = C'(p_1)$ □

Remark 4.15 (The Extension space conjecture and the Baues problem)

The above theorem strongly relates two important open problems: the conjecture that all the extensions of a realizable oriented matroid are connected by mutations and the conjecture that all the triangulations of a point configuration are connected by bistellar flips. Actually, the first conjecture becomes equivalent to the restriction of the second to Lawrence polytopes. Equally, the negative answer to the first conjecture in the non-realizable case produces:

Corollary 4.16 *There is a (non-realizable) rank 23 Lawrence polytope $\Lambda(\mathcal{M})$ with 38 elements and a triangulation of it which admits no bistellar flip. In particular, the triangulations of $\Lambda(\mathcal{M})$ are not connected by bistellar flips.*

Proof: With the previous theorem, using the fact that there is a uniform oriented matroid of rank 4 on 19 elements and an extension of it which cannot be mutated (Theorem 2.3 of Richter-Gebert [23]). □

Even more, it is obvious from our results that the poset of extensions of $\mathcal{M}^*$ ordered by weak maps (or by "perturbation" in our nomenclature) is isomorphic to the poset of subdivisions of $\Lambda(\mathcal{M})$ ordered by refinement. In the realizable case, these two posets appear in particular cases of the so-called "generalized Baues problem": given two polytopes $P \subset \mathbf{R}^p$ and $Q \subset \mathbf{R}^q$ and a projection map $\pi : \mathbf{R}^p \to \mathbf{R}^q$ with $\pi(P) = Q$, Billera et al. have defined the concept of a *subdivision of* Q *induced by* π *from* P and asked whether the order poset of these induced subdivisions ordered by refinement has always the homotopy type of a sphere of dimension $dim(P) - dim(Q) - 1$ [4]. The Baues problem has been answered with a counterexample by Rambau and Ziegler (see [24]), but the following particular cases are specially interesting and still open:

(i) If P is a simplex, then the π-coherent subdivisions of Q are all the polytopal subdivisions of Q which use (perhaps not all of) the image points by π of the vertices of P. The Baues problem asks whether the order poset of all proper subdivisions has the homotopy type of a sphere.

(ii) If P is a cube (which implies that Q is a zonotope) then, the π-coherent subdivisions of Q are the so-called *zonotopal subdivisions* (see [7, page 60]) or *zonotopal tilings* (see [30, Section 7.5]) of the zonotope Q. The Bohne-Dress Theorem on zonotopes (see [7, Theorem 2.2.13] or [30, Theorem 7.32]) implies that the zonotopal subdivisions of Q are in bijective correspondence with the extensions of the associated oriented matroid, with refinement of subdivisions corresponding to perturbation of extensions. That is, the zonotopal case of the Generalized Baues problem is equivalent to the *extension space conjecture* posed by Sturmfels and Ziegler [29], which conjectures that the extension space of a realizable oriented matroid $\mathcal{M}$ of rank r has the homotopy type of a $(r-1)$-sphere.

Thus, Theorem 4.14 has the following important consequence in the Baues context:

Corollary 4.17 *The extension space conjecture is equivalent to the following one: for any realized Lawrence polytope* Λ *of dimension* d *with* n *vertices, the order poset of proper polytopal subdivisions of* Λ *ordered by refinement has the homotopy type of a* $(n-d-2)$*-sphere.* □

4.3 A "reoriented" Lawrence construction

Here we introduce a reoriented version of the Lawrence construction. The construction is interesting because, applied to an acyclic non-polytopal oriented matroid $\mathcal{M}$ produces a matroid polytope $\Sigma(\mathcal{M})$ with exactly the same collection of triangulations as $\mathcal{M}$. In other words, the "polytopal case" cannot not be considered simpler than the "acyclic case" when dealing with triangulations (unless we are interested in a fixed rank).

Let $\mathcal{M}$ be an oriented matroid of rank r on n elements which we identify with $E := \{1, \ldots, n\}$. Assign a positive integer k_i to each element i of n. Let k be the sum of these integers. We consider the oriented matroid $\Sigma(\mathcal{M})^*$ constructed from the dual $\mathcal{M}^*$ of $\mathcal{M}$ by substituting each element i for k_i parallel copies of it. The dual $\Sigma(\mathcal{M})$ of this oriented matroid has rank $k+r-n$ and k elements. If $k_i = 2$ for every i, then $\Sigma(\mathcal{M})$ is a reorientation of the

Lawrence polytope $\Lambda(\mathcal{M})$. It will be good for us to allow the full generality of arbitrary k_i's in connection to weighted unimodular configurations which will appear in Example 5.8.

The elements of $\Sigma(\mathcal{M})$ lie in n equivalence classes $\Sigma(1), \ldots, \Sigma(n)$ with $k_1, \ldots, k_n$ elements respectively, each class corresponding to an element of $\mathcal{M}$. Two elements e and f of $\Sigma(\mathcal{M})$ are *co-parallel* (i.e., $(\{e\}, \{f\})$ is a covector) if and only if they lie in the same class or in classes corresponding to co-parallel elements of $\mathcal{M}$. The interesting point of this construction is that it induces a bijection between the triangulations of $\mathcal{M}$ and those of $\Sigma(\mathcal{M})$:

Theorem 4.18 *Let $\mathcal{M}$ be an oriented matroid of rank r on n elements which we identify with $E := \{1, \ldots, n\}$. Let $k_1, \ldots, k_n$ be positive integers. The oriented matroid $\Sigma(\mathcal{M})$ of rank $k+r-n$ on k elements just defined has the following properties.*

(i) *$\Sigma(\mathcal{M})$ is acyclic if and only if $\mathcal{M}$ is acyclic. If this is the case, then an element of $\Sigma(\mathcal{M})$ is a vertex (face of rank 1) if and only if the corresponding element i of $\mathcal{M}$ was a vertex of $\mathcal{M}$, or if k_i is greater than 1. In particular, $\Sigma(\mathcal{M})$ is polytopal if and only if $\mathcal{M}$ is acyclic and $k_i > 1$ for every non-vertex element of $\mathcal{M}$.*

(ii) *There is a natural bijective correspondence between the triangulations of $\mathcal{M}$ and the triangulations of $\Sigma(\mathcal{M})$ which preserves the features of being lifting or lexicographic.*

Before going into the proof, the following simple example may help to clarify the construction. Suppose that $\mathcal{A}$ is the point configuration in the plane consisting of the vertices of a convex polygon P plus an interior point p. We are going to show the construction $\Sigma(\mathcal{A})$ applied to $\mathcal{A}$ with all the parameters k_i equal to 1 except the one of the interior point p which will be equal to 2. The resulting configuration $\Sigma(A)$ in $\mathbf{R}^3$ consists of the vertices of a bipyramid, with the equator of the bipyramid being the polygon P and in such a way that the intersection of the axis with the equatorial plane of the bipyramid coincides with the point p. The reader should try to visualize in this example the correspondence between triangulations of $\mathcal{A}$ and of $\Sigma(\mathcal{A})$ exhibited in the following proof.

Proof: (i) $\Sigma(\mathcal{M})^*$ is clearly totally cyclic if and only if $\mathcal{M}^*$ is totally cyclic, which proves the first part of (i). For the second part, assume that $\Sigma(\mathcal{M})^*$ and $\mathcal{M}^*$ are totally cyclic. An element of an acyclic oriented matroid is a vertex if and only if the contraction at this element is acyclic. Thus, an element of $\Sigma(\mathcal{M})$ is a vertex if and only if its deletion in $\Sigma(\mathcal{M})^*$ is totally cyclic. This happens if and only if its equivalence class has at least another element or the deletion of the corresponding element of $\mathcal{M}^*$ is totally cyclic.

(ii) It is obvious how to relate lifting and lexicographic triangulations of $\mathcal{M}$ and $\Sigma(\mathcal{M})$, since the extensions of $\mathcal{M}^*$ and $\Sigma(\mathcal{M})^*$ are "the same" and an extension is interior (resp. in general position) in $\mathcal{M}^*$ if and only if it is as well in $\Sigma(\mathcal{M})^*$. Studying what this correspondence between extensions of $\mathcal{M}^*$ and

$\Sigma(\mathcal{M})^*$ looks like in terms of the extension vectors we conclude the following heuristic rule for obtaining a triangulation $\Sigma(T)$ of $\Sigma(\mathcal{M})$ from a triangulation T of $\mathcal{M}$. Let $\Sigma(i)$ denote the equivalence class in the set of elements of $\Sigma(\mathcal{M})$ of the element $i \in E$. For each subset $\sigma \in E$ define the following collection of subsets of the elements of $\Sigma(\mathcal{M})$:

$$\Sigma(\sigma) = \{S \mid \#(S \cap \Sigma(i)) = k_i - 1 + \#(\sigma \cap \{i\})\}.$$

In other words, the subsets S appearing in $\Sigma(\sigma)$ are those which contain the equivalence classes associated to the elements of σ and miss exactly one element from the other equivalence classes. It follows that the complements of the so-defined sets S are independent (resp. spanning) in $\Sigma(\mathcal{M})^*$ if and only if σ is independent (resp. spanning) in $\mathcal{M}^*$. Thus, the following is a collection of maximal simplices of $\Sigma(\mathcal{M})$:

$$\Sigma(T) = \cup_{\sigma \in T} \Sigma(\sigma).$$

It follows from the definition of $\Sigma(T)$ that a triangulation T of $\mathcal{M}$ is the lifting triangulation corresponding to an extension $\mathcal{M}^* \cup p$ if and only if $\Sigma(T)$ is the lifting triangulation corresponding to "the same" extension $\Sigma(\mathcal{M})^* \cup p$ of $\Sigma(\mathcal{M})^*$. This proves that the correspondence $T \to \Sigma(T)$ restricts to a bijection between lifting (resp. lexicographic) triangulations of $\mathcal{M}$ and $\Sigma(\mathcal{M})$.

We will show that the correspondence is bijective on the set of all triangulations of $\mathcal{M}$ and $\Sigma(\mathcal{M})$ using the characterization of triangulations which appears in Theorem 3.8. If e and f are elements of $\Sigma(\mathcal{M})^*$ in the equivalence class of a non-loop element of $\mathcal{M}$, then the signed set $(\{f\}, \{e\})$ is a circuit of $\Sigma(\mathcal{M})^*$. A collection of maximal simplices of $\Sigma(\mathcal{M})$ satisfies the circuit equations of Theorem 3.8 for all the circuits of this type if and only if it is a union of collections of simplices of the form $\Sigma(\sigma)$ for different maximal simplices σ of $\mathcal{M}$. Thus we can restrict our attention to collections of maximal simplices of $\Sigma(\mathcal{M})^*$ which are of this type.

Also, it is obvious that the triangulations of $\Sigma(\mathcal{M})^*$ are all obtained from the triangulations of $\mathcal{M}^*$ by choosing a representative of each equivalence class of elements. This implies that a collection $\Sigma(T)$ of simplices of $\Sigma(\mathcal{M})$ obtained as the union of the $\Sigma(\sigma)$ corresponding to a collection T of simplices of $\mathcal{M}$ satisfies the duality equations $\langle v_{\Sigma(T)}, v\rangle = 1$ for all the incidence vectors v of triangulations of $\Sigma(\mathcal{M})^*$ if and only if T itself satisfies the equations for the incidence vectors of triangulations of $\mathcal{M}^*$.

Finally, all the circuits of $\Sigma(\mathcal{M})^*$ are either of the form $(\{e\}, \{f\})$ for equivalent elements e and f or obtained from circuits of $\mathcal{M}^*$ by choosing a representative in $\Sigma(\mathcal{M})^*$ for each element of $\mathcal{M}$. Thus, a collection T of simplices of $\mathcal{M}$ satisfies the circuit equations of Theorem 3.8 if and only $\Sigma(T)$ satisfies the circuit equations as well. This finishes the proof of part (ii). □

Remark 4.19 ($\Lambda(\mathcal{M})$ and $\Sigma(\mathcal{M})$ for graphic oriented matroids).

An oriented matroid $\mathcal{M}$ is called *graphic* (see [22]) if it is isomorphic to the cycle oriented matroid of a directed graph G. In other words, if the elements of $\mathcal{M}$ correspond to the edges of G and the signed circuits of $\mathcal{M}$ correspond to the cycles of G, with the signing given by the orientation of the edges (see [7, Section 1.1]).

Graphic oriented matroids form a very restricted class, strictly contained in the so-called *binary* or *regular* oriented matroids; for example, uniform oriented matroids are graphic only if they have rank or corank at most one. For this reason, directed graphs are not normally considered a good model for oriented matroids. (The situation is quite different in matroid theory, in which many results and constructions can be interpreted in terms of graphs; see, for example, [22]). However, the Lawrence construction (both in its original and reoriented versions) has the following very simple interpretation for graphic oriented matroids:

- Let $\mathcal{M}$ be a graphic oriented matroid corresponding to the graph G. Then, the oriented matroid $\Lambda(\mathcal{M})$ is the graphic oriented matroid corresponding to the graph $\Lambda(G)$ obtained from G by subdividing each edge into two parts by the addition of a vertex and giving opposite directions to the two parts.

- Let $\mathcal{M}$ be as above, let $1, \ldots, n$ denote the edges of the graph (i.e., the elements of the oriented matroid) and let $k_1, \ldots, k_n$ be positive integers. Then the oriented matroid $\Sigma(\mathcal{M})$ corresponding to this choice of k_i's is the graphic oriented matroid corresponding to the graph $\Sigma(G)$ obtained from G by subdividing each edge i into k_i new edges and giving to all of them the same direction as the old edge had.

Observe that the constructions above are consistent with the fact that $\Lambda(\mathcal{M})$ is acyclic and invariant under reorientation of $\mathcal{M}$, while $\Sigma(\mathcal{M})$ is not invariant and is acyclic only if $\mathcal{M}$ is acyclic. The graph $\Sigma(G)$ is uniquely defined by G (and the parameters k_i), but in the graph $\Lambda(G)$ we have the choice of which of the two parts of each edge of G gets each of the two directions. However, since every cycle of $\Lambda(G)$ will contain either both or none of the two sub-edges, the graphic oriented matroid obtained is the same independently of this choice. Actually, we can choose all orientations of the sub-edges to go from the old vertex of the sub-edge to the new one. With this choice the graph $\Lambda(G)$ is bipartite, which has the following interesting consequence:

Proposition 4.20 *Let $\mathcal{M}$ be an oriented matroid of rank r on n elements. Then, the following properties are equivalent:*

(i) *$\mathcal{M}$ is graphic.*

(ii) *$\Lambda(\mathcal{M})$ is a full-rank restriction of the affine dependences oriented matroid of the product $\Delta_{n-1} \times \Delta_r$ of two simplices of dimensions $n-2$ and $r-1$.*

Proof: We recall the following elementary facts from matroid theory: the graphic oriented matroid corresponding to a connected graph $G = (V, E)$ has $|E|$ elements and rank $|V| - 1$. Also, that every graphic (oriented) matroid can be represented by a connected (directed) graph.

It is known that the cycle oriented matroid of the complete bipartite graph $K_{n,r+1}$ directed from one part to the other equals the affine dependences oriented matroid of the product $\Delta_{n-1} \times \Delta_r$ of two simplices. This follows, for example, from the description of $\Delta_{n-1} \times \Delta_r$ which appears in [15, pages 246–251].

The remarks before the statement imply that whenever $\mathcal{M}$ is graphic, $\Lambda(\mathcal{M})$ is the cycle oriented matroid corresponding to a connected restriction of the graph $K_{n,r+1}$ which uses all the vertices of the graph. That is, $\Lambda(\mathcal{M})$ is a full-rank restriction of the cycle oriented matroid of $K_{n,r+1}$. This proves (i)⇒(ii).

The other implication is trivial, since every minor of a graphic oriented matroid is graphic and $\mathcal{M}$ is a contraction of $\Lambda(\mathcal{M})$. □

5 Lifting triangulations

5.1 Some properties. Lifting versus regular triangulations.

We start by showing the relation between *regular triangulations* of a point configuration and lifting triangulations of the underlying oriented matroid.

Examples 5.1 (Regular and lifting triangulations)

Let $\mathcal{M}$ be an oriented matroid realized as a point (or vector) configuration $\mathcal{A}$. We think of $\mathcal{A}$ as being a matrix with n columns and r rows, where n is the number of elements of $\mathcal{M}$ and r its rank. A *Gale transform* $\mathcal{A}^*$ of $\mathcal{A}$ is an $n \times (n-r)$ matrix whose row space is the orthogonal complement of the row space of $\mathcal{A}$. It is well-known that $\mathcal{A}^*$ realizes the dual oriented matroid $\mathcal{M}^*$. Any point lying in the convex hull (more generally, in the positive span) of the columns of $\mathcal{A}^*$ defines an interior (and realizable) extension of $\mathcal{M}^*$. If the point is not in the convex hull of any non-full-dimensional geometric simplex with vertices in $\mathcal{A}^*$, then the extension defines a lifting triangulation of $\mathcal{M}$. The triangulations obtained in this way are called *regular triangulations* of $\mathcal{A}$ [19, Definition 1] (some authors use the word *coherent* [15, Chapter 7] or *convex*). Observe that they can be alternatively defined as those which agree with the projection of the lower envelope of a certain orthogonal lift of $\mathcal{A}$; the equivalence between the two definitions is the "realized-case" analogue of Proposition 4.2. Thus, regular triangulations of a realized oriented matroid are a class in-between lifting and lexicographic triangulations. Their main draw-back in the context of this paper is that regularity depends on the specific realization of $\mathcal{M}$ and not only on the oriented matroid.

Two points in the convex hull of $\mathcal{A}^*$ define the same regular triangulation of $\mathcal{A}$ if and only if they are contained in the same collection of convex hulls of geometric simplices of $\mathcal{A}^*$; that is, if the two points are in the same full-dimensional cell of the common refinement of all the triangulations of $\mathcal{A}^*$. This common refinement is sometimes called the chamber complex of $\mathcal{A}^*$. The bijective correspondence between regular triangulations of a configuration $\mathcal{A}$ and maximal chambers of of its Gale transform $\mathcal{A}^*$ was explored in [3] and generalized in [11] to include a correspondence between non-regular triangulations of $\mathcal{A}$ and "virtual" chambers of $\mathcal{A}^*$, by means of the realized version of Theorem 3.8.

The relation between regular and lifting triangulations divides the triangulations of a realizable oriented matroid $\mathcal{M}$ in four categories, with different degrees of "realizability". First, there are the triangulations which are regular for any realization of $\mathcal{M}$; this includes lexicographic triangulations but also some non-lexicographic ones, as the example shown in part (c) of Figure 4. Second, there are triangulations which are regular or non-regular depending on the realization of $\mathcal{M}$, as the ones in parts (a) and (b), in which the oriented matroid is the same but only the triangulation in part (a) is regular. These triangulations correspond to extensions of the dual oriented matroid $\mathcal{M}^*$ which are realizable but not as an extension of an arbitrary realization of $\mathcal{M}^*$.

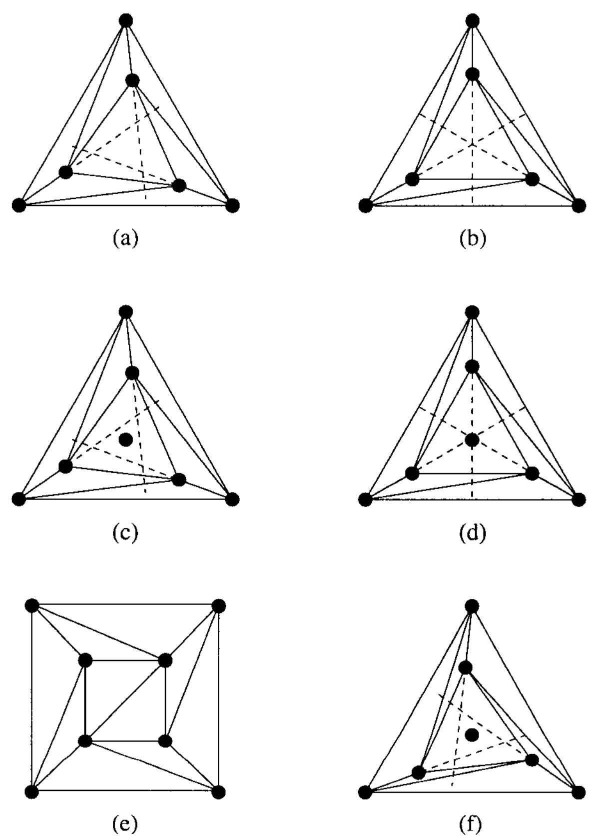

Figure 4: Some lifting triangulations.

Finally, the triangulations which are not regular for any realization of $\mathcal{M}$, as the ones in parts (d), (e) and (f) of Figure 4, can still be lifting triangulations (corresponding to non-realizable lifts/extensions) or not. The three in the figure are lifting triangulations, as follows from the following consequence of Lemma 7.3.2 of [7]: if the hyperplanes (flats of corank 1) of an oriented matroid $\mathcal{M}$ which are dependent are all circuits then an arbitrary perturbation of them into bases is an oriented matroid. Applying the lemma to the lift of (d) into a triangular prism with an interior point we get lifts for the triangulations (d) and (f). For (e) we do the same with the lift into a cube. In Examples 5.5 we will construct two very simple non-lifting triangulations and in Section 5.2 we will show some more complicated ones.

We address now the following problem: suppose that we are given an oriented matroid $\mathcal{M}$ of rank r, an element a of $\mathcal{M}$ and a triangulation of either $\mathcal{M}/a$ or $\mathcal{M} \setminus a$; we want to extend it to $\mathcal{M}$. More precisely, for a triangulation T' of $\mathcal{M} \setminus a$ we want to find a triangulation T of $\mathcal{M}$ with $T' \subset T$. For a triangulation T'' of $\mathcal{M}/a$ we want to find a triangulation T of $\mathcal{M}$ with $T'' = link_T(a)$. We also want to know if good lifting properties of T' and T'' can be inherited by T.

Corollary 2.11 was a first result in this direction: a triangulation of $\mathcal{M} \setminus a$ can always be extended to $\mathcal{M}$. However, the property fails in general for triangulations of $\mathcal{M}/a$ as the following example of an oriented matroid of rank 4 in 7 elements shows: Consider the point configuration in part (f) of Figure 4, and "lift it" to a point configuration in $\mathbf{R}^3$ by giving three different heights to the seven points; put the three vertices of the outer triangle on the bottom, the interior point on top, and the three vertices of the inner triangle in the middle, very close to the bottom. Let a be the point on top and consider the seven tetrahedra obtained by coning a to the seven triangles which appear in the figure. This collection of tetrahedra is known not to be completable to a triangulation of the point configuration. Observe that the link of the top point in the non-completable collection of simplices is a lifting (but non-regular) triangulation of the vertex figure. In fact, it is the same triangulation of parts (a) and (b) of Figure 4, in a different realization of the oriented matroid.

This same example has appeared in Lemma 2.1 in [11], and different versions of it have appeared in other places, going back to Schönhardt [27]. The mentioned Lemma 2.1 of [11] also says that *regular* triangulations of both the deletion and the contraction of $\mathcal{M}$ can be extended to regular triangulations of $\mathcal{M}$. Since lifting triangulations are in some sense the oriented matroid analogue of regular triangulations (see Examples 5.1), one could expect that the same holds for lifting triangulations. The example above shows that this is not the case and the following proposition tells us what is true.

Proposition 5.2 *Let $\mathcal{M}$ be an oriented matroid of rank r on a set E and let $a \in E$ be one of its elements.*

(i) *Let T' be a lifting triangulation of the contraction $\mathcal{M}/a$. Suppose that either T' is a lexicographic triangulation or $\mathcal{M}^*$ is a lexicographic extension of $\mathcal{M}^* \setminus a$. Then, there is a lifting triangulation T of $\mathcal{M}$ such that for every simplex τ in T' the simplex $\{a\} \cup \tau$ is in T. Moreover, if T' is lexicographic, then T can also be taken lexicographic.*

(ii) *Let T' be a lifting triangulation of the deletion $\mathcal{M} \setminus a = \mathcal{M}(E \setminus a)$. Then, there is a lifting triangulation T of $\mathcal{M}$ with $T' \subset T$. Moreover, if T' is lexicographic, then T can also be taken lexicographic.*

Proof: (i) The lifting triangulation T' of $\mathcal{M}/a$ corresponds to an extension $(\mathcal{M}^*\setminus a)\cup p$ of $\mathcal{M}^*/a$. Thus, we have two extensions of $\mathcal{M}^*\setminus a$ (by the elements p and a). It suffices to show that they are "compatible"; that is, that there is a two-element extension $(\mathcal{M}^*\setminus a)\cup\{p,a\}$ whose deletions by p and a coincide respectively with $\mathcal{M}^*$ and $(\mathcal{M}^*\setminus a)\cup p$. This is true if one of the extensions is lexicographic, which is our hypothesis, by Lemma 1.8.

(ii) Let $\mathcal{M}^*$ be the dual oriented matroid to $\mathcal{M}$. Then $\mathcal{M}^*/a$ is the dual of the deletion $\mathcal{M}\setminus a$. Let $(\mathcal{M}^*/a)\cup p$ be the interior extension in general position which defines the triangulation T'. We need to find an interior extension $\mathcal{M}^*\cup p'$ in general position of $\mathcal{M}^*$ such that whenever τ is a maximal simplex of $\mathcal{M}^*/a$ having p in its convex hull, $\tau\cup\{a\}$ is a maximal simplex of $\mathcal{M}^*$ having p' in its convex hull. Such an extension was constructed in Lemma 1.9. □

We will work now towards obtaining a combinatorial characterization of lifting triangulations. Recall that a *lift* of an oriented matroid $\mathcal{M}$ is an oriented matroid $\mathcal{M}\cup p$ such that $(\mathcal{M}\cup p)/p=\mathcal{M}$. For an element a in $\mathcal{M}$ let us consider the lift $(\mathcal{M}^*\cup\overline{a})^*$ by a "co-antiparallel" element $\overline{a}$, meaning by this that the dual $\mathcal{M}^*\cup\overline{a}$ is the extension of $\mathcal{M}^*$ by an element antiparallel to a. The Lawrence polytope $\Lambda(\mathcal{M})$ associated to $\mathcal{M}$ is obtained by performing a sequence of these type of lifts, one for each element of $\mathcal{M}$. These lifts are sometimes called *Lawrence lifts* and their duals Lawrence extensions. In the following statement we exploit this iterative construction of the Lawrence polytope:

Proposition 5.3 *Let T be a triangulation of an oriented matroid $\mathcal{M}$ on a set E. Let $\Lambda(\mathcal{M})$ be the Lawrence polytope associated with $\mathcal{M}$. Recall that $\Lambda(\mathcal{M})$ has element set $E\cup\overline{E}$. The following conditions are equivalent:*

(a) *T is a lifting triangulation.*

(b) *There is a triangulation $\Lambda(T)$ of $\Lambda(\mathcal{M})$ which satisfies $link_{\Lambda(T)}(\overline{E})=T$.*

Proof:

(b)⇒(a) Since $\Lambda(T)$ is a lifting triangulation (Theorem 4.14) we only have to prove that a link in a lifting triangulation is a lifting triangulation as well. Using recursion, we only have to prove this for the link of a single vertex of the triangulation. Let a be a vertex of a triangulation T of $\mathcal{M}$, and suppose that T is the lifting triangulation corresponding to the interior extension in general position $\mathcal{M}^*\cup p$ of $\mathcal{M}^*$. Then, there is a maximal simplex in $\mathcal{M}^*$ which has a in its complement and p in its relative interior. This implies that $(\mathcal{M}^*\cup p)\setminus a$ is an interior extension of $\mathcal{M}^*\setminus a$ in general position. The reader can verify that the lifting triangulation corresponding to this extension is precisely the link of a in T.

(a)⇒(b) Here we have to prove the reciprocal of the previous assertion. Namely, that a triangulation T of an oriented matroid $\mathcal{M}$ extends to a triangulation T' of every Lawrence lift $(\mathcal{M}^*\cup\overline{a})^*$ of $\mathcal{M}$. In the dual, this is equivalent to the fact that every interior extension $\mathcal{M}^*\cup p$ in general position of $\mathcal{M}^*$ extends to an interior extension in general position of the extension $\mathcal{M}^*\cup\overline{a}$. In other words, that the two extensions $\mathcal{M}^* \cup p$ and $\mathcal{M}^* \cup \overline{a}$ of $\mathcal{M}^*$ are "compatible". This follows from Lemma 1.8 since the extension $\mathcal{M}^* \cup \overline{a}$ is lexicographic. (Observe that this is a particular case of part (i) of Proposition 5.2). □

We want to translate this result into a more combinatorial characterization of lifting triangulations. This will be done in Theorem 5.13 in Section 5.3, but here we will prove a weak version of it in order to show examples of non-lifting triangulations. It will be convenient to consider a triangulation as being a simplicial complex; that is, to take account of the non-full-rank simplices. We use the following notations, where T is a collection of subsets of a set E and $A \subset E$:

$$\mathcal{P}(T) := \{\tau \subset E \mid \tau \subset \sigma \text{ for some } \sigma \in T\}$$

$$T|_A := \{\tau \in T \mid \tau \subset A\}$$

We say that a collection S of simplices with vertices in E extends to a triangulation of an oriented matroid $\mathcal{M}$ on the ground set E if there is a triangulation T of $\mathcal{M}$ with $S \subset \mathcal{P}(T)$.

Proposition 5.4 *If T is a lifting triangulation of an oriented matroid $\mathcal{M}$ on a set E, then $\mathcal{P}(T)|_A$ (in particular, $T|_A$) can be extended to a triangulation of $\mathcal{M}(A)$, for every $A \subset E$.* □

Proof: If T is a lifting triangulation, let $\widehat{\mathcal{M}}$ be an acyclic lift of $\mathcal{M}$ on the set $E \cup \{\widehat{p}\}$ which defines the lifting triangulation T of $\mathcal{M}$. Then, $\mathcal{P}(T)$ is the collection of faces of $\widehat{\mathcal{M}}$ which do not contain $\widehat{p}$.

We can assume the lift to be "in general position", meaning by this that its dual is an extension in general position of $\mathcal{M}^*$; that is, that any hyperplane of $\widehat{\mathcal{M}}$ not containing $\widehat{p}$ is a simplex. For every $A \subset E$ the restriction $\widehat{\mathcal{M}}(A)$ is an acyclic simplicial lift of $\mathcal{M}(A)$ and defines a triangulation $T(A)$ of $\mathcal{M}(A)$. It is clear that the triangulation $T(A)$ extends the simplicial complex $\mathcal{P}(T)|_A$. □

From this result it is easy to conclude the existence of non-lifting triangulations, as will be done in Examples 5.5. It is an interesting question whether every non-lifting triangulation can be proved to be non-lifting using Proposition 5.4 or not. That is, whether the condition in the corollary is already a characterization of non-lifting triangulations.

Examples 5.5 (Two non-lifting triangulations)

We have seen in Examples 5.1 that all the triangulations in Figure 4 are lifting triangulations. However, we are going to use Proposition 5.4 to construct non-lifting triangulations of point configurations in $\mathbf{R}^3$.

Let $\mathcal{A}$ be the point configuration in $\mathbf{R}^3$ obtained by giving three different heights to the seven points of Figure 4(d): the inner triangle at the bottom, the outer triangle in the middle and the middle point on top. Call $\mathcal{M}$ the rank 4 oriented matroid with 7 elements obtained. Let T be the triangulation of $\mathcal{A}$ (and of $\mathcal{M}$) obtained coning each triangle of the planar triangulation to the top point. Proposition 5.4 implies that T is not lifting, since removing the top point we get a triangulation of a part of the boundary of a triangular prism which cannot be extended to the whole prism.

Another proof of the fact that T is non-lifting is as follows: in any realization of $\mathcal{M}$, the geometric link in T of the top point is precisely the triangulation of part (b) of Figure 4, with the three dashed lines converging in one point thanks to Desargues theorem. Thus, this link is non-regular. A triangulation with a non-regular link is itself non-regular, by Lemma 2.1 of [11]. Thus, T is non-regular in every realization of $\mathcal{M}$. Since $\mathcal{M}$ has rank 4 and 7 points, every extension of its dual is realizable. Thus, every lifting triangulation of $\mathcal{M}$ is regular for some realization of $\mathcal{M}$, which proves that T is not a lifting triangulation.

The same construction of a non-lifting triangulation applies to part (f) of the figure, except that now three additional simplices have to be added in order to fill completely the convex hull of the lifted point configuration. The non-lifting triangulation obtained in this case has the strong property that it is non-lifting for any oriented matroid of which it is a triangulation (the other one does not, as follows from the same construction applied to the regular triangulation (c), which produces a combinatorially equivalent regular triangulation). A triangulation with this same property appears in page 410 of [7].

5.2 Three interesting non-lifting triangulations

Example 5.6 (The Edmonds-Fukuda-Mandel oriented matroid.)

We consider here the oriented matroid **EFM**(8) which appeared in [13, 14]. A detailed study of it can be found also in pages 461–468 of [7]. With this oriented matroid we will show counterexamples to the following two situations:

- Suppose that two full-rank disjoint simplices σ_1 and σ_2 of an oriented matroid $\mathcal{M}$ are "strongly separated"; by this we mean that there is a covector which is positive in one and negative in the other. Then, there is a vector in the dual oriented matroid $\mathcal{M}^*$ which is positive and negative respectively in their complements σ_1^* and σ_2^*. If $\mathcal{M}^*$ is realizable (more generally, if it has an adjoint, see Sections 5.3 and 7.5 of [7]) then there is an extension of $\mathcal{M}^*$ which lies in the relative interior of both σ_1^* and σ_2^* and, thus, there is a lifting triangulation of $\mathcal{M}$ containing the simplices σ_1 and σ_2. We will show that this does not happen in **EFM**(8) (part (i) of Proposition 5.7).

- We have mentioned in Remark 2.5(v) that the good-intersection property for two simplices σ_1 and σ_2 of an oriented matroid satisfying the "Generalized Euclidean intersection property" ($\mathtt{IP}_2$), (see Definition 7.5.2 in [7]) is equivalent to a strong "no-overlapping-circuit" property, namely that there is no circuit with its positive part contained in σ_1 and its negative part contained in σ_2. We show the necessity of the property ($\mathtt{IP}_2$) by a counterexample to this equivalence in the dual **EFM**(8)* (part (ii) of Proposition 5.7).

$\mathtt{EFM}(8)$ is a rank 4 non-realizable oriented matroid on eight elements $\{1, 2, 3, 4, 5, 6, f, g\}$. As a way of definition, we show in Figure 5 the contractions of $\mathtt{EFM}(8)$ at the six first points; these contractions are acyclic oriented matroids of rank 3 on seven elements, realizable as point configurations in the plane. The cocircuits of $\mathtt{EFM}(8)$ can be read from the figure: the contraction $\mathtt{EFM}(8)/a$ permits to read the cocircuits which do not have a on the support, and no cocircuit can have all the six points 1, 2, 3, 4, 5, and 6 on its support. The interested reader can check that the cocircuits read from the figure coincide with the ones listed in page 464 of [7]. The figure shows that $\mathtt{EFM}(8)$ has symmetry group isomorphic to S_3, generated by the permutations $(16)(24)(35)(fg)$ and $(123)(456)$.

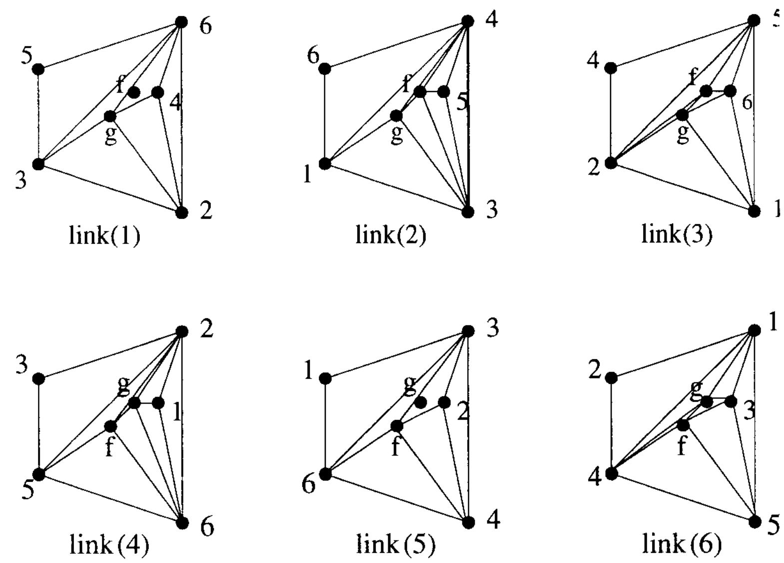

Figure 5: Links of a triangulation of $\mathtt{EMF}(8)$.

Actually, in the figure we have drawn a certain triangulation of each contraction. The six triangulations are the links of the following collection of simplices of $\mathtt{EFM}(8)$:

$$T := \{\{1624\}, \{2435\}, \{3516\},$$
$$\{235f\}, \{245f\}, \{356f\}, \{456f\},$$
$$\{146g\}, \{136g\}, \{124g\}, \{123g\},$$
$$\{24fg\}, \{46fg\}, \{63fg\}, \{32fg\}\}.$$

Claim 1: T is a triangulation of $\mathtt{EFM}(8)$.

Proof: Any interior rank-3 simplex τ (recall that this means dimension 2) contains at least one point $a \in \{1, 2, 3, 4, 5, 6\}$. The oriented pseudo-manifold property for $\tau \setminus \{a\}$ in $link_T(a)$ implies the property for τ in T. Thus, T has the oriented pseudo-manifold property. Also, since 1 is a vertex of $\mathtt{EFM}(8)$, any simplex of T which covers a lexicographic interior extension in general position starting by $[1^+, \ldots]$ must contain 1. The fact that $link_T(1)$ is a triangulation implies that all such extensions are covered exactly once. □

Claim 2: T is a non-lifting triangulation of EFM(8).

Proof: We look at the restriction of EFM(8) to the points $\{1,2,3,4,5,6\}$. The restriction of T contains the simplices $\{1624\}$, $\{2435\}$ and $\{3516\}$. Suppose that there is a triangulation T' of the restriction containing these three simplices. For the link of T' at 1 to be a triangulation it is necessary that $\{1346\}$ and $\{1234\}$ be in T'. But in the contraction at point 3 we see that $\{1234\}$ intersects $\{2345\}$ improperly.

That is, the restriction of T to the six points cannot be extended to a triangulation without using additional points. Proposition 5.4 implies that T is not a lifting triangulation. □

Claim 3: T is the only triangulation of EFM(8) containing the simplices $\{146g\}$ and $\{235f\}$.

Proof: We will show that the only way to complete $\{146g\}$ and $\{235f\}$ to a triangulation of EFM(8) is using precisely the simplices of T.

- the presence of $\{146g\}$ implies (see the contraction at point 1) the presence of the simplex $\{1246\}$ and the absence of any simplex containing $\{1f\}$. With similar arguments at 5 we conclude the presence of $\{2345\}$ and the absence of $\{5g\}$.

- then, the fact that 5 and g do not lie on the same simplex in the contraction at point 1 implies the presence of the simplices $\{136g\}$ and $\{1356\}$. A similar argument at 5 shows the presence of $\{356f\}$.

- with this, the only way to complete the links at 3 and 6 to subdivisions is the inclusion of the simplices $\{36fg\}$, $\{23fg\}$, $\{123g\}$, $\{46fg\}$ and $\{456f\}$.

- the simplex $\{124g\}$ completes the link at 1, the simplex $\{245f\}$ completes the link at 5 and then $\{24fg\}$ completes the links at 2 and 4. □

The three claims together imply that no lifting triangulation of EFM(8) contains the two simplices $\{146g\}$ and $\{235f\}$. Part (ii) of the following statement has the following stronger implication: no lifting subdivision of EFM(8) has $\{146g\}$ and $\{235f\}$ contained in two different cells.

Proposition 5.7 (i) *No lifting triangulation of* EFM(8) *contains the two simplices* $\{146g\}$ *and* $\{235f\}$*, although* $(\{146g\}, \{235f\})$ *is a covector.*

(ii) *The dual oriented matroid* $\text{EFM}(8)^*$ *has no extension* $\text{EFM}(8)^* \cup p$ *with* $p \in conv_{\text{EFM}(8)^* \cup p}(\{146g\}) \cap conv_{\text{EFM}(8)^*}(\{235f\})$. *In particular, the two simplices* $\{146g\}$ *and* $\{235f\}$ *intersect properly, although* $(\{16g\}, \{35\})$ *is a circuit. No triangulation of* $\text{EFM}(8)^*$ *contains both simplices.*

Proof: (i) The covector $(\{16g\}, \{235f\})$ can be read from the contraction at point 4 in Figure 5. Its composition with any covector positive at 4 shows that

$(\{146g\}, \{235f\})$ is a covector. The rest of the statement follows from claims 2 and 3.

(ii) That $(\{16g\}, \{35\})$ is a cocircuit of $\mathtt{EFM}(8)$ can be read from the contraction at either 2 or 4.

There is no triangulation of $\mathtt{EFM}(8)^*$ containing $\{146g\}$ and $\{235f\}$ because such a triangulation would contain the complements of two simplices of the triangulation T of $\mathtt{EFM}(8)$, which is impossible by part (g) of Theorem 2.4. We now prove that the two simplices intersect properly; even more, that there is no extension $\mathtt{EFM}(8)^* \cup p$ with $p \in conv(\{146g\}) \cap conv(\{235f\})$.

If there was one such extension $\mathtt{EFM}(8)^* \cup p$, then there would be two circuits $(\tau_1, \{p\})$ and $(\tau_2, \{p\})$ in $\mathtt{EFM}(8)^* \cup p$ with $\tau_1 \subset \{146g\}$ and $\tau_2 \subset \{235f\}$. Elimination of the element p would imply that (τ_1, τ_2) is a vector of $\mathtt{EFM}(8)^* \cup p$.

If one of τ_1 and τ_2 (say τ_1) has four elements, then p is in the relative interior of $\{146g\}$. The perturbation $p' := [p^+, 2^+, 3^+, 5^+, f^+]$ will produce an extension of $\mathtt{EFM}(8)^*$ interior, in general position and in the relative interior of both $\{146g\}$ and $\{235f\}$. This is impossible since then the associated lifting triangulation of $\mathtt{EFM}(8)$ contains both simplices, in contradiction with claims (1) and (2).

If both τ_1 and τ_2 have at most three elements, then one of them (say τ_1) has three elements. Since (τ_1, τ_2) is a vector of $\mathtt{EFM}(8)^*$, τ_2 is a positive vector in the rank-1 contraction $\mathtt{EFM}(8)^*/\tau_1$. In particular, there is an element $a \in \tau_2$ such that the cocircuit vanishing on τ_1 has the same sign on a and in the element $\{146g\} \setminus \tau_1$. Then, the perturbation $p' := [p^+, a^+]$ is still in the convex hull of τ_2 and in the relative interior of $\{146g\}$. This is the previous case. □

Another interesting feature of the example $\mathtt{EFM}(8)$ is the following. Its deletion at point f is realizable by the columns of the following matrix, as shown in page 461 of [7]:

$$\begin{pmatrix} -1 & \epsilon & -\epsilon & 1 & 0 & 0 & 0 \\ -\epsilon & -1 & \epsilon & 0 & 1 & 0 & 0 \\ \epsilon & -\epsilon & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A Gale transform of this configuration, rescaled at the point g (in particular, reoriented) is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \epsilon & -\epsilon & 1/3 \\ 0 & 1 & 0 & -\epsilon & \epsilon & 1 & 1/3 \\ 0 & 0 & 1 & \epsilon & 1 & -\epsilon & 1/3 \end{pmatrix},$$

which can be viewed as a point configuration $\mathcal{A}$ in the plane $x + y + z = 1$ of $\mathbf{R}^3$. $\mathcal{A}$ consists of the six vertices of a (non-regular) hexagon and an interior point in it. Using the pushing-pulling characterization of lexicographic triangulations (see Remark 4.4) it is easy to conclude that all the triangulations of $\mathcal{A}$ are lexicographic (and thus regular). However, the dual oriented matroid has non-realizable extensions (such as the reorientation at g of $\mathtt{EFM}(8)$). The apparent contradiction between the fact that $\mathcal{A}$ has only lexicographic triangulations while its dual has non-realizable interior extensions in general position is not so, because different extensions of the dual can correspond to the same triangulation of $\mathcal{A}$. In particular, the triangulation of $\mathcal{A}$ produced by the extension `EFM`(8) can coincide with the one produced by some lexicographic extension.

Example 5.8 (A non-lifting triangulation of a unimodular polytope)

A real matrix is called *unimodular* if all its maximal minors lie in $\{0, 1, -1\}$. It is a classical result of matroid theory that a matroid can be represented over the rationals by a unimodular matrix if and only if it can be represented over any field [22]; if the matroid is orientable this is equivalent to the absence of any rank 2 uniform minor on 4 elements and to the matroid being representable over the field with two elements. Such orientable matroids are called *binary*.

Unimodular vector or point configurations play an important role in different branches of discrete mathematics. In connection with triangulations, it was somewhat unexpected that unimodular configurations can have non-regular triangulations. This was shown by de Loera [9] who constructed a non-regular triangulation of the product $\Delta_3 \times \Delta_3$ of two 3-dimensional simplices. Later on Sturmfels [28, Theorem 10.15] constructed a non-regular triangulation of $\Delta_2 \times \Delta_5$. It seems not to be a coincidence the fact that $\Delta_{r-1} \times \Delta_{n-r-1}$ has non-regular triangulations if and only if there are non-realizable oriented matroids on n elements of rank r. Actually, Sturmfels has shown that the two cited examples of non-regular triangulations can be "derived" from the Vamos and Pappus matroids. We consider an interesting problem to decide whether a product of two simplices (or a minor of it) can have non-lifting triangulations.

Before going into detail let us make another consideration. A point configuration $\mathcal{A}$ in $\mathbf{R}^d$ is called *unimodular* if all the d-simplices with vertices in $\mathcal{A}$ have the same volume. This occurs if and only if the configuration can be represented by a *homogeneous* unimodular matrix, where "homogeneous" means that all the column vectors of the matrix lie in an affine hyperplane. However, any acyclic unimodular matrix can be considered to represent a *weighted* unimodular point configuration as follows.

Let $v_1, \ldots, v_n \in \mathbf{R}^r$ be the columns of a rank r acyclic unimodular matrix. Choose a linear functional f on $\mathbf{R}^r$ positive in all the columns of the matrix (which exists since the configuration is acyclic). Dividing each vector v_i by the value $f(v_i)$ we get a homogeneous matrix and thus a point configuration in $\mathbf{R}^{r-1}$. We say that this configuration is *weighted unimodular* with weights $f(v_1), \ldots, f(v_n)$ because the volume of any maximal simplex of the configuration multiplied by the product of weights of its vertices equals one. If all the weights are rational (which can be achieved by a rational choice of f if the matrix $\mathcal{A}$ itself has rational entries), the lifting procedure exhibited in Section 4.3 allows to construct a unimodular configuration with "the same" triangulations as $\mathcal{A}$. This follows from the following result:

Proposition 5.9 *Let $\{P_1, \ldots, P_n\}$ be a weighted unimodular point configuration in $\mathbf{R}^{r-1}$ with positive integer weights $w_1, \ldots, w_n$. Let $\mathcal{M}$ be the oriented matroid of affine dependences of the point configuration and consider the oriented matroid $\Sigma(\mathcal{M})$ of Section 4.3 taking each k_i to be precisely the weight w_i. Then, $\Sigma(\mathcal{M})$ is the oriented matroid of affine dependences of a certain unimodular point configuration with $\sum w_i$ points in dimension $\sum w_i - n + r - 1$.*

Proof: We will explicitly construct the matrix of a certain point configuration of the stated rank and number of columns, and then show that it is unimodular, homogeneous and that it realizes $\Sigma(\mathcal{M})$. Let $v_1, \ldots, v_n$ be a non-homogeneous unimodular vector configuration in $\mathbf{R}^r$ which homogenizes to $P_1, \ldots, P_n$. Let $\{v_1^*, \ldots, v_n^*\} \in \mathbf{R}^{n-r}$ be a Gale transform of it. That is, assume that the matrices A and A' having as columns the vectors v_i and v_i' respectively have orthogonally complementary row spaces.

For each $i = 1, \ldots, n$ we consider the following three matrices, of sizes $w_i \times (\omega_i - 1)$, $w_i \times r$ and $w_i \times (n-r)$ respectively. $I_{\omega_i - 1}$ represents the identity matrix of size $\omega_i - 1$ and $1_{\omega_i - 1}$ the column vector $(1, 1, \ldots, 1)$ of length $w_i - 1$:

$$S_i := (-I_{\omega_i-1} \mid 1_{\omega_i-1}) \qquad V_i := (v_i/w_i \ \cdots \ v_i/w_i) \qquad V_i^* := (v_i^* \ \cdots \ v_i^*).$$

Consider then the following two matrices:

$$\Sigma(A) := \begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_n \\ V_1 & V_2 & \cdots & V_n \end{pmatrix} \qquad \Sigma(A)^* := (\, V_1^* \quad \cdots \quad V_n^* \,).$$

It is obvious that $\Sigma(A)^*$ realizes the oriented matroid $\Sigma(\mathcal{M})^*$. It is also easy to check that $\Sigma(A)$ and $\Sigma(A)^*$ are Gale transforms of one another: the lower rows of $\Sigma(A)$ are orthogonal to the rows of $\Sigma(A)^*$ because A and A^* are Gale transforms of one another, while the upper rows of $\Sigma(A)$ are orthogonal to the rows of $\Sigma(A)^*$ because the sum of the entries in each row of each S_i is zero while the rows of each V_i^* are constant. Thus, $\Sigma(A)$ realizes the oriented matroid $\Sigma(\mathcal{M})$.

Let $f = (f_1, \ldots, f_r)$ be a linear functional on $\mathbf{R}^r$ which gives the weights $w_i = f(v_i)$. Then, the linear functional $(0, \ldots, 0, f_1, \ldots, f_r)$ with a string of $\sum w_i - n$ zeroes shows that the columns of $\Sigma(A)$ are homogeneous.

Unimodularity of $\Sigma(A)$ can be checked directly, or deduced from the fact that a Gale transform of a unimodular matrix is unimodular as well. This implies that A^* is unimodular, which clearly shows unimodularity for $\Sigma(A)^*$ and in turn for $\Sigma(A)$. □

We now construct the desired non-lifting triangulation of a homogeneous polytope. As a first step we construct a non-lifting triangulation of a weighted unimodular point configuration $\mathcal{A}$ with 9 points in $\mathbf{R}^3$. The configuration in question is given by the columns of the following homogeneous rank 4 matrix:

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 1/4 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1/2 & 1/2 & 1/4 \end{pmatrix}.$$

Geometrically, the configuration $\mathcal{A}$ consists of the four vertices of a tetrahedron Δ, four mid-poins of the edges of Δ and the barycenter of Δ. It is weighted unimodular with weights $(1, 1, 1, 1, 2, 2, 2, 2, 4)$, as can be easily checked. The fact that the oriented matroid admits a unimodular representation also follows from the fact that its dual is graphic, and hence binary. Indeed, the dual is the graphic oriented matroid associated to the complete bipartite graph $K_{3,3}$ with the orientation shown in Figure 6. The labels on the edges refer to the indices of the columns of A.

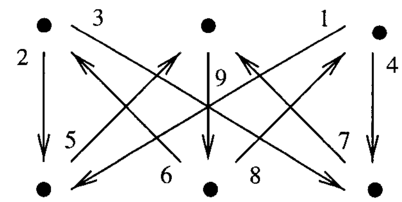

Figure 6: A graph whose cycle matroid is dual to the configuration $\mathcal{A}$.

Let us consider the following triangulation ∂T of the boundary of the tetrahedron:

$$T := \{\{3,6,7\},\{2,6,7\},\{2,4,7\},\{2,5,6\},\{1,5,6\},\{1,3,6\},$$

$$\{1,5,8\},\{4,5,8\},\{2,4,5\},\{4,7,8\},\{3,7,8\},\{1,3,8\}\}.$$

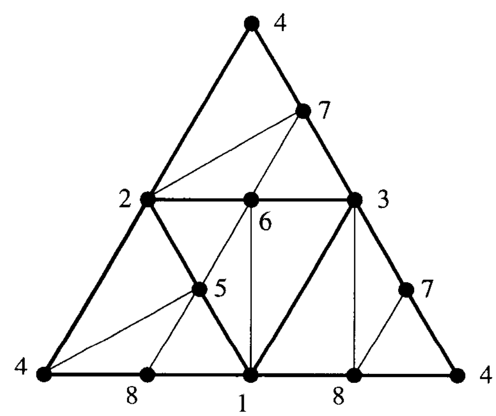

Figure 7: A triangulation of the boundary of a tetrahedron.

This triangulation is displayed in Figure 7, where the boundary of the tetrahedron appears "unfolded". ∂T cannot be completed to a triangulation $\widehat{T}$ of $\mathcal{A}$ without using the interior point 9: if it could, the triangle $\sigma = \{1,3,6\} \in T$ should be joined in $\widehat{T}$ to one of the three points 4, 7 or 8 which do not lie on the facet of σ. It cannot be joined to neither 4 nor 7 because the edges $\{1,4\}$ and $\{1,7\}$ are not edges of ∂T. Thus, we conclude that it should be joined to 8 and, in particular, that $\{6,8\}$ should be an edge of $\widehat{T}$. With the same arguments we conclude that $\{7,5\}$ should also be an edge, but this is impossible because these two edges intersect improperly at the barycenter of the tetrahedron.

By Proposition 5.4, this implies that the triangulation T of $\mathcal{A}$ obtained coning ∂T to the central point 9 is not a lifting triangulation of $\mathcal{A}$. As a consequence:

Proposition 5.10 *$\mathcal{A}$ is a weighted unimodular configuration with 9 points in $\mathbf{R}^3$ which has a non-lifting triangulation. There exists a unimodular polytope with 16 points in $\mathbf{R}^{10}$ which has a non-lifting triangulation.*

Proof: The first part has already been shown. For the second part we apply Proposition 5.9 to $\mathcal{A}$ to get a unimodular point configuration with 16 points in $\mathbf{R}^{10}$. The configuration is polytopal by Theorem 4.18, since all the non-vertices of $\mathcal{A}$ have weight at least 2. □

Incidentally, T is a triangulation with only 4 bistellar flips, supported on the 4 quadrilaterals that appear in the facets of the tetrahedron; that is, the number of flips is less than the dimension $n - d - 1 = 5$ of the associated *secondary polytope* (see definition in [3] or [15]. A combinatorially equivalent triangulation was constructed by de Loera et al. [10] and is the smallest known example of a triangulation with less bistellar flips than the dimension of the secondary polytope.

Example 5.11 (A non-lifting triangulation of the 4-cube)

Jesus de Loera [9] has shown that the 4-dimensional cube has non-regular triangulations. We go further and construct a non-lifting triangulation of the 4-cube.

We consider the 4-cube as realized by the point configuration C_4 in $\mathbf{R}^4$ whose 16 points are all the 0-1 vectors on 4 coordinates. The contraction of C_4 at any of its points is realized by the point configuration in $\mathbf{R}^3$ consisting of the barycenters of the fifteen faces of a tetrahedron, including the vertices and the tetrahedron itself as faces but excluding the empty face. We observe that the nine-point configuration $\mathcal{A}$ of the previous example is a subconfiguration of this. Figure 8 shows the boundary of the triangulation T of the previous example but with the points labeled as coming from the contraction of the 4-cube at the vertex (0000).

This provides a non-lifting triangulation with 12 simplices of the contraction $C_4/(0000)$. We denote this triangulation here by $T(C_4/(0000))$. Since it is a non-lifting triangulation, no triangulation of C_4 can have it as a link (we saw in the proof of Proposition 5.3 that any link in a lifting triangulation is a lifting triangulation as well). Thus, we only need to show how to complete the join $T(C_4/(0000)) \cdot (0000)$ to a triangulation of C_4.

For doing this, observe that all the simplices in $T(C_4/(0000)) \cdot (0000)$ use the point (1111) (this point is the barycenter of the tetrahedron in the link at (0000). Thus, we can look at $T(C_4/(0000)) \cdot (0000)$ in the link at point (1111). This link is again the barycenters of the faces of a tetrahedron, and the triangulation obtained as the link at (1111) of $T(C_4/(0000)) \cdot (0000)$ is depicted in Figure 9. It is the triangulation of a polytope P whose vertices are the four barycenters of the facets of the tetrahedron (the four points labeled (1000),

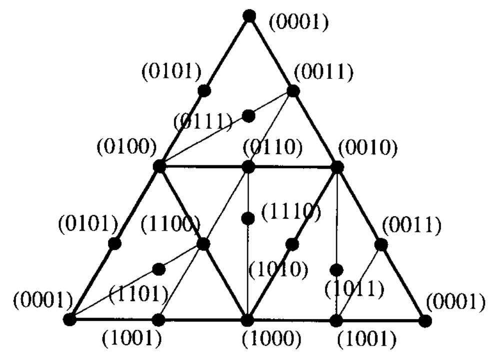

Figure 8: A triangulation of a link of the 4-cube.

(0100), (0010) and (0001)) and four of the six mid-points of the edges of the tetrahedron (the points labeled (1100), (0110), (0011) and (1001)). P has four triangular facets and four square ones and is triangulated with 12 tetrahedra.

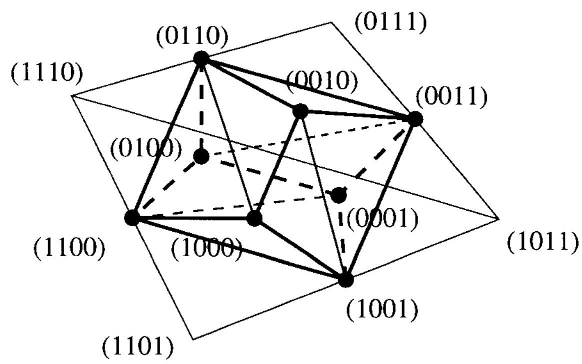

Figure 9: A triangulation of a polytope P contained in the tetrahedron.

We will complete the triangulation of P to a triangulation of the whole tetrahedron as follows. The mid-point (1010) of the segment [(1011), (1110)] sees four boundary triangles of the triangulation of the polytope P. We add the four tetrahedra obtained joining them to (1010). In the same way, we add the joins of (0101) to the four boundary triangles of the triangulation of P seen from it. More precisely, we are adding the following eight tetrahedra to the triangulation of P:

$$\{\{(1100),(1000),(0110)\},\{(1000),(0110),(0010)\},$$
$$\{(0011),(0010),(1001)\},\{(0010),(1001),(1000)\},\}\cdot(1010)$$

and

$$\{\{(0110),(0100),(0011)\},\{(0100),(0011),(0001)\},$$
$$\{(1001),(0001),(1100)\},\{(0001),(1100),(0100)\},\}\cdot(0101).$$

This produces a triangulation with 20 tetrahedra of the octahedron whose vertices are the six mid-points of the edges of the tetrahedron $C_4/(1111)$. It is now easy to complete this to a triangulation of the tetrahedron $C_4/(1111)$ with 24 tetrahedra, by adding four tetrahedra. Namely, for each of the vertices (1110), (1101), (1011) and (0111) of the tetrahedron $C_4/(1111)$ we add its cone to the only boundary triangle visible from it.

Thus, we have triangulated $C_4/(1111)$ with 24 tetrahedra. Let us denote by $T(C_4/(1111))$ this triangulation. In C_4, we consider the collection of 24 full-dimensional simplices $T(C_4/(1111)) \cdot (1111)$. It is clear that these 24 simplices intersect properly (in the usual geometric sense) since all them have (1111) as a vertex and their links at (1111) intersect properly. In the other hand, any triangulation of the 4-cube has at most $4! = 24$ simplices; thus, the 24 simplices in question must cover the 4-cube (again in a geometric sense). We conclude that the 24 simplices provide a triangulation of the 4-cube.

5.3 A characterization of lifting triangulations.

In Section 5.1 we have characterized lifting triangulations of an oriented matroid $\mathcal{M}$ as those which can be "lifted" to triangulations of the Lawrence polytope $\Lambda(\mathcal{M})$ (Proposition 5.3) and we have shown a combinatorial condition which is necessary for a triangulation to be lifting (Proposition 5.4). Here we use the first fact to convert the second in a necessary and sufficient condition for a triangulation to be lifting.

We will use the notation $\mathcal{P}(T)$ and $T|_A$ introduced before Proposition 5.4, and also the following one introduced before Proposition 2.10: if B is a collection of subsets of a set E and $b \in E$, we denote

$$B \cdot b := \{\sigma \cup \{b\} | \sigma \in B\}.$$

Given a lifting triangulation T of an oriented matroid $\mathcal{M}$, in the proof of Corollary 3.3 we constructed a lifting triangulation $T(A)$ of any restriction $\mathcal{M}(A)$ of $\mathcal{M}$, with the property that $T(A)$ extends $\mathcal{P}(T)|_A$ (in particular, $T(E) = T$). The construction is based in the simple fact that a lift $\widehat{\mathcal{M}}$ of an oriented matroid $\mathcal{M}$ induces a lift of every restriction of $\mathcal{M}$. It is easy to verify (and we will do it in the proof of Theorem 5.13) that the collection of triangulations $T(A)$ obtained in this way satisfies the following "compatibility" properties. For every $A \subset E$ and $b, c \in A$:

- $\mathcal{P}(T(A))|_{A\setminus\{b\}} \subset \mathcal{P}(T(A \setminus \{b\}))$, and
- $T(A \setminus \{b\}) \cap T(A \setminus \{c\}) \subset T(A)|_{A\setminus\{b,c\}}$.

It is the goal of this section to prove that a triangulation T of an oriented matroid $\mathcal{M}$ is a lifting triangulation *if and only if* for every restriction $\mathcal{M}(A)$ of $\mathcal{M}$ there is a triangulation $T(A)$ in such a way that the triangulations of the restrictions satisfy these compatibility properties. One direction of the proof is just checking that the triangulations obtained in the proof of Proposition 5.4 satisfy the properties. The other direction will be an iterative use of the following technical (and difficult) lemma:

Lemma 5.12 *Let $\mathcal{M}$ be an oriented matroid on a set E. Suppose that for each subset $A \subset E$ we are given a triangulation $T(A)$ of the restricted oriented matroid $\mathcal{M}(A)$ and that the following properties are satisfied, for every $A \subset E$ and $b, c \in A$:*

$$\mathcal{P}(T(A))|_{A\setminus\{b\}} \subset \mathcal{P}(T(A \setminus \{b\}))$$
$$T(A \setminus \{b\}) \cap T(A \setminus \{c\}) \subset T(A)$$

Let $a \in E$ be one of the elements of $\mathcal{M}$ and consider the Lawrence lift $\widehat{\mathcal{M}} := (\mathcal{M}^* \cup \overline{a})^*$ of $\mathcal{M}$ on the element a, where $\mathcal{M}^* \cup \overline{a}$ denotes the lexicographic extension of $\mathcal{M}^*$ by the element $\overline{a} := [a^-]$ antiparallel to a. For every $A \subset E \setminus a$ consider the following four collections of subsets of $E \cup \overline{a}$:

$$\widehat{T}(A) := T(A), \qquad \widehat{T}(A \cup \{a\}) := T(A) \cdot a, \qquad \widehat{T}(A \cup \{\overline{a}\}) := T(A) \cdot \overline{a},$$

$$\widehat{T}(A \cup \{a, \overline{a}\}) := T(A \cup \{a\}) \cdot \overline{a} \;\cup\; (T(A) \setminus T(A \cup a)) \cdot a.$$

In these conditions,

(i) *For any $B \subset E \cup \{\overline{a}\}$, $\widehat{T}(B)$ is a triangulation of the restriction $\widehat{\mathcal{M}}(B)$.*

(ii) *For any $B \subset E \cup \{\overline{a}\}$ and any $b \in B$*

$$\mathcal{P}(\widehat{T}(B))|_{B\setminus\{b\}} \subset \mathcal{P}(\widehat{T}(B \setminus \{b\})).$$

(iii) *For any $B \subset E \cup \{\overline{a}\}$ and any $b, c \in B$*

$$\widehat{T}(B \setminus \{b\}) \cap \widehat{T}(B \setminus \{c\}) \subset \widehat{T}(B).$$

Figure 10 shows an example of the triangulation $\widehat{T}(A \cup \{a, \overline{a}\})$ where A is a point configuration with 4 points in a line and a is an extra point on the line.

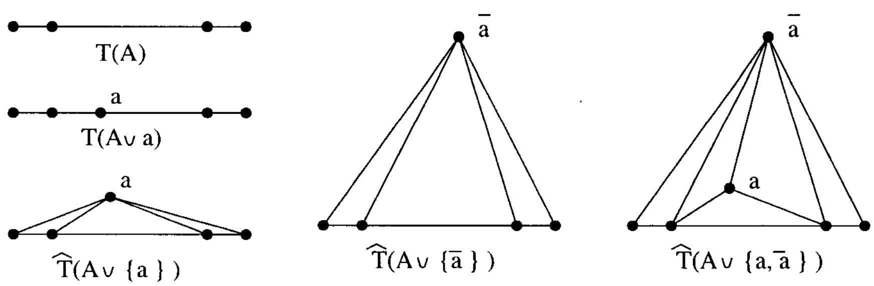

Figure 10: The lifting procedure of Lemma 5.12.

Proof: We first prove the following

Claim: For any $A \subset E \setminus a$, $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(\overline{a})$ and $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(a)$ are triangulations of $\widehat{\mathcal{M}}(A \cup \{a, \overline{a}\})/\overline{a}$ and $\widehat{\mathcal{M}}(A \cup \{a, \overline{a}\})/a$, respectively.

The oriented matroid $\widehat{\mathcal{M}}(A \cup \{a, \overline{a}\})/\overline{a}$ is precisely $\mathcal{M}(A \cup \{a\})$, and from the definition of $\widehat{T}(A \cup \{a, \overline{a}\})$ it follows that $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(\overline{a})$ is the triangulation

$T(A \cup a)$. In the same way, the oriented matroid $\widehat{\mathcal{M}}(A \cup \{a, \overline{a}\})/a$ equals the reorientation $\mathcal{M}' := \mathcal{M}(A \cup \{a\}) \cup \overline{a} \setminus a$ of $\mathcal{M}(A \cup \{a\})$ at the element a. The link $link_{\widehat{T}(A \cup \{a,\overline{a}\})}(a)$ can be rewritten as

$$T' := link_{\widehat{T}(A \cup \{a,\overline{a}\})}(a) = link_{T(A \cup a)}(a) \cdot \overline{a} \quad \cup \quad (T(A) \setminus T(A \cup a)).$$

We will prove that T' satisfies the oriented pseudo-manifold property and covers some interior extension exactly once.

Let τ be an interior $(r-1)$-simplex of $\mathcal{M}'$. Since the link of $\overline{a}$ in T' is a triangulation (it equals the link of a in $link_{\widehat{T}(A \cup \{a,\overline{a}\})}(\overline{a})$) we only need to prove the oriented pseudo-manifold property of T' in the case $\overline{a} \notin \tau$. In this case $\tau \in \mathcal{P}(T(A))$, since it lies either in a simplex of $T(A)$ or in one of $T(A \cup a)$ (or both). That is, there is at least one r-simplex $\tau \cup \{b_1\}$ containing τ in $T(A)$.

Suppose first that τ is interior in $\mathcal{M}(A)$. Then, there are exactly two r-simplices $\tau \cup b_1$ and $\tau \cup b_2$ in $T(A)$ containing τ, with b_1 and b_2 in opposite sides of the cocircuit vanishing on τ. If both $\tau \cup \{b_1\}$ and $\tau \cup \{b_2\}$ were in $T(A \cup \{a\})$, then τ could not be in T', according to the formula for T' above. If none of $\tau \cup b_1$ and $\tau \cup b_2$ is in $T(A \cup \{a\})$, then they are both in T'. Moreover, in this case $\tau \cup \{a\}$ cannot be a simplex of $T(A \cup \{a\})$ because then there would be a second simplex in $T(A \cup \{a\})$ containing τ, which could only be either $\tau \cup \{b_1\}$ or $\tau \cup \{b_2\}$. Thus, $\tau \cup \{b_1\}$ and $\tau \cup \{b_2\}$ are the only r-simplices in T' containing τ and the oriented pseudo-manifold property is satisfied. If one of them, say $\tau \cup b_1$, is in $T(A \cup \{a\})$ and the other is not, then there has to be another r-simplex in $T(A \cup \{a\})$ containing τ and it can only be $\tau \cup \{a\}$ (otherwise it would be in $T(A \cup \{a\})|_A \subset T(A)$). This implies that $\tau \cup \{\overline{a}\}$ and $\tau \cup \{b_2\}$ are the only r-simplices in T' containing τ. The oriented pseudo-manifold property is satisfied since b_1 and a lie on opposite sides of the cocircuit vanishing on τ.

Now suppose that τ is not interior in $\mathcal{M}(A)$. Then there is a unique simplex $\tau \cup \{b_1\}$ containing τ in $T(A)$. Since τ is interior in $\mathcal{M}'$, $\overline{a}$ and b_1 lie in opposite sides of τ. That is, a and b_1 lie on the same side of τ in $\mathcal{M}$. This implies that τ is not interior in $\mathcal{M}(A \cup \{a\})$. Let $F \subset A \cup \{a\}$ denote the facet of $\mathcal{M}(A \cup \{a\}$ which contains τ, which is also a facet of $\mathcal{M}(A)$. Recall from Corollary 2.12 that the restriction of a triangulation to a facet is a triangulation of the restricted oriented matroid.

Let $\tau := \{a_1, \ldots, a_{r-1}\}$ and consider the lexicographic extension by the point $p := [a_1^+, \ldots, a_{r-1}^+]$, which is interior and in general position in $\mathcal{M}(F)$. This lexicographic extension is covered by τ in $T(A)$ and by some $(r-1)$-simplex τ' in $T(A \cup \{a\})$. Since $a \notin F$, we have $\tau' \in \mathcal{P}(T(A \cup \{a\}))|_A \subset \mathcal{P}(T(A))$. That is, tau' is a facet of a simplex of $T(A)$ and, thus, $\tau = \tau'$. In other words, τ is a facet of a (unique, because τ is not interior) simplex of $T(A \cup \{a\})$. We cannot have $\tau \cup \{b_1\} in T(A \cup \{a\}$, because then $\tau \notin T'$. The containment $\mathcal{P}(T(A \cup \{a\}))|_A \subset \mathcal{P}(T(A))$ implies that the only other possibility is $\tau \cup \{a\} \in T(A \cup \{a\})$. Then, the simplices of T' containing τ will be $\tau \cup \{b_1\}$ and $\tau \cup \{\overline{a}\}$, which agrees with the oriented pseudo-manifold property.

We finally have to check that T' covers some interior extension in general position exactly once. For this consider any simplex $\sigma = \{\overline{a}, a_1, \ldots, a_{r-1}\}$ in T' containing $\overline{a}$ and the lexicographic extension by $[\overline{a}^+, a_1^+, \ldots, a_{r-1}^+]$. It is covered by σ and by no other simplex of T' containing $\overline{a}$: this is so because $link_{T'}(\overline{a})$ is a triangulation (it is a link of $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(\overline{a})$). If a simplex $\sigma' \in T'$ not containing $\overline{a}$ covers the extension, then $\sigma' \in T(A) \setminus T(A\cup\{a\})$. Then the extension is interior in $\mathcal{M}(A)$ and thus in $\mathcal{M}(A\cup\{a\})$, and it has to be covered by a simplex $\sigma'' \in T(A\cup\{a\})$. But this is impossible: if $A \in \sigma''$ then σ'' cannot cover a lexicographic extension of the type $[\overline{a}^+,\ldots]$. If $a \notin \sigma''$ then $\sigma'' \in T(A)$ and the extension is covered twice in $T(A)$. This finishes the proof of the claim.

(i) Let $A \in E\backslash a$. The collections $\widehat{T}(A)$, $\widehat{T}(A\cup\{a\})$ and $\widehat{T}(A\cup\{\overline{a}\})$ are clearly triangulations; the first one is the triangulation $T(A)$ of $\mathcal{M}(A)$ and the other two are cones over it. For the case of $T(A\cup\{a,\overline{a}\})$, observe that A is a facet of $\widehat{\mathcal{M}}(A\cup\{a,\overline{a}\})$. Thus, all the interior $(r-1)$-simplices of $\widehat{\mathcal{M}}(A\cup\{a,\overline{a}\})$ contain either a or $\overline{a}$ (or both) the fact that the two links in part (i) are triangulations implies that $\widehat{T}(A\cup\{a,\overline{a}\})$ has the oriented pseudo-manifold property.

If a is not used in the triangulation $T(A\cup\{a\})$, then $T(A) = T(A\cup\{a\})$ and $T(A\cup\{a,\overline{a}\}) = T(A\cup\{\overline{a}\})$. Otherwise, let $\sigma = \{a, a_1, \ldots, a_{r-1}\}$ be a simplex of $T(A\cup\{a\})$ which uses a. The lexicographic extension of $\widehat{\mathcal{M}}(A\cup\{a,\overline{a}\}$ given by the expression $[a^+, \overline{a}^+, a_1^+, \ldots, a_{r-1}^+]$ is covered by σ and cannot be covered by any other r-simplex of $\widehat{T}(A\cup\{a,\overline{a}\}$ containing either a or $\overline{a}$, because then the contracted extension would be covered twice either in $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(\overline{a})$ or in $link_{\widehat{T}(A\cup\{a,\overline{a}\})}(a)$. Since any triangulation of $\widehat{T}(A\cup\{a,\overline{a}\}$ contains either a or $\overline{a}$, we have finished.

(ii) If B does not contain either a or $\overline{a}$ then the checking is fairly easy. Thus, suppose that $a, \overline{a} \in B$. We distinguish the three cases $b = a$, $b = \overline{a}$ and $b \notin \{a, \overline{a}\}$. We denote by $A = B \setminus \{a, \overline{a}\}$. That is, $\mathcal{P}(\widehat{T}(B)) = \mathcal{P}(T(A\cup\{a\}) \cdot \overline{a}) \cup \mathcal{P}((T(A) \setminus T(A\cup\{a\})) \cdot a)$.

If $b = a$, then both parts of $\mathcal{P}(\widehat{T}(B))|_{A\cup\{\overline{a}\}}$ are in $\mathcal{P}(T(A)\cdot\overline{a}) = \mathcal{P}(\widehat{T}(B\backslash\{a\})$.

If $b = \overline{a}$, then the second part of $\mathcal{P}(\widehat{T}(B))|_{A\cup\{\overline{a}\}}$ is clearly in $\mathcal{P}(T(A)\cdot a) = \mathcal{P}(\widehat{T}(B\setminus\{\overline{a}\}))$. For the first part we have to verify that $\mathcal{P}(T(A\cup\{a\})) \subset \mathcal{P}(T(A)\cdot a$. Let $\tau \in \mathcal{P}(T(A\cup\{a\}))$. Then, either $a \notin \tau$ and $\tau \in T(A)$ or $a \in \tau$ and $\tau \setminus \{a\} \in T(A)$. In both cases $\tau \in \mathcal{P}(T(A)\cdot a$.

If $b \notin \{a, \overline{a}\}$ then

$$\mathcal{P}(\widehat{T}(B\backslash\{b\})) = \mathcal{P}(T(A\cup\{a\}\backslash\{b\})\cdot\overline{a}) \quad \cup \quad \mathcal{P}((T(A\backslash\{b\})\backslash T(A\cup\{a\}\backslash\{b\}))\cdot a)$$

Let ρ be a maximal set in $\mathcal{P}(\widehat{T}(B))|_{B\backslash\{b\}}$. If $\rho \in \mathcal{P}(T(A\cup\{a\})\cdot\overline{a})|_{B\backslash\{b\}}$ then clearly

$$\rho \in \mathcal{P}(T(A\cup\{a\}\setminus\{b\})\cdot\overline{a}) \subset \mathcal{P}(\widehat{T}(B\setminus\{b\})).$$

Otherwise $\rho \in \mathcal{P}((T(A)\setminus T(A\cup\{a\}))\cdot a)|_{B\backslash\{b\}}$. By our maximality assumption, either $\rho = \sigma\cup\{a\}$ where $\sigma \in T(A)\setminus T(A\cup\{a\})$ or $\rho = \tau\cup\{a\}$ where $\tau\cup\{b\} \in T(A)\setminus T(A\cup\{a\})$.

In the first case we have $\sigma \in T(A)$ and $\sigma \notin T(A\cup\{a\})$. Since $b \notin \sigma$ we have $\sigma \in T(A\setminus\{b\})$. Also, since $T(A)\cap T(A\cup\{a\}\setminus\{b\}) \subset T(A\cup\{a\})$, we must have $\sigma \notin T(A\cup\{a\}\setminus\{b\})$. Thus, $\sigma \in T(A\setminus\{b\})\setminus T(A\cup\{a\}\setminus\{b\})$.

In the second case, we will prove that either $\tau\cup\{a\} \in \mathcal{P}(T(A\cup\{a\}\setminus\{b\}))$ or $\tau \notin \mathcal{P}(T(A\cup\{a\}\setminus\{b\}))$. This finishes the proof since the two things imply that $\tau\cup\{a\}$ is respectively in the first or the second parts of $\mathcal{P}(\widehat{T}(B\setminus\{b\}))$.

If $rank(A \cup \{a\}) > rank(A)$, then $T(A \cup \{a\})$ is the cone of the triangulation $T(A)$ with apex a, which implies $\tau \cup \{a\} \in \mathcal{P}(T(A \cup \{a\}))$. Otherwise, the assumption that there is a unique simplex $\tau \cup \{b\}$ in $T(A) \setminus T(A \cup \{a\})$ containing τ implies that either $\tau \cup \{a\} \in T(A(\cup\{a\})|_{A\cup\{a\}\setminus\{b\}} \subset T(A \cup \{a\} \setminus \{b\})$ or the hyperplane containing τ is a facet F of $\mathcal{M}(A)$. We assume the second and will consider the three possible situations of a respect to the hyperplane of $A \cup \{a\}$ passing through τ: on the same side as b, on the opposite side, or on the hyperplane.

- If a and b lie on the same side, then F is also a facet of $A \cup \{a\}$ and $T(A \cup \{a\})$ restricted to F is a triangulation of $\mathcal{M}(F)$ (by Corollary 2.12) which has to coincide with the restriction of $T(A)$ to F. Thus, $\tau \in \mathcal{P}(T(A \cup \{a\})$ which in turn implies $\tau \cup \{a\} \in T(A \cup \{a\})$ and $\tau \cup \{a\} \in T(A \cup \{a\} \setminus \{b\})$.

- If a lies on the hyperplane, then $F \cup \{a\})$ is a facet of $\mathcal{M}(A \cup \{a\})$ and a face of $\mathcal{M}(A \cup \{a\} \setminus \{b\})$. The restrictions of $T(A \cup \{a\})$ and $T(A \cup \{a\} \setminus \{b\})$ to $F \cup \{a\}$ coincide and, thus, either $\tau \notin \mathcal{P}(T(A \cup \{a\} \setminus \{b\}))$ or $\tau \in \mathcal{P}(T(A \cup \{a\}))$. The second possibility is impossible, because it would imply that $\tau \cup \{c\} \in T(A \cup \{a\})$ for some $c \notin \{a, b\}$ and thus $\tau \cup \{c\} \in T(A)$ and $\tau \cup \{b\} \in T(A)$ with b and c on the same side of the hyperplane containing τ.

- If a and b lie on opposite sides of the hyperplane and $\tau \in \mathcal{P}(T(A \cup \{a\} \setminus \{b\})$, then clearly $\tau \cup \{a\} \in T(A \cup \{a\} \setminus \{b\})$, because a is the only point of $A \cup \{a\} \setminus \{b\}$ in that side.

(iii) Again, if B does not contain either a or $\overline{a}$, then the property is easy to check. If $a, \overline{a} \in B$ we have four possibilities, according to whether $\{b, c\}$ contains a, $\overline{a}$, none of them or both of them. We denote by $A = B \setminus \{a, \overline{a}\}$.

- If $\{b, c\} = \{a, \overline{a}\}$, then we have to prove that
$$\widehat{T}(A \cup \{a\}) \cap \widehat{T}(A \cup \{\overline{a}\}) \subset \widehat{T}(A \cup \{a, \overline{a}\}).$$
This is trivial since the intersection on the left hand side is empty.

- If $c = a$ and $b \neq \overline{a}$, then we have to prove that
$$\widehat{T}(A \cup \{\overline{a}\}) \cap \widehat{T}(A \cup \{a, \overline{a}\} \setminus \{b\}) \subset \widehat{T}(A \cup \{a, \overline{a}\}).$$
The left-hand side equals $(T(A) \cap T(A \cup \{a\} \setminus \{b\})) \cdot \overline{a}$, which is contained in $T(A \cup \{a\}) \cdot \overline{a}$ and thus in $\widehat{T}(A \cup \{a, \overline{a}\})$.

- If $c = \overline{a}$ and $b \neq a$, then we have to prove that
$$\widehat{T}(A \cup \{a\}) \cap \widehat{T}(A \cup \{a, \overline{a}\} \setminus \{b\}) \subset \widehat{T}(A \cup \{a, \overline{a}\}).$$
The left-hand side equals $(T(A) \cap (T(A \setminus \{b\}) \setminus T(A \cup \{a\} \setminus \{b\}))) \cdot a$. This is clearly contained in $(T(A) \setminus T(A \cup \{a\} \setminus \{b\}))|_{A\setminus\{b\}} \cdot a$. None of its simplices $\sigma \cup \{a\}$ can be contained in $T(A \cup \{a\}) \cdot a$, because then $\sigma \in T(A \cup \{a\})|_{A\cup\{a\}\setminus\{b\}} \subset T(A \cup \{a\} \setminus \{b\})$. This finishes this case.

- If $a, \overline{a} \notin \{b, c\}$, we have to prove that

$$\widehat{T}(A \cup \{a, \overline{a}\} \setminus \{c\}) \cap \widehat{T}(A \cup \{a, \overline{a}\} \setminus \{b\}) \subset \widehat{T}(A \cup \{a, \overline{a}\}).$$

The left hand side equals

$$(T(A \cup \{a\} \setminus \{c\}) \cap T(A \cup \{a\} \setminus \{b\})) \cdot \overline{a} \quad \cup$$

$$((T(A \setminus \{c\}) \setminus T(A \cup \{a\} \setminus \{c\})) \cap (T(A \setminus \{b\}) \setminus T(A \cup \{a\} \setminus \{b\}))) \cdot a.$$

The first part is contained in $T(A \cup \{a\}) \cdot \overline{a}$. The second part is contained in $(T(A) \setminus T(A \cup \{a\})) \cdot a$. That is, both parts are contained in $\widehat{T}(A \cup \{a, \overline{a}\})$.
□

Theorem 5.13 *Let T be a triangulation of an oriented matroid $\mathcal{M}$ on a set E. Then, T is a lifting triangulation if and only if there is a collection of triangulations $\{T(A) | A \subset E\}$ for $A \subset E$ with:*

- *$T(E) = T$ and $T(A)$ is a triangulation of the restriction $\mathcal{M}(A)$,*
- *For every $A \subset E$ and $b, c \in A$:*

$$\mathcal{P}(T(A))|_{A \setminus \{b\}} \subset \mathcal{P}(T(A \setminus \{b\}))$$

$$T(A \setminus \{b\}) \cap T(A \setminus \{c\}) \subset T(A)|_{A \setminus \{b,c\}}.$$

Proof: If T is a lifting triangulation, let $\widehat{\mathcal{M}}$ be an acyclic lift of $\mathcal{M}$ on the set $E \cup \{\widehat{p}\}$ which defines the lifting triangulation T of $\mathcal{M}$. Then, $\mathcal{P}(T)$ is the collection of faces of $\widehat{\mathcal{M}}$ which do not contain $\widehat{p}$.

We can assume the lift to be "in general position", meaning by this that its dual is an extension in general position of $\mathcal{M}^*$; that is, that any hyperplane of $\widehat{\mathcal{M}}$ not containing $\widehat{p}$ is a simplex. For every $A \subset E$ the restriction $\widehat{\mathcal{M}}(A)$ is an acyclic simplicial lift of $\mathcal{M}(A)$ and defines a triangulation $T(A)$ of $\mathcal{M}(A)$. If $A \subset B$ all the faces of $\widehat{\mathcal{M}}(B)$ contained in A are faces of $\widehat{\mathcal{M}}(A)$ and, thus, $\mathcal{P}(T(B))|_A \subset \mathcal{P}(T(A))$.

Also, if $\sigma \in T(A \setminus \{b\}) \cap T(A \setminus \{c\})$, then σ is a simplicial facet of both $\widehat{\mathcal{M}}(A \setminus \{b\})$ and $\widehat{\mathcal{M}}(A \setminus \{c\})$. This implies that the two oriented matroids have the same rank, equal to the rank of $\widehat{\mathcal{M}}(A)$, and that σ is a facet of $\widehat{\mathcal{M}}(A)$. Thus, $T(A \setminus \{b\}) \cap T(A \setminus \{c\}) \subset T(A)$.

Reciprocally, suppose that T is a triangulation in the conditions of the statement. We will use Proposition 5.3 to prove that T is a lifting triangulation. Remember that $\Lambda(\mathcal{M})$ is obtained from $\mathcal{M}$ by a sequence of lifts of the type $\widehat{\mathcal{M}} := (\mathcal{M}^* \cup \{\overline{a}\})^*$, where $\mathcal{M}^* \cup \{\overline{a}\}$ denotes the extension of $\mathcal{M}^*$ by an element antiparallel to a. In Lemma 5.12 we have seen that a collection of triangulations in the conditions of the statement can be lifted to a collection of triangulations of $\widehat{\mathcal{M}}$ in the same conditions and with $link_{\widehat{T}(E \cup \overline{a})}(\overline{a}) = T$. Recursively, we obtain a triangulation $\Lambda(T)$ of $\Lambda(\mathcal{M})$ with $link_{\Lambda(T)}(\overline{E}) = T$. □

References

[1] **Laura Anderson**, All Euclidean combinatorial differential manifolds are PL manifolds, preprint, 1995. See also: **L. Anderson**, *Topology of Combinatorial Differential Manifolds*, Ph. D. Thesis, M.I.T., 1994.

[2] **Margaret M. Bayer and Bernd Sturmfels**, Lawrence Polytopes, *Canadian J. Math.*, **42** (1990), 62–79.

[3] **Louis J. Billera, Paul Filliman and Bernd Sturmfels**, Constructions and complexity of secondary polytopes, *Adv. in Math.*, **83** (1990), 155-179.

[4] **Louis J. Billera, Mikhail M. Kapranov and Bernd Sturmfels**, Cellular Strings on Polytopes, *Proc. Amer. Math. Soc.*, **122**(2) (1994), 549–555.

[5] **Louis J. Billera and Beth S. Munson**, Triangulations of oriented matroids and convex polytopes, *SIAM J. Algebraic Discrete Methods*, **5** (1984), 515–525.

[6] **Louis J. Billera and Beth S. Munson**, Polarity and inner products in oriented matroids, *European J. Combinatorics*, **5** (1984), 293–308.

[7] **Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White and Günter Ziegler**, *Oriented Matroids*, Cambridge University Press, Cambridge, 1992.

[8] **Jesús A. de Loera**, *Triangulations of polytopes and computational algebra*, Ph. D. Thesis, Cornell University, 1995.

[9] **Jesús A. de Loera**, Nonregular triangulations of products of simplices, *Discrete. Comput. Geometry*, **15** (1996), 253–264.

[10] **Jesús A. de Loera, Francisco Santos and Jorge Urrutia**, On the number of geometric bistellar neighbors of a triangulation, preprint, 1996.

[11] **Jesús A. de Loera, Serkan Hoşten, Francisco Santos and Bernd Sturmfels**, On the polytope of all triangulations of a point configuration, *Documenta Mathematica*, **1** (1996), 103–119. Accessible by WWW at http://www.mathematik.uni-bielefeld.de/documenta

[12] **Paul H. Edelman and Victor Reiner**, Visibility complexes and the Baues problem in the plane, preprint, 1996.

[13] **Jack Edmonds and Komei Fukuda**, *Oriented matroid programming*, Ph. D. Thesis of K. Fukuda, University of Waterloo, 1982.

[14] **Jack Edmonds and Armando Mandel**, *Topology of oriented matroids*, Ph. D. Thesis of A. Mandel, University of Waterloo, 1982.

[15] **Israel M. Gel'fand, Mikhail M. Kapranov and Andrei V. Zelevinsky**, *Multidimensional Determinants, Discriminants and Resultants*, Birkhäuser, Boston, 1994.

[16] **Israel M. Gel'fand and Robert D. MacPherson**, A combinatorial formula for the Pontrjagin classes, *Bull. Amer. Math. Soc.*, **26** (1992), 304–309.

[17] **Michel Las Vergnas**, Extensions ponctuelles d'une géométrie combinatoire orienté, in: *Problémes combinatoires et théorie des graphes (Actes Coll. Orsay 1976)*, Colloques Internationaux, C.N.R.S., No. 260, pp. 265–270.

[18] **C.L. Lawson**, Software for C^1-interpolation, in *Mathematical Software III* (John Rice ed.), Academic Press, New York, 1977.

[19] **Carl W. Lee**, Regular triangulations of convex polytopes, in: *Applied Geometry and Discrete Mathematics-The Victor Klee Festschrift*, (P. Gritzmann and B. Sturmfels eds.), Dimacs Series in Discrete Math. and Theoretical Comp. Science 4, 1991, pp. 443–456.

[20] **Carl W. Lee**, *Triangulations of Polytopes*, preprint, 1995, 19 pages; CRC Handbook on "Discrete and Computational Geometry" (J.E. Goodman, J. O'Rourke, eds.), to appear.

[21] **Robert MacPherson**, Combinatorial Differential Manifolds, in: *Topological Methods in Modern Mathematics: a Symposium in Honor of John Milnor's Sixtieth Birthday*, Publish or Perish, 1993.

[22] **James G. Oxley**, *Matroid Theory*, Oxford University Press, 1992.

[23] **Jürgen Richter-Gebert**, Oriented Matroids with few mutations, *Discrete Comput. Geom.*, **10** (1993), 251–269.

[24] **Jörg Rambau and Günter M. Ziegler**, Projections of polytopes and the Generalized Baues Conjecture, *Discrete Comput. Geom.*, **16** (1996), 215–237.

[25] **Jörg Rambau**, Triangulations of cyclic polytopes and higher Bruhat orders, technical report 496/1996, Technische Universität Berlin. To appear in *Mathematika*.

[26] **Colin P. Rourke and Brian J. Sanderson**, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, 1982.

[27] **E. Schönhardt**, Über die Zerlegung von Dreieckspolyedern in Tetraeder, *Mathematische Annalen*, **98** (1928), 309–312.

[28] **Bernd Sturmfels**, *Gröbner bases and convex polytopes*, University Lecture Series, **8**, American Mathematical Society, Providence, 1995.

[29] **Bernd Sturmfels and Günter Ziegler**, Extension spaces of oriented matroids, *Discrete Comput. Geom.*, **10** (1993), 23–45.

[30] **Günter Ziegler**, *Lectures on Polytopes*, Springer-Verlag, New York, 1994.