Realizable but not strongly euclidean oriented matroids *

Francisco Santos
Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria, E-39005, Santander, SPAIN.
santos@matesco.unican.es

March 2000

Abstract

The *extension space conjecture* of oriented matroid theory claims that the space of all (non-zero, non-trivial, single-element) extensions of a realizable oriented matroid of rank r is homotopy equivalent to an $(r-1)$-sphere.

In 1993, Sturmfels and Ziegler proved the conjecture for the class of *strongly Euclidean oriented matroids*, which includes those of rank at most 3 or corank at most 2. They did not provide any example of a realizable but not strongly Euclidean oriented matroid. Here we produce two such examples for the first time, one with rank 4 and one with corank 3. Both have 12 elements.

Introduction

The *extension space conjecture* is one of the central open questions in oriented matroid theory. It asserts that the extension space of any realizable oriented matroid $\mathcal{M}$, with its natural poset topology, is homotopy equivalent to a sphere of dimension $\mathrm{rank}(\mathcal{M}) - 1$. The conjecture was proved in rank three and in corank two by Sturmfels and Ziegler [15], but the second edition of [4] suggests that "there are substantial grounds for pessimism on this conjecture". This paper is one new reason for pessimism, since it contains the first realizable oriented matroids which do not have the properties needed for the Sturmfels-Ziegler proof to work.

More precisely, Sturmfels and Ziegler introduced the class of *strongly Euclidean oriented matroids*. They proved that it contains all oriented matroids of rank at most three or corank at most two (and some others, such as the alternating oriented matroids) and they showed that the extension space of any strongly Euclidean oriented matroid has the appropriate homotopy type. In this paper we construct realizable but not strongly Euclidean oriented matroids of

*This research was partially supported by grant PB97–0358 of the Spanish Dirección General de Enseñanza Superior e Investigación Científica.

rank four (Section 3, Theorem 3.1) and of corank three (Section 4, Theorem 4.1). The latter is the dual of the root system of type A_3.

Our techniques to detect non-Euclideanness are based on the concept of *lifting triangulations*. The interest of this method is that we can (sometimes) check that an oriented matroid of rank r is non-Euclidean by looking at a triangulation of a point configuration of rank $r-1$, i.e. of dimension $r-2$. Details are given in Section 2. Section 1 contains some known facts concerning extensions and Euclideanness of oriented matroids, including the definition of strong Euclideanness.

The extension space conjecture is closely related to the *Baues problem* [11]. Given a polytope projection $\pi : P \to Q$ between two polytopes P and Q, the *Baues poset of* π, vaguely consists of all polyhedral subdivisions of Q whose cells are projections of faces of P. The original *Generalized Baues conjecture* (GBP) motivated by the work of Billera et al. [3] asserted that this poset is always homotopy equivalent to a sphere of dimension $\dim(P) - \dim(Q) - 1$. A counterexample with $\dim(P) = 5$ and $\dim(Q) = 2$ was found in [10].

If P is a cube (and hence Q is a *zonotope*) then the Baues poset of π consists of all *zonotopal tilings* of Q (see, e.g. [16]). The GBP in this case is equivalent to the extension space conjecture, via the *Bohne-Dress theorem* on zonotopal tilings [4, 5, 7, 13, 16]. This theorem says that given a vector configuration $\mathcal{V}$ with oriented matroid $\mathcal{M}$, there is a 1-to-1 order preserving bijection between the zonotopal tilings of the zonotope $Z(\mathcal{V})$ generated by $\mathcal{V}$ and the extensions of the dual oriented matroid $\mathcal{M}^*$.

Interesting in this context is the geometric approach to strong Euclideanness contained in [2]. Athanasiadis defines *stackability* of a zonotopal tiling, which is related to Euclideanness of the corresponding extension of the dual oriented matroid, and *strong stackability* of a zonotope $Z(\mathcal{V})$, which he proves equivalent to strong Euclideanness of the dual oriented matroid $\mathcal{M}^*$. Among other things he shows that every zonotopal tiling of a strongly stackable zonotope is shellable. This suggests the possibility that the non-Euclidean extensions constructed in this paper correspond via the Bohne-Dress theorem to non-shellable zonotopal tilings (it is an open question whether non-shellable zonotopal tilings exist). In particular, the extension constructed in Theorem 4.1 has good chances to produce a non-shellable zonotopal tiling, since it produces a zonotopal tiling which is not stackable with respect to the direction of any of the generators of the zonotope.

Finally, the extension space conjecture is the case $k = d-1$ of the following far-reaching conjecture by MacPherson, Mnëv and Ziegler [11, Conjecture 11]: the poset of all strong images of rank k of any realizable oriented matroid $\mathcal{M}$ of rank d (the *OM-Grassmannian of rank* k of $\mathcal{M}$) is homotopy equivalent to the usual real Grassmannian $G^k(\mathbb{R}^d)$. This conjecture is relevant in the context of matroid bundles [1] and the *combinatorial differential geometry* introduced by MacPherson [8].

Acknowledgements The first example of a realizable but not strongly Euclidean oriented matroid was obtained after an inspiring conversation on Euclideanness with Jürgen Richter-Gebert and Jesús de Loera while I was visiting ETH Zürich in August 1999.

1 Extensions and Euclideanness in oriented matroids

In this section we recall some definitions and results concerning extensions and Euclideanness in oriented matroid theory. We refer to [4] for proofs and details.

Let $\mathcal{M}$ be an oriented matroid of rank r on a ground set E. A *(single-element, non-trivial) extension* of $\mathcal{M}$ is an oriented matroid $\widetilde{\mathcal{M}}$ of the same rank on a ground set $\widetilde{E}$ such that $\widetilde{E} \setminus E$ has exactly one element and the restriction of $\widetilde{\mathcal{M}}$ to E is M.

We will use the notation $\mathcal{M} \cup p$ for a single-element extension of $\mathcal{M}$, where $\{p\} = \widetilde{E} \setminus E$. Any cocircuit C of $\mathcal{M}$ extends uniquely to a cocircuit $\widetilde{C}$ of $\widetilde{\mathcal{M}}$ which agrees with C on E [4, Proposition 7.1.4]. The function σ : cocircuits$(\mathcal{M}) \to \{+, 0, -\}$ defined by $\sigma(C) = \widetilde{C}(p)$ is called the *signature function* of the extension, and determines it. We will say that the extension is *positive, zero or negative* at a cocircuit, accordingly.

Every oriented matroid can be extended by a loop. We call this the *zero extension.* The extension poset $\mathcal{E}(\mathcal{M})$ of $\mathcal{M}$ is the set of all non-zero, non-trivial, single-element extensions of $\mathcal{M}$ ordered by weak maps [4, Section 7.2]. When referring to the topology of $\mathcal{E}(\mathcal{M})$, we mean that of the *order complex* of this poset, a simplicial complex whose vertices are the elements of $\mathcal{E}(\mathcal{M})$ and whose simplices are the chains in the poset [4, Section 4.7]. In this sense, $\mathcal{E}(\mathcal{M})$ is also called the *extension space* of $\mathcal{M}$. We will normally drop the attributes "single-element", "non-zero" and "non-trivial" from our exposition, since all our extensions will have these properties.

An *oriented matroid program* is any triple $(\mathcal{M}, g, f)$ where $\mathcal{M}$ is an oriented matroid on a ground set E and $f, g \in E$ are elements such that $\{f, g\}$ has rank 2 both in $\mathcal{M}$ and in the dual $\mathcal{M}^*$. (This is slightly more restrictive than [4, Definition 10.1.3], where g is only required to be a non-loop and f a non-coloop. By requiring both to be neither loops nor coloops we only loose trivial cases and the definition becomes symmetric in f and g).

An oriented matroid program $(\mathcal{M}, g, f)$ is *Euclidean* if for any given cocircuit of $\mathcal{M}$ there is an extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup p$ of $\mathcal{M}$ which is zero at that cocircuit and such that the set $\{f, g, p\}$ has rank 2 in $\widetilde{\mathcal{M}}$ [4, Definition 10.5.2].

Non-Euclideanness is related to the existence of cycles in a certain graph, which we now introduce. Let $(\mathcal{M}, g, f)$ be an oriented matroid program. Let $\mathcal{C}_g^*$ be the set of cocircuits of $\mathcal{M}$ which are positive at g. The *graph of cocircuits of* $(\mathcal{M}, g, f)$ [4, Definition 10.1.16], which we denote $G_{(\mathcal{M},g,f)}$, is a partially directed graph whose vertex set is $\mathcal{C}_g^*$, whose edges are the pairs of cocircuits whose complement hyperplanes meet in a corank 2 subset of $\mathcal{M}$ and where edges are directed according to the following rule: Let $X, Y \in \mathcal{C}_g^*$ be such that $X^0 \cap Y^0$ has corank 2. Let Z be the unique cocircuit vanishing on $(X^0 \cap Y^0) \cup \{g\}$ and positive on $(Y^+ \setminus X^+) \cup (X^- \setminus Y^-)$ (obtained by eliminating g from $-X$

and Y). The edge joining X and Y is directed if and only if $f \notin Z^0$ (i.e. if $(X^0 \cap Y^0) \cup \{f, g\}$ is a spanning set). It is directed from X to Y if $f \in Z^+$ and from Y to X if $f \in Z^-$.

A *directed cycle* in a partially directed graph such as $G_{(\mathcal{M},g,f)}$ is a cycle with at least one directed edge and in which all the directed edges are directed in the direction of the cycle.

Proposition 1.1 (Edmonds-Fukuda-Mandel) (i) *An oriented matroid program $(\mathcal{M}, g, f)$ is Euclidean if and only if $G_{(\mathcal{M},g,f)}$ contains no directed cycles [4, Theorem 10.5.5].*

(ii) *An oriented matroid program $(\mathcal{M}, g, f)$ is Euclidean if and only if the dual program $(\mathcal{M}^*, f, g)$ is Euclidean [4, Corollary 10.5.9].* □

The orientations in the graph G_f are easier to describe if $\mathcal{M}$ is realizable. Since the definition is invariant under reorientation of any element other than f or g, we can suppose that $\mathcal{M}$ is acyclic and let it be realized by a point configuration $\mathcal{A}$ in which g is a vertex of $\text{conv}(\mathcal{A})$. Each cocircuit X of C_g^* is uniquely identified with a hyperplane H_X spanned by elements of $\mathcal{A}$ and not passing through g. Then, the edge XY in $G_{(\mathcal{M},g,f)}$ is directed from X to Y (resp. from Y to X) if and only if the directed line passing through g and f in that order crosses H_X before (resp. after) H_Y. It is undirected if the line and the two hyperplanes meet in a point.

Sturmfels and Ziegler [15, Definition 3.8] define an oriented matroid $\mathcal{M}$ to be *strongly Euclidean* if it has rank 1, or if it possesses an element g such that $\mathcal{M}/g$ is strongly Euclidean and the program $(\widetilde{\mathcal{M}}, g, f)$ is Euclidean for every extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$. They prove:

Theorem 1.2 ([15, Theorem 1.2]) *Let $\mathcal{M}$ be a strongly Euclidean rank r oriented matroid. Then the extension poset $\mathcal{E}(\mathcal{M})$ is homotopy equivalent to the $(r-1)$-sphere S^{r-1}.*

The *extension space conjecture* asserts that in this result the condition *strongly Euclidean* can be replaced by *realizable*. The result is known to be false for non-realizable oriented matroids starting in rank 4: Two rank 4 oriented matroids with disconnected extension space appear in [9]. The smaller one has 19 elements and is based on a construction from [12]. In [6] this construction is improved to 16 elements.

Here we construct for the first time realizable oriented matroids which are not strongly Euclidean. Observe that a minimal non-strongly Euclidean oriented matroid $\mathcal{M}$ has the property that for every element g there is an extension $\mathcal{M} \cup f$ such that the program $(\mathcal{M} \cup f, g, f)$ is not Euclidean. In Theorem 4.1 we construct a realizable oriented matroid of corank 3 with the stronger property that the same extension $\mathcal{M} \cup f$ makes the program $(\mathcal{M} \cup f, g, f)$ not Euclidean for every element g.

2 Euclideanness and lifting triangulations

In our examples we will use lifting triangulations as a tool to recognize non-Euclideanness of an oriented matroid program $(\mathcal{M}, g, f)$ by looking at the contraction $\mathcal{M}/g$. Although this is not a general procedure we consider it interesting in itself.

Recall that the Las Vergnas face lattice of an oriented matroid $\mathcal{M}$ is the poset of positive covectors. If $\mathcal{M}$ is acyclic then this lattice is a cell decomposition of a $(\mathrm{rank}(\mathcal{M}) - 2)$-sphere [4, Proposition 9.1.1]. If $\mathcal{M}$ is realized by a point configuration $\mathcal{A}$ then the Las Vergnas lattice equals the face lattice of the polytope $\mathrm{conv}(\mathcal{A})$ [4, Example 4.1.6(1)].

Throughout this section let $\mathcal{A}$ be a point configuration and let $\mathcal{M}$ be the (acyclic, realizable) oriented matroid of affine dependences in $\mathcal{A}$. A *triangulation* of $\mathcal{A}$ is a collection of bases of $\mathcal{M}$ (i.e affine bases contained in $\mathcal{A}$) whose convex hulls form a geometric simplicial complex covering $\mathrm{conv}(\mathcal{A})$.

A (single-element) *lift* of $\mathcal{M}$ is any oriented matroid $\widehat{\mathcal{M}}$ with an element g such that $\widehat{\mathcal{M}}/g = \mathcal{M}$. We will always assume that g is neither a loop nor a coloop, i.e. that $\widehat{\mathcal{M}}^*$ is a single-element, non-zero, non-trivial extension of $\mathcal{M}^*$. We call the lift *simplicial* if for every positive cocircuit X of $\widehat{\mathcal{M}}$ not vanishing at g, X^0 is independent, i.e. if the antistar of g in the Las Vergnas face lattice of $\widehat{\mathcal{M}}$ is a simplicial complex.

Every lift of an acyclic oriented matroid is acyclic and has g as a vertex (i.e. complement of a positive covector). If the lift is simplicial then the antistar of g in the Las Vergans face lattice of $\widehat{\mathcal{M}}$ is a triangulation of $\mathcal{A}$. The triangulations which can be obtained in this fashion are called *lifting triangulations* [4, 14]. (If a lift is not simplicial, then it induces a non-simplicial polyhedral decomposition of $\mathrm{conv}(\mathcal{A})$, called a *lifting subdivision*).

The *adjacency graph* of a triangulation T of $\mathcal{A}$ has as vertices the maximal simplices of T. Two maximal simplices $\sigma, \tau \in T$ form an edge if and only if they are *adjacent* i.e. if $\sigma \cap \tau$ is a simplex of codimension 1. For a fixed element $f \in \mathcal{A}$ it is natural to define the following orientations of edges in the graph:

- The edge $\sigma\tau$ is directed if and only if the hyperplane spanned by $\sigma \cap \tau$ does not contain f.
- In this case, the edge is directed "from f", i.e. it is directed from σ to τ if and only if f and $\sigma \setminus \tau$ lie on the same side of the hyperplane spanned by $\sigma \cap \tau$.

We call this partially directed graph the *adjacency graph of T directed from f* and denote it $G_{T,f}$.

Lemma 2.1 *Let $\widehat{\mathcal{M}}$ be a lift of $\mathcal{M}$ with $\widehat{\mathcal{M}}/g = \mathcal{M}$ and let $f \in \mathcal{A}$. Let T be the lifting triangulation of $\mathcal{A}$ induced by $\widehat{\mathcal{M}}$.*

(i) *$G_{T,f}$ is a subgraph of $G_{(\widehat{\mathcal{M}},g,f)}$.*

(ii) *Hence, if $G_{T,f}$ has a directed cycle, then $(\widehat{\mathcal{M}}, g, f)$ is not Euclidean.*

Proof: It is clear from the definition of lifting triangulation that if σ is a simplex in T then there is a unique positive cocircuit in $\widehat{\mathcal{M}}$ vanishing on σ and positive at g. In this sense the vertices of $G_{T,f}$ are also vertices of $G_{(\widehat{\mathcal{M}},g,f)}$.

Let σ and τ be two adjacent simplices in T. Let X and Y be the cocircuits of $\widehat{\mathcal{M}}$ with $X^0 = \sigma$, $Y^0 = \tau$ and positive at g. It is clear that $X^0 \cap Y^0$ has corank two (in $\widehat{\mathcal{M}}$), since $\sigma \cap \tau$ has corank one (in $\mathcal{M}$). Let Z be a cocircuit of $\widehat{\mathcal{M}}$ obtained by elimination of g from $-X$ and Y. Z vanishes on $(X^0 \cap Y^0) \cup \{g\}$ and, since X and Y are positive cocircuits, $X^0 \setminus Y^0 = Y^+ \setminus X^+ \subset Z^+$. In other words, Z corresponds to the cocircuit of $\mathcal{M}$ vanishing on $\sigma \cap \tau$ and oriented so that σ is in the positive side and τ in the negative side.

If $f \in Z^0$ then the edges XY in $G_{(\widehat{\mathcal{M}},g,f)}$ and $\sigma\tau$ in $G_{T,f}$ are both undirected. If this is not the case, assume without loss of generality that $f \in Z^+$ (otherwise, exchanging the roles of X and Y we can reverse Z). The edge XY of $G_{(\widehat{\mathcal{M}},g,f)}$ is directed from X to Y by definition, and the edge $\sigma\tau$ of $G_{T,f}$ is directed from σ to τ. This finishes the proof of (i). Part (ii) is a trivial consequence of Proposition 1.1(i). $\square$

Figure 1 shows three triangulations (in part (c) the central quadrangle can be triangulated arbitrarily) in which the graph $G_{T,f}$ has a directed cycle. The three of them are lifting (see [14, Examples 5.1]). By Lemma 2.1, any lifts producing these triangulations provide examples of non-Euclidean oriented matroid programs of rank 4 with 8 (in parts (a) or (b)) and 10 (in part (c)) elements. The ones with 8 elements are minimal in both rank and corank. Part (b) is a (reoriented) contraction of the non-Euclidean oriented matroid RS8. More precisely, starting with the description of RS8 in [4, Section 1.5], the reorientation of it at the elements 1 and 5, contracted at 8, gives the rank 3 point configuration and lifting triangulation in part (b) of the Figure.

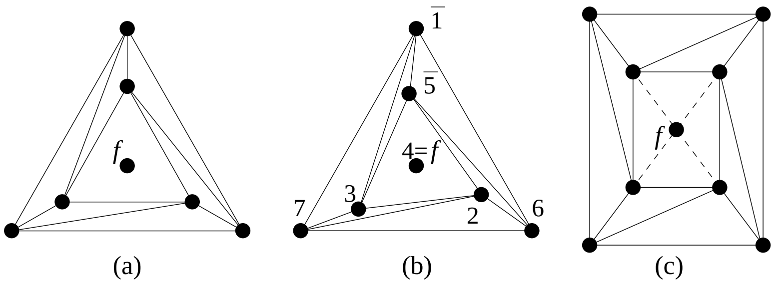

Figure 1: Contractions of non-Euclidean oriented matroid programs.

The following lemma is a version of [4, Corollary 7.3.2]

Lemma 2.2 *Let $\mathcal{M}$ be an oriented matroid of rank r and let $C = \{x_1, \ldots, x_r\}$ be a hyperplane with exactly r elements and spanned by any $r-1$ of them (in other words, a circuit of corank 1 such that no other element lies in the flat spanned by it). Then, the chirotope obtained from $\mathcal{M}$ by perturbing C to be a basis with any two of the possible signs is still an oriented matroid.*

Proof: Let $\mathcal{M}'$ denote the perturbed chirotope and suppose that it is not an oriented matroid. Then, the change of the sign of $\chi(x_1, x_2, x_3, \ldots, x_r)$ from zero to non-zero creates a violation of some 3-term Grassmann-Plücker relation in $\mathcal{M}'$ (see [4, Theorem 3.6.2]). I.e., there exist y_1, y_2 such that

$$\chi(y_1, x_2, x_3, \ldots, x_r) \cdot \chi(x_1, y_2, x_3, \ldots, x_r) \geq 0,$$

$$\chi(y_2, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, x_1, x_3, \ldots, x_r) \geq 0, \text{but}$$

$$\chi(x_1, x_2, x_3, \ldots, x_r) \cdot \chi(y_1, y_2, x_3, \ldots, x_r) < 0$$

In $\mathcal{M}$, the third line equals zero and the Grassman-Plücker relations imply that the first two lines cannot be both positive. Hence, one of y_1 or y_2 lies in the hyperplane spanned by C. □

3 A realizable but not strongly Euclidean oriented matroid of rank 4

Let $\mathcal{A}_0 = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3\}$ be a point configuration with 12 points in $\mathbb{R}^3$ in the following conditions:

- The three lines $\{a_i, b_i\}$, $(i = 1, 2, 3)$ meet in a point p outside $\text{conv}(\mathcal{A}_0)$, with a_i lying between p and b_i.
- The three lines $\{c_i, d_i\}$, $(i = 1, 2, 3)$ meet in a point q outside $\text{conv}(\mathcal{A}_0)$, with d_i lying between q and c_i.
- The six quadruples $\{a_i, a_j, b_j, b_i\}$ and $\{c_i, c_j, d_j, d_i\}$ are the only affinely dependent quadruples in $\mathcal{A}$, and they are the vertex sets of six quadrangular facets of $\text{conv}(\mathcal{A}_0)$. In particular, $\mathcal{A}_0$ is in convex position.

Let $\mathcal{M}_0$ be the oriented matroid of $\mathcal{A}_0$.

$\mathcal{A}_0$ can be constructed as follows: let P_1 be the triangular prism (with bases of different size) given by the vertices $\{(-2, -2, k), (2, 0, k), (0, 2, k), (-1, -1, k+1), (1, 0, k+1), (0, 1, k+1)\}$, for a sufficiently big k. Let P_2 be the prism, congruent to P_1, with vertices $\{(-2, -2, -k), (2, 0, -k), (0, 2, -k), (-1, -1, -k-1), (1, 0, -k-1), (0, 1, -k-1)\}$. The big bases of P_1 and P_2 see each other and are parallel. After performing a generic rotation of P_1 or P_2 along the z axis, the twelve vertices of P_1 and P_2 satisfy all the required conditions, with $p = (0, 0, k+2)$ and $q = (0, 0, -k-2)$. See Figure 2.

We now perturb the point configuration $\mathcal{A}_0$ by slightly rotating the triangles $\{a_1, a_2, a_3\}$ and $\{d_1, d_2, d_3\}$ in the way that creates the following twelve triangles in the boundary of the convex hull: $\{a_i, b_i, a_{i+1}\}$, $\{b_i, a_{i+1}, b_{i+1}\}$, $\{d_i, c_i, d_{i+1}\}$, and $\{c_i, d_{i+1}, c_{i+1}\}$, $(i = 1, 2, 3$ and indices are regarded modulo 3). Let $\mathcal{A}$ be the perturbed point configuration, whose oriented matroid we denote by $\mathcal{M}$.

Theorem 3.1 *$\mathcal{M}$ is a realizable, uniform, not strongly Euclidean oriented matroid of rank four on twelve elements.*

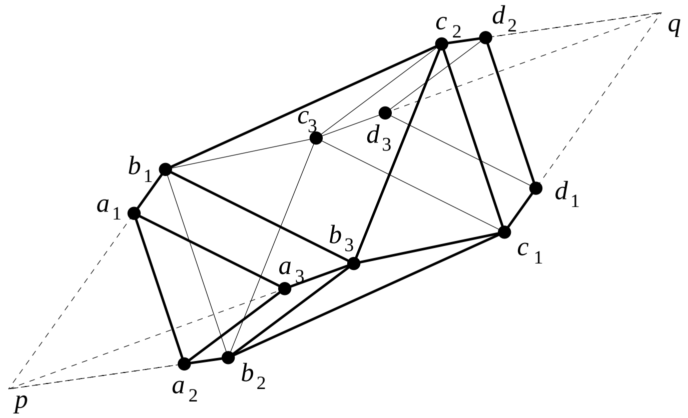

Figure 2: A realizable but not strongly Euclidean oriented matroid of rank 4.

Proof: We will prove that for any element g of $\mathcal{M}$ there is an extension $\mathcal{M} \cup f$ such that the program $(\mathcal{M} \cup f, g, f)$ is not Euclidean. Let $g = a_i$ for $i \in \{1,2,3\}$ (the discussion is completely analogous with the elements b_i, c_i or d_i).

We start by considering again the point configuration $\mathcal{A}_0$. Let f be a point in the segment joining a_i to q, but otherwise generic. $\mathcal{A}_0 \cup \{f\}$ has nine coplanar quadruples: the six present in $\mathcal{A}_0$ and the three quadruples $\{f, a_i, c_j, d_j\}$, for $j = 1,2,3$. Since they span different hyperplanes, Lemma 2.2 allows us to perturb them into bases independently and still have the chirotope of an oriented matroid. In particular, we can perturb the first six as in $\mathcal{A}$ and leave the last three unperturbed. Let $\widetilde{\mathcal{M}}$ be the oriented matroid so obtained, which is an extension of $\mathcal{M}$ and a simplicial lift of its contraction $\widetilde{\mathcal{M}}/a_i$.

In the induced lifting triangulation of $\widetilde{\mathcal{M}}/a_i$, the seven points $\{f, c_1, c_2, c_3, d_1, d_2, d_3\}$ are exactly as in part (a) of Figure 1. Hence, by Lemma 2.1, the oriented matroid program $(\widehat{\mathcal{M}}, a_i, f)$ is not Euclidean (we are implicitly using the fact that every minor of a Euclidean program is Euclidean [4, Corollary 10.5.6]). This finishes the proof that $\mathcal{M}$ is not strongly Euclidean. $\square$

The same idea, but with a lifting subdivision instead of a triangulation, can be used to prove that the non-uniform oriented matroid $\mathcal{A}_0$ is not strongly Euclidean either.

4 A realizable but not strongly Euclidean oriented matroid of corank 3

Let $\mathcal{A}_3$ be the root system of type A_3, i.e. the rank three vector configuration consisting of the twelve vectors $\pm e_i \pm e_j$, with $i, j \in \{1,2,3\}$ and $i \neq j$ (e_1, e_2 and e_3 denote the standard basis vectors in $\mathbb{R}^3$).

Theorem 4.1 *The realizable corank three oriented matroid $\mathcal{A}_3^*$ on twelve elements whose dual is realized by the root system of type A_3 has an extension f*

such that for any $g \in \mathcal{A}_3^*$ the oriented matroid program $(\mathcal{A}_3^*, f, g)$ is not Euclidean. In particular, $\mathcal{A}_3^*$ is not strongly Euclidean.

Proof: Let $\widetilde{\mathcal{A}_3}$ denote the same twelve elements of $\mathcal{A}_3$, but now regarded as a rank 4 point configuration, i.e. the vertex set of a regular cuboctahedron. Let $o = (0,0,0)$ denote the origin, so that $(\widetilde{\mathcal{A}_3} \cup o)/o = \mathcal{A}_3$. See a cuboctahedron in the left part of Figure 3. Its vertices have been labelled with the following notation: i and $\bar{\imath}$ represent the vectors/points e_i and $-e_i$, and two adjacent symbols are meant to represent the addition of the two corresponding vectors.

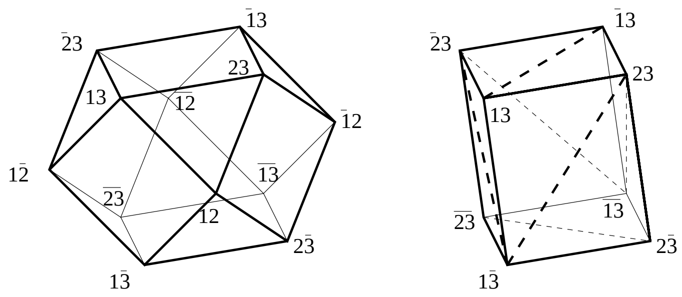

Figure 3: A cuboctahedron (left) and a square prism inscribed in it, with diagonals inserted in its facets (right).

The cuboctahedron $\mathrm{conv}(\widetilde{\mathcal{A}_3})$ has three pairs of opposite square facets. The eight vertices of each pair form a square prism. One of the three square prisms is depicted in the right part of Figure 3. Let us call it P.

The 18 facets of these three square prisms are the only dependent quadruples contained in $\widetilde{\mathcal{A}_3} \cup \{o\}$ which span a plane not passing through o. Each of the 18 spans a different plane, so Lemma 2.2 allows us to convert them into bases independently and still have an oriented matroid. We choose to perturb the six facets of P in the way that creates as new boundary edges the six dashed lines shown in the right part of Figure 3, and we perturb the other two square prisms accordingly (i.e. so that the whole family of perturbations is symmetric under cyclic permutation of coordinates). Let us call $\widetilde{\mathcal{M}}$ the (non-realizable) oriented matroid on 13 elements with rank 4 obtained by perturbing $\widetilde{\mathcal{A}_3} \cup o$ in this way.

For every $i, j \in \{1,2,3\}$, the eight points of $\widetilde{\mathcal{A}_3}$ other than ij, $i\bar{\jmath}$, $\overline{ij}$ and $\bar{\imath}j$ form one of the three square prisms, and its facets have been perturbed forming a "whirlwind" with respect to the two axes $\{ij, \overline{ij}, o\}$ and $\{i\bar{\jmath}, \bar{\imath}j, o\}$. See Figure 3 for the case $\{i,j\} = \{1,2\}$. Hence, for any element g in $\widetilde{\mathcal{M}}$ other than o, $\widetilde{\mathcal{M}}$ contains a minor whose contraction at g produces as lifting triangulation the one depicted in part (c) in Figure 1. In particular, the oriented matroid program $(\widetilde{\mathcal{M}}, o, g)$ is not Euclidean.

By Proposition 1.1(ii), the dual program $(\widetilde{\mathcal{M}}^*, o, g)$ is not Euclidean either. But $\widetilde{\mathcal{M}}/o = (\widetilde{\mathcal{A}_3} \cup o)/o = \mathcal{A}_3$ because our perturbations do not involve the element o. Hence, $\widetilde{\mathcal{M}}^* \setminus o = \mathcal{A}_3^*$ and $\widetilde{\mathcal{M}}^*$ is an extension of $\mathcal{A}_3^*$, as desired. $\square$

Our first hope when devising this example was that the "strongly non-Euclidean" extension of $\mathcal{A}_3^*$ constructed in the previous theorem might not be in the same connected component as the realizable extensions of $\mathcal{A}_3^*$, mimicking the construction of a disconnected extension space in [9, Section 1]. Unfortunately this does not happen, so we do not expect $\mathcal{A}_3{}^*$ to have a disconnected extension space:

Proposition 4.2 *The non-Euclidean extension $\widetilde{\mathcal{M}}^*$ of $\mathcal{A}_3^*$ constructed in the proof of Theorem 4.1 is a minimal element in the extension poset $\mathcal{E}(\mathcal{A}_3^*)$ and connected in it to realizable extensions.*

Proof: An extension $\mathcal{N} \cup f$ is minimal in $\mathcal{E}(\mathcal{N})$ if and only if every circuit of $\mathcal{N} \cup f$ involving $\mathcal{N}$ is spanning (such extensions are called "in general position" in [14]). In our case, $\widetilde{\mathcal{M}}^*$ has this property for the element o since $\widetilde{\mathcal{M}}$ has the property that the complement of any cocircuit not involving o is independent.

$\widetilde{\mathcal{M}}^*$ is clearly connected in $\mathcal{E}(\mathcal{A}_3^*)$ to the realizable oriented matroid $(\widetilde{\mathcal{A}}_3 \cup o)^*$ of which it is a weak map. But, moreover, we can prove that it is connected by mutations to a realizable minimal element of $\mathcal{E}(\mathcal{A}_3^*)$: the 18 new bases we have introduced in $\widetilde{\mathcal{M}}$ can be mutated arbitrarily to provide new oriented matroids. By slightly moving the twelve elements of $\widetilde{\mathcal{A}}_3$ into generic positions along the lines passing through them and o we can get one which is realizable. In the dual, the mutation of any of these bases corresponds to mutating bases containing o. $\square$

Incidentally, the realizable and not strongly Euclidean oriented matroid $\mathcal{A}_3^*$ is realized by the vertex set of the Lawrence polytope over the rank three vector configuration $\{e_1, e_2, e_3, e_1+e_2, e_1+e_3, e_1+e_2+e_3\}$ (a complete quadrilateral in the terminology of projective geometry).

References

[1] L. Anderson, Matroid bundles and sphere bundles, in: *New Perspectives in Algebraic Combinatorics* (L.J. Billera et al. eds), MSRI publications **38**, Cambridge University Press, New York, 1999, pp. 1–21.

[2] C.A. Athanasiadis, Zonotopal subdivisions of cyclic zonotopes, preprint 2000.

[3] L. Billera, M.M. Kapranov and B. Sturmfels, Cellular Strings on Polytopes, *Proc. Amer. Math. Soc.* **122**(2) (1994), 549–555.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented Matroids*, Cambridge University Press, Cambridge 1992. See also 2nd. edition, 1999.

[5] J. Bohne, *Eine kombinatorische Analyse zonotopaler Raumaufteilungen*, Dissertation, Fachbereich Mathematik, Universität Bielefeld, 1992, 100 pp.

[6] J. Bokowski and H. Rohlfs, On a mutation problem of oriented matroids, preprint, 2000.

[7] B. Huber, J. Rambau and F. Santos, The Cayley Trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings, preprint 1999. To appear in *J. Eur. Math. Soc. (JEMS).*

[8] R. MacPherson, Combinatorial Differential Manifolds, in *Topological Methods in Modern Mathematics: a Symposium in Honor of John Milnor's Sixtieth Birthday*, Publish or Perish, 1993.

[9] N. E. Mnëv and J. Richter-Gebert, Two constructions of oriented matroids with disconnected extension space, *Discrete Comput. Geom.* **10** (1993), 271–285.

[10] J. Rambau and G.M. Ziegler, Projections of polytopes and the generalized Baues conjecture, *Discrete Comput. Geom.* (3) **16** (1996), 215–237.

[11] V. Reiner, The generalized Baues problem, in: *New Perspectives in Algebraic Combinatorics* (L.J. Billera et al. eds), MSRI publications **38**, Cambridge University Press, New York, 1999, pp. 293–336.

[12] J. Richter-Gebert, Oriented Matroids with few mutations, *Discrete Comput. Geom.* **10** (1993), 251–269.

[13] J. Richter-Gebert and G. M. Ziegler, Zonotopal tilings and the Bohne-Dress theorem, in *Jerusalem Combinatorics 1993* (H. Barceló and G. Kalai eds.), Contemp. Math. **178** (1994), Amer. Math. Soc., Providence, RI, pp. 211-232.

[14] F. Santos, Triangulations of Oriented Matroids, preprint 1997. To appear in *Mem. Amer. Math. Soc.*

[15] B. Sturmfels and G. M. Ziegler, Extension spaces of oriented matroids, *Discrete Comput. Geom.* **10** (1993), 23–45.

[16] G. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York, 1994.