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Stability index of real varieties

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0. Introduction

Let R be a real closed field, for example the field of real numbers. For an affine R-variety V (the use of the word "variety" does not imply irreducibility) and polynomial functions $f_1, \ldots, f_r \in R[V] = \Gamma(V, \mathcal{O}_V)$ we put

$$S(f_1, \ldots, f_r) := S_V(f_1, \ldots, f_r) := \{x \in V(R) \colon f_1(x) > 0, \ldots, f_r(x) > 0\} \qquad (*)$$

and

$$\bar{S}(f_1, \ldots, f_r) := \bar{S}_V(f_1, \ldots, f_r) := \{x \in V(R) \colon f_1(x) \geqq 0, \ldots, f_r(x) \geqq 0\}. \qquad (**)$$

The semi-algebraic sets of the form $(*)$ constitute a natural basis for the strong topology on $V(R)$ and are therefore called *basic open*. It is a fact that also each closed semi-algebraic subset of $V(R)$ is a finite union of sets of the form $(**)$ (Finiteness Theorem, see e.g. [BCR, Théorème 2.7.1]), so the latter are called the *basic closed* sets. Given a fixed basic open $S \subset V(R)$, one denotes the minimal number r of inequalities necessary to describe S as in $(*)$ by $s(S)$ (or by $s_V(S)$, if the surrounding variety is to be emphasized); similarly, for basic closed $F \subset V(R)$ the minimal number of inequalities required for F as in $(**)$ is denoted by $\bar{s}(F) = \bar{s}_V(F)$.

It had been well known for some time that for fixed V the supremum of all the numbers $s_V(S)$, S running over the non-empty basic open subsets of $V(R)$, is finite. It is denoted by $s(V)$ and is called the *(geometric) stability index* of the variety V (concerning this terminology compare the remarks in § 1). Also the supremum of the numbers $\bar{s}_V(F)$ with $\emptyset \neq F \subset V(R)$ basic closed is finite and is denoted by $\bar{s}(V)$. (The only purpose of excluding the empty set in these definitions is to get $s(V) = 0$ in case that $V(R)$ contains exactly one point. To make the definition formally complete let us put $s(V) = -1$ if $V(R) = \emptyset$.)

But much more than finiteness of the numbers $s(V)$ and $\bar{s}(V)$ was known, since L. Bröcker proved that there are upper bounds for them depending only on the dimension of V. Assuming that V is real (i.e. that $V(R)$ is Zariski dense in V) he showed that $s(V) = \dim V = n$ if $1 \leqq n \leqq 3$, and that in general

$$n \leqq s(V) \leqq \begin{cases} 2^{m-1} \cdot 3 & \text{if } n = 2m \\ 2^m & \text{if } n = 2m-1 \end{cases}$$

holds for every $n \geqq 1$ ([B4], compare also [B2], [B3], [Mah2]). On the basic closed side, he proved in [B5] that $\bar{s}(V)=\frac{1}{2}n(n+1)$ for $n=1, 2$ and that

$$n+2 \leqq \bar{s}(V) \leqq \tfrac{1}{2}n(n+1) \quad \text{for } n \geqq 3.$$

To the best of my knowledge, it seems that up to now no one had computed the exact value of $s(V)$ for any particular (real) R-variety V of dimension at least four. Also there was no example of an R-variety of dimension larger than two whose $\bar{s}$-invariant was known. However it had already been conjectured for some time that $s(V)=\dim V$ should hold for real V of arbitrary (positive) dimension. The main result of this paper (Theorem 2) confirms this conjecture. In fact its statement is much more precise, and it applies to far more general situations. The other main result (Theorem 1) deals with the basic closed case. Also this question is solved completely: The invariant $\bar{s}(V)$ of an n-dimensional real affine R-variety V $(n>0)$ is given by $\bar{s}(V)=\frac{1}{2}n(n+1)$.

Our proofs make permanent use of the real spectrum of a ring, as developed by M. Coste and M.-F. Roy. The other main ingredient is reduced quadratic form theory over fields and, more generally, over semilocal rings, together with its abstract generalization, Marshall's theory of spaces of orderings. Here we rely heavily on work of E. Becker, L. Bröcker, M. Knebusch, M. Marshall and many others. Both of our main results are just natural applications of these theories, they do not require any new tools. This seems to indicate the power of these concepts – they lead to results of a completely basic and elementary nature for which no other proofs are known.

The first section a more detailed introduction to the problem of the stability index together with a sketch of the essential idea of our proof of Theorem 2. In §2 we recall that part of the theory of spaces of orderings which is relevant for our purposes, and add some (well-known) facts about real places between fields. Section 3 is concerned with basic closed sets. After recalling Bröcker's proof of the upper estimation $\bar{s}(V) \leqq \frac{n}{2}(n+1)$ we only have to show that there are in fact basic closed sets which require this number of inequalities. This is done rather explicitly, and we illustrate the construction by an example in affine n-space. Finally, the last section contains the main theorem on basic open sets; for more details see the outline in §1.

After the author had informed L. Bröcker about the proof of $s(V)=n$ in the smooth case together with a rough sketch of this proof, Bröcker also found a proof for $s(V)=n$ in the general case. It will be contained in a forthcoming article by Bröcker.

I wish to express my sincere gratitude to Prof. Knebusch for his encouragement and support during the preparation of this paper, and to Jan van Geel for lots of stimulating discussions around the stability index that we had in summer 1987. I also want to thank the referee for his valuable suggestions which helped me to improve the presentation of these results.

1. Some basic notions and an outline of the proof of the main result

First recall the notion of the real spectrum $\operatorname{Sper} A$ of a ring A, due to M. Coste and M.-F. Roy ([CR]; see also [Be], [BCR] and [KS]). It consists of all pairs $x=(p, \alpha)$ with $p \in \operatorname{Spec} A$ and α an ordering of its residue field $\kappa(p)$. Denoting the real closure of $\kappa(p)$ with respect to α by $k(x)$, one has natural homomorphisms $\rho_x: A \to k(x)$ for $x \in \operatorname{Sper} A$. Usually one writes $f(x)$ instead of $\rho_x(f)$ $(f \in A)$. The topology on $\operatorname{Sper} A$ is defined by the open basis consisting of the sets

$$S_A(f_1, \ldots, f_r) := \{x \in \operatorname{Sper} A: f_1(x)>0, \ldots, f_r(x)>0\}$$

$(r \geqq 1, f_i \in A)$; it makes $\operatorname{Sper} A$ a spectral topological space (in the sense of [Ho]). By obvious reasons, subsets of the form $S=S_A(f_1, \ldots, f_r)$ are called *basic open* (constructible) subsets of $\operatorname{Sper} A$, and similarly to the "geometric" case before one writes $s_A(S)$ for the least possible $r \geqq 0$. (In an analogous manner one defines basic closed constructibles and integers $\bar{s}_A(\cdot)$ in the abstract setting, but we will not be concerned with them here.) The invariant

$$\operatorname{st}(A) := \sup\{s_A(S): \emptyset \neq S \subset \operatorname{Sper} A \text{ is basic open constructible}\}$$

($\operatorname{st}(A) := -1$ if $\operatorname{Sper} A=\emptyset$) of the ring A had already been introduced in [Mah2] (there denoted by $s(A)$) and was investigated in [ABR]. In the latter paper the name "stability index of A" was used for it. However, L. Mahé pointed out to me that it might be more appropriate to reserve this name for another invariant, namely for the least integer s (or ∞) such that 2^s annihilates the cokernel of the global signature map (cf. [Mah1])

$$W(A) \to C(\operatorname{Sper} A, \mathbf{Z})$$

(according to [Mah1, Théorème 3.2] this cokernel is always 2-primary torsion). He also proposed to call the above $\operatorname{st}(A)$ the *geometric stability index* of A, a terminology which seems very reasonable. We shall only be concerned with this geometric stability index $\operatorname{st}(A)$ in this paper. If $A=k$ happens to be a field, $\operatorname{Sper} k$ is just the boolean space of all orderings of k, and both stability indices from above coincide (see §2). In this particular case this number is also known as the "reduced stability index of k", compare [L1, §13].

Returning to an affine variety V over the real closed base field R, it is easy to see that the set $V(R)$ of R-rational points, equipped with the "strong topology" (coming from the ordering of R) is contained in $\operatorname{Sper} R[V]$ as a topological subspace. The assignment $Z \mapsto V(R) \cap Z$ defines a bijection from the constructible subsets Z of $\operatorname{Sper} R[V]$ to the semi-algebraic subsets M of $V(R)$; its inverse is commonly denoted by $M \mapsto \tilde{M}$ (see e.g. [BCR], [KS]). So we may also write $\widetilde{V(R)}$ for $\operatorname{Sper} R[V]$. It is obvious that the basic open semi-algebraic subsets $S \subset V(R)$ and the basic open constructibles of $\operatorname{Sper} R[V]$ correspond to each other under this bijection, and that $s_V(S)=s_{R[V]}(\tilde{S})$ holds, hence also $s(V)=\operatorname{st}(R[V])$. This justifies one to call $s(V)$ the (geometric) stability index of the variety V. The advantage of this translation of the original problem into the real spectrum language is that in this way it becomes more algebraic and hence more tractable.

For simplicity, let us henceforth assume that V is irreducible; denote its function field by $R(V)$ and put $n := \dim V$. It is a fundamental fact that for any n-dimensional formally real function field K over R one has $\operatorname{st}(K)=n$. This has first been shown by Bröcker [B1, Satz 4.8]; later Mahé gave in [Mah2,

p. 62] a particularly elegant proof which shows that even the "non-reduced stability index" of K (cf. [EL]) is equal to n. From $\mathrm{st}(K)=n$ one draws the following geometric consequence: Given any basic open $S\subset V(R)$, there is some basic open $S'=S(f_1,\ldots,f_n)$ (with $n=\dim V$) such that the symmetric difference $(S\cup S')\setminus(S\cap S')$ is not Zariski dense in V. If V is non-singular, this may also be expressed by saying that the symmetric difference has empty interior in $V(R)$. Let us write $S\approx S'$ for this situation and refer to such an S' or to $(f_1,\ldots,f_n)$ as a *generic* presentation of S (by n inequalities). It is easy to see that such an S' does also exist with $S'\subset S$.

Having presented a given S generically by n inequalities in this way, one may form the Zariski closure W of $S\setminus S'$ in V. Reasoning as before, one can generically present $S\cap W'(R)$ in W' by $n-1$ inequalities for every irreducible component W' of W, and this procedure can be continued. In order to obtain a true presentation of S from these approximations one needs some pasting techniques to glue together these presentations on subvarieties. Such techniques have been developed by Bröcker [B4], [ABR] and are essentially consequences of the Łojasiewicz inequality. If S is presented on a closed subvariety by r and on its open complement by s inequalities, they show that S can be presented by rs and also by $r+s$ inequalities. Applying these pasting lemmas together with a refined method of finding generic presentations (which approximates a given basic open set up to codimension two), Bröcker arrived at the upper bounds for $s(V)$ and $\bar{s}(V)$ mentioned in the introduction. Of course the lower bound $s(V)\geqq n$ comes from $\mathrm{st}(R(V))=n$.

It was obvious that, in order to establish the conjecture $s(V)=n$, a new idea had to be found since the bounds obtained by using pasting lemmas are too weak *in principle*. Instead it turned out to be more fruitful to improve the generic presentations. Here reduced quadratic form theory comes inevitably into play, and more generally its abstract generalization, the theory of spaces of ordering which is due to M. Marshall. It seems that this theory is in fact needed in its generality here since one has to apply it to rather general semilocal rings, and even to arbitrary (constructible) subspaces of their spaces of orderings.

In order to sketch the main lines of the proof for $s(V)=n$, let us in addition suppose that V is non-singular. Assume we are given some basic open $S\subset V(R)$. Among all the generic presentations $S\approx S'=S(f_1,\ldots,f_n)$ with $f_i\in R[V]$ it suffices to find one with $S\subset S'$ (i.e. with $f_i|S>0$ for $i=1,\ldots,n$). As a first step, how can we show that there is one with at least $f_1|S>0$? To do so we translate the question into quadratic form theory. Fix an arbitrary generic presentation $S\approx S(g_1,\ldots,g_n)$ and put $\phi:=\langle\langle g_1,\ldots,g_n\rangle\rangle=:\phi'\perp\langle 1\rangle$, a quadratic form over $R(V)$. Then the n-tuples $(f_1,\ldots,f_n)$ of functions $f_i\in R(V)^*$ with $S\approx S(f_1,\ldots,f_n)$ are just those n-tuples for which $\phi\equiv\langle\langle f_1,\ldots,f_n\rangle\rangle$ modulo torsion in the Witt ring $W(R(V))$. In particular, the functions f_1 occuring in such an n-tuple are those functions which are represented by some multiple $N\times\phi'$, $N\geqq 1$, and hence they are closed under addition. So a simple argument shows that for the first step it is enough to prove that all the $f\in R[V]$ occuring in a generic length n presentation of S do not have a common zero in S.

This leads one to consider spaces of orderings of local rings. Given $p\in\bar{S}$, one has to show that $s_{\mathcal{O}_{V,p}}(\tilde{S}\cap\mathrm{Sper}\,\mathcal{O}_{V,p})\leqq n$ holds, that is, that there is a generic length n presentation $S'\approx S$ for which p is not contained in the Zariski closure of the symmetric difference of S and S'. The Representation Theorem for spaces of orderings, due to Becker und Bröcker [BB] (in the case of fields) and Marshall [Mar] (for general spaces of orderings) is a kind of local global principle for the existence of forms with a given distribution of signatures on a space of orderings, the "local" objects being certain finite subspaces called fans. Applying this theorem and crucially using regularity of the local ring $\mathcal{O}_{V,p}$ one arrives at the desired conclusion. Altogether this has given us the existence of $f_1,\ldots,f_n\in R[V]$ with $S\approx S(f_1,\ldots,f_n)$ and $f_1|S>0$.

Now this step can, in principle, be iterated. For example, one has next to look at generic presentations of length $n-1$ of S *on the subspace* $S_{R(V)}(f_1)$ of $\mathrm{Sper}\,R(V)$. Although at a first glance this seems to be a different kind of problem, it does in fact make no difference if one uses the language of spaces of orderings. In this way one arrives after n steps at some $S'=S(f_1,\ldots,f_n)\approx S$ with $S\subset S'$, as desired.

Actually the proof just sketched shows much more than $s(V)=n$. Whenever a basic open $S\subset V(R)$ admits just a *generic* presentation by $m\geqq 1$ inequalities - which it always does for some $m\leqq n$ -, it already has the form $S=S(f_1,\ldots,f_m)$! Returning now to arbitrary (possibly singular) affine R-varieties it turns out that the proof and also the assertion just made carry over to them. However the situation becomes slightly more complicated since it may not be sufficient to consider generic presentations in the real spectrum of the function field (or ring of rational functions). Instead $R(V)$ has to be replaced by an appropriate semilocalization of $R[V]$ which takes care of the (real) singularities of V. If one calls two basic open sets generically equal if they agree on the real spectrum of this semilocalization, then it remains true that $s_V(S)$ is just the smallest length of a *generic* presentation of S.

But the proof applies even to a considerably larger class of situations. It yields in fact a similar result for the basic open constructibles in the real spectrum of every noetherian ring A whose real singularities do not behave too bad. For instance, it is enough that, for every real prime ideal p of A, the real singular locus of A/p should be contained in a proper closed subscheme of $\mathrm{Spec}\,A/p$. This condition is so weak that it seems to be difficult to find a neotherian ring which does not satisfy it. In this general form the theorem has also other geometric applications, e.g. to the problems studied in [ABR]. Moreover several interesting corollaries can be deduced from it, for example it reduces the determination of the stability index of a ring to the stability indices of its residue fields. Therefore the theorem and its proof are presented in this more general setup, the more so since the proof does hardly become more complicated than in the geometric situation.

1. Spaces of orderings and real places

Let A be a ring (always commutative with unit). By A^* we denote its group of units. The residue field A_p/pA_p of $p\in\mathrm{Spec}\,A$ is denoted by $\kappa(p)$. The real spectrum $\mathrm{Sper}\,A$ has already been defined. If $x=(p,\alpha)$ is a point of $\mathrm{Sper}\,A$, we shall denote its *support* by σx, so $\sigma x := p \in \operatorname{Spec} A$. If $x, y \in \operatorname{Sper} A$ and $y \in \overline{\{x\}}$, then y is called a specialization of x and x a generalization of y, as usual. A remarkable feature of the real spectrum is that the closure of each point becomes totally ordered by the specialization relation. If $\Sigma \subset A$ is a multiplicative subset, then the natural homomorphism $i: A \to \Sigma^{-1} A$ induces a map $i^*: \operatorname{Sper} \Sigma^{-1} A \to \operatorname{Sper} A$ which is a homeomorphism onto its image. Moreover the field embedding $k(i^* z) \to k(z)$ is an isomorphism for every $z \in \operatorname{Sper} \Sigma^{-1} A$. Hence $\operatorname{Sper} \Sigma^{-1} A$ will frequently be identified with a subspace of $\operatorname{Sper} A$, to get rid of overburded notation. In the same way, $\operatorname{Sper} A/I$ is identified with a closed subspace of $\operatorname{Sper} A$ for every ideal $I \subset A$. In particular, $\operatorname{Sper} \kappa(p)$ will be regarded as the subspace $\{x \in \operatorname{Sper} A : \sigma x = p\}$ of $\operatorname{Sper} A$ for every $p \in \operatorname{Spec} A$.

We'll have to apply Marshall's theory of spaces of orderings. It may be regarded as a formalization of reduced quadratic form theory over fields which however applies to a much larger variety of cases, in particular to semilocal rings (see below). All the general facts we require can be found in [Mar]; for the case of preorderings of fields see also Lam's lectures [L 1]. Let us just briefly recall the most important definitions and facts.

A *space of orderings* is a pair (X, G) consisting of a discrete group G of exponent ≤ 2 and a closed subset X of its character group $\hat{G} = \operatorname{Hom}(G, \{\pm 1\})$ which (topologically) generates $\hat{G}$. In G there has to be a (necessarily unique) distinguished element -1 such that $x(-1) = -1$ for every $x \in X$. (Hence $X = \emptyset$ iff G is the trivial group.) In order to formulate the last and most important axiom one has to introduce some more terminology: An n-dimensional *form* $(n \geq 1)$ over (X, G) is an n-tuple $\varphi \in G^n$, written $\varphi = \langle f_1, \ldots, f_n \rangle$. It is convenient to introduce also a unique form of dimension zero (the "empty form") satisfying obvious rules. Out of two forms φ and ψ one can build their direct sum $\varphi \perp \psi$ and their tensor product $\varphi \otimes \psi$, formally in the same way as one does with ordinary diagonalized quadratic forms. The *signature* of φ at a point $x \in X$ is the integer $x(\varphi) := x(f_1) + \ldots + x(f_n)$; the continuous map $\hat{\varphi}: X \to \mathbb{Z}$ defined by $\hat{\varphi}(x) := x(\varphi)$ is called the *total signature* of φ. The subring $\{\hat{\varphi}: \varphi$ a form over $X\}$ of $C(X, \mathbb{Z})$ is called the *Witt ring* $W(X)$ of (X, G). One commonly speaks of the elements of $W(X)$ as of those functions in $C(X, \mathbb{Z})$ which can be *represented over* X. Two forms φ and ψ are said to be *isomorphic* if $\dim \varphi = \dim \psi$ and $\hat{\varphi} = \hat{\psi}$. The form φ is said to *represent* an element $f \in G$ iff $\varphi \cong \langle f \rangle \perp \psi$ for some form ψ; the set of elements represented by φ is written $D(\varphi)$ or $D_X(\varphi)$. Now the last axiom for (X, G) to be a space of orderings runs as follows:

(O_4) *Given non-empty forms* φ *and* ψ *and an element* $h \in D(\varphi \perp \psi)$, *there are elements* $f \in D(\varphi)$ *and* $g \in D(\psi)$ *such that* $h \in D(\langle f, g \rangle)$.

A form φ is said to be *isotropic* if there is a form ψ with $\varphi \cong \langle 1, -1 \rangle \perp \psi$, otherwise *anisotropic*. φ is isotropic iff it is *universal*, i.e. iff $D(\varphi) = G$. Every form φ has a *Witt decomposition* $\varphi \cong \varphi_0 \perp n \cdot H$, unique up to isomorphism, with φ_0 anisotropic, $n \geq 0$ and $H = \langle 1, -1 \rangle$. Two forms φ and ψ have the same image in $W(X)$ (i.e. their global signatures coincide) iff their anisotropic kernels φ_0 and ψ_0 are isomorphic.

If (X, G) is a space of orderings, a subset $Y \subset X$ will be called a *subspace* of X if $Y = X \cap Y^{\perp\perp}$ holds; in this case the pair $(Y, G/Y^{\perp})$ becomes a space of orderings by itself. In the context of spaces of orderings the word "subspace" will be reserved for this situation. The constructible (=clopen) subspaces of X are precisely the subsets of X of the form

$$Y = X \cap \{f_1, \ldots, f_r\}^{\perp} = \{x \in X : x(f_1) = \ldots = x(f_r) = 1\}$$

with $f_i \in G$, $r \geq 0$. Note that Y is just the set of $x \in X$ in which the n-fold *Pfister form* $\langle\langle f_1, \ldots, f_n \rangle\rangle := \langle 1, f_1 \rangle \otimes \ldots \otimes \langle 1, f_n \rangle$ has positive signature (equal to 2^n). By abuse of language (however suggestive!) let us say in this case that Y can be *described by r inequalities in* X. The least such $r \geq 0$ is denoted by $s_X(Y)$ (by definition, $s_X(X) = 0$). *The stability index* $\operatorname{st}(X)$ of the space of orderings (X, G) can be defined as the supremum of all the $s_X(Y)$ with Y running over the non-empty constructible subspaces of X; put $\operatorname{st}(\emptyset) := -1$.

The first class of examples is provided by fields. If k is a field, char $k \neq 2$, then $(\operatorname{Sper} k, k^*/\Sigma k^{*2})$ is a space of orderings (where Σk^{*2} is the group of non-zero sums of squares in k). Axiom (O_4) is a consequence of Pfister's local global principle. The (non-empty) subspaces correspond bijectively to the preorderings of k. To indicate that $\operatorname{Sper} k$ is viewed as a space of orderings we will use the notation X_k for $\operatorname{Sper} k$. The Witt ring of X_k is isomorphic to the reduced Witt ring of the field k, by means of the global signature map. Note that $\operatorname{st}(X_k) = \operatorname{st}(k)$, i.e. the two notions of stability index coincide for fields.

A space of orderings (X, G) is called a *fan* if $X = \{x \in \hat{G}: x(-1) = -1\}$. This kind of spaces of orderings is of particular importance because of the

Representation theorem [BB], [Mar, Theorem 5.5]. *Let* (X, G) *be a space of orderings and* $f: X \to \mathbb{Z}$ *a continuous map. Then* f *can be represented over* X *(that is,* $f \in W(X)$*) if and only if* $f | Y$ *can be represented over* Y *for every finite subspace* Y *of* X *which is a fan. It is also equivalent to say that for each such* Y *the congruence*

$$\sum_{y \in Y} f(y) \equiv 0 \bmod |Y|$$

is valid.

Using this theorem one deduces that (for $X \neq \emptyset$) the stability index $\operatorname{st}(X) =: s$ is also described by

$$2^s = \text{exponent of the group } C(X, \mathbb{Z})/W(X)$$

(in fact this is the original definition). Another important characterization of $s = \operatorname{st}(X)$ deduced from this theorem is

$$2^s = \text{maximal size of a fan in } X$$

(of course, $2^\infty := \infty$).

We will have to apply the following easy consequence of the Representation Theorem, for which we have not found an appropriate reference (but compare the argument in [Mar, p. 517]):

Lemma 2.1. *Let* (X, G) *be a space of orderings and* $Y \subset X$ *a constructible* (=*clopen*) *subspace of the space of orderings* X. *Let* $m \geq 1$. *Then* Y *can be described by*

m inequalities in X if and only if for every finite fan $F \subset X$ with $|F| = 2^k > 2^m$ one has

$$|F \cap Y| \equiv 0 \bmod 2^{k-m}.$$

Proof. We may assume $Y \neq \emptyset$. Since Y is a clopen subspace, there are $p \geqq 1$ and $a_1, \ldots, a_p \in G$ such that $\phi = \langle\langle a_1, \ldots, a_p \rangle\rangle$ has signature $\hat{\phi} = 2^p \cdot 1_Y$. To prove the non-trivial direction assume that the above congruences are satisfied. By the Representation Theorem this implies $2^m \cdot 1_Y \in W(X)$. We may assume $p > m$. Then also $2^{p-1} \cdot 1_Y \in W(X)$, hence there is an anisotropic form τ with $\hat{\phi} = 2\hat{\tau}$, and even $\phi \cong 2\tau$ since both forms are anisotropic. Since $1 \in D(\phi) = D(2\tau) = D(\tau)$ we can write $\tau \cong \langle 1 \rangle \perp \tau'$, hence $\phi \cong \langle 1, 1 \rangle \perp 2\tau'$. This gives a $(p-1)$-fold Pfister form ψ with $\phi \cong 2\psi$ [Mar, Lemma 6.3]. Iterating this procedure one arrives at an m-fold Pfister form ψ with $\phi \cong 2^{p-m} \cdot \psi$, i.e. $\hat{\psi} = 2^m \cdot 1_Y$, whence the assertion. □

It will be important to know how a semilocal ring A gives rise to a space of orderings (compare [K 2] for the following). We put $X_A := (\mathrm{Sper}\, A)^{\max}$ (:= the set of closed points of Sper A) and $G_A := A^*/\{u \in A^* : u | \mathrm{Sper}\, A > 0\}$. Then X_A may be identified with a (closed) subset of $\hat{G}_A$, and (X_A, G_A) is a space of orderings. The topologies on X_A inherited from Sper A on the one hand and from $\hat{G}_A$ on the other coincide, and we will always regard X_A as a topological subspace of Sper A as well. Knebusch showed that for every non-trivial fan $F \subset X_A$ (i.e. $|F| \geqq 4$) there is a prime ideal p with $\sigma x = p$ for every $x \in F$ [K 2, Theorem 7.4]. As a consequence one deduces that

$$\mathrm{st}\, X_A \leqq \sup(\{1\} \cup \{\mathrm{st}(\kappa(p)) : p \in \mathrm{Spec}\, A\})$$

holds [K 2, Theorem 9.5]. For example, if V is an n-dimensional affine variety over a real closed field and A is any semilocalization of its coordinate ring, then $\mathrm{st}(X_A) \leqq n$ by the theorem of Bröcker mentioned in § 1. Observe however that, contrary to the case of fields, one only has st $X_A \leqq$ st A for semilocal rings, and in general equality will fail to hold.

We now collect together some facts concerning the interplay between fields, their spaces of orderings and places between them.

Let B be a valuation ring with field of fractions K and residue field k, and let $\lambda: K \to k, \infty$ be the associated place. Let $Y \subset \mathrm{Sper}\, k$ be a subspace of the space of orderings $X_k = \mathrm{Sper}\, k$. The pullback of Y with respect to λ (see [L 1, p. 22]) is a subspace of $X_K = \mathrm{Sper}\, K$ which may be described as

$$\{x \in \mathrm{Sper}\, K : x \text{ and } \lambda \text{ are compatible, and the ordering on } k \text{ induced by } x \text{ lies in } Y\}$$

or, equivalently, as

$$\{x \in \mathrm{Sper}\, K : \overline{\{x\}} \cap Y \neq \emptyset\},$$

where Sper K and Sper k are regarded as topological subspaces of Sper B and the closure is formed in Sper B. Let us denote this subspace by $\lambda^* Y$. It is easy to see, but of some importance, that if Y is a fan then also $\lambda^* Y$ is a fan. The Baer-Krull theorem (see e.g. [L 1, Theorem 3.10], [BCR, Théorème 10.1.10], [KS, II, § 7]) tells us that the unique map $s: \lambda^* Y \to Y$ satisfying $s(x) \in \overline{\{x\}}$ $(x \in \lambda^* Y)$ is surjective, more precisely that there is a natural free and transitive action of the character group of $\Gamma \otimes (\mathbb{Z}/2\mathbb{Z})$ on each of the fibres of s (here Γ denotes the value group of λ). If Y is a fan there is a subfan F of $\lambda^* Y$ such that the restriction $s | F: F \to Y$ is bijective.

Let A be a ring and $q \subset A$ a prime ideal. Later we'll have to "lift back" a given fan from $X_{\kappa(q)}$ to $X_{\kappa(p)}$, for some prime ideal p properly contained in q. That is the point where some control over the (real) singularities of A is required. The following simple and well-known fact (see e.g. [K 1, p. 285]) stands behind this:

Lemma 2.2. *If A is a regular noetherian local domain with field of fractions K and residue field k, there is a valuation ring B of K which dominates A such that $k \to B/m_B$ is an isomorphism. Moreover, if d is the dimension of A, one can achieve that B has value group $\mathbb{Z}^d$, ordered lexicographically.*

Proof. For the reader's convenience we'll include the proof. Assume $d \geqq 1$. There is a prime ideal $p \subset A$ of height one such that A/p is again regular (for example, $p = (f_1)$ if $f_1, \ldots, f_d$ is a regular system of parameters in A, see [Mat, Theorem 36]). By induction on d we may assume the lemma holds for A/p, i.e. we find a place $\lambda: \kappa(p) \to k, \infty$ which is finite on the image of A such that the composition $A \to \kappa(p) \xrightarrow{\lambda} k$ is the residue map $A \to k$, and such that λ has value group $\mathbb{Z}^{d-1}$ with lexicographic order. Since A_p is a rank one discrete valuation ring of K, we may compose the associated place $K \to \kappa(p), \infty$ with λ to get the desired result. □

Corollary 2.3. *Let A be a noetherian domain with field of fractions K, let $p \in \mathrm{Spec}\, A$ such that A_p is regular of dimension d, and let G be a fan in $X_{\kappa(p)} = \mathrm{Sper}\, \kappa(p)$. Then there is a fan F in $X_K = \mathrm{Sper}\, K$ together with a map $s: F \to G$ such that $s(x) \in \overline{\{x\}}$ holds (in Sper A) for every $x \in F$, and such that $s^{-1}(y)$ has cardinality 2^d for every $y \in G$. There is also a subfan F' of F such that s restricted to F' is bijective.*

Proof. Take a place $\lambda: K \to \kappa(p), \infty$ which is finite on A_p such that $\lambda | A_p$ is the residue homomorphism, and put $F := \lambda^* G$. □

Corollary 2.4 [ABR, Lemma 7.5]. $\mathrm{st}(K) \geqq d + \mathrm{st}(k)$.

Proof. Use the characterization of the stability index by means of the maximal size of a fan. □

Another direct consequence of Corollary 2.3 is

Corollary 2.5. *If A is a noetherian domain and if $y \in \mathrm{Sper}\, A$ is such that the local ring $A_{\sigma y}$ is regular, then y has a generalization x in Sper A with $\sigma x = (0)$.* □

3. Basic closed sets

Let R be a fixed real closed field; all varieties considered here are assumed to be reduced affine R-varieties. Given one, say V, we write $R[V] := \Gamma(V, \mathcal{O}_V)$. By $R(V)$ we denote the total ring of fractions of $R[V]$ (i.e. its semilocalization in the set of minimal prime ideals). The R-rational points $V(R)$ are Zariski dense in V iff $R(V)$ is a direct product of formally real fields, in which case V is called *real*. For $f \in R[V]$ put $Z(f) := \{x \in V(R): f(x) = 0\}$. Given a closed subvariety $T \subset V$, there is always some $t \in R[V]$ with $T(R) = Z(t)$, and any such t will be called an *equation for* $T(R)$.

We start by recalling the upper bound for $\bar{s}$ which has previously been obtained by Bröcker [B5]. We also include the proof since it turns out to be useful later.

Proposition 3.1 (Bröcker). *Let V be an n-dimensional affine variety, $n \geqq 1$. Then* $\bar{s}(V) \leqq \frac{1}{2} n(n+1)$.

Proof (see [B5]). Induction on n. Since the case $n=1$ is easy and the arguments are not used in the sequel we suppress it. So let $n \geqq 2$, and let $F \subset V(R)$ be basic closed. Then $\tilde{F} \cap \operatorname{Sper} R(V)$ is a subspace of $X_{R(V)} = \operatorname{Sper} R(V)$. Since $\operatorname{st}(X_{R(V)}) \leqq n$, there are $f_1, \ldots, f_n \in R[V]$ which are not zero divisors in $R[V]$ such that

$$\tilde{F} \cap \operatorname{Sper} R(V) = S(f_1, \ldots, f_n)^{\sim} \cap \operatorname{Sper} R(V) = \bar{S}(f_1, \ldots, f_n)^{\sim} \cap \operatorname{Sper} R(V).$$

Put $F_1 := \bar{S}(f_1, \ldots, f_n) \subset V(R)$, and let T be the Zariski closure (in V) of the symmetric difference $(F \cup F_1) \setminus (F \cap F_1)$ of F and F_1. Then $\dim T < n$, and $F \setminus T(R) = F_1 \setminus T(R)$. Moreover there are $g_1, \ldots, g_m \in R[V]$ with $F \cap T(R) = \bar{S}(g_1, \ldots, g_m) \cap T(R)$ and $m \leqq \bar{s}(T) \leqq \frac{n}{2}(n-1)$, by the induction hypothesis. We now get $\bar{s}_V(F) \leqq n + \frac{n}{2}(n-1) = \frac{n}{2}(n+1)$ by applying

Lemma 3.2. *Let $T \subset V$ be a closed subvariety and let $F \subset V(R)$, $G \subset T(R)$ be closed semi-algebraic subsets such that $F \cap T(R) \subset G$ holds and such that there exist $a_1, \ldots, a_k$, $b_1, \ldots, b_l \in R[V]$ with $F \setminus T(R) = \bar{S}(a_1, \ldots, a_k) \setminus T(R)$ and $G = \bar{S}(b_1, \ldots, b_l) \cap T(R)$. Then $F \cup G$ is basic closed, and $\bar{s}_V(F \cup G) \leqq k + l$.*

Proof. Let $t \in R[V]$ be an equation for $T(R)$. For $j = 1, \ldots, l$ put $X_j := F \cap \bar{S}(-b_j)$. An application of the inequality of Łojasiewicz ([B4, Lemma 6.1] or [BCR, Lemme 7.7.10]) gives ε_j, $p_j \in R[V]$ with $\varepsilon_j \geqq 0$, $p_j > 0$ on $V(R)$, such that the functions $b'_j := p_j t^2 + \varepsilon_j b_j$ satisfy

$$\operatorname{sgn} b'_j = \operatorname{sgn} t^2 \text{ on } X_j, \quad \text{and} \quad Z(\varepsilon_j) = \overline{X_j \cap T(R)}^Z,$$

hence $Z(\varepsilon_j) = \overline{F \cap T(R) \cap \bar{S}(-b_j)}^Z \subset \overline{G \cap \bar{S}(-b_j)}^Z \subset Z(b_j)$ $(j = 1, \ldots, l)$. (Here we have written $\bar{M}^Z$ for the set of R-rational points in the Zariski closure of a subset $M \subset V(R)$.) One clearly has $b'_j | F \geqq 0$, and since also $\bar{S}(b_j) \cap T(R) = \bar{S}(b'_j) \cap T(R)$ holds $(j = 1, \ldots, l)$, this implies $F \cup G = \bar{S}(t^2 a_1, \ldots, t^2 a_k, b'_1, \ldots, b'_l)$. $\square$

We now show that the bound given in Proposition 3.1 is already best possible, that is, $\bar{s}(V) = \frac{1}{2} n(n+1)$ for every real affine variety V of dimension $n > 0$. It is immediate to check that for any open affine subvariety W one has $\bar{s}(W) \leqq \bar{s}(V)$. Hence it is enough to prove $\bar{s}(V) \geqq \frac{1}{2} n(n+1)$ for every non-singular irreducible real V with $\dim V = n$. This will be done by induction on n, the start of the induction being clear.

So let V be an n-dimensional non-singular irreducible real affine variety, $n \geqq 2$, and choose a real prime divisor $H \subset V$ (i.e. a closed irreducible real subvariety of codimension one). By induction hypothesis there is a basic closed subset $G \subset H(R)$ with $\bar{s}_H(G) = \frac{1}{2} n(n-1)$. We need that $\tilde{G} \cap \operatorname{Sper} R(H) \neq \emptyset$ (Sper $R(H)$ being identified with the set of points in $\widetilde{V(R)}$ with support H). This can be achieved for $n = 2$ by taking $G = \bar{S}_H(h)$ for some $h \in R[H]$ such that neither h nor $-h$ is a sum of squares in $R(H)$, and is automatically satisfied for $n > 2$: Indeed, otherwise we would have $G \subset W(R)$ for a proper subvariety W of H, and hence $\bar{s}_H(G) \leqq \bar{s}_W(G) + 1 \leqq \frac{1}{2}(n-1)(n-2) + 1 < \frac{n}{2}(n-1) = \bar{s}_H(G)$, a contradiction.

Let H_{reg} denote the regular locus of H. Since $\tilde{G} \cap \operatorname{Sper} R(H)$ is open in $\operatorname{Sper} R(H)$ and non-empty, the interior of $G \cap H_{\text{reg}}(R)$ in $H(R)$ is non-empty. Pick any point p in this set and choose a place $\lambda: R(H) \to R, \infty$ over R which dominates the local ring $\mathcal{O}_{H,p}$ and which has value group $\mathbb{Z}^{n-1}$ (this is possible by Lemma 2.2). The pullback of the ordering of R with respect to λ is a fan Z in $X_{R(H)} = \operatorname{Sper} R(H)$ with $|Z| = 2^{n-1}$ and $Z \subset \tilde{G}$. Using the discrete rank one valuation of $R(V)$ associated to H we pull back Z even further to get a fan Y in $X_{R(V)} = \operatorname{Sper} R(V)$ of size 2^n. There is a basic closed subset C of $V(R)$ with $|\tilde{C} \cap Y| = 1$ and $C \cap H(R) \subset G$. Indeed, fixing some $y \in Y$ and its specialization z in Z, we have

$$\widetilde{H(R)} \cap \bigcap_{f \in R[V]:\, f(y) > 0} \widetilde{\bar{S}(f)} = \widetilde{H(R)} \cap \overline{\{y\}} = \overline{\{z\}} \subset \tilde{G}$$

(here the fact was used that $\overline{\{y\}}$ forms a chain in $\widetilde{V(R)}$). Since the constructible topology on $\widetilde{V(R)}$ is compact, there are finitely many $f_1, \ldots, f_r \in R[V]$ with $\bar{S}(f_1, \ldots, f_r)^{\sim} \cap Y = \{y\}$ and $\bar{S}(f_1, \ldots, f_r) \cap H(R) \subset G$, so we may take $C = \bar{S}(f_1, \ldots, f_r)$. By Lemma 3.2, the subset $F := G \cup C$ of $V(R)$ is also basic closed. We claim that $\bar{s}_V(F) = \frac{n}{2}(n+1)$.

To see this, let $F = \bar{S}(g_1, \ldots, g_N)$ with $g_i \in R[V]$. If g_i doesn't vanish identically on H, we have $g_i | Z > 0$ (since $Z \subset \tilde{F}$), hence also $g_i | Y > 0$. Since $|\tilde{F} \cap Y| = 1$, at least n of the g_i must vanish identically on H. Since $F \cap H(R) = G$ there must be at least $\bar{s}_H(G) = \frac{n}{2}(n-1)$ further g_i's which do not vanish identically on H. Altogether we get $N \geqq n + \frac{n}{2}(n-1) = \frac{1}{2} n(n+1)$, and we have proved

Theorem 1. *For an affine real variety V of dimension $n > 0$ one has*

$$\bar{s}(V) = \tfrac{1}{2} n(n+1). \quad \square$$

In order to give an idea how basic closed sets look like which require the maximal number of inequalities, here is an

Example. Consider affine n-space $\mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$ $(n \geqq 1)$ and the following semi-algebraic subsets of R^n:

$$F_1 = \{x \in R^n: x_1 \geqq 0,\ x_2 = \ldots = x_n = 0\},$$
$$F_2 = \{x \in R^n: x_1 \geqq 1,\ x_2 \geqq 0,\ x_3 = \ldots = x_n = 0\},$$
$$F_3 = \{x \in R^n: x_1 \geqq 2,\ x_2 \geqq 1,\ x_3 \geqq 0,\ x_4 = \ldots = x_n = 0\},$$
$$\ldots$$
$$F_n = \{x \in R^n: x_1 \geqq n-1,\ x_2 \geqq n-2,\ \ldots,\ x_n \geqq 0\}.$$

Put $F=F_1\cup\ldots\cup F_n$. Since the construction of F follows exactly the device of the proof above, we conclude that F is basic closed and that at least $\frac{n}{2}(n+1)$ simultaneous inequalities are needed to describe F. An explicit system of such inequalities in provided by the presentation

$$F=\bar{S}((x_i+i-j)\,x_j^2;\ 1\leq i\leq j\leq n).$$

4. Basic open sets

Let A be a ring. The following are equivalent:

(i) $a_1^2+\ldots+a_n^2=0$ *implies* $a_1=\ldots=a_n=0$, *for* $a_i\in A$;

(ii) *A is reduced, and every minimal prime ideal of A has a (formally) real residue field*;

(iii) $A=0$, *or A admits an injective homomorphism into a direct product of a family of (formally) real fields.*

If they are satisfied, the ring A is called *real* (see [L2, §1], [KS, III, §2]). An ideal I of a ring A is said to be *real* if the ring A/I is real. The set of all real prime ideals of A (i.e. the set of $p\in\operatorname{Spec} A$ for which $\kappa(p)$ is formally real) is denoted by $(\operatorname{Spec} A)_{re}$. More generally, for an arbitrary subset X of Spec A we'll write $X_{re}:=X\cap(\operatorname{Spec} A)_{re}$, and we equip X_{re} with the topology it inherits from the Zariski topology on Spec A.

Now let A be a noetherian ring. Note that X_{re} is a noetherian topological space for every subset X of Spec A. Write $\operatorname{Reg} A:=\{p\in\operatorname{Spec} A: A_p \text{ is regular}\}$ for the regular locus of Spec A and put $\operatorname{Sing} A:=(\operatorname{Spec} A)\setminus(\operatorname{Reg} A)$.

Let us consider the following property of a noetherian ring A:

(R) $(\operatorname{Reg} A/p)_{re}$ *is open in* $(\operatorname{Spec} A/p)_{re}$ *for every prime ideal p of A.*

Condition (R) is in fact equivalent to the apparently weaker condition that $(\operatorname{Reg} A/p)_{re}$ contains a non-empty open subset of $(\operatorname{Spec} A/p)_{re}$ for every real prime ideal p of A. In fact, $(\operatorname{Reg} A/p)_{re}=\emptyset$ if $p\in\operatorname{Spec} A$ is not real (Corollary 2.5), and for real p one can argue as in [EGA IV, 6.12.2]. However what we are really going to use in the proof of Theorem 2 is

(R') *There is a finite subset D of* $(\operatorname{Spec} A)_{re}$ *containing the minimal elements of* $(\operatorname{Spec} A)_{re}$ *such that, for every* $q\in(\operatorname{Spec} A)_{re}$ *and every $p\in D$ which is maximal in D under $p\subset q$, the local ring A_q/pA_q is regular.*

Lemma 4.1. (R) *implies* (R') *for every noetherian ring A.*

Proof. We first construct a sequence $\{D_i\}_{i\geq 0}$ of finite subsets of $(\operatorname{Spec} A)_{re}$. Let D_0 be the (finite) set of minimal elements of $(\operatorname{Spec} A)_{re}$. Assuming that $D_0,\ldots,D_i$ have already been constructed we define D_{i+1} as follows. For every $p\in D_i$ there is a real ideal $J(p)$ of A with $p\subset J(p)$, $p\neq J(p)$, such that A_q/pA_q is regular for each $q\in(\operatorname{Spec} A)_{re}$ with $p\subset q$ and $J(p)\not\subset q$. Let $D_{i+1}(p)\subset(\operatorname{Spec} A)_{re}$ be the set of minimal prime ideals of A lying over $J(p)$ (thus $D_{i+1}(p)=\emptyset$ iff $J(p)=A$) and put $D_{i+1}:=\bigcup_{p\in D_i} D_{i+1}(p)$.

Since one has

$$\bigcap_{p\in D_i} p\subsetneqq\bigcap_{p\in D_i} J(p)=\bigcap_{p\in D_{i+1}} p \quad \text{if } D_i\neq\emptyset,$$

only finitely many of the D_i can be non-empty, since A is noetherian. Their union D has the required property–indeed, if q and p are as in (R'), say $p\in D_i$, then $J(p)\not\subset q$ since otherwise $p\subsetneqq p'\subset q$ for some $p'\in D_{i+1}(p)\subset D_{i+1}$, and hence A_q/pA_q is regular. $\square$

The proof shows that there is a canonical choice for the finite set D in case A satisfies (R) above: D can be taken to be the smallest subset of $(\operatorname{Spec} A)_{re}$ containing the minimal real prime ideals of A and containing with each p also the minimal elements of $(\operatorname{Sing} A/p)_{re}$. One should bear in mind that property (R') is very weak, and all noetherian rings one usually encounters should satisfy it.

Now let A be a noetherian ring satisfying (R'), and fix some finite subset D of $(\operatorname{Spec} A)_{re}$ with this property. We suppose in the sequel that Sper A is not empty.

Theorem 2. *Let $S=S_A(h_1,\ldots,h_N)$ be a basic open subset of* Sper A. *Then there is a subset $D'=D'(S)$ of D with $h_i\notin p$ for every $p\in D'$ and $i=1,\ldots,N$, such that for the semilocalization B of A in D'*

(a) $s_A(S)=s_{X_B}(S\cap X_B)$ *or*

(b) Sper $B\subset S$, *in which case $S=S_A(f^2)$ for some $f\in A$ (and hence $s_A(S)\leq 1$) holds.*

Note that $S\cap X_B$ is a subspace of the space of orderings X_B since the h_i are invertible in B. The subset D' of D is readily described: It consists (in case $S\neq\emptyset$) of those $p\in D$ which are maximal under "$p\subset\sigma x$ for some $x\in S$".

Corollary 1. *Let A be a noetherian ring satisfying (R'). Then*

$$\operatorname{st}(A)=\sup\{\operatorname{st}(\kappa(p)): p\in\operatorname{Spec} A\}=\max\{\operatorname{st}(\kappa(p)): p\in D\},$$

except when each $\kappa(p)$, $p\in\operatorname{Spec} A$, has at most one ordering and Sper A *contains at least two elements, in which case* $\operatorname{st}(A)=1$.

Proof. Let B be a semilocalization of A. Then $\operatorname{st} X_B\leq\sup\{\operatorname{st}(\kappa(q)): q\in(\operatorname{Spec} B)_{re}\}$ or $\operatorname{st} X_B=1$, by Knebusch's results (see §2). But for $q\in(\operatorname{Spec} B)_{re}\subset(\operatorname{Spec} A)_{re}$ there is some $p\in D$ with $p\subset q$ for which A_q/pA_q is regular, and hence with $\operatorname{st}(\kappa(q))\leq\operatorname{st}(\kappa(p))$ (Corollary 2.4). The opposite inequality $\operatorname{st}(\kappa(p))\leq\operatorname{st}(A)$ for $p\in\operatorname{Spec} A$ is trivial. $\square$

In particular, this answers a question of E. Becker in the positive (see [ABR, Remark 7.9]):

Corollary 2. *Let A be a noetherian ring satisfying (R'). Then* $\operatorname{st}(A)=\max\{\operatorname{st}(A_m)$: *$m$ is a maximal ideal of A}, unless* $\operatorname{st}(A)=1$ *and the right hand side is zero.* $\square$

Corollary 3. *Let A be a regular noetherian domain with field of fractions K. If S is a proper basic open subset of* Sper A, *then* $s_A(S)=s_K(S\cap \text{Sper } K)$ *unless* Sper $K\subset S$, *in which case* $s_A(S)=1$. *In particular, one has* $\text{st}(A)=\text{st}(K)$.

In the geometric situation (R a real closed field) this means

Corollary 3′. *Let V be an affine R-variety without R-rational singular points and let S be a basic open semi-algebraic subset of $V(R)$. If S admits a generic presentation by $m\geqq 1$ inequalities, then already $s_V(S)\leqq m$.*

(Here *generic* may be read as "up to a subset with empty interior".) □

Corollary 4. *Let $n>0$. Then $s(V)=n$ for every real n-dimensional affine R-variety V.* □

Proof of the theorem. We may assume $S\neq\emptyset$. Let $D^0=\{p\in D\colon p\subset\sigma x$ for some $x\in S\}$ and D' the set of maximal elements of D^0, moreover B the semilocalization of A in D'. Note that $D'\neq\emptyset$ since $S\neq\emptyset$. Let I denote the intersection of all $p\in D\setminus D^0$; thus $I\not\subset\sigma x$ for every $x\in S$. Remember that we identify Sper B with $\{x\in\text{Sper } A\colon\sigma x\subset p$ for some $p\in D'\}$, and similarly for the Zariski spectrum. We observe

(1) *Every $y\in$ Sper A with $I\not\subset\sigma y$ has a generalization contained in* Sper B. *In particular, this applies to every $y\in S$.*

(2) *For each $p\in D'$ there is $x\in S$ with $p=\sigma x$.*

(For (1) choose $p\in D$ maximal under $p\subset\sigma y$. By Corollary 2.5, y has a generalization x with $\sigma x=p$ since $A_{\sigma y}/pA_{\sigma y}$ is regular by (R'). Moreover $p\in D^0$ since $I\not\subset p$, and so $p\in$ Spec B. In (2) we have $p\subset\sigma y$ for some $y\in S$ by definition. Since $A_{\sigma y}/pA_{\sigma y}$ is regular by (R'), there is a generalization x of y with support p.)

The natural homomorphism $A\to B$ is denoted by φ_B. Note that any $a\in A$ which does not vanish anywhere on S becomes a unit in B, by (2). In particular, for any basic open $U\subset$ Sper A which contains S, the subset $U\cap$ Sper B is open and closed in Sper B, and $U\cap X_B$ is a subspace of the space of orderings X_B.

Main Lemma 4.2. *Let U be a basic open subset of* Sper A *containing S. Let $m\geqq 1$ and assume*

$$S\cap X_B=S_B(u_1,\ldots,u_m)\cap U\cap X_B \qquad (\#)$$

for some $u_i\in B^*$. Then there are $f_1,\ldots,f_m\in A$ such that $f_i|S>0$ and such that (#) holds with $u_i=\varphi_B(f_i)$, $i=1,\ldots,m$.

Before we prove the lemma we show how to deduce Theorem 2 from it. Let $m:=s_{X_B}(S\cap X_B)$, a non-negative integer. Applying the Main Lemma we find $f_1,\ldots,f_m\in A$ such that the basic open set $S':=S_A(f_1,\ldots,f_m)$ contains S and $S\cap X_B=S'\cap X_B$ holds. (Note that simply $S'=$ Sper A if $m=0$.) Since $S\cap$ Sper B and $S'\cap$ Sper B are basic open subsets of Sper B defined by units of B, they are both clopen in Sper B. Moreover their intersections with $(\text{Sper } B)^{\max}=X_B$ coincide, so we must have $S\cap$ Sper $B=S'\cap$ Sper B. I claim $S'\subset\bar{S}\cup\{x\in\text{Sper } A\colon I\subset\sigma x\}$. To see this, pick $y\in S'\setminus S$. If $I\not\subset\sigma y$, there is a generalization x of y in $S'\cap$ Sper $B=S\cap$ Sper B, by (1). Hence $y\in\overline{\{x\}}\subset\bar{S}$.

Let $a_1,\ldots,a_M$ be generators of I and let $S=S_A(h_1,\ldots,h_N)$ be any presentation of S. Putting $a:=a_1^2+\ldots+a_M^2$ and $h:=h_1\ldots h_N$ we have

$$S=S_A(ah^2f_1,f_2,\ldots,f_m)$$

if $m>0$, and $S=S_A(ah^2)$ in case $m=0$, since h vanishes on $\bar{S}\setminus S$.

So it remains to prove Lemma 4.2. It suffices to find $f_1\in A$ with $f_1|S>0$ such that there is a presentation (#) with $u_1=\varphi_B(f_1)$, since then this argument may again be applied with U replaced by $U\cap S_A(f_1)$ and m by $m-1$, etc. First note that (#) implies

(3) $S\cap\text{Sper } B=S_B(u_1,\ldots,u_m)\cap U$,

by the same argument as before (both sides are defined by units of B, hence are clopen in Sper B, and their intersections with $(\text{Sper } B)^{\max}=X_B$ coincide).

Denote the subspace $U\cap X_B$ of X_B by Y. Given $u'_1,\ldots,u'_m\in B^*$, eq. (#) holds with the u_i replaced by the u'_i if and only if

$$\langle\!\langle u_1,\ldots,u_m\rangle\!\rangle\cong\langle\!\langle u'_1,\ldots,u'_m\rangle\!\rangle \text{ as forms over } Y.$$

Writing $\phi:=\langle\!\langle u_1,\ldots,u_m\rangle\!\rangle=:\langle 1\rangle\perp\phi'$, it is well-known that (completely analogous to ordinary quadratic form theory) for $u\in B^*$, one has $\phi\cong_Y\langle\!\langle u,\ldots\rangle\!\rangle$ if and only if u is represented by ϕ' over Y [Mar, Lemma 6.3]. Thus if we put

$$L:=\{f\in A\colon\varphi_B(f)\in B^* \text{ and } \varphi_B(f)\in D_Y(\phi')\}$$
$$=\{f\in A\colon\varphi_B(f)\in B^*, \text{ and there is a presentation (\#) with } u_1=\varphi_B(f)\},$$

we have to find some $f\in L$ with $f|S>0$ in order to prove the lemma.

First observe that

(4) *Every $f\in L$ is non-negative on S.*

(If $f(x)<0$ for some $x\in S$, then we also find such an x contained in Sper B, by (1). The closed specialization $\bar{x}$ of x in Sper B lies in $S\cap X_B$, and $f(\bar{x})<0$. This is a contradiction since every unit of B represented by ϕ over Y is positive on $S\cap X_B$.)

The following fact is elementary:

If B is a semilocal ring, Y is a subspace of X_B, φ is a Y-form and $b,b'\in B^*$ are represented by φ over Y, then also $b+b'$ is represented by φ over Y provided that $b+b'\in B^*$.

(We can write $\varphi\cong_Y\langle b\rangle\perp\psi\cong_Y\langle b'\rangle\perp\psi'$, and since $D_Y(2\varphi)=D_Y(\varphi)$ it is enough to show $b+b'\in D_Y(\langle b,b'\rangle)$. But this follows from the obvious isomorphism $\langle b,b'\rangle\cong\langle b+b',bb'(b+b')\rangle$.)

This leads to the following important observation:

(5) $L+L\subset L$.

(Let $f,g\in L$; by the above we only have to show $\varphi_B(f+g)\in B^*$. Thus let $p\in D'$ and choose $x\in S$ with $\sigma x=p$, using (2). By (4) we have $f(x)>0$ and $g(x)>0$, and hence $f+g\notin p$.)

To prove the lemma it suffices to find for each $x \in S$ some $f \in L$ with $f(x) \neq 0$. For if this can be done one argues as follows: Start with an arbitrary $f_1 \in L$ and let $x_1 \in S$ be a zero of f_1 (if there is none, we're ready by (4)). Pick $g_1 \in L$ with $g_1(x_1) \neq 0$ and put $f_2 := f_1 + g_1 \in L$. By (4), f_2 has strictly less zeros on S than f_1. Iterating this step one arrives after finitely many steps at some $f_n \in L$ vanishing nowhere on S, since A is noetherian.

So let $x \in S$. If $x \in \operatorname{Sper} B$, then $f(x) > 0$ for each $f \in L$, so we may assume to the contrary that $\sigma x \not\subset p$ for every $p \in D'$. Let C be the semilocalization of A in σx and in each $p \in D'$ whith $p \not\subset \sigma x$. Using the canonical homomorphisms $\psi: C \to B$ and $\varphi_C: A \to C$ (satisfying $\psi \circ \varphi_C = \varphi_B$) we identify $\operatorname{Sper} B$ (and in particular X_B) with a topological subspace of $\operatorname{Sper} C$. Again $S \cap \operatorname{Sper} C$ and $U \cap \operatorname{Sper} C$ are clopen in $\operatorname{Sper} C$, and $S \cap X_C$, $U \cap X_C$ are subspaces of the space of orderings X_C.

It is enough to show $s_{U \cap X_C}(S \cap X_C) \leq m$. For, if $S \cap X_C = S_C(v_1, \ldots, v_m) \cap U \cap X_C$ with $v_i \in C^*$, then also $S \cap \operatorname{Sper} C = S_C(v_1, \ldots, v_m) \cap U$ by the same argument as in (3). Multiplying by squares of denominators we may assume $v_i = \varphi_C(f_i)$ with $f_i \in A$; since $\psi(v_i) = \varphi_B(f_i) \in B^*$ and $S \cap X_B = S_B(\varphi_B(f_1), \ldots, \varphi_B(f_m)) \cap U \cap X_B$, we have $f_i \in L$, and clearly $f_i(x) > 0$, $i = 1, \ldots, m$.

So it remains to show that the subspace $S \cap X_C$ of the space of orderings $U \cap X_C$ can be described by m inequalities. This can be checked by inspecting finite non-trivial fans of $U \cap X_C$, according to Lemma 2.1 which was a corollary to the Representation Theorem. Hence let $G \subset U \cap X_C$ be a (non-trivial) fan with $|G| = 2^k > 2^m$; we have to show $|S \cap G| \equiv 0 \bmod 2^{k-m}$. There is $q \in (\operatorname{Spec} A)_{re}$ such that $\sigma y = q$ for every $y \in G$ (where G is considered also to be contained in $\operatorname{Sper} A$), and G is also a fan in the space of orderings $X_{\kappa(q)} = \operatorname{Sper} \kappa(q)$ [K2, Theorem 7.4]. Of course $I \not\subset q$, hence we find $p \in \operatorname{Spec} B$ with $p \subset q$ such that A_q/pA_q is regular (1). By Corollary 2.3 there is a fan $F \subset X_{\kappa(p)}$ with $|F| = |G| = 2^k$ together with a bijective specialization map $s: F \to G$ satisfying $s(x) \in \overline{\{x\}}$ for $x \in F$. Regarding F as a subset of $U \cap \operatorname{Sper} B$ we can describe $S \cap F$ by m inequalities inside F because of (3). This means $|S \cap F| \equiv 0 \bmod 2^{k-m}$. But for $x \in F$ one has $x \in S \Leftrightarrow s(x) \in S$, since $S \cap \operatorname{Sper} C$ is clopen in $\operatorname{Sper} C$, and thus $|S \cap G| = |S \cap F|$.

This completes the proof of the lemma and of the theorem. □

References

[ABR] Andradas, C., Bröcker, L., Ruiz, J.M.: Minimal generation of basic open semianalytic sets. Invent. Math. **92**, 409–430 (1988)

[Be] Becker, E.: On the real spectrum of a ring and its applications to semialgebraic geometry. Bull. Am. Math. Soc. (N.S.) **15**, 19–60 (1986)

[BB] Becker, E., Bröcker, L.: On the description of the reduced Witt ring. J. Algebra **52**, 328–346 (1978)

[BCR] Bochnak, J., Coste, M., Roy, M.-F.: Géométrie Algébrique Réelle. Berlin-Heidelberg-New York: Springer 1987

[B1] Bröcker, L.: Zur Theorie der quadratischen Formen über formal reellen Körpern. Math. Ann. **210**, 233–256 (1974)

[B2] Bröcker, L.: Minimale Erzeugung von Positivbereichen. Geom. Dedicata **16**, 335–350 (1984)

[B3] Bröcker, L.: Spaces of orderings and semialgebraic sets. In: Canadian Mathematical Society Conference Proceedings, vol.4, 1984, pp. 231–248

[B4] Bröcker, L.: Description of semialgebraic sets by few polynomials. Lecture CIMPA, Nice, 1985

[B5] Bröcker, L.: Characterization of basic semialgebraic sets. Preprint 1987

[CR] Coste, M., Roy, M.-F.: La topologie du spectre réel. Contemp. Math. **8**, 27–59 (1982)

[EL] Elman, R., Lam, T.Y.: Quadratic forms over formally real fields and pythagorean fields. Am. J. Math. **94**, 1155–1194 (1972)

[EGA IV] Grothendieck, A.: Éléments de Géométrie Algébrique IV. Publ. Math., Inst. Hautes Etud. Sci. **20**, **24**, **28**, **32** (1964–1967)

[Ho] Hochster, M.: Prime ideal structure in commutative rings. Trans. Am. Math. Soc. **142**, 43–60 (1969)

[K1] Knebusch, M.: Specialization of quadratic and symmetric bilinear forms, and a norm theorem. Acta Arith. **24**, 279–299 (1973)

[K2] Knebusch, M.: On the local theory of signatures and reduced quadratic forms. Abh. Math. Sem. Univ. Hamburg **51**, 149–195 (1981)

[KS] Knebusch, M., Scheiderer, C.: Einführung in die reelle Algebra. Wiesbaden: Vieweg, (to appear)

[L1] Lam, T.Y.: Orderings, valuations and quadratic forms. Am. Math. Soc. Conference Board of the Mathematical Sciences **52**, (1983)

[L2] Lam, T.Y.: An introduction to real algebra. Rocky Mt. J. Math. **14**, 767–814 (1984)

[Mah1] Mahé, L.: Signatures et composantes connexes. Math. Ann. **260**, 191–210 (1982)

[Mah2] Mahé, L.: Théorème de Pfister pour les variétés et anneaux de Witt réduits. Invent. Math. **85**, 53–72 (1986)

[Mar] Marshall, M.: The Witt ring of a space of orderings. Trans. Am. Math. Soc. **258**, 505–521 (1980)

[Mat] Matsumura, H.: Commutative algebra. Second Edition. Reading, MA.: Benjamin 1980

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