ON THE ALEKSANDROV–FENCHEL INEQUALITY

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1. Introduction

For convex bodies $K_1, \ldots, K_n$ in n-dimensional Euclidean space, the Aleksandrov–Fenchel inequality says

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n), \quad (1.1)$$

where V denotes the mixed volume. Aleksandrov's [1, 2] proofs are reproduced in the books of Busemann [6] and Leichtweiss [15]. This inequality, special cases of which go back to Minkowski, has numerous applications to extremal and other problems in the geometry of convex bodies. In recent years new interest in this inequality has arisen from different sides. Surprising connections with algebraic geometry have been discovered, which have led to new proofs of (1.1) by Teissier [21] and Hovanski [11] (the proof of Fedotov [8], reproduced by Burago and Zalgaller in [5], seems to be incomplete). Also, some unusual applications of (1.1) or the closely related inequality for mixed discriminants were found, of which we mention Egorychev's and Falikman's proofs of the van der Waerden conjecture on permanents (see Lagarias [13] for references and further discussion) and Stanley's [20] results of a combinatorial nature.

Despite fresh interest in the Aleksandrov–Fenchel inequality and the new approaches to it, there is one major problem connected with it which has remained open for decades, namely, the characterization of the equality case. Without very restrictive assumptions on the bodies, it is not known when equality occurs in (1.1). None of the known proofs yields general information in this respect, since (1.1) is first proved for special convex bodies (very smooth bodies in Aleksandrov's second proof, and special polytopes in the other cases) and then generally by approximation, which blurs the cases of equality. Not even a plausible conjecture concerning equality in (1.1) has been reported in the literature. It is the purpose of the present paper to recall the known results in that direction, to formulate a conjecture on the general case of equality, in part due to Loritz, and to collect some partial results in favor of it. Maybe a published conjecture will stimulate further study of this question.

2. Preliminaries

By $\mathbb{R}^n$ we denote n-dimensional Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$, by B its unit ball with center at the origin 0, and by $\Omega = \partial B$ its unit sphere (∂ stands for the boundary). The volume of B is denoted by κ_n, and the usual Lebesgue measure on Ω by ω. $\mathcal{K}^n$ is the space of convex bodies (nonempty, compact, convex subsets) in $\mathbb{R}^n$. For mixed volumes and for some of the results used in the following, one may consult Bonnesen and Fenchel [4] and Leichtweiss [15]. $S(K_1,$

$\ldots, K_{n-1}; \cdot)$ is the mixed area fu
measure on Ω and it satisfies

$nV(K, K_2, \ldots, K$

for all $K \in \mathcal{K}^n$, where $h(K, \cdot)$ is the
For $K \in \mathcal{K}^n$ and $u \in \Omega$, let K^u
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volume of $n-1$ bodies $K_1^u, \ldots, K$
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abbreviation

$V(K_1, \ldots, K_r, \mathcal{C}$

and similarly for mixed area fun
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Now let $\mathcal{C}$ be a fixed ($n-2$
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Lemma 2.5. *Suppose that the*
dim $M = n$ *and*

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Then

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The following assumptions are eq
(a) *equality in* (2.4),
(b) *equality in* (2.7),
(c) $V(K, K, \mathcal{C})/V(L, L, \mathcal{C}) =$
(d) *the measures* $S(K, \mathcal{C}; \cdot)$

$\ldots, K_{n-1}; \cdot)$ is the mixed area function of $K_1, \ldots, K_{n-1}$; this is a positive Borel measure on Ω and it satisfies

$$nV(K, K_2, \ldots, K_n) = \int_\Omega h(K, u)\, dS(K_2, \ldots, K_n; u) \tag{2.1}$$

for all $K \in \mathscr{K}^n$, where $h(K, \cdot)$ is the support function of K.

For $K \in \mathscr{K}^n$ and $u \in \Omega$, let K^u be the image of K under orthogonal projection onto the hyperplane through 0 orthogonal to u. The $(n-1)$-dimensional mixed volume of $n-1$ bodies $K_1^u, \ldots, K_{n-1}^u$ in such a hyperplane is denoted by $v(K_1^u, \ldots, K_{n-1}^u)$. Since $v(K_1^u, \ldots, K_{n-1}^u) = nV(K_1, \ldots, K_{n-1}, U)$, where U is a segment of length one parallel to u (Bonnesen and Fenchel [4, p. 45]), **(2.1)** gives

$$v(K_1^u, \ldots, K_{n-1}^u) = \frac{1}{2}\int_\Omega |\langle u, v\rangle|\, dS(K_1, \ldots, K_{n-1}; v). \tag{2.2}$$

Integration of **(2.2)** over all u together with an application of Fubini's theorem results in the generalized Kubota formula

$$\int_\Omega v(K_1^u, \ldots, K_{n-1}^u)\, d\omega(u) = n\kappa_{n-1} V(K_1, \ldots, K_{n-1}, B). \tag{2.3}$$

If $\mathscr{C} = (K_{r+1}, \ldots, K_n)$ is an $(n-r)$-tuple of convex bodies, we will often use the abbreviation

$$V(K_1, \ldots, K_r, \mathscr{C}) := V(K_1, \ldots, K_r, K_{r+1}, \ldots, K_n).$$

and similarly for mixed area functions and in lower dimensions. We will also write $\mathscr{C}^u := (K_{r+1}^u, \ldots, K_n^u)$.

Now let $\mathscr{C}$ be a fixed $(n-2)$-tuple of convex bodies. We are interested in the convex bodies K, L for which equality holds in the inequality

$$V(K, L, \mathscr{C})^2 \geq V(K, K, \mathscr{C})V(L, L, \mathscr{C}). \tag{2.4}$$

We may exclude the case where one of the mixed volumes vanishes, since in that case the equality is easily discussed by means of a well-known criterion (Bonnesen and Fenchel [4, p. 41]). The following lemma collects useful information on the equality case in **(2.4)**.

LEMMA 2.5. *Suppose that the $(n-2)$-tuple $\mathscr{C}$ and the convex bodies K, L, M satisfy* $\dim M = n$ *and*

$$V(K, K, \mathscr{C}) > 0, \qquad V(L, L, \mathscr{C}) > 0. \tag{2.6}$$

Then

$$\frac{V(K, K, \mathscr{C})}{V(K, M, \mathscr{C})^2} - 2\frac{V(K, L, \mathscr{C})}{V(K, M, \mathscr{C})V(L, M, \mathscr{C})} + \frac{V(L, L, \mathscr{C})}{V(L, M, \mathscr{C})^2} \leq 0. \tag{2.7}$$

The following assumptions are equivalent:

(a) *equality in* **(2.4)**,
(b) *equality in* **(2.7)**,
(c) $V(K, K, \mathscr{C})/V(L, L, \mathscr{C}) = V(K, M, \mathscr{C})^2/V(L, M, \mathscr{C})^2$,
(d) *the measures* $S(K, \mathscr{C}; \cdot)$ *and* $S(L, \mathscr{C}; \cdot)$ *are proportional.*

Parts of this lemma go back to Favard [7] and Fenchel [9]; for the general case see Fenchel and Jessen [10, p. 25]; see also Leichtweiss [14; 15, Section 24]. A different proof of the implication (a) $\Rightarrow$ (d) is due to Aleksandrov [1].

3. Conjectures

Suppose now that equality holds in (**2.4**), where (**2.6**) is satisfied. Then Lemma 2.5 shows that, after a suitable dilatation of K or L,

$$S(K, \mathscr{C}; \cdot) = S(L, \mathscr{C}; \cdot). \tag{3.1}$$

Therefore, one has to find out what this equality implies for the convex bodies K and L. The first conjecture below, as well as the lemma following it (for the special case $L = B$), is due to Loritz [16]. Here supp μ denotes the support of the measure μ.

Conjecture 3.2. Equality (**3.1**) holds if, and under the assumption (**2.6**) only if, after a suitable translation of K or L,

$$h(K, u) = h(L, u) \qquad \text{for each } u \in \operatorname{supp} S(B, \mathscr{C}; \cdot). \tag{3.3}$$

According to Aleksandrov [2], Conjecture 3.2 is true if $\mathscr{C}$ consists only of bodies with analytic support functions and positive radii of curvature.

A convex body is called *regular* if at each of its boundary points it has only one supporting hyperplane.

Lemma 3.4. *If $L \in \mathscr{K}^n$ is regular and strictly convex, then*

$$\operatorname{supp} S(K, \mathscr{C}; \cdot) \subset \operatorname{supp} S(L, \mathscr{C}; \cdot)$$

for all $K \in \mathscr{K}^n$.

Proof. Suppose there exists $u_0 \in \operatorname{supp} S(K, \mathscr{C}; \cdot)\backslash \operatorname{supp} S(L, \mathscr{C}; \cdot)$. Then there is an open neighborhood α of u_0 in Ω such that $S(L, \mathscr{C}; \alpha) = 0$. Let

$$\tilde{L} := \{x \in L: \langle x, u_0 \rangle \leq h(L, u_0) - \varepsilon\},$$

where $\varepsilon > 0$ is so small that every exterior unit normal vector to L at a point of $\operatorname{cl}(\partial L \backslash \tilde{L})$ belongs to α. Such an ε exists since L is regular. For $u \in \Omega\backslash\alpha$, the unique boundary point of L at which u is attained as an exterior normal vector, is also a boundary point of $\tilde{L}$. We deduce that $S(L, \mathscr{C}; \beta) = S(\tilde{L}, \mathscr{C}; \beta)$ for every Borel set $\beta \subset \Omega\backslash\alpha$. This yields

$$\begin{aligned} 0 &\leq V(B, L, \mathscr{C}) - V(B, \tilde{L}, \mathscr{C}) \\ &= \int_\Omega h(B, u)\, dS(L, \mathscr{C}; u) - \int_\Omega h(B, u)\, dS(\tilde{L}, \mathscr{C}; u) \\ &= S(L, \mathscr{C}; \alpha) - S(\tilde{L}, \mathscr{C}; \alpha) \end{aligned}$$

and hence $S(\tilde{L}, \mathscr{C}; \alpha) = 0$. Thus we have $S(L, \mathscr{C}; \cdot) = S(\tilde{L}, \mathscr{C}; \cdot)$ and therefore $V(L, K, \mathscr{C}) = V(\tilde{L}, K, \mathscr{C})$ for any K by (**2.1**). On the other hand, we can choose

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$S(K, \mathscr{C}; \cdot)$, we have $S(K, \mathscr{C}; \beta) >$

$$nV(L, K, \mathscr{C}) = \int$$

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is not unique); call it $N(K, u)$. F
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$H_{n-1} \subset \mathbb{R}^n$ with $N(K_i, u) \subset H_i$
This notion generalizes that of a
a *p-extreme* normal vector of the
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Conjecture 3.5. $\operatorname{supp} S(K_1, \ldots$
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This is true at least in the foll
(a) if $K_1, \ldots, K_{n-1}$ are poly
(b) if $K_1 = \cdots = K_p$ and t
convex, for some $p \in \{0, \ldots, n -$
Suppose first that $K_1, \ldots, K_n$

$$F(K, u) :=$$

an open neighborhood β of u_0 such that $h(L, u) > h(\tilde{L}, u)$ for $u \in \beta$. Since $u_0 \in \operatorname{supp} S(K, \mathscr{C}; \cdot)$, we have $S(K, \mathscr{C}; \beta) > 0$, which implies

$$nV(L, K, \mathscr{C}) = \int_\Omega h(L, u)\, dS(K, \mathscr{C}; u)$$
$$> \int_\Omega h(\tilde{L}, u)\, dS(K, \mathscr{C}; u) = nV(\tilde{L}, K, \mathscr{C}),$$

a contradiction. This proves LEMMA 3.4. ∎

LEMMA 3.4 shows, in particular, that in **(3.3)** the unit ball B can be replaced by any other regular and strictly convex body, without changing the condition. It also shows that one direction of Conjecture 3.2 is true, namely, that (3.3) implies **(3.1)**: Suppose that **(3.3)** holds. Then for any convex body M we have $\operatorname{supp} S(M, \mathscr{C}; \cdot) \subset \operatorname{supp} S(B, \mathscr{C}; \cdot)$ and thus

$$\int_\Omega h(M, u)\, d[S(K, \mathscr{C}; u) - S(L, \mathscr{C}; u)]$$
$$= nV(M, K, \mathscr{C}) - nV(M, L, \mathscr{C})$$
$$= \int_\Omega [h(K, u) - h(L, u)]\, dS(M, \mathscr{C}; u) = 0.$$

Since M was arbitrary, **(3.1)** follows.

In order to interpret **(3.3)** geometrically, one would next require a geometric description of the support of the measure $S(B, \mathscr{C}; \cdot)$. For the formulation of a corresponding conjecture we need some more notation.

For $K \in \mathscr{R}^n$ and $u \in \Omega$, we consider a boundary point $x \in \partial K$ where u occurs as an exterior unit normal vector. Let N be the cone (with apex 0) of normal vectors to K at x. The vector u is contained in a unique relatively open face of the convex cone N (see, e.g., Theorem 18.2 of Rockafellar [17]). This maximal relatively open convex cone of normal vectors does not depend upon the choice of the point x (if this point is not unique); call it $N(K, u)$. For $K_1, \ldots, K_{n-1} \in \mathscr{R}^n$, we now say that the vector u is $(K_1, \ldots, K_{n-1})$-*extreme* if there exist $(n-1)$-dimensional linear subspaces $H_1, \ldots, H_{n-1} \subset \mathbb{R}^n$ with $N(K_i, u) \subset H_i$ for $i = 1, \ldots, n-1$ and $\dim H_1 \cap \ldots \cap H_{n-1} = 1$. This notion generalizes that of a p-extreme normal vector: The vector $u \in \Omega$ is called a *p-extreme* normal vector of the convex body K if $\dim N(K, u) \le p + 1$. Thus, u is a p-extreme normal vector of K if and only if u is

$$(\underbrace{K, \ldots, K}_{n-1-p}, \underbrace{B, \ldots, B}_{p})\text{-extreme.}$$

Conjecture 3.5. $\operatorname{supp} S(K_1, \ldots, K_{n-1}; \cdot)$ is the closure of the set of $(K_1, \ldots, K_{n-1})$-extreme unit vectors.

This is true at least in the following cases:

(a) if $K_1, \ldots, K_{n-1}$ are polytopes,

(b) if $K_1 = \cdots = K_p$ and the bodies $K_{p+1}, \ldots, K_{n-1}$ are regular and strictly convex, for some $p \in \{0, \ldots, n-1\}$.

Suppose first that $K_1, \ldots, K_{n-1}$ are polytopes. Writing

$$F(K, u) := \{x \in \mathbb{R}^n : \langle x, u \rangle = h(K, u)\}$$

for the face of K with exterior unit normal vector u, we have

$$S(K_1, \ldots, K_{n-1}; \{u\}) = v(F(K_1, u), \ldots, F(K_{n-1}, u)),$$

and $S(K_1, \ldots, K_{n-1}; \cdot)$ is concentrated on the finite set of those u for which this is positive. Hence, $u \in \operatorname{supp} S(K_1, \ldots, K_{n-1}; \cdot)$ if and only if segments $S_i \subset F(K_i, u)$, $i = 1, \ldots, n-1$, can be found with linearly independent directions (Bonnesen and Fenchel [4, p. 41]). Now $F(K_i, u)$ contains a (nondegenerate) segment S_i if and only if $N(K_i, u)$ is contained in the $(n-1)$-dimensional linear subspace orthogonal to S_i. The truth of Conjecture 3.5 for polytopes is now easy to see.

Now let $K \in \mathscr{K}^n$ and $p \in \{0, \ldots, n-1\}$ be given. Then

$$\operatorname{supp} S(\underbrace{K, \ldots, K}_{p}, \underbrace{B, \ldots, B}_{n-1-p}; \cdot)$$

is the closure of the set of $(n-p-1)$-extreme normal vectors of K, as shown by Schneider [18]. By the remark made above, this is also the closure of the set of

$$(\underbrace{K, \ldots, K}_{p}, \underbrace{B, \ldots, B}_{n-1-p})\text{-extreme}$$

vectors. If here the last $n-1-p$ arguments are replaced by any regular and strictly convex bodies, neither the set of $(K, \ldots, K, \cdot, \ldots, \cdot)$-extreme vectors nor, by LEMMA 3.4, the support of $S(K, \ldots, K, \cdot, \ldots, \cdot; \cdot)$ changes. Hence, Conjecture 3.5 is also true in case (b).

Putting LEMMA 2.5 and Conjectures 3.2 and 3.5 together, we now end up with the following conjecture. Here a supporting hyperplane of a convex body is called *$(B, \mathscr{C})$-extreme* if its exterior unit normal vector is $(B, \mathscr{C})$-extreme.

Conjecture 3.6. If $V(K, K, \mathscr{C}) > 0$ and $V(L, L, \mathscr{C}) > 0$, then equality in **(2.4)** holds if and only if suitable homothets of K and L have the same $(B, \mathscr{C})$-extreme supporting hyperplanes.

Under each of the following assumptions, it is known that equality in **(2.4)** [**(2.6)** being satisfied] implies that K and L are homothetic:

(a) $\mathscr{C}$ consists of balls,

(b) $\mathscr{C}$ consists of strongly combinatorially isomorphic simple polytopes, and K, L are polytopes having the same system of normal vectors to the facets as the polytopes of $\mathscr{C}$,

(c) $\mathscr{C}$ consists of convex bodies with twice continuously differentiable support functions and positive radii of curvature, and K and L have twice continuously differentiable support functions.

For (a), see Bonnesen and Fenchel [4, p. 93]; (b) and (c) are due to Aleksandrov [1, 2] (compare Busemann [6]). In cases (a) and (c), the set of $(B, \mathscr{C})$-extreme vectors coincides with Ω, while in case (b) it is the set of exterior unit normal vectors to the facets of the polytopes in $\mathscr{C}$. Thus, Conjecture 3.6 is in agreement with these classical cases.

For the case $\mathscr{C} = (K, \ldots, K)$, Bol [3] (see also Knothe [12] for a special case with a different proof) proved that equality in **(2.4)** implies that K is homothetic to a $(n-2)$-tangential body of L, which means that for a suitable homothet L' of L, each 1-extreme supporting hyperplane of K supports L'. Since the 1-extreme vectors of K are precisely the $(B, \mathscr{C})$-extreme vectors, Conjecture 3.6 holds also in this case.

4. Special Results

We shall now prove a few new results which confirm Conjecture 3.6 in additional special cases. Our first theorem implies a generalization of cases (a) and (c) above. The convex body $\tilde{M}$ is called a *summand* of the convex body M if $M = \tilde{M} + M'$ for a suitable convex body M'.

Theorem 4.1. *Let $\mathscr{C} = (K_3, \ldots, K_n)$ and suppose that K_i has an n-dimensional summand $\tilde{K}_i$ $(i = 3, \ldots, n)$. If* (**2.6**) *is satisfied and equality holds in* (**2.4**), *then also*

$$V(K, L, \tilde{\mathscr{C}})^2 = V(K, K, \tilde{\mathscr{C}})V(L, L, \tilde{\mathscr{C}})$$

with $\tilde{\mathscr{C}} = (\tilde{K}_3, \ldots, \tilde{K}_n)$.

Proof. If the assumptions are satisfied, then by Lemma 2.5 we may assume, after some dilatation of K or L, that

$$S(K, \mathscr{C}; \cdot) = S(L, \mathscr{C}; \cdot). \tag{4.2}$$

Write $\mathscr{C}' = (K_4, \ldots, K_n)$ (for $n = 2$ there is nothing to prove, and for $n = 3$ we modify the proof by just omitting $\mathscr{C}'$). Let Q be a convex body of dimension n. By (**2.7**) (replace K_3 by Q and M by K_3), we have

$$\frac{V(K, K, Q, \mathscr{C}')}{V(K, Q, \mathscr{C})^2} - 2\frac{V(K, L, Q, \mathscr{C}')}{V(K, Q, \mathscr{C})V(L, Q, \mathscr{C})} + \frac{V(L, L, Q, \mathscr{C}')}{V(L, Q, \mathscr{C})^2} \le 0. \tag{4.3}$$

Since

$$V(K, Q, \mathscr{C}) = V(L, Q, \mathscr{C}) \tag{4.4}$$

by (**2.1**) and (**4.2**), we get

$$V(K, K, Q, \mathscr{C}') - 2V(K, L, Q, \mathscr{C}') + V(L, L, Q, \mathscr{C}') \le 0. \tag{4.5}$$

By approximation, this inequality holds also if Q is a lower-dimensional convex body.

Now suppose that $K_3 = \tilde{K}_3 + K_3'$ with convex bodies $\tilde{K}_3$, K_3'. Writing down (**4.5**) for $Q = \tilde{K}_3$ and for $Q = K_3'$ and adding the two inequalities, we get

$$V(K, K, K_3, \mathscr{C}') - 2V(K, L, K_3, \mathscr{C}') + V(L, L, K_3, \mathscr{C}') \le 0.$$

By (**4.2**) the equality sign must hold here; hence, in particular,

$$V(K, K, \tilde{K}_3, \mathscr{C}') - 2V(K, L, \tilde{K}_3, \mathscr{C}') + V(L, L, \tilde{K}_3, \mathscr{C}') = 0.$$

Together with (**4.4**) for $Q = \tilde{K}_3$ this shows that (**2.7**), with K_3 replaced by $\tilde{K}_3$ and M by K_3, holds with equality; hence,

$$V(K, L, \tilde{K}_3, \mathscr{C}')^2 = V(K, K, \tilde{K}_3, \mathscr{C}')V(L, L, \tilde{K}_3, \mathscr{C}')$$

by Lemma 2.5, provided that $\dim \tilde{K}_3 = n$. Thus we have replaced K_3 by an n-dimensional summand and still have equality in (**2.4**). Repeating the argument with $K_4, \ldots, K_n$, we arrive at the conclusion of Theorem 4.1. ∎

Corollary 4.6. *Suppose that $\mathscr{C}$ consists only of outer parallel bodies (i.e., K_i has some ball as a summand, $i = 3, \ldots, n$). If* (**2.6**) *is satisfied and equality holds in* (**2.4**), *then K and L are homothetic.*

This follows from THEOREM 4.1 and result (a) mentioned after Conjecture 3.6. Thus, the corollary is a common generalization of the classical cases (a) and (c) after Conjecture 3.6, since it is well known that every convex body with a twice continuously differentiable support function and with positive radii of curvature has some ball as a summand. Since under the assumption of COROLLARY 4.6 every unit vector is $(B, K_3, \ldots, K_n)$-extreme, COROLLARY 4.6 is in agreement with Conjecture 3.6.

Another specialization of Conjecture 3.6 which should be tested is the case where each component of $\mathscr{C}$ is equal to either K or L. It was conjectured that equality in

$$V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i})^2 \geq V(\underbrace{K, \ldots, K}_{n-i+1}, \underbrace{L, \ldots, L}_{i-1}) V(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i+1}) \tag{4.7}$$

$(i = 1, \ldots, n-1)$ holds only if K is homothetic to an $(n-i-1)$-tangential body of L or L is homothetic to an $(i-1)$-tangential body of K (see Bonnesen and Fenchel [4, p. 92]). Recall that, for $i \in \{0, \ldots, n-1\}$, a convex body $K \in \mathscr{K}^n$ is called an $(n-i-1)$-*tangential body* of the convex body L if every i-extreme supporting hyperplane of K also supports L (for equivalent definitions, see Schneider [19]). For $i = 1$ and $i = n-1$ this was settled by Bol [3], but he conjectured [3, p. 56] that for $i \neq 1, n-1$ these are not the only cases of equality. With the further specialization $L = B$, (4.7) is the inequality

$$W_i(K)^2 \geq W_{i-1}(K) W_{i+1}(K) \tag{4.8}$$

$(i = 1, \ldots, n-1)$ for the quermassintegrals of K. For these we have the following result.

THEOREM 4.9. *If* $K \in \mathscr{K}^n$ *is centrally symmetric and* $i \in \{1, \ldots, n-1\}$, *then equality in* (4.8) *holds if and only if either* $\dim K < n - i$ *or* K *is an* $(n-i-1)$-*tangential body of a ball.*

It is very probable that the assumption of central symmetry, which is made for a technical reason, can be omitted. In that case our result would show that Conjecture 3.6 is true in the special case

$$\mathscr{C} = (\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i-1}).$$

The proof generalizes a method of Favard [7] (compare also Leichtweiss [14, 15]). The first part is stated as a separate lemma since it extends to a more general situation.

LEMMA 4.10. *Let* $\mathscr{C}$ *be an* $(n-3)$-*tuple of convex bodies in* $\mathbb{R}^n$ $(n \geq 3)$; *let* K, $L \in \mathscr{K}^n$ *satisfy*

$$V(K, K, B, \mathscr{C}) > 0, \qquad V(L, L, B, \mathscr{C}) > 0 \tag{4.11}$$

and

$$S(K, B, \mathscr{C}; \cdot) = S(L, B, \mathscr{C}; \cdot). \tag{4.12}$$

Then

$$v(K^u, L^u, \mathscr{C}^u)^2 = v(K^u, K^u, \mathscr{C}^u) v(L^u, L^u, \mathscr{C}^u) \tag{4.13}$$

for each $u \in \Omega$.

Proof. In the following, E_u denotes the $(n-1)$-dimensional linear subspace orthogonal to $u \in \Omega$. Let $u \in \Omega$. Inequality **(2.7)**, applied in E_u, shows that

$$\frac{v(K^u, K^u, \mathscr{C}^u)}{v(K^u, B^u, \mathscr{C}^u)^2} - 2\,\frac{v(K^u, L^u, \mathscr{C}^u)}{v(K^u, B^u, \mathscr{C}^u)v(L^u, B^u, \mathscr{C}^u)} + \frac{v(L^u, L^u, \mathscr{C}^u)}{v(L^u, B^u, \mathscr{C}^u)^2} \leq 0,$$

provided that $v(K^u, K^u, \mathscr{C}^u) > 0$ and $v(L^u, L^u, \mathscr{C}^u) > 0$. By **(2.3)** and **(4.11)**, this holds for ω-almost all u. From **(2.2)** and **(4.12)** we get

$$v(K^u, B^u, \mathscr{C}^u) = v(L^u, B^u, \mathscr{C}^u); \tag{4.14}$$

hence,

$$v(K^u, K^u, \mathscr{C}^u) - 2v(K^u, L^u, \mathscr{C}^u) + v(L^u, L^u, \mathscr{C}^u) \leq 0 \tag{4.15}$$

for almost all u and then by continuity for all $u \in \Omega$. In view of **(2.3)**, integration yields

$$V(K, K, B, \mathscr{C}) - 2V(K, L, B, \mathscr{C}) + V(L, L, B, \mathscr{C}) \leq 0.$$

By **(2.1)** and **(4.12)**, here the equality sign is valid; hence, equality holds in **(4.15)** for all $u \in \Omega$. Together with **(4.14)** this implies **(4.13)** by LEMMA 2.5. ∎

Proof of THEOREM 4.9. Since $W_i(K) > 0$ if and only if $\dim K \geq n - i$, we may assume that $\dim K \geq n - i + 1$. That equality in **(4.8)** holds for $(n-i-1)$-tangential bodies of balls, is known (Schneider [19, Theorem (3.8)]). We prove by induction with respect to n that equality in **(4.8)** and the assumption $\dim K \geq n - i + 1$ imply that K is an $(n-i-1)$-tangential body of a ball ($i \in \{1, \ldots, n-1\}$). For $n = 2$ this is well known. Assume that $n \geq 3$; the assertion has been proven in dimensions less than n, and equality holds in **(4.8)** for some $i \in \{1, \ldots, n-1\}$, where $\dim K \geq n - i + 1$. For $i = 1$, K is an $(n-2)$-tangential body of a ball by the result of Bol mentioned above; hence, we may assume that $i \geq 2$. According to LEMMA 2.5, after a suitable dilatation of K we may assume that

$$S(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i-1}; \cdot) = S(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i}; \cdot). \tag{4.16}$$

By LEMMA 4.10 this implies

$$v_{n-i}(K^u)^2 = v_{n-i+1}(K^u)v_{n-i-1}(K^u)$$

for $u \in \Omega$, where we have written

$$v_k(K^u) = v(\underbrace{K^u, \ldots, K^u}_{k}, \underbrace{B^u, \ldots, B^u}_{n-k-1}).$$

By the induction hypothesis, K^u is an $(n-i-1)$-tangential body of a homothet of B^u. If this homothet has radius r_u, then $r_u^{-1}K^u$ is an $(n-i-1)$-tangential body of a unit ball; hence (Schneider [19, Theorem (3.8)]),

$$v_{n-i}(r_u^{-1}K^u) = v_{n-i-1}(r_u^{-1}K^u).$$

Since **(2.2)** and **(4.16)** imply

$$v_{n-i}(K^u) = v_{n-i-1}(K^u),$$

we deduce that $r_u = 1$.

Let B_r be the maximal ball (r its radius) in K with center at the symmetry center of K. K and B_r have at least one common supporting hyperplane H, and by symmetry a second one parallel to H, say H'. Choose a unit vector u parallel to H. The intersection of H with E_u is a common supporting hyperplane (in E_u) of B_r^u and K^u and hence an extreme supporting hyperplane of K^u. The same holds true for the intersection of H' and E_u. Since both hyperplanes must support some ball (in E_u) of radius 1, it follows that $r = 1$. Hence, we may assume that $B_r = B$.

Now let $u \in \Omega$ be arbitrary. The projection K^u is an $(n - i - 1)$-tangential body of a unit ball, but since $B^u \subset K^u$, this is necessarily the ball B^u (observe that a convex body is the intersection of its supporting half spaces bounded by extreme supporting hyperplanes).

Finally, let H be an i-extreme supporting hyperplane of K and u its exterior unit normal vector. Choose a vector v orthogonal to u and, if dim $N(K, u) \geq 2$, in the linear hull of $N(K, u)$. The intersection of H with E_v is then an $(i - 1)$-extreme supporting hyperplane of K^v (relative to E_v). Since K^v is an $(n - i - 1)$-tangential body of B^v, that hyperplane supports B^v; hence, H supports B. Thus, K is an $(n - i - 1)$-tangential body of B. ∎

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