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ON THE PROJECTIONS OF A CONVEX POLYTOPE

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It is shown that in the class of all centrally symmetric convex bodies in E^d a polytope is uniquely determined, up to a translation, by its brightness (or certain similar functionals) in a suitable, though "arbitrarily small", set of directions.

It is well known that a centrally symmetric convex body (compact, convex set with interior points) in d-dimensional Euclidean space $E^d (d \geqq 3)$ is, up to a translation, uniquely determined by its brightness function. To formulate a more general result, let $S^{d-1} := \{x \in E^d : \|x\| = 1\}$ be the unit sphere in E^d; for a convex body $K \subset E^d$ and a unit vector $u \in S^{d-1}$ let $K(u)$ be the convex set that arises by orthogonal projection of K on to the $(d-1)$-dimensional linear subspace orthogonal to u. For $p \in \{0, 1, \cdots, d-2\}$ let $v_p(K, u)$ denote the p-th cross-section measure (Quermassintegral; for a definition see Bonnesen-Fenchel [2, p. 49], or Hadwiger [5, p. 209]) of dimension $d-1$ of the set $K(u)$. Thus, e.g., $v_0(K, u)$ is the brightness of K in the direction u, and $v_{d-2}(K, u)$ is, up to a factor depending only on d, the mean width of $K(u)$. The following theorem has been proved by A. D. Aleksandrov [1]:

If $K, \bar{K} \subset E^d$ are centrally symmetric convex bodies satisfying $v_p(K, u) = v_p(\bar{K}, u)$ for each $u \in S^{d-1}$ and for some $p \in \{0, 1, \cdots, d-2\}$, then $\bar{K}$ is a translate of K.

For another proof and a generalization see Chakerian [3].

One might ask whether in Aleksandrov's theorem it is really necessary to assume the equality $v_p(K, u) = v_p(\bar{K}, u)$ for the set of all directions u or whether some nondense subset thereof might suffice. The latter is, however, not true in general. In fact, given a centrally symmetric convex body $K \subset E^d$ with sufficiently smooth boundary and a symmetric subset $A \subset S^{d-1}$ which is not dense in S^{d-1}, there exists a centrally symmetric convex body $\bar{K} \subset E^d$, not a translate of K, which satisfies $v_0(K, u) = v_0(\bar{K}, u)$ for each $u \in A$. Examples to this effect have been constructed in [7, § 4]. The object of the present note is to exhibit a contrary situation: In case K is a centrally symmetric polytope, there exist sets $A \subset S^{d-1}$ of arbitrarily small (positive) measure such that the assumption

$$v_p(K, u) = v_p(\bar{K}, u) \quad \text{for each} \quad u \in A$$

forces the centrally symmetric convex body $\bar{K}$ to be a translate of K. More precisely, we shall prove the following

THEOREM. *Let $K \subset E^d$ be a centrally symmetric convex polytope. Let $p \in \{0, 1, \cdots, d-2\}$, and let $A \subset S^{d-1}$ be an open set which contains, corresponding to each $(d-1-p)$-dimensional face of K, a vector which is parallel to that face. If $\bar{K} \subset E^d$ is a centrally symmetric convex body which satisfies*

$$v_p(K, u) = v_p(\bar{K}, u) \quad \textit{for each} \quad u \in A\,,$$

then $\bar{K}$ is a translate of K.

For $p \leqq d-3$ there exist universal sets A with the properties demanded in the theorem. For instance, if A is a neighborhood of an "equator sphere" of S^{d-1}, then A contains, corresponding to any $(d-1-p)$-face F of any convex polytope, a vector which is parallel to F.

The following remarks are preparatory to the proof of the theorem. For a convex body $K \subset E^d$ let $\mu_p(K, \cdot)$, $p = 1, \cdots, d-1$, be its p-th surface area function; thus μ_p is a positive Borel measure on S^{d-1} which may be characterized by the fact that

$$(1) \qquad V(\bar{K}, \underbrace{K, \cdots, K}_{p}, \underbrace{B, \cdots, B}_{d-1-p}) = \frac{1}{d}\int_{S^{d-1}} \bar{h}(v)\mu_p(K, dv)$$

for every convex body $\bar{K} \subset E^d$ (see Fenchel-Jessen [4]); here the left side is a mixed volume, B is the ball bounded by S^{d-1}, and $\bar{h}$ is the support function of $\bar{K}$. As a special case of (1) we have the representation

$$(2) \qquad v_p(K, u) = \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle| \mu_{d-1-p}(K, dv)\,, \qquad u \in S^{d-1}\,.$$

For a convex polytope $P \subset E^d$ and $p \in \{1, \cdots, d-1\}$ let $\sigma_p(P) \subset S^{d-1}$ be the spherical image of the p-faces of P, thus, by definition, $u \in \sigma_p(P)$ if and only if the supporting hyperplane of P with exterior normal vector u contains a p-face of P. We assert that the measure $\mu_p(P, \cdot)$ is concentrated on $\sigma_p(P)$. In fact, if $\omega \in S^{d-1}$ is a Borel set having empty intersection with $\sigma_p(P)$, then $\mu_p(P, \omega) = 0$ as may be seen from the last formula of Fenchel-Jessen [4] and an easy estimate of the measure of the "brush set" corresponding to ω.

We shall need two lemmas concerning expressions of the type occurring in (2). Let μ be a positive Borel measure on S^{d-1} which is symmetric (i.e., attains the same value at antipodal sets). Then

$$H(x) := \int_{S^{d-1}} |\langle x, v\rangle| \mu(dv) \tag{3}$$

is, for $x \in E^d$, a (symmetric) convex function. Let $H'(x; y)$ for $y \in E^d \setminus \{0\}$ denote the directional derivative (see Bonnesen-Fenchel [2, p. 19]) of H at x in the direction y.

LEMMA 1. *If H is given by* (3) *with symmetric μ, then*

$$H'(x; y) = 2\int_{S_x} \langle y, v\rangle \mu(dv) + \int_{\omega_x} |\langle y, v\rangle| \mu(dv)$$

where

$$S_x := \{v \in S^{d-1} : \langle x, v\rangle > 0\} ,$$

$$\omega_x := \{v \in S^{d-1} : \langle x, v\rangle = 0\} .$$

For the easy computation, see [6, Lemma 6.1].

LEMMA 2. *If μ is a symmetric signed Borel measure on S^{d-1} which satisfies*

$$\int_{S^{d-1}} |\langle u, v\rangle| \mu(dv) = 0 \quad \textit{for each} \quad u \in S^{d-1} ,$$

then $\mu = 0$.

Essentially, this has been proved by Aleksandrov [1, §8]. In proving his theorem quoted in the introduction, he showed the assertion of Lemma 2 to be true in the case where μ is a difference of two $(d-1-p)$-th surface area functions of convex bodies; but this assumption is not needed in the proof. To be sure, this is not a special case, since from the well known existence theorem of Minkowski, Aleksandrov, and Fenchel-Jessen [4, p. 16], it follows that every symmetric Borel measure on S^{d-1} is the difference of the $(d-1)$-st surface area functions of two appropriate centrally symmetric convex bodies; hence Lemma 2 follows also directly from Aleksandrov's theorem cited earlier. For further references and a generalization of Lemma 2, see [6].

We proceed now to the proof of the theorem. It is convenient to write $d - 1 - p = q$. The assumptions of the theorem together with formula (2) give

$$\int_{S^{d-1}} |\langle u, v\rangle| \mu_q(K, dv) = \int_{S^{d-1}} |\langle u, v\rangle| \mu_q(\bar{K}, dv) \tag{4}$$

for each $u \in A$. Let F be a q-dimensional face of the polytope K. We have assumed that the set A contains a vector f which is parallel to F. Since A is an open set it contains a neighborhood of f. If equation (4) holds for a unit vector u, it holds also for every αu, $\alpha > 0$; thus there is an open set U of E^d containing f such that (4) holds for each $u \in U$. Therefore the convex functions which are defined by the left and the right side of (4), respectively, must have equal directional derivatives at f with respect to every direction y. Then Lemma 1 yields

$$\begin{aligned} 2\int_{S_f} \langle y, v\rangle \mu_q(K, dv) &+ \int_{\omega_f} |\langle y, v\rangle| \, \mu_q(K, dv) \\ &= 2\int_{S_f} \langle y, v\rangle \mu_q(\bar{K}, dv) + \int_{\omega_f} |\langle y, v\rangle| \, \mu_q(\bar{K}, dv) \end{aligned}$$

for each $y \in E^d$. If we replace y by $-y$ and add the resulting equation to the former one we see that

$$\int_{\omega_f} |\langle y, v\rangle| \, \mu_q(K, dv) = \int_{\omega_f} |\langle y, v\rangle| \, \mu_q(\bar{K}, dv) \,. \tag{5}$$

Since K and $\bar{K}$ are centrally symmetric, the measures $\mu_q(K, \cdot)$ and $\mu_q(\bar{K}, \cdot)$ are symmetric. We can now apply Lemma 2 with the dimension d replaced by $d-1$, with S^{d-1} replaced by ω_f, and with μ replaced by the restriction of $\mu_q(K, \cdot) - \mu_q(\bar{K}, \cdot)$ to ω_f. We deduce that

$$\mu_q(K, \omega \cap \omega_f) = \mu_q(\bar{K}, \omega \cap \omega_f) \tag{6}$$

for every Borel set ω of S^{d-1}. Now observe that the vector f has been chosen parallel to the q-face F. Thus every unit vector which is orthogonal to F is contained in ω_f, hence ω_f contains the spherical image of the face F. Therefore equation (6) is especially true if ω_f is replaced by the spherical image of F. Now F is an arbitrary q-face of K, hence the the additivity of the measures allows us to further replace the spherical image of F by the union of the spherical images of the q-faces of K:

$$\mu_q(K, \omega \cap \sigma_q(K)) = \mu_q(\bar{K}, \omega \cap \sigma_q(K)) \,. \tag{7}$$

It has already been noticed that the measure $\mu_q(K, \cdot)$ is concentrated on $\sigma_q(K)$, therefore to intersect ω with $\sigma_q(K)$ on the left side of (7) is indeed superfluous; we have

$$\mu_q(K, \omega) = \mu_q(\bar{K}, \omega \cap \sigma_q(K)) \tag{8}$$

for every Borel set ω on S^{d-1}. Write

$$\nu(\omega) := \mu_q(\bar{K}, \omega) - \mu_q(K, \omega) ,$$

then (8) gives

$$\nu(\omega) = \mu_q(\bar{K}, \omega \cap [S^{d-1} \setminus \sigma_q(K)])$$

so that ν is still a positive measure. Hence the function

$$H(x) := \int_{S^{d-1}} |\langle x, v\rangle| \, \nu(dv) ,$$

defined for $x \in E^d$, is the support function of a compact convex set C. By (4) we have $H(u) = 0$ for each $u \in A$, where A is an open set on S^{d-1}, and since H is even, we have also $H(u) = 0$ for each u in the set antipodal to A. Thus C cannot contain a point different from 0. This gives $H(x) = 0$ for each $x \in E^d$, and another application of Lemma 2 shows that ν, being symmetric, must vanish identically. We have proved that the convex bodies K and $\bar{K}$ have the same q-th surface area function, hence they differ at most by a translation (Aleksandrov [1], Fenchel-Jessen [4]).

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