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CLASSIFICATION OF LOCALLY TOROIDAL REGULAR POLYTOPES

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Abstract. A central problem in classical geometry is the classification of all regular polytopes and tessellations in spherical, euclidean or hyperbolic space. When asked within the theory of abstract regular polytopes, the classification problem must necessarily take a different form, because a priori an abstract polytope is not embedded into the geometry of an ambient space. The appropriate substitute now calls for the classification of abstract regular polytopes by their local or global topological type. The classical theory of regular polytopes is concerned with, and solves, the spherical case. In recent years, much work has been done on the toroidal case and a complete classification is now within reach.

Key words: Abstract regular polytopes, Toroidal polytopes, Reflection groups, Coxeter groups.

1. Introduction

Symmetry of geometric figures is a fascinating phenomenon which makes a powerful appearance in the classical theory of regular polytopes (Coxeter [9]). These figures have an outstanding history of study unmatched by almost any other geometric object. For a more detailed discussion on this history the reader is referred to the article by Peter McMullen in this volume.

In the past 15 years this area of classical geometry has been extended in several directions which are all centered around an abstract combinatorial polytope theory and a combinatorial notion of regularity. Abstract regular polytopes generalize the classical notion of a regular polytope and tessellation to more complicated combinatorial structures with a distinctive geometric and topological flavour. The notion of an abstract polytope was introduced in Grünbaum [18] and Danzer & Schulte [12], with a more systematic approach starting in [36]. For related concepts which occurred earlier or at about the same time, we also refer to Buekenhout [3], Dress [13], McMullen [21], Tits [46] and Vince [47]. See again the article by Peter McMullen for more details on the history of this subject.

A central problem in the classical theory is the complete classification of all regular polytopes and tessellations in spherical, euclidean or hyperbolic space. The solution to this problem is well-known and is closely related to the classification of Coxeter groups of spherical, euclidean or hyperbolic type; see Coxeter & Moser [10], Humphreys [19], or the article by Arjeh Cohen in this volume. When asked within the theory of abstract polytopes, the classification problem must necessarily take

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a different form, because a priori an abstract polytope is not embedded into the geometry of an ambient space. The appropriate substitute now calls for the classification of abstract regular polytopes by their local or global topological type. In the first place this requires to associate with abstract polytopes a natural topology, a problem which is very subtle and which in general cannot be uniquely solved. On the other hand, many polytopes admit a natural topology and so are subject to the classification with respect to this topology.

The classical theory of regular polytopes is concerned with, and solves, the spherical case. Convex polytopes and tessellations are locally spherical in the sense that their local building blocks (facets or tiles, and vertex-figures) are topologically spheres; and convex polytopes are also globally spherical ([17]). Using terminology introduced further below we can restate this by saying that the only universal abstract regular polytopes which are locally spherical are the classical regular tessellations in spherical, euclidean or hyperbolic space; among those, only the spherical regular tessellations (convex regular polytopes) are finite and globally spherical ([8, 9, 21, 14]).

A major concern is now to extend this classification to polytopes with more sophisticated topologies like that of arbitrary spherical, euclidean or hyperbolic space-forms (Wolf [52]). In this generality the classification problem is wide open, yet significant progress has been made in the case where the euclidean space-form is a torus (McMullen & Schulte [25, 27, 28, 29, 30]). In this paper we shall mainly discuss this toroidal case. For a clarification of what classification means in this context we refer to Section 3.

In the 70's, Grünbaum [18] triggered the theory of abstract polytopes by posing the challenging problem, as yet unsolved, of completely classifying the locally toroidal regular polytopes in each rank $n \geq 4$. As a first step this requires, for each rank n, the classification of the globally toroidal regular polytopes, the toroids; see Section 4. For rank 3 the toroids are the well-known regular (reflexible) maps on the 2-torus (Coxeter & Moser [10]). An abstract polytope of rank n is now called locally toroidal if its facets and vertex-figures are (globally) spherical or toroidal, with at least one kind toroidal. Locally toroidal regular polytopes can exist in ranks 4, 5 and 6 alone, because in higher ranks there are no suitable hyperbolic honeycombs to derive them from.

The situation is currently best understood in ranks 4 and 5. In rank 4, the classification involves analysis of the Schläfli types $\{4,4,r\}$ with $r = 3,4$, $\{6,3,p\}$ with $p = 3,4,5,6$, and $\{3,6,3\}$, and their duals. A complete classification is known for all types except $\{4,4,4\}$ and $\{3,6,3\}$ ([25, 27, 28]). For $\{4,4,4\}$ the classification is almost complete, and for $\{3,6,3\}$ partial results were obtained. The picture is particularly satisfactory for the types $\{6,3,p\}$ and the known cases of $\{3,6,3\}$ ([25]). Here the structure of the polytopes is governed by a complex hermitian form. In particular, the polytope is finite if and only if the corresponding form is positive definite. This generalizes the well-known classical situation where the structure of a regular convex polytope or regular tessellation is determined by a real quadratic form which determines the geometry of the ambient space; this correspondence sets up a beautiful link between geometry and algebra ([9]).

In rank 5, only one Schläfli type occurs, $\{3,4,3,4\}$ (and its dual). The locally toroidal polytopes of this type are completely classified ([29, 30]).

In rank 6, the types are $\{3,3,3,4,3\}$, $\{3,3,4,3,3\}$ and $\{3,4,3,3,3\}$, and their duals. There is a list of known finite polytopes which is conjectured to be complete ([29, 30]). In general, these polytopes are huge and wild, and so their classification is difficult. Honoring our old friends who used to inhabit our planet millions of years ago, we may wish to call them dinotopes. However, in contrast to those, they are still giving us a hard time.

On the group level, the classification of toroidal and locally toroidal polytopes amounts to the classification of certain groups which are defined in terms of generators and relations. These groups are quotients of euclidean or hyperbolic Coxeter groups and are obtained from those by either one or two extra relations. These extra relations force the toroidal structure upon the whole polytope or its facets or vertex-figures.

In contrast to regular polytopes, relatively little is known about chiral polytopes (Schulte & Weiss [39, 40], Monson & Weiss [33], Nostrand [34]). While regular polytopes have maximal symmetry with respect to (combinatorial) reflection, chiral polytopes are abstract polytopes with maximal rotational symmetry. Chirality of polytopes is a fascinating phenomenon which does not occur in the classical theory. In rank 3, the chiral polytopes are the irreflexible maps on surfaces ([10]); there are infinitely many such maps of genus 1 but for higher genus the occurrence is rather sporadic. For ranks $n \geq 4$ there are no chiral toroids, so the classification of locally toroidal chiral polytopes makes sense in rank 4 alone. However, here our knowledge is rather incomplete and is complicated by the fact that chiral polytopes occur in two enantiomorphic (mirror image) forms. The methods used for the construction of chiral polytopes of rank 4 all employ representations of hyperbolic rotation groups as projective linear groups over finite rings.

2. Basic Notions

In this section we give a brief introduction to the theory of abstract regular and chiral polytopes. For more details the reader is referred to [31, 39] or the article by Peter McMullen in this volume.

An *(abstract) polytope of rank n*, or simply an *n-polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function with range $\{-1,0,\dots,n\}$. The elements of rank i are called the *i-faces* of $\mathcal{P}$, or *vertices*, *edges* and *facets* of $\mathcal{P}$ if $i = 0, 1$ or $n-1$, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ all contain exactly $n+2$ faces, including the unique minimal face F_{-1} and unique maximal face F_n of $\mathcal{P}$. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$, which are such that Φ_{i-1} and Φ_i are *adjacent* (differ by just one face), and such that $\Phi \cap \Psi \subseteq \Phi_i$ for each i. Finally, if F and G are an $(i-1)$-face and an $(i+1)$-face with $F < G$, then there are exactly *two* i-faces H such that $F < H < G$.

If F and G are faces with $F \leq G$, we call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$. We can usually safely identify a face F with the section F/F_{-1}. For a face F, the section F_n/F is called the *coface of $\mathcal{P}$ at F*, or the *vertex-figure at F* if F is a vertex.

An abstract n-polytope $\mathcal{P}$ is *regular* if its (*combinatorial automorphism*) *group* $A(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \ldots, F_n\}$ be a fixed or *base* flag of $\mathcal{P}$. The group $A(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$ (*with respect to* Φ), where ρ_i is the unique automorphism which keeps all but the i-face of Φ fixed. These generators satisfy relations

$$(\rho_i\rho_j)^{p_{ij}} = \varepsilon \quad (i,j = 0,\ldots,n-1) \tag{1}$$

with

$$p_{ii} = 1\,,\; p_{ij} = p_{ji} \geq 2\;(i \neq j)\,, \tag{2}$$

and

$$p_{ij} = 2 \;\text{ if } |i-j| \geq 2\,. \tag{3}$$

Here the numbers $p_{i+1} := p_{i,i+1}$ determine the (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. Further, $A(\mathcal{P})$ has the *intersection property* (*with respect to the distinguished generators*), namely

$$\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle \;\text{ for all } I, J \subset \{0, \ldots, n-1\}\,. \tag{4}$$

By a *C-group* we mean a group which is generated by involutions such that (1), (2) and (4) hold. If in addition (3) holds, then the group is called a *string C-group.* The group of a regular polytope is a string C-group. Conversely, given a string C-group there is an associated regular polytope of which it is the automorphism group ([36]). Note that Coxeter groups are examples of C-groups ([19]).

Each string C-group is a quotient of the Coxeter group $[p_1, \ldots, p_{n-1}]$ with the string diagram

(5)

with the integers p_j defined as above. For any $p_1, \ldots, p_{n-1} \geq 2$, $[p_1, \ldots, p_{n-1}]$ is the group of the *universal* regular polytope $\{p_1, \ldots, p_{n-1}\}$ ([36, 46]). This polytope is denoted by $\{p_1, \ldots, p_{n-1}\}$ and covers any other regular polytope of type $\{p_1, \ldots, p_{n-1}\}$.

For a regular polytope $\mathcal{P}$ the rotations

$$\sigma_j := \rho_j\rho_{j-1} \quad (j = 1, \ldots, n-1)$$

generate the *rotation subgroup* $A^+(\mathcal{P})$ of $A(\mathcal{P})$, which is of index at most 2. These rotations σ_j fix all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permute consecutive j-faces of $\mathcal{P}$ in the section F_{j+1}/F_{j-2} of $\mathcal{P}$ of rank 2. A regular polytope $\mathcal{P}$ is called *directly regular* if $A^+(\mathcal{P})$ has index 2 in $A(\mathcal{P})$. For a regular polytope $\mathcal{P}$, direct regularity is equivalent to orientability of its *order complex* $\Delta(\mathcal{P})$, the simplicial complex whose simplices are given by the totally ordered subsets of $\mathcal{P}$ not containing F_{-1} and F_n ([45]). Note that for $\mathcal{P} = \{p_1, \ldots, p_{n-1}\}$ the rotation subgroup $A^+(\mathcal{P})$ is the even subgroup $[p_1, \ldots, p_{n-1}]^+$ of $[p_1, \ldots, p_{n-1}]$ ([10]).

Now let $\mathcal{P}$ be a polytope of rank $n \geq 3$. Then $\mathcal{P}$ is said to be *chiral* if $\mathcal{P}$ is not regular, but if for some base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$ there still exist

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3. The Classification

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automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathcal{P}$ such that σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permutes consecutive j-faces of $\mathcal{P}$ in the section F_{j+1}/F_{j-2} of rank 2. These automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ (when suitably oriented) are called the *distinguished generators* of $A(\mathcal{P})$. Then a polytope $\mathcal{P}$ is chiral if and only if its group $A(\mathcal{P})$ has precisely two orbits on the flags with adjacent flags belonging to different orbits.

Each chiral polytope occurs in two *enantiomorphic forms*, in a sense in a *right* and a *left* version. In terms of groups and generators, these can be represented by two distinct systems of generators for $A(\mathcal{P})$, $\{\sigma_1, \ldots, \sigma_{n-1}\}$ and $\{\sigma_1{}^{-1}, \sigma_2\sigma_1{}^2, \sigma_3, \ldots, \sigma_{n-1}\}$, belonging to Φ and its adjacent flag with another vertex, respectively. Note that for a directly regular polytope $\mathcal{P}$ the corresponding systems are equivalent under conjugation in $A(\mathcal{P})$ by the "reflection" ρ_0; that is, there is no distinction between a left and right version of $\mathcal{P}$ or, equivalently, the two enantiomorphic forms are the same. An *oriented chiral* or *oriented directly regular* polytope is a chiral or directly regular polytope together with a distinguished enantiomorphic form; in the chiral case there are two "orientations", in the directly regular case only one. We shall often drop the qualification "oriented" when confusion is not possible.

3. The Classification Problem

A main thrust in regular polytopes is the amalgamation of polytopes of lower rank. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two polytopes of rank n such that the vertex-figures of $\mathcal{P}_1$ are isomorphic to the facets of $\mathcal{P}_2$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are regular, we denote by $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ the *class* of all regular polytopes $\mathcal{P}$ of rank $n+1$ whose facets are isomorphic to $\mathcal{P}_1$ and whose vertex-figures are isomorphic to $\mathcal{P}_2$. Each non-empty class $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ contains a member, denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}$, which is *universal* in the sense that it covers any other polytope in the class $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ ([38]). By $[\mathcal{P}_1, \mathcal{P}_2]$ we denote the group of $\{\mathcal{P}_1, \mathcal{P}_2\}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are directly regular, then so is $\{\mathcal{P}_1, \mathcal{P}_2\}$. Note that there are examples where $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ is empty.

These universal polytopes are our main object of study. The following simple example illustrates some natural questions about these polytopes. Assume that we wish to construct a triangulated surface in which every vertex of the triangulation is contained in 5 triangles; that is, the vertex-figures are pentagons $\{5\}$. This can be done in only two ways both leading to finite triangulations. If the triangulation is "freely" generated , then the resulting surface is the 2-sphere and the triangulation is isomorphic to the icosahedron $\{3,5\}$. However, if additional identifications are allowed to be made, we can also construct the hemi-icosahedron $\{3,5\}/2$, the triangulation of the real projective plane obtained from $\{3,5\}$ by identifying antipodal points. In the above notation, $\{3,5\}$ and $\{3,5\}/2$ are members of $\langle\{3\},\{5\}\rangle$, and $\{3,5\} = \{\{3\},\{5\}\}$, the universal 3-polytope with triangular facets and pentagonal vertex-figures. The important point to make here is that this universal polytope is finite.

The picture changes completely if we require exactly 6 triangles around a vertex. Now there are many ways to generate triangulations including the maps $\{3,6\}_{(b,c)}$ on the torus (described in Section 4) and the (freely generated) triangular tessellation

$\{3,6\}$ in the euclidean plane. All these are members of $\langle\{3\},\{6\}\rangle$, and $\{3,6\} = \{\{3\},\{6\}\}$ which is now infinite.

These examples address the following problems about general universal polytopes $\{\mathcal{P}_1,\mathcal{P}_2\}$ for given regular n-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$.

- When is $\langle\mathcal{P}_1,\mathcal{P}_2\rangle \neq \emptyset$? Or, equivalently, when does $\{\mathcal{P}_1,\mathcal{P}_2\}$ exist?
- When is $\{\mathcal{P}_1,\mathcal{P}_2\}$ finite? (That is, when does it behave like a convex polytope, when like an infinite tessellation?)
- Identify the group $[\mathcal{P}_1,\mathcal{P}_2]$ of $\{\mathcal{P}_1,\mathcal{P}_2\}$. (That is, construct $\{\mathcal{P}_1,\mathcal{P}_2\}$ explicitly.)

In this paper, when we use the term "classification" of polytopes, then in the given context we mean the classification of all the *finite universal* polytopes.

Given $\mathcal{P}_1$ and $\mathcal{P}_2$ the search for the universal polytope in $\langle\mathcal{P}_1,\mathcal{P}_2\rangle$ involves analysis of the group A generated by involutions $\rho_0,\ldots,\rho_n$ subject to the relations dictated by $A(\mathcal{P}_1)$ (for $\rho_0,\ldots,\rho_{n-1}$) and $A(\mathcal{P}_2)$ (for $\rho_1,\ldots,\rho_n$) together with $(\rho_0\rho_n)^2 = \varepsilon$ ([38]). This group is a quotient of the free amalgamated product of $A(\mathcal{P}_1)$ and $A(\mathcal{P}_2)$ with amalgamation along their joint subgroup which is the group of the vertex-figure of $\mathcal{P}_1$ (and the facet of $\mathcal{P}_2$), the quotient being defined by the additional relation $(\rho_0\rho_n)^2 = \varepsilon$. Now, the universal polytope $\{\mathcal{P}_1,\mathcal{P}_2\}$ exists if and only if this group A has the intersection property (4) and its subgroups $\langle\rho_0,\ldots,\rho_{n-1}\rangle$ and $\langle\rho_1,\ldots,\rho_n\rangle$ are isomorphic to $A(\mathcal{P}_1)$ and $A(\mathcal{P}_2)$, respectively. It is usually difficult to verify these conditions.

It is easy to see that in rank 3 the universal polytopes $\{\mathcal{P}_1,\mathcal{P}_2\}$ are precisely the regular tessellations $\{p,q\}$ on the 2-sphere, in the euclidean plane or in the hyperbolic plane. However, in higher ranks the structure of abstract regular polytopes is far less obvious and is complicated by the lack of easily accessible non-classical examples. To give an example in rank 4, let $\mathcal{P}_1$ be the torus map $\{6,3\}_{(s,s)}$ and $\mathcal{P}_2$ the tetrahedron $\{3,3\}$. Then $\{\mathcal{P}_1,\mathcal{P}_2\} = \{\{6,3\}_{(s,s)},\{3,3\}\}$ is a 4-polytope with toroidal facets and spherical vertex-figures. Its group $[\{6,3\}_{(s,s)},\{3,3\}]$ has the presentation

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = \varepsilon\ ,$$

$$(\rho_0\rho_1)^6 = (\rho_1\rho_2)^3 = (\rho_2\rho_3)^3 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2 = \varepsilon\ ,$$

$$(\rho_0(\rho_1\rho_2)^2)^{2s} = \varepsilon\ .$$

The relations in the first two rows are the standard relations for the Coxeter group $[6,3,3]$, and the one extra relation in the third row corresponds to (7) below and causes the collapse of $\{6,3\}$ to the torus map $\{6,3\}_{(s,s)}$. In Section 6.2 we shall use hermitian forms to study these groups and the related universal polytopes $\{\{6,3\}_{(s,s)},\{3,3\}\}$.

For chiral polytopes the definition of classes is more subtle and involves taking care of the two enantiomorphic forms in which a polytope can occur. More precisely, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are oriented chiral or directly regular n-polytopes, then $\langle\mathcal{P}_1,\mathcal{P}_2\rangle^{ch}$ denotes the *class* of all oriented chiral $(n+1)$-polytopes $\mathcal{P}$ with (oriented) facets isomorphic to $\mathcal{P}_1$ and (oriented) vertex-figures isomorphic to $\mathcal{P}_2$. Again, if $\mathcal{P}_1$ or $\mathcal{P}_2$ is chiral and the class $\langle\mathcal{P}_1,\mathcal{P}_2\rangle^{ch}$ is non-empty, then it also contains a universal member denoted by $\{\mathcal{P}_1,\mathcal{P}_2\}^{ch}$. Note that if the orientations of both $\mathcal{P}_1$ and $\mathcal{P}_2$ get changed, then the orientations of all members in the class get changed; and hence that of $\{\mathcal{P}_1, \mathcal{P}_2\}^{ch}$. However, the classes seem to be unrelated if the orientations of only one polytope is changed ([40]).

An abstract n-polytope $\mathcal{P}$ is called *spherical* if it is isomorphic to the partially ordered set of faces of a spherical complex on the euclidean $(n-1)$-sphere $\mathbb{S}^{n-1}$ ([17]). If a spherical polytope $\mathcal{P}$ has a Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ (with $p_1, \ldots, p_{n-1} \geq 3$), then it is regular and is isomorphic to the face-lattice of a regular convex polytope ([21, 14]). In particular this rules out the existence of chiral spherical polytopes.

A *toroidal* polytope or, more briefly, a *toroid*, of rank $n+1$ is an abstract $(n+1)$-polytope which is the quotient of a periodic tessellation $\mathcal{T}$ of euclidean n-space $\mathbb{E}^n$ by a subgroup Λ of its translational symmetries generated by n independent translations; the resulting toroid is written $\mathcal{T}/\Lambda$ ([29]). For a regular (resp. chiral) toroid $\mathcal{P}$ we may also assume $\mathcal{T}$ to be regular and then Λ must be normal in the symmetry group $A(\mathcal{T})$ of $\mathcal{T}$ (resp. the rotation subgroup $A^+(\mathcal{T})$ of $A(\mathcal{T})$). We shall also refer to Λ as the *identification lattice* for $\mathcal{P}$.

For a classification of the regular and chiral toroids see Section 4. It would be interesting to extend this classification to polytopes on arbitrary spherical, euclidean or hyperbolic space-forms ([52]). In rank 3 this (essentially) amounts to the classification of regular and chiral maps on surfaces; in the orientable case such a classification is known up to genus 6 ([10, 44, 16]). For higher ranks see also Section 12.

Let $\mathcal{P}$ be an abstract polytope. We call $\mathcal{P}$ *locally spherical* if both its facets and vertex-figures are spherical. We say that $\mathcal{P}$ is *locally toroidal* if its facets and vertex-figures are spherical or toroidal, with at least one kind toroidal. Our use of the term "locally of some type" always refers to the sections of rank $n-1$ of the polytope. More general terminology may only require the minimal sections which are not spherical to be of the required topological type; see for instance ([23]).

In general it is a very subtle problem to define the global topology of an abstract polytope $\mathcal{P}$. Clearly, since $\mathcal{P}$ is a partially ordered set, its order complex $\Delta(\mathcal{P})$ is a simplicial complex which provides a topological space $|\Delta(\mathcal{P})|$ on which the full automorphism group $A(\mathcal{P})$ acts as a group of homeomorphisms ([45]). However, unless all facets and vertex-figures of $\mathcal{P}$ are spherical, this space distorts some of the topological features of $\mathcal{P}$ which we may wish to preserve.

For example, if $\mathcal{P}$ is a locally toroidal 4-polytope in $\langle\{6,3\}_{(s,s)}, \{3,3\}\rangle$, then in $|\Delta(\mathcal{P})|$ each facet is realized as a cone over the 2-torus but not as a solid torus as may be desirable. However, we can overcome this problem at the price of ambiguity. In fact, given $\mathcal{P}$ we can construct a closed real 3-manifold $\mathcal{M}$ and a decomposition $\mathcal{P}'$ of $\mathcal{M}$ into solid tori, each equipped with a map $\{6,3\}_{(s,s)}$ on its boundary, such that $\mathcal{P}'$ is isomorphic to $\mathcal{P}$. In a sense, $\mathcal{P}'$ is a combinatorially regular generalized Heegaard splitting of $\mathcal{M}$ of genus 1 ([41]). But as for ordinary Heegaard splittings of genus 1 (which involve only two tori), there are many different ways in which the solid tori can be glued together to give a manifold. (In fact, for two tori the resulting manifolds are known to be $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$ and the lens spaces.)

These examples illustrate some of the difficulties in defining the global topological type of an abstract polytope. For a more systematic approach to these problems as well as some classification results on the possible manifolds $\mathcal{M}$, the reader is referred to Brehm, Kühnel & Schulte [2]. Examples with $\mathcal{M} = \mathbb{S}^3$ were also discovered in Grünbaum [18], Coxeter & Shephard [11].

It is worth noting that our definition of spherical or toroidal polytopes avoids any of the problems just mentioned. In fact, by definition a spherical or toroidal polytope $\mathcal{P}$ has spherical facets and vertex-figures, and so the sphere or torus is its natural topological space (which also coincides with $|\Delta(\mathcal{P})|$). A more general notion of spherical or toroidal polytope $\mathcal{P}$ may also allow the sphere or torus to be decomposed into handlebodies which are then the facets of $\mathcal{P}$. This wider use of terminology adds on considerable complications and it may well be that the corresponding classification is then completely intractable. However, in this paper we shall not pursue these lines.

4. The Toroids

The regular and chiral toroids of rank 3 are well-known, and have been much discussed in the literature ([10]). They are the reflexible and irreflexible maps on the 2-torus and are of types $\{3,6\}$, $\{6,3\}$ and $\{4,4\}$. We begin with the first type.

Consider the euclidean plane tessellation $\mathcal{T} = \{3,6\}$. Its translation group is generated by translations τ_1, τ_2 along unit vectors x_1, x_2 inclined at $\pi/3$. If $A(\mathcal{T}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $A^+(\mathcal{T}) = \langle \sigma_1, \sigma_2 \rangle$, we can take

$$\tau_1 = (\rho_2\rho_1\rho_0)^2 = \sigma_2^2\sigma_1^{-1}, \quad \tau_2 = (\rho_0\rho_2\rho_1)^2 = \sigma_2\sigma_1^{-1}\sigma_2 . \tag{6}$$

For each pair $s = (b,c)$ of non-negative integers the fundamental region of the subgroup $\Lambda_s = \Lambda_{(b,c)} := \langle \tau_1^b\tau_2^c, \tau_1^{-b}\tau_2^{b+c} \rangle$ is a parallelogram with vertices (b,c), $(0,0)$, $(-c, b+c)$, $(b-c, b+2c)$ (with coordinates relative to x_1, x_2). We define $\{3,6\}_s = \{3,6\}_{(b,c)} := \mathcal{T}/\Lambda_s$, the quotient of $\mathcal{T}$ by Λ_s. If $(b,c) \neq (1,0), (0,1)$, this is a toroid of rank 3, which is regular if $bc(b-c) = 0$ and chiral otherwise; in the excluded cases the map on the torus is not a polytope in our sense. We give the details of the toroids in Table 1. The most important things we need subsequently are the numbers v of their vertices and f of their facets, and the orders g of their groups. In the regular case we usually write $s = (b,c)$ in the form $s = (s^k, 0^{2-k})$ with $s \geq 1$ and $k = 1$ or 2. In the chiral case the maps $\{3,6\}_{(b,c)}$ and $\{3,6\}_{(c,b)}$ are enantiomorphic.

We shall also write $[3,6]_s$ for the group of $\{3,6\}_s$, and $[3,6]_s^+$ for its rotation subgroup. Then we have

Theorem 1 (a) *For each $s = (s^k, 0^{2-k})$ with $s \geq 2, k = 1$ or $s \geq 1, k = 2$, the group $[3,6]_s$ of the regular toroid $\{3,6\}_s$ is the Coxeter group $[3,6] = \langle \rho_0, \rho_1, \rho_2 \rangle$, factored out by the relation*

$$\begin{cases} (\rho_0\rho_1\rho_2)^{2s} = \varepsilon & \text{if } k = 1, \\ (\rho_0(\rho_1\rho_2)^2)^{2s} = \varepsilon & \text{if } k = 2. \end{cases} \tag{7}$$

(b) *For each $s = (b,c)$ with $b, c \geq 0$ and $(b,c) \neq (0,0), (1,0), (0,1)$, the rotation subgroup $[3,6]_{(b,c)}^+$ of the (regular or chiral) toroid $\{3,6\}_{(b,c)}$ is the even subgroup $[3,6]^+ = \langle \sigma_1, \sigma_2 \rangle$ (defined by $\sigma_1^3 = \sigma_2^6 = (\sigma_1\sigma_2)^2 = \varepsilon$) of the Coxeter group $[3,6]$,*

s	v	f	g
$(s,0)$	s^2	$2s^2$	$12s^2$
(s,s)	$3s^2$	$6s^2$	$36s^2$

Table 1. The regular toroids $\{3,6\}_s$

factored out by the relations

$$(\sigma_2^2\sigma_1^{-1})^b(\sigma_2\sigma_1^{-1}\sigma_2)^c = \varepsilon. \qquad (8)$$

For Theorem 1 and similar situations below, note that the extra relations correspond to the defining translation, here in the direction of $s = (b,c)$; this translation and its conjugates in the group span the identification lattice.

The toroid $\{6,3\}_{(b,c)}$ is the dual of $\{3,6\}_{(b,c)}$. The corresponding presentations for $[6,3]_{(b,c)}$ and $[6,3]^+_{(b,c)}$ can be obtained by dualizing the above relations; that is, by replacing ρ_0, ρ_1, ρ_2 by ρ_2, ρ_1, ρ_0, and σ_1, σ_2 by $\sigma_2^{-1}, \sigma_1^{-1}$.

The toroids of type $\{4,4\}$ are constructed in a similar way from the euclidean plane tessellation $\mathcal{T} = \{4,4\}$ with vertex-set $\mathbb{Z}^2$, the set of points with integer cartesian coordinates. Now the translation group is generated by the unit translations τ_1, τ_2 along the cartesian axes. If $A(\mathcal{T}) = \langle\rho_0,\rho_1,\rho_2\rangle$ and $A^+(\mathcal{T}) = \langle\sigma_1,\sigma_2\rangle$, we can take

$$\tau_1 = \rho_0\rho_1\rho_2\rho_1 = \sigma_1^{-1}\sigma_2, \quad \tau_2 = \rho_2\rho_1\rho_0\rho_1 = \sigma_2\sigma_1^{-1}.$$

For each pair $s = (b,c)$ of non-negative integers we set $\{4,4\}_s = \{4,4\}_{(b,c)} := \mathcal{T}/\Lambda_s$, with $\Lambda_s = \Lambda_{(b,c)} := \langle\tau_1^b\tau_2^c, \tau_1^{-c}\tau_2^b\rangle$ whose fundamental region is the square with vertices (b,c), $(0,0)$, $(-c,b)$ and $(b-c, b+c)$. If $(b,c) \neq (0,0), (1,0), (0,1), (1,1)$ this is a toroid of rank 3, which is regular if $bc(b-c) = 0$ and chiral otherwise. In the chiral case, $\{4,4\}_{(b,c)}$ and $\{4,4\}_{(c,b)}$ are enantiomorphic.

The regular toroids $\{4,4\}_s$ with $s = (s^k, 0^{2-k})$ with $s \geq 2$, $k = 1,2$ are the first instance of a series of toroids $\{4,3^{n-2},4\}_s$ of rank $n+1$. These will be discussed further below.

Theorem 2 *For each $s = (b,c)$ with $b,c \geq 0$ and $(b,c) \neq (0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, the rotation subgroup $[4,4]^+_{(b,c)}$ of the (regular or chiral) toroid $\{4,4\}_{(b,c)}$ is the even subgroup $[4,4]^+ = \langle\sigma_1,\sigma_2\rangle$ (defined by $\sigma_1^4 = \sigma_2^4 = (\sigma_1\sigma_2)^2 = \varepsilon$) of the Coxeter group $[4,4]$, factored out by the relations*

$$(\sigma_1^{-1}\sigma_2)^b(\sigma_2\sigma_1^{-1})^c = \varepsilon. \qquad (9)$$

For a detailed discussion of the toroids of higher rank we refer to [29]. Here we recall some important facts. We begin with the following observation.

Theorem 3 *There are no chiral toroids of rank greater than 3.*

For notational reasons, in the remainder of this section we prefer to denote the rank by $n+1$. To construct a regular toroid of rank $n+1 \geq 4$, we must begin

s	v	f	g
$(s,0^{n-1})$	s^n	s^n	$(2s)^n\cdot n!$
$(s^2,0^{n-2})$	$2s^n$	$2s^n$	$2^{n+1}s^n\cdot n!$
(s^n)	$2^{n-1}s^n$	$2^{n-1}s^n$	$2^{2n-1}s^n\cdot n!$

Table 2. The regular toroids $\{4,3^{n-2},4\}_s$

with a regular honeycomb of $\mathbb{E}^n$. Except for $n=4$, the only such honeycomb is the tessellation $\{4,3^{n-2},4\}$ of $\mathbb{E}^n$ by cubes; here and below, r^k will be used to denote a string of k consecutive r's. In $\mathbb{E}^4$, there are two other regular honeycombs $\{3,3,4,3\}$ and $\{3,4,3,3\}$, which are duals.

We first consider the cubic tessellation $\{4,3^{n-2},4\}$ (with $n\geq 2$). Its vertex set may be taken to be $\mathbb{Z}^n$; this set can also be regarded as its translation group. Because we wish the resulting toroid to be regular, if the translation by $s\in\mathbb{Z}^n$ occurs in the identification lattice Λ, then so must all its conjugates under the group $[4,3^{n-2},4]$ of the honeycomb, or, what amounts to the same thing, under the group $[3^{n-2},4]$ of its vertex-figure, which consists of all permutations of the coordinates of vectors with all changes of signs. We shall write Λ_s for the translation group generated by $s:=(s^k,0^{n-k})$ and its images under permutation and changes of sign of coordinates, where $s\geq 1$ is an integer and $1\leq k\leq n$. We shall see that the only allowed values of k are $k=1,2$ or n. The regular polytope which results by this factorization is denoted by $\{4,3^{n-2},4\}_s:=\{4,3^{n-2},4\}/\Lambda_s$. In order that the corresponding group, which we write as $[4,3^{n-2},4]_s$, satisfy the intersection property, we must actually have $s\geq 2$, but otherwise there are no further restrictions; see Table 2.

Theorem 4 *For each $n\geq 2$, and $s=(s^k,0^{n-k})$ with $s\geq 2$ and $k=1,2$ or n, there is a (self-dual) regular toroid $\{4,3^{n-2},4\}_s$ of rank $n+1$. Its group $[4,3^{n-2},4]_s$ is the Coxeter group $[4,3^{n-2},4]=\langle\rho_0,\ldots,\rho_n\rangle$, factored out by the single extra relation*

$$(\rho_0\rho_1\cdots\rho_n\rho_{n-1}\cdots\rho_k)^{sk}=\varepsilon\,. \tag{10}$$

As we said above, the only other toroids are dual pairs derived from $\{3,3,4,3\}$ and $\{3,4,3,3\}$. We just consider the former. We may take the vertex set to be $\mathbb{Z}^4\cup(\mathbb{Z}^4+(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}))$, the set of points of $\mathbb{E}^4$ whose cartesian coordinates are all integers or all halves of odd integers. These points also correspond to the integer quaternions; in this context, the symmetry group $[3,3,4,3]$ consists of the mappings $z\mapsto q_1zq_2+h$ and $z\mapsto q_1\bar{z}q_2+h$, where q_1,q_2 are unit integer quaternions, h is an integer quaternion, and $\bar{z}$ is the (quaternion) conjugate of z ([15]). Much the same analysis as above applies, and, initially bearing only the vertices of $\{3,3,4,3\}$ in $\mathbb{Z}^4$ in mind, we conclude that the identification is by a vector $(s^k,0^{4-k})$ (and its images under permutation and changes of sign of cordinates) for some integer $s\geq 2$ and some $k=1,2$ or 4. However, taking the full group of symmetries of $\{3,3,4,3\}$ into account, we observe that (s^4) is equivalent to $(2s,0^3)$, and so the last case has already been counted. Using the same notation as for the cubic toroids, and denoting the dual by the same suffix, we thus obtain

s	v	f	g
$(s,0,0,0)$	s^4	$3s^4$	$1152s^4$
$(s,s,0,0)$	$4s^4$	$12s^4$	$4608s^4$

Table 3. The regular toroids $\{3,3,4,3\}_s$

Theorem 5 *For each $s=(s^k,0^{4-k})$ with $s\geq 2$ and $k=1$ or 2, there is a regular toroid $\{3,3,4,3\}_s$ (and its dual $\{3,4,3,3\}_s$) of rank 5. The group $[3,3,4,3]_s$ is the Coxeter group $[3,3,4,3]=\langle\rho_0,\ldots,\rho_4\rangle$, factored out by the extra relation*

$$\begin{cases}(\rho_0\sigma\tau\sigma)^s=\varepsilon & \text{if } k=1,\\ (\rho_0\sigma\tau)^{2s}=\varepsilon & \text{if } k=2,\end{cases}\tag{11}$$

where $\sigma:=\rho_1\rho_2\rho_3\rho_2\rho_1$, $\tau:=\rho_4\rho_3\rho_2\rho_3\rho_4$.

We list the details of these polytopes in Table 3. However, since the number of vertices of $\{3,3,4,3\}_s$ is the same as the number of facets of its dual $\{3,4,3,3\}_s$, and vice versa, we need only consider the former.

There are various quotient and subgroup relations between the groups of these toroids ([29]). The quotient relations arise from corresponding subgroup relations between the translation groups Λ_s. For $[4,3^{n-2},4]_s$ we have

$$\Lambda_{(2s,0^{n-1})}\leq\begin{Bmatrix}\Lambda_{(s^n)}\\ \Lambda_{(s^2,0^{n-2})}\end{Bmatrix}\leq\Lambda_{(s,0^{n-1})},$$

for all $s\geq 2$. If n is even, there is also the relation

$$\Lambda_{(s^n)}\leq\Lambda_{(s^2,0^{n-2})}.$$

Moreover, if p is an odd prime, we obviously have $\Lambda_{ps}\leq\Lambda_s$ for every s. It may be seen that every other subgroup relationship is a consequence of these. We deduce

Theorem 6 *Let $n\geq 3$. For each $s\geq 2$, there are coverings*

$$\{4,3^{n-2},4\}_{(2s,0^{n-1})}\searrow\begin{Bmatrix}\{4,3^{n-2},4\}_{(s^n)}\\ \{4,3^{n-2},4\}_{(s^2,0^{n-2})}\end{Bmatrix}\searrow\{4,3^{n-2},4\}_{(s,0^{n-1})}.$$

In addition, if n is even, there is a covering

$$\{4,3^{n-2},4\}_{(s^n)}\searrow\{4,3^{n-2},4\}_{(s^2,0^{n-2})}.$$

Lastly, for each $s=(s^k,0^{n-k})$ (with $s\geq 2$ and $k=1,2$ or n) and every odd prime p, there is a covering

$$\{4,3^{n-2},4\}_{ps}\searrow\{4,3^{n-2},4\}_s.$$

Exactly similar considerations apply to the polytopes of type $\{3,3,4,3\}$, and we obtain

Theorem 7 *Let $s \geq 2$. Then there are coverings*

$$\{3,3,4,3\}_{(2s,0,0,0)} \searrow \{3,3,4,3\}_{(s,s,0,0)} \searrow \{3,3,4,3\}_{(s,0,0,0)}\,.$$

Further, if p is an odd prime, there is a covering

$$\{3,3,4,3\}_{p\mathbf{s}} \searrow \{3,3,4,3\}_{\mathbf{s}}\,,$$

with $\mathbf{s} = (s,0,0,0)$ or $(s,s,0,0)$.

5. Hyperbolic Honeycombs

In preparation for our investigation of the locally toroidal regular polytopes, we now recall some facts about regular honeycombs in hyperbolic space $\mathbb{H}^n$ of dimension $n \geq 3$ ([8]). Since the facets and vertex-figures of a locally toroidal polytope are spherical or quotients of euclidean tessellations, the polytope itself must necessarily be a quotient of a hyperbolic honeycomb with spherical or euclidean facets or vertex-figures.

In $\mathbb{H}^3$, there are 15 regular honeycombs. The honeycombs $\{3,4,4\}$, $\{3,3,6\}$, $\{4,3,6\}$ and $\{5,3,6\}$ have spherical facets and have all their vertices on the absolute. Their duals, $\{4,4,3\}$, $\{6,3,3\}$, $\{6,3,4\}$ and $\{6,3,5\}$ have spherical vertex-figures and all their facets are inscribed in horospheres instead of finite spheres. The self-dual honeycombs $\{4,4,4\}$, $\{6,3,6\}$ and $\{3,6,3\}$ have both their vertices at infinity and their facets inscribed in horospheres. All these eleven types occur as Schläfli symbols of locally toroidal regular polytopes of rank 4. The remaining four honeycombs $\{3,5,3\}$, $\{4,3,5\}$, $\{5,3,4\}$ and $\{5,3,5\}$ are locally spherical and are (locally finite) tessellations in $\mathbb{H}^3$.

In $\mathbb{H}^4$, there are 7 regular honeycombs. Of those, only $\{3,4,3,4\}$ and its dual $\{4,3,4,3\}$ are not locally spherical and can occur as the type of some locally toroidal regular polytope of rank 5. The first has 24-cells as facets and its vertices are all on the absolute, and the second has 24-cells as vertex-figures and its facets are cubic tessellations inscribed into horospheres.

In $\mathbb{H}^5$, there are 5 regular honeycombs, all of which are not locally spherical and have euclidean tessellations as facets inscribed into horospheres or all their vertices at infinity. These are $\{3,3,3,4,3\}$, $\{4,3,3,4,3\}$, $\{3,3,4,3,3\}$, $\{3,4,3,3,4\}$ and $\{3,4,3,3,3\}$. Only the first has spherical facets (which are crosspolytopes), and only the last, the dual of the first, has spherical vertex-figures (which are cubes). These are the only types for locally toroidal regular polytopes of rank 6.

In $\mathbb{H}^n$ with $n \geq 6$, there are no regular honeycombs. As a consequence, locally toroidal regular polytopes can exist in ranks 4, 5 and 6 alone.

6. Polytopes of Rank 4

In constructing regular polytopes from groups, the following *twisting technique* has proved to be extremely useful ([24, 30]).

Let W be a group generated by k involutions $\sigma_1, \ldots, \sigma_k$; usually W is a C-group, for example, a Coxeter group or unitary reflection group. A *twisting operation* shall only be defined for those groups W which admit certain automorphisms τ permuting the generators σ_i. If these automorphisms τ are themselves involutions, we can augment W by their addition and in suitable cases obtain a new group A with certain distinguished generators $\rho_0, \ldots, \rho_{n-1}$. Writing B for the group of automorphisms of W generated by these τ, we have $A = W \ltimes B$, the semi-direct product of W by B. We shall write such a twisting operation as

$$(\sigma_1, \ldots, \sigma_k; \tau\text{'s}) \mapsto (\rho_0, \ldots, \rho_{n-1}) .$$

In applications, B may be of order 2 generated by just one involutory automorphism τ (as in (12)), or, in the other extreme case, may itself be the group of any regular polytope of higher rank suitably acting on W. In many examples the group W can actually be represented by a diagram and the automorphism τ can be realized by symmetries of this diagram.

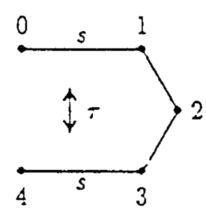
We now discuss the locally toroidal regular polytopes of rank 4 and begin with those of Schläfli type $\{4,4,r\}$ (or $\{r,4,4\}$) with $r = 3$ or 4.

6.1. Types $\{4,4,r\}$

The universal regular 4-polytopes

$$_1\mathcal{T}_s^4 := \{\{4,4\}_s, \{4,3\}\} ,$$

$s = (s^k, 0^{2-k})$ with $s \geq 2$ and $k = 1$ or 2, can be constructed directly from twisting operations on Coxeter groups. Since these are simple, we shall include them here. If $k = 1$, we can simply take the group $W = \langle\sigma_0, \ldots, \sigma_4\rangle$ with diagram



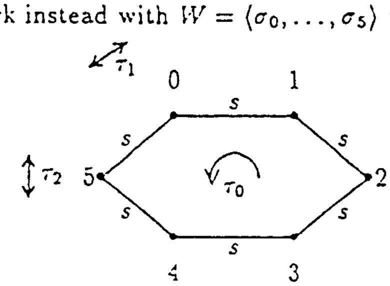
(12)

and apply the operation

$$(\sigma_0, \ldots, \sigma_4; \tau) \mapsto (\sigma_0, \tau, \sigma_3, \sigma_2) =: (\rho_0, \ldots, \rho_3). \tag{13}$$

It is straightforward to verify the defining relations for the corresponding group, which in turn implies the universality of the polytope.

If $k = 2$, we can work instead with $W = \langle\sigma_0, \ldots, \sigma_5\rangle$ with diagram



(14)

and use the operation

$$(\sigma_0, \ldots, \sigma_5; \tau_0, \tau_1, \tau_2) \mapsto (\tau_0, \sigma_2, \tau_1, \tau_2) =: (\rho_0, \ldots, \rho_3) .$$

This proves

Theorem 8 *The regular 4-polytope* ${}_1\mathcal{T}_s^4 = \{\{4,4\}_s, \{4,3\}\}$ *exists for all* $s = (s^k, 0^{2-k})$ *with* $s \geq 2$ *and* $k = 1,2$. *The only finite instances occur for* $s = (2,0), (3,0)$ *and* $(2,2)$, *with groups* $D_4 \ltimes S_4$ *of order* 192, $S_6 \times C_2$ *of order* 1440, *and* $C_2 \wr D_6$ *(wreath product) of order* 768, *respectively.*

The classification of the universal regular 4-polytopes

$${}_2\mathcal{T}_{s,t}^4 := \{\{4,4\}_s, \{4,4\}_t\} ,$$

$s = (s^k, 0^{2-k}), t = (t^l, 0^{2-l})$ with $s,t \geq 2$ and $k,l = 1$ or 2, is more difficult and requires more sophisticated twisting operations ([27]). The classification is complete except when $k = l = 1$ and s,t are odd and distinct.

Theorem 9 *The regular 4-polytope* ${}_2\mathcal{T}_{s,(t,t)}^4 = \{\{4,4\}_s, \{4,4\}_{(t,t)}\}$ *exists for all* $s = (s^k, 0^{2-k})$ *with* $s \geq 2$ *and* $k = 1,2$ *and all* $t \geq 2$. *The only finite instances occur for:* $s = (2,0), t \geq 2$; $s = (3,0), t = 2$; *and* $s = (2,2), t = 2$ *or* 3. *The corresponding groups are:* $(D_t \times D_t \times C_2 \times C_2) \ltimes (C_2 \times C_2)$ *of order* $64t^2$; $(S_4 \times S_4) \ltimes (C_2 \times C_2)$ *of order* 2304; $C_2^4 \ltimes [4,4]_{(2,2)}$ *of order* 1024; *and* $C_2^6 \ltimes [4,4]_{(3,3)}$ *of order* 9216, *respectively.*

Theorem 10 *Let* $2 \leq s \leq t$, *and let* s,t *not be both odd and distinct. Then the regular 4-polytope* ${}_2\mathcal{T}_{(s,0),(t,0)}^4 = \{\{4,4\}_{(s,0)}, \{4,4\}_{(t,0)}\}$ *exists except when* $s = 2$ *and* t *is odd. The only finite instances occur for* $s = 2$ *and* $t = 2m$ *even, and* $(s,t) = (3,3)$ *or* $(3,4)$. *The corresponding groups are* $(D_m \times D_m) \ltimes [4,4]_{(2,0)}$ *of order* $128m^2$ *(with* $D_1 = C_2$ *if* $m = 1$*),* $S_6 \times C_2$ *of order* 1440, *and* $C_2 \wr [4,4]_{(3,0)}$ *of order* 36864, *respectively.*

In the exceptional case when s,t are odd and distinct, the cut method of Section 7 below supports the following

Conjecture 1 *Let* $s,t \geq 3$ *be odd and distinct. Then the regular 4-polytope* ${}_2\mathcal{T}_{(s,0),(t,0)}^4 = \{\{4,4\}_{(s,0)}, \{4,4\}_{(t,0)}\}$ *exists and is infinite if* $(s,t) \neq (3,5), (5,3)$.

Note that an application of the Coxeter-Todd coset enumeration algorithm suggests that the polytope is also infinite in the two cases excluded in the conjecture (even though the corresponding cuts are finite).

6.2. Types $\{6,3,p\}$

In this section we classify the universal regular 4-polytopes

$${}_p\mathcal{T}_s^4 = \{\{6,3\}_s, \{3,p\}\}$$

with $p = 3,4,5$ and $s = (s^k$

with $s = (s^k, 0^{2-k}), t = (t^l,$
that the left suffix (3, 4, 5
Schläfli symbol. We write
[6, 3, 6] which is defined by
Then, if ${}_p\mathcal{T}_s^4$ and ${}_6\mathcal{T}_{s,t}^4$ exis
with the notation as in Se
In classifying these poly
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2. Construct a "locally u
$f : W \mapsto GL_m(\mathbb{C})$ (say
This representation f
3. Use ⟨ , ⟩ to determine
The construction of W a
flection groups (Shephard
Consider the group [1
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$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 =$$

This group can be repres

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rightmost extra relation.
act as indicated by the
both simple examples of

$$(\sigma_1,$$

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$$(\sigma_1, \sigma$$

gives $[6,3]_{(s,s)} \cong [1\ 1\ 1]'$
Geometrically the ger
be the canonical base of
by

with $p = 3, 4, 5$ and $s = (s^k, 0^{2-k})$, with $s \geq 2$ if $k = 1$ and $s \geq 1$ if $k = 2$, as well as

$$_6\mathcal{T}^4_{s,t} = \{\{6,3\}_s, \{3,6\}_t\}$$

with $s = (s^k, 0^{2-k}), t = (t^l, 0^{2-l})$, with $s, t \geq 2$ if $k, l = 1$ and $s, t \geq 1$ if $k, l = 2$. Note that the left suffix (3, 4, 5 or 6) in our notation is the same as the last entry in the Schläfli symbol. We write $_pA^4_s$ and $_6A^4_{s,t}$ for the quotient $\langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ of $[6, 3, p]$ or $[6, 3, 6]$ which is defined by the extra relations for $[6, 3]_s$ and $[3, 6]_t$, see Theorem 1. Then, if $_p\mathcal{T}^4_s$ and $_6\mathcal{T}^4_{s,t}$ exist, then $_pA^4_s = [\{6,3\}_s, \{3,p\}]$ and $_6A^4_{s,t} = [\{6,3\}_s, \{3,6\}_t]$, with the notation as in Section 3.

In classifying these polytopes $\mathcal{P}$ the following strategy proved to be successful.

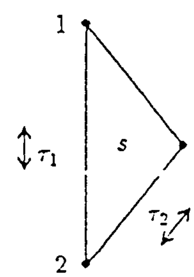
1. Find a "suitable" normal subgroup W of A $(= {_pA^4_s}, {_6A^4_{s,t}})$ of finite index.
2. Construct a "locally unitary" representation of W over the complex numbers $\mathbb{C}$, $f : W \mapsto GL_m(\mathbb{C})$ (say) with m determined by the vertex-figure $\{3, p\}$ or $\{3, 6\}_t$. This representation f will support a hermitian form $\langle \, , \, \rangle$ on $\mathbb{C}^m$.
3. Use $\langle \, , \, \rangle$ to determine the structure of $\mathcal{P}$ and A.

The construction of W and f is based on the following observation on unitary reflection groups (Shephard & Todd [43], Coxeter [7], Cohen [4]).

Consider the group $[1\ 1\ 1]^s$ $(s \geq 2)$ which is generated by involutions $\sigma_1, \sigma_2, \sigma_3$ and abstractly defined by the presentation

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = (\sigma_1\sigma_3)^3 = (\sigma_1\sigma_2\sigma_3\sigma_2)^s = \varepsilon. \qquad (15)$$

This group can be represented by a triangular diagram



(16)

(the underlying Coxeter diagram), with a mark s inside the triangle to indicate the rightmost extra relation. Now, using the two group automorphisms τ_1 and τ_2 which act as indicated by the diagram symmetries, we can extend $[1\ 1\ 1]^s$ in two ways, both simple examples of twisting operations. First, the operation

$$(\sigma_1, \sigma_2, \sigma_3; \tau_1) \mapsto (\tau_1, \sigma_2, \sigma_3) =: (\rho_0, \rho_1, \rho_2)$$

recognizes the group $[6, 3]_{(s,0)}$ as $[1\ 1\ 1]^s \ltimes C_2$. And second,

$$(\sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2) \mapsto (\sigma_1, \tau_1, \tau_2) =: (\rho_0, \rho_1, \rho_2)$$

gives $[6, 3]_{(s,s)} \cong [1\ 1\ 1]^s \ltimes S_3$.

Geometrically the generators σ_i can be described as follows ([7, 25]). Let e_1, e_2, e_3 be the canonical base of complex 3-space $\mathbb{C}^3$. Define the linear mapping $S_i : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$zS_i = z - 2\langle z, e_i \rangle e_i \quad (z \in \mathbb{C}^3), \qquad (17)$$

where $\langle\ ,\ \rangle$ is a hermitian form on $\mathbb{C}^3$ defined by

$$\langle z, y\rangle = \sum_{i=1}^{3} z_i \overline{y_i} \ - \ \tfrac{1}{2} \sum_{i,j=1, i\neq j}^{3} c_{ij} z_i \overline{y_j}\ . \tag{18}$$

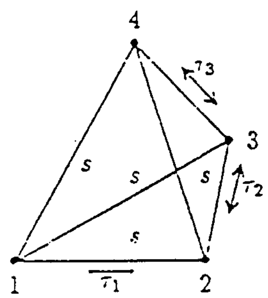
There are several choices for the coefficients c_{ij} each of which gives a positive definite form $\langle\ ,\ \rangle$ (defining a unitary geometry) such that $\sigma_i \mapsto S_i$ $(i = 1, 2, 3)$ defines a unitary representation of $[1\ 1\ 1]^s$. Write $\gamma_{123} := c_{12}c_{23}c_{31}$ and $c_s := e^{2\pi i/s}$. Then one such choice requires that both each c_{ij} and γ_{123} are equal to c_s or $\overline{c_s}$. Note that this is a symmetrical version of the choice in [7].

For each of the groups ${}_pA_s^4$ and ${}_6A_{s,t}^4$ it is now possible to identify the group W and representation f. In each case the choice depends on the parameters s, p and t. We shall illustrate the method by an example rather than discussing the construction in full generality.

Consider the group ${}_3A_{(s,s)}^4$ of ${}_3\mathcal{T}_{(s,s)}^4 = \{\{6,3\}_{(s,s)}, \{3,3\}\}$. Then the vertex-figure is a tetrahedron $\{3,3\}$ and has 4 vertices. Take the group $W = W_{(s,s)}$ with 4 generators $\sigma_1, \ldots, \sigma_4$ abstractly defined by

$$\sigma_i^2 = (\sigma_i\sigma_j)^3 = (\sigma_i\sigma_j\sigma_k\sigma_j)^s = \varepsilon \quad (1 \le i, j, k \le 4;\ \text{distinct}). \tag{19}$$

This group can be represented by the tetrahedral diagram



(20)

in which each triangular 2-face is marked by s. (The number of generators is 4 because the vertex-figure has 4 vertices, not because the rank of the locally toroidal polytope is 4. If the vertex-figure is an icosahedron $\{3,5\}$, then there are 12 generators and the hermitian form has 12 variables.) Now W admits three group automorphisms τ_1, τ_2, τ_3 each represented by a transposition. Adjoining these to W and using the twisting operation

$$(\sigma_1, \ldots, \sigma_4; \tau_1, \tau_2, \tau_3) \mapsto (\sigma_1, \tau_1, \tau_2, \tau_3) =: (\rho_0, \ldots, \rho_3) \tag{21}$$

we can now recognize ${}_3A_{(s,s)}^4$ as $W_{(s,s)} \ltimes S_4$; in fact, the defining relations for the two groups correspond to each other.

Next we construct a complex representation $f : W_{(s,s)} \mapsto GL_4(\mathbb{C})$ which supports a hermitian form $\langle\ ,\ \rangle$ on $\mathbb{C}^4$. We define S_i as in (17) (with $\mathbb{C}^3$ replaced $\mathbb{C}^4$) and $\langle z, y\rangle$ as in (18) (with 3 replaced by 4). Writing $\gamma_{ijk} := c_{ij}c_{jk}c_{ki}$ $(i, j, k$ distinct) we impose the condition that each c_{ij} and each γ_{ijk} is equal to c_s or $\overline{c_s}$. (In the case of an arbitrary vertex-figure modifications to this rule are needed for index sets $\{i, j\}$

or $\{i,j,k\}$ which are non-edges or non-faces of the vertex-figure.) For instance, we can take

$$c_{12} = c_{34} = c_{31} = c_s\,,\ c_{23} = c_{24} = c_{41} = \overline{c_s}\,.$$

The condition on the c_{ij}'s and γ_{ijk}'s implies that any restriction of $\langle\ ,\ \rangle$ to 3 variables is a positive definite form; that is, h is *locally unitary*. In particular,

$$\langle \sigma_i, \sigma_j, \sigma_k\rangle \cong \langle S_i, S_j, S_k\rangle \cong [1\,1\,1]^s$$

for all $\{i,j,k\}$, and $\langle S_1,\ldots,S_4\rangle$ acts irreducibly on $\mathbb{C}^4$.

We can now decide which groups $W_{(s,s)}$ are finite. If h is a non-degenerate and indefinite form, then $\langle S_1,\ldots,S_4\rangle$ acts irreducibly on $\mathbb{C}^4$ and is infinite; it follows that $W_{(s,s)}$ must also be infinite. If h is positive definite, then $\langle S_1,\ldots,S_4\rangle$ is a finite unitary reflection group and the representation f is faithful. (In the general case it is not known if f is always faithful.) It follows that $W_{(s,s)}$ is finite if and only if h is positive definite. The same is now also true for ${}_3A^4_{(s,s)}$ and its polytope ${}_3\mathcal{T}^4_{(s,s)}$. But

$$det(h) = (-9 - 16\cos(2\pi/s) - 2\cos(4\pi/s))/16\,,$$

so that positive definiteness occurs exactly for $s = 2$; in particular, $W_{(2,2)} \cong S_5$ and ${}_3A^4_{(2,2)} \cong S_5 \times S_4$, so we actually have real groups here.

In a similar fashion we can classify (almost) all universal locally toroidal regular 4-polytopes of types $\{6,3,p\}$ with $p = 3,4,5,6$. We now summarize the results. For notation for unitary reflection groups we refer to [7, 25].

Theorem 11 *The regular 4-polytope* ${}_3\mathcal{T}^4_s = \{\{6,3\}_s, \{3,3\}\}$ *exists for all* $s = (s^k, 0^{2-k})$ *with* $s \geq 2$ *and* $k = 1,2$ *(but not for* $s = 1, k = 2$*). The only finite instances occur for* $s = (2,0)$, $(3,0)$, $(4,0)$ *and* $(2,2)$. *In the first case, its group is* $S_5 \times C_2$ *of order* 240, *and in the last case it is* $S_5 \times S_4$ *of order* 2880. *In the second and third case (where* $s = 3,4$*), the group is* $[1\,1\,2]^s \ltimes C_2$, *of order* 1296 *or* 25360 *respectively; here* $[1\,1\,2]^s$ *is the finite unitary reflection group in* $\mathbb{C}^4$ *whose diagram is obtained by attaching at vertex* 3 *of* (16) *a tail consisting of one unmarked branch.*

Note that Theorem 11 confirms a conjecture of Grünbaum [18] on the finiteness of these polytopes (which he denoted by $\mathcal{H}_{(s,0)}$ and $\mathcal{H}_{(s,s)}$, respectively). See also Altshuler [1] for the construction of 3-dimensional simplicial complexes whose vertex links are preassigned torus maps; the duals of the corresponding face lattices are abstract 4-polytopes with toroidal facets and simplicial vertex-figures; however, in general, these are neither regular or chiral.

Theorem 12 *The regular 4-polytope* ${}_4\mathcal{T}^4_s = \{\{6,3\}_s, \{3,4\}\}$ *exists for all* $s = (s^k, 0^{2-k})$, *with* $s \geq 2$ *if* $k = 1$ *and* $s \geq 1$ *if* $k = 2$. *The only finite instances occur for* $s = (1,1)$ *and* $(2,0)$, *with groups* $S_3 \ltimes [3,4]$ *of order* 288 *and* $[3,3,4] \ltimes C_2$ *of order* 768, *respectively.*

Theorem 13 *The regular 4-polytope* ${}_5\mathcal{T}^4_s = \{\{6,3\}_s, \{3,5\}\}$ *exists for all* $s = (s^k, 0^{2-k})$ *with* $s \geq 2$ *and* $k = 1,2$ *(but not for* $s = 1, k = 2$*). The only finite instance occurs for* $s = (2,0)$, *in which case the group is* $[3,3,5] \ltimes C_2$ *of order* 28800.

Theorem 14 *(a) The regular 4-polytope* $_6\mathcal{T}^4_{(s,s),t} = \{\{6,3\}_{(s,s)}, \{3,6\}_t\}$ *exists for all* $s \geq 1$ *and all* $t = (t^l, 0^{2-l})$, *except when* $s = l = 1$ *and* $3 \not| t$. *The only finite instances occur for* $s = 1$, *or* $s = 2$ *and* $t = (2,0)$. *In the first case the group is* $S_3 \ltimes [3,6]_t$ *which is of order* $72t^2$ *if* $l = 1$ *or* $216t^2$ *if* $l = 2$, *and in the second case the group is* $S_5 \times S_4 \times C_2$ *of order* 5760.
(b) The regular 4-polytope $_6\mathcal{T}^4_{(s,0),(t,0)} = \{\{6,3\}_{(s,0)}, \{3,6\}_{(t,0)}\}$ *exists for all* s, t *with* $s, t \geq 2$. *The only finite instances occur for* $t = 2 \leq s \leq 4$ *(or* $s = 2 \leq t \leq 4$*), in which case the group is* $[1\,1\,2]^s \ltimes (C_2 \times C_2)$, *of order* 480, 2592 *and* 30720 *respectively, with* $[1\,1\,2]^s$ *as in Theorem 12.*

Many polytopes in the above theorems admit geometric realizations in euclidean spaces, and for several finite examples explicit coordinates of the vertices of these realizations are known ([24, 25]). For a general discussion on realizations we refer to [22] or the article by Peter McMullen in this volume.

6.3. Type $\{3,6,3\}$

Relatively little is known about the universal regular 4-polytopes

$$_7\mathcal{T}^4_{s,t} = \{\{3,6\}_s, \{6,3\}_t\}$$

with $s = (s^k, 0^{2-k}), t = (t^l, 0^{2-l})$, with $s, t \geq 2$ if $k, l = 1$ and $s, t \geq 1$ if $k, l = 2$. Except for some specific parameter values like those mentioned in [5, 49], the only results known are those obtained by the method in the previous section and some variants of this ([25, 28]). However, these methods are not strong enough to settle the general case for the type $\{3,6,3\}$. In particular, one can prove

Theorem 15 *The regular 4-polytopes* $_7\mathcal{T}^4_{(s,s),(s,0)} = \{\{3,6\}_{(s,s)}, \{6,3\}_{(s,0)}\}$ *and* $_7\mathcal{T}^4_{(s,s),(3s,0)} = \{\{3,6\}_{(s,s)}, \{6,3\}_{(3s,0)}\}$ *exist for all* $s \geq 2$, *the latter (but not the former) also for* $s = 1$. *Among these, the only finite instances are* $_7\mathcal{T}^4_{(1,1),(3,0)}$ *with group* $[1\,1\,1]^3 \ltimes S_3$ *of order* 324 *and* $_7\mathcal{T}^4_{(2,2),(2,0)}$ *with group* $S_5 \times S_3$ *of order* 720.

There are various quotient and subgroup relations between the locally toroidal regular 4-polytopes of types $\{6,3,3\}$, $\{6,3,4\}$, $\{6,3,6\}$ and $\{3,6,3\}$; see [28] for a detailed discussion. These are based on relations between the symmetry groups of the corresponding hyperbolic honeycombs. For example, the polytopes in the next theorem are related to $_3\mathcal{T}^4_{(s,0)} = \{\{6,3\}_{(s,0)}, \{3,3\}\}$.

Theorem 16 *The regular 4-polytope* $_7\mathcal{T}^4_{(s,0),(s,0)} = \{\{3,6\}_{(s,0)}, \{6,3\}_{(s,0)}\}$ *exists at least for all* s *with* $3 \not| s$. *It is infinite when* $s \geq 5$ *and* $3 \not| s$ *(and most likely also when* $s = 4$*). If* $s = 2$ *it is finite and its group is* $S_5 \times C_2$ *of order* 240.

7. The Cut Method

Before we proceed with the discussion in ranks 5 and 6 we illustrate a powerful geometric method, the *cut method*, which sheds some light on why certain parameter

s	v	f	g
(2,0,0)	24	8	9216
(2,2,0)	48	32	36864
(2,2,2)	1536	2048	2359296

Table 4. The finite polytopes $\mathcal{T}_s^5 = \{\{3,4,3\},\{4,3,4\}_s\}$

It would be very helpful to be able to preserve some of the arguments in this example and prove universality of the cut $\mathcal{M}$ without using an explicit construction for $\mathcal{P}$. In fact, this would immediately imply non-finiteness results for $\mathcal{P}$, which would be especially useful in higher ranks.

8. Polytopes of Rank 5

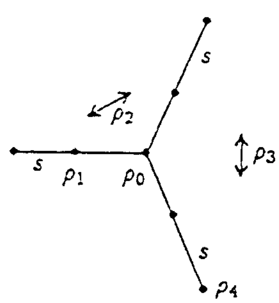
The only candidates for regular 5-polytopes whose facets and vertex-figures are spherical or toroidal (with at least one of the latter kind) are those of type $\{3,4,3,4\}$ and their duals. Confining our attention to the first of each dual pair, we shall write

$$\mathcal{T}_s^5 := \{\{3,4,3\},\{4,3,4\}_s\},$$

with the convention that $s = (s^k, 0^{3-k})$ with $s \geq 2$ and $k = 1, 2$ or 3. Then we have ([29, 30])

Theorem 17 *The universal regular 5-polytope $\mathcal{T}_s^5 = \{\{3,4,3\},\{4,3,4\}_s\}$ exists for all $s = (s^k, 0^{3-k})$ with $s \geq 2$ and $k = 1,2,3$. It is finite when $s = 2$, and infinite when $s \geq 3$. If $s = 2$ and $k = 1,2,3$, the corresponding groups are $C_2^3 \ltimes [3,4,3]$ of order 9216, $C_2^5 \ltimes [3,4,3]$ of order 36864, and $(C_2^6 \ltimes C_2^5) \ltimes [3,4,3]$ of order 2359296 respectively.*

For $k = 1$ the polytopes can be constructed by a direct twisting argument, as indicated in



(23)

If W_s is the underlying Coxeter group, then $A(\mathcal{T}_{(s,0,0)}^5) \cong W_s \ltimes S_3$. Hence, finiteness occurs exactly for $s = 2$. We list all the finite polytopes in Table 4.

Let us also note that, when $k = 1$, we have a cut $\{\{3,4\},\{4,4\}_{\bar{s}}\}$ of $\mathcal{T}_s^5$, where $\bar{s} := (s^k, 0^{2-k})$ with $s \geq 2$, induced by a corresponding cut of $\{3,4,3,4\}$. In fact,

s	v	f	g
(2,0,0,0)	20	960	368640
(2,2,0,0)	160	30720	11796480
(3,0,0,0)	780	189540	72783360

Table 5. The known finite polytopes $_1\mathcal{T}^6_s = \{\{3,3,3,4\},\{3,3,4,3\}_s\}$

9.2. Type $\{3,3,4,3,3\}$

The situation for the remaining two types of locally toroidal regular polytopes of rank 6 is somewhat similar. We can appeal in each case to known results about which of the regular 4-polytopes of type $\{4,4,4\}$ exists and is finite, but since these do not, at present, cover all possibilities, our knowledge of the polytopes of rank 6 is correspondingly incomplete ([29]).

Consider the universal regular 6-polytope

$$_2\mathcal{T}^6_{s,t} := \{\{3,3,4,3\}_s,\{3,4,3,3\}_t\},$$

with $s=(s^k,0^{4-k}), t=(t^l,0^{4-l})$ with $s,t\geq 2$ and $k,l=1,2$. Now, if $A(_2\mathcal{T}^6_{s,t}) = \langle\rho_0,\ldots,\rho_5\rangle$, $\sigma := \rho_1\rho_2\rho_3\rho_2\rho_1$ and $\tau := \rho_4\rho_3\rho_2\rho_3\rho_4$, then the operation

$$(\rho_0,\ldots,\rho_5)\mapsto(\rho_0,\sigma,\tau,\rho_5) =: (\varphi_0,\ldots,\varphi_3)$$

yields a cut of $_2\mathcal{T}^6_{s,t}$ (with group $\langle\varphi_0,\ldots,\varphi_3\rangle$) in the class $\langle\{4,4\}_{\bar{s}},\{4,4\}_{\bar{t}}\rangle$ (as usual, $\bar{s}=(s^k,0^{2-k})$ when $s=(s^k,0^{4-k})$, and so on). Evidence indicates that this cut is indeed $_2\mathcal{T}^4_{\bar{s},\bar{t}} = \{\{4,4\}_{\bar{s}},\{4,4\}_{\bar{t}}\}$ (that is, the cut is universal), but so far we have not been able to prove this. Now, most cases of the regular polytopes of type $\{4,4,4\}$ are completely settled, and as a consequence, we have the following conjecture and subsequent theorem. The known finite polytopes are listed in Table 6.

Conjecture 3 *The regular 6-polytope $_2\mathcal{T}^6_{s,t} = \{\{3,3,4,3\}_s,\{3,4,3,3\}_t\}$ exists for each $s=(s^k,0^{4-k}), t=(t^l,0^{4-l})$ with $s,t\geq 2$ and $k,l=1,2$, except when $s=(2,0,0,0)$ and t is odd, or $t=(2,0,0,0)$ and s is odd.*

Theorem 18 *Under the assumption that the cut above is universal, then if the polytope $_2\mathcal{T}^6_{s,t}$ exists, it is infinite in at least the following cases:*
a) $s=(s,0,0,0)$, $t=(t,t,0,0)$ and $\frac{1}{s}+\frac{1}{2t}\leq\frac{1}{2}$.
b) $s=(s,s,0,0)$, $t=(t,t,0,0)$ and $\frac{1}{s}+\frac{1}{2t}\leq\frac{1}{2}$ or $\frac{1}{2s}+\frac{1}{t}\leq\frac{1}{2}$.
c) $s=(s,0,0,0)$, $t=(t,0,0,0)$, with s or t even, or $s=t$ odd, and $\frac{1}{s}+\frac{1}{t}\leq\frac{1}{2}$.

We shall denote by $_1A^6_s$ and $_2A^6_{s,t}$ the groups abstractly defined by the presentation belonging to the polytopes $_1\mathcal{T}^6_s$ and $_2\mathcal{T}^6_{s,t}$; these are the quotients of $[3,3,3,4,3]$ and $[3,3,4,3,3]$ defined by the extra relations for the facets or vertex-figures. If the two polytopes exist, then $_1A^6_s$ and $_2A^6_{s,t}$ are their groups, respectively.

It is known that $[3,3,4,3,3]$ is a subgroup of index 5 in $[3,3,3,4,3]$ ([29, 30]). The corresponding relationship between the groups of the locally toroidal polytopes

s
(2,0,0,0)
(2,0,0,0)
(2,2,0,0)
(3,0,0,0)

Table 6. The k

Theorem 19 *Under t*
2, $_2A^6_{(s,s,0,0),(s,0,0,0)}$ *is a*
is a subgroup of index 5

9.3. Type $\{3,4,3,3,4\}$

In the case of the univers

with $s=(s^k,0^{4-k}), t=$
again use a cut of type
$\{\{4,3,3,4\}_s,\{3,4,3,3\}_t\}$
operation

$$(\rho_0,\ldots$$

and, if $l=1$ or 2, belongs
is that introduced earlier
that it is isomorphic to
cut is not universal. Nov
supports the following co

Conjecture 4 *The re*
all $s=(s^k,0^{4-k}), t=(t^l,$
(a) $s=(2,0,0,0)$ and $t=$
(b) $t=(2,0,0,0)$ and $s=$
(c) $t=(2,2,0,0)$ and s c

Note that only the fir
which do not give 4-poly
([29, 30]).
Conjecture 4 was con
a rather sophisticated tw
toroid of rank 5 whose fa
gram $\mathcal{D}_{s,m}$ whose nodes
one connects antipodal v
and the other connects p
of $\mathcal{K}$ and its branches ar

s	t	v	f	g
(2,0,0,0)	(t,0,0,0) (t even)	32	$2t^4$	$36864t^4$
(2,0,0,0)	(t,t,0,0) (t even)	32	$8t^4$	$147476t^4$
(2,2,0,0)	(2,2,0,0)	2048	2048	150994944
(3,0,0,0)	(3,0,0,0)	2340	2340	218350080

Table 6. The known finite polytopes ${}_2\mathcal{T}^6_{s,t} = \{\{3,3,4,3\}_s, \{3,4,3,3\}_t\}$

Theorem 19 *Under the assumption that Conjectures 2 and 3 hold, for each* $s \geq 2$, ${}_2A^6_{(s,s,0,0),(s,0,0,0)}$ *is a subgroup of index* 5 *in* ${}_1A^6_{(s,s,0,0)}$, *while* ${}_2A^6_{(2s,0,0,0),(s,s,0,0)}$ *is a subgroup of index* 5 *in* ${}_1A^6_{(2s,0,0,0)}$.

9.3. Type {3, 4, 3, 3, 4}

In the case of the universal regular 6-polytope

$$ {}_3\mathcal{T}^6_{s,t} := \{\{3,4,3,3\}_s, \{4,3,3,4\}_t\} $$

with $s = (s^k, 0^{4-k}), t = (t^l, 0^{4-l})$, with $s, t \geq 2$, $k = 1, 2$ and $l = 1, 2, 4$, we can again use a cut of type $\{4,4,4\}$ but now, strictly speaking, of the dual $({}_3\mathcal{T}^6_{s,t})^* = \{\{4,3,3,4\}_t, \{3,3,4,3\}_s\}$. If $A(({}_3\mathcal{T}^6_{s,t})^*) = \langle \rho_0, \ldots, \rho_5 \rangle$, this cut is induced by the operation

$$ (\rho_0, \ldots, \rho_5) \mapsto (\rho_0, \rho_1, \rho_2\rho_3\rho_4\rho_3\rho_2, \rho_5\rho_4\rho_3\rho_4\rho_5) $$

and, if $l = 1$ or 2, belongs to the class $\langle \{4,4\}_{\vec{t}}, \{4,4\}_{\vec{s}} \rangle$, where the notation for suffixes is that introduced earlier. In fact, we conjecture again that the cut is universal, so that it is isomorphic to ${}_7\mathcal{T}^4_{\vec{t},\vec{s}} = \{\{4,4\}_{\vec{t}}, \{4,4\}_{\vec{s}}\}$; on the other hand, if $l = 4$, this cut is not universal. Now, employing the results on polytopes of type $\{4,4,4\}$ this supports the following conjecture.

Conjecture 4 *The regular 6-polytope* ${}_3\mathcal{T}^6_{s,t} := \{\{3,4,3,3\}_s, \{4,3,3,4\}_t\}$ *exists for all* $s = (s^k, 0^{4-k}), t = (t^l, 0^{4-l})$ *with* $s, t \geq 2$, $k = 1, 2$ *and* $l = 1, 2, 4$, *except when*
(*a*) $s = (2,0,0,0)$ *and* $t = (t,0,0,0)$ *with* t *odd,*
(*b*) $t = (2,0,0,0)$ *and* $s = (s,0,0,0)$ *with* s *odd,*
(*c*) $t = (2,2,0,0)$ *and* s *odd.*

Note that only the first two excluded cases of this conjecture correspond to cuts which do not give 4-polytopes. The third case of collapse can be proved directly ([29, 30]).

Conjecture 4 was confirmed in [30] for all s, t with $l = 1$ and $t = 2m$ even using a rather sophisticated twisting argument. To explain this, let $\mathcal{K} := \{3,3,4,3\}_s$, a toroid of rank 5 whose facets are 4-crosspolytopes $\{3,3,4\}$. Consider a Coxeter diagram $\mathcal{D}_{s,m}$ whose nodes are the vertices of $\mathcal{K}$ and whose branches are of two types: one connects antipodal vertices of facets of $\mathcal{K}$ and its branches are marked $m (\geq 2)$, and the other connects pairs of vertices which are not vertices of a common facet of $\mathcal{K}$ and its branches are marked ∞. Then the group of the universal 6-polytope

s	t	v	f	g
$(s,0,0,0)$ (s even)	$(2,0,0,0)$	$3s^4$	16	$18432s^4$
$(s,s,0,0)$	$(2,0,0,0)$	$12s^4$	16	$73728s^4$
$(s,0,0,0)$ (s even)	$(2,2,0,0)$	$6s^4$	64	$73728s^4$
$(s,s,0,0)$ (s even)	$(2,2,0,0)$	$24s^4$	64	$294912s^4$
$(2,0,0,0)$	$(2,2,2,2)$	384	1024	18874368
$(2,0,0,0)$	$(4,0,0,0)$	12288	65536	1207959552
$(3,0,0,0)$	$(3,0,0,0)$	2340	780	72783360

Table 7. The known finite polytopes ${}_3\mathcal{T}^6_{s,t} = \{\{3,4,3,3\}_s, \{4,3,3,4\}_t\}$

$({}_3\mathcal{T}^6_{s,(2m,0^3)})^*$ can be constructed by a twisting operation on the corresponding Coxeter group $W_{s,m}$ with diagram $\mathcal{D}_{s,m}$, using the fact that $A(\mathcal{K})$ acts on $\mathcal{D}_{s,m}$ as a group of diagram automorphisms; in particular, $W_{s,m} \ltimes A(\mathcal{K})$ is the group of the resulting polytope. The case $m = 1$ requires a variant of this construction.

Theorem 20 *The 6-polytope* ${}_3\mathcal{T}^6_{s,(2m,0^3)} = \{\{3,4,3,3\}_s, \{4,3,3,4\}_{(2m,0^3)}\}$ *exists for each* $s = (s^k, 0^{4-k})$ *with* $s \geq 2$ *and* $k = 1,2$ *and each* $m \geq 1$. *The only finite instances occur for* $m = 1$, *and* $s = (2,0,0,0), m = 2$. *In the first case the group is* $C_2^4 \ltimes [3,3,4,3]_s$, *of order* $18432s^4$ *or* $73728s^4$ *if* $k = 1$ *or* 2, *respectively; in the second case it is* $C_2^{16} \ltimes [3,3,4,3]_{(2,0,0,0)}$, *of order* 1207959552.

Table 7 lists all the known finite polytopes ${}_3\mathcal{T}^6_{s,t}$, and it is conjectured that this list is complete. The table entries were checked (or obtained) by an application of the Coxeter-Todd coset enumeration algorithm. For all polytopes, except the one in the last row, the structure of the group is explicitly known ([30]). The first two rows and the next to last row are covered by Theorem 20. The group in the third and fourth row is $C_2^6 \ltimes [3,3,4,3]_s$ with $s = (s,0,0,0)$ or $(s,s,0,0)$, respectively; and in the fifth row it is $C_2^{10} \ltimes [3,3,4,3]_{(2,0,0,0)}$. See [30] for more general results on polytopes with small facets or vertex-figures like those discussed here.

Writing ${}_3A^6_{s,t}$ for the group abstractly defined by the presentation belonging to the polytope ${}_3\mathcal{T}^6_{s,t}$, and using the fact that $[3,4,3,3,4]$ is a subgroup of index 10 in $[3,3,3,4,3]$, we now have

Theorem 21 *Under the assumption that Conjectures 2 and 4 hold, for each* $s \geq 2$, ${}_3A^6_{(s,0,0,0),(s,s,0,0)}$ *is a subgroup of index* 10 *in* ${}_1A^6_{(s,s,0,0)}$, *while* ${}_3A^6_{(s,s,0,0),(2s,0,0,0)}$ *is a subgroup of index* 10 *in* ${}_1A^6_{(2s,0,0,0)}$.

Concerning the polytope in the last row of Table 7, it is interesting to note that the two groups ${}_1A^6_{(3,0,0,0)}$ and ${}_3A^6_{(3,0,0,0),(3,0,0,0)}$ are isomorphic and that the corresponding polytopes ${}_1\mathcal{T}^6_{(3,0,0,0)}$ and ${}_3\mathcal{T}^6_{(3,0,0,0),(3,0,0,0)}$ have the same number of vertices. Further, ${}_1A^6_{(3,0,0,0)}$ is a quotient of ${}_2A^6_{(3,0,0,0),(3,0,0,0)}$ by a normal subgroup of order 3, so that the number of vertices of ${}_1\mathcal{T}^6_{(3,0,0,0)}$ is only one third of that of ${}_2\mathcal{T}^6_{(3,0,0,0),(3,0,0,0)}$.

10. Finite Quotients

It is well known that the
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Let U be any group.
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Theorem 22 *Let* $\mathcal{P}_1$
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Clearly the groups o
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Theorem 23 *Let* $\{p,$
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Theorem 24 *Let* $\mathcal{P}_1$
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Then $(\mathcal{P}_1, \mathcal{P}_2)$ *contains*

11. Chiral Polytopes

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10. Finite Quotients

It is well known that the euclidean and hyperbolic regular tessellations $\{p, q, r\}$ are the universal coverings for infinitely many finite regular tessellations of the same type on closed compact 3-manifolds. This generalizes also to many classes of abstract regular polytopes. The previous sections were aiming at the classification of all the finite polytopes among the universal locally toroidal regular polytopes $\{\mathcal{P}_1, \mathcal{P}_2\}$. The question remains whether the infinite polytopes of this kind cover in fact infinitely many finite regular polytopes in the same class, namely $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$. For most classes $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ this can indeed be proved by generalizing a technique used in Vince [47]; see [26]. This seems to indicate that in general it is very hard to fully describe these classes.

Let U be any group. Then U is called *residually finite* if for each finite subset $\{\varphi_1, \ldots, \varphi_m\}$ of $U \setminus \{\varepsilon\}$ there exists a homomorphism f of U onto a finite group such that $\varphi_j f \neq \varepsilon$ for $j = 1, \ldots, m$. By a central result in the theory of linear groups, due to Malcev [20], every finitely generated linear group is residually finite; see also [48]. In particular, every (finitely generated) Coxeter group is residually finite.

Theorem 22 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be finite regular n-polytopes with $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle \neq \emptyset$. Let $\mathcal{P}$ be an infinite regular $(n+1)$-polytope in $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ whose group $A(\mathcal{P})$ is residually finite. Then $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ contains infinitely many regular $(n+1)$-polytopes which are finite and are covered by $\mathcal{P}$.*

Clearly the groups of regular polytopes are finitely generated. But then, by Malcev's result, Theorem 22 applies if $\mathcal{P}$ is an infinite member in $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ whose group is isomorphic to a linear group. This is the form in which Theorem 22 can be applied to many classes of polytopes. We give two examples in rank 4, the first being the well-known classical case. There are similar results for other types and for higher ranks.

Theorem 23 *Let $\{p, q, r\} = \{4, 3, 4\}, \{3, 5, 3\}, \{5, 3, 5\}, \{4, 3, 5\}$ or $\{5, 3, 4\}$. Then $\langle \{p, q\}, \{q, r\} \rangle$ contains infinitely many regular polytopes which are finite.*

Theorem 24 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be regular toroidal maps for which the universal $\{\mathcal{P}_1, \mathcal{P}_2\}$ exists, is infinite, and is of type $\{4, 4, 3\}$ or $\{6, 3, p\}$ with $p = 3, 4, 5$ or 6. Then $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ contains infinitely many regular polytopes which are finite.*

11. Chiral Polytopes

Relatively little is known about locally toroidal chiral polytopes. By Theorem 3, there is no chiral toroid of rank ≥ 4, and so the classification of such polytopes makes sense in rank 4 alone. The corresponding Schläfli types $\{p, q, r\}$ are the same as for regular 4-polytopes but now the parameter vectors s and t for the facets and vertex-figures are less restricted; see Section 4. Another complication is added through enantiomorphism.

We shall introduce notation similar to that of universal regular polytopes, except that now we drop the superfix "4" for the rank and replace it by "ch". For example, we write

$${}_1\mathcal{T}^{ch}_s := \{\{4,4\}_s, \{4,3\}\}^{ch},$$

$s = (b,c)$ with $bc(b-c) \neq 0$, for the oriented universal chiral 4-polytope whose oriented facets are maps $\{4,4\}_s$ and whose vertex-figures are (oriented) cubes $\{4,3\}$. (Recall that for directly regular polytopes the two orientations can canonically be identified.) Note that interchanging b and c in s changes ${}_1\mathcal{T}^{ch}_s$ to the other enantiomorphic form, $\overline{{}_1\mathcal{T}^{ch}_s}$ (say), of the same underlying polytope. In short, if $\bar{s} := (c,b)$ when $s = (b,c)$, then ${}_1\mathcal{T}^{ch}_{\bar{s}} = \overline{{}_1\mathcal{T}^{ch}_s}$. The situation is similar for the chiral polytopes

$${}_p\mathcal{T}^{ch}_s := \{\{6,3\}_s, \{3,p\}\}^{ch}$$

with $p = 3,4,5$ and $s = (b,c)$, $bc(b-c) \neq 0$; that is, ${}_p\mathcal{T}^{ch}_{\bar{s}} = \overline{{}_p\mathcal{T}^{ch}_s}$ ([39, 40]).

For the three remaining chiral 4-polytopes

$${}_2\mathcal{T}^{ch}_{s,t} := \{\{4,4\}_s, \{4,4\}_t\}^{ch}, \quad {}_6\mathcal{T}^{ch}_{s,t} := \{\{6,3\}_s, \{3,6\}_t\}^{ch},$$

$${}_7\mathcal{T}^{ch}_{s,t} := \{\{3,6\}_s, \{6,3\}_t\}^{ch},$$

with $s = (b,c), t = (d,e)$ and $bc(b-c) \neq 0$ or $de(d-e) \neq 0$, the situation is more complicated, because now both the facets and vertex-figures can be chiral. In general, interchanging the components in only one parameter vector, s (say), does not simply change ${}_i\mathcal{T}^{ch}_{s,t}$ to the other enantiomorphic form $\overline{{}_i\mathcal{T}^{ch}_{s,t}}$. In fact, in general it seems that the two polytopes ${}_i\mathcal{T}^{ch}_{s,t}$ and ${}_i\mathcal{T}^{ch}_{\bar{s},t}$ are unrelated. However, if the components in both parameter vectors are interchanged, then only the enantiomorphic form is changed; that is, ${}_i\mathcal{T}^{ch}_{\bar{s},\bar{t}} = \overline{{}_i\mathcal{T}^{ch}_{s,t}}$ for $i = 2,6,7$.

Problem 1 *Classify all the universal chiral 4-polytopes* ${}_i\mathcal{T}^{ch}_s$ *for each* $i = 1,3,4,5$ *and* ${}_i\mathcal{T}^{ch}_{s,t}$ *for each* $i = 2,6,7$.

Here the term "classification" is used in the same sense as in Section 3. Except for the existence part of the problem and solutions for a few sporadic cases ([5]), no general classification results are known. As usual the existence of the universal polytope can be deduced from the fact that the corresponding class of polytopes is non-empty.

For all seven types $\{p,q,r\}$ of locally toroidal 4-polytopes, chiral polytopes have been constructed from representations of the hyperbolic rotation groups $[p,q,r]^+$ as projective linear groups over finite rings (Weiss [51], Schulte & Weiss [40], Nostrand & Schulte [35]). See also Nostrand [34] for similar such examples of (locally spherical) types $\{3,5,3\}$ and $\{5,3,4\}$. We shall not give the details of these constructions but instead illustrate a typical result for the type $\{3,6,3\}$, which relates to prime decomposition in the ring $\mathbb{Z}[\omega]$ of Eisenstein integers. Let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$, the ring of integers modulo m.

Theorem 25 *Let* m *be a positive integer, let* $m = p_1^{e_1} \cdot \ldots \cdot p_k^{e_k}$ *be its prime decomposition, and let* $p_j \equiv 1 \pmod 3$ *for each* $j = 1,\ldots,k$. *Let* b,c *be positive integers such that* $m = b^2 + bc + c^2$, $(b,c) = 1$. *Then there exists a self-dual chiral 4-polytope in* $\langle\{3,6\}_{(b,c)}, \{6,3\}_{(b,c)}\rangle^{ch}$ *whose group is*

(a) $PSL_2(\mathbb{Z}_m)$ *if* $p_j \equiv 1 \pmod{12}$ *for all* j;

(b) $PSL_2(\mathbb{Z}_m) \ltimes C_2$ *if* $p_j \equiv 7 \pmod{12}$ *for at least one* j.

12. Other Local To

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12. Other Local Topological Types

As remarked in the introduction, there is as yet no comprehensive study of abstract regular or chiral polytopes which are locally of some topological type that is not spherical or toroidal. However, many interesting examples are known, and here we shall mention a few to point out some possible direction of further research.

In rank 4, call a regular (or chiral) polytope $\mathcal{P}$ *locally of genus g* if both its facets and vertex-figures are maps on orientable surfaces of genus at most g, with at least one kind of genus exactly g. For $g = 1$ this gives the locally toroidal polytopes. Variants of this terminology could also include maps on non-orientable surfaces.

Problem 2 *For small $g \geq 2$, classify all the regular 4-polytopes which are locally of genus g.*

Problem 3 *For small $g \geq 2$, classify all the regular 4-polytopes whose facets are of genus g and whose vertex-figures are spherical.*

Clearly, a solution for Problem 2 includes one for Problem 3. Note that the polytopes in Problem 3 are of interest also because they can be realized as combinatorially regular decompositions of certain topological 3-manifolds into handlebodies of genus g ([2]).

An interesting special case is $g = 3$. In [32, 33], Monson and Weiss construct a family of regular or chiral 4-polytopes of type $\{p, 3, 3\}$ with $p \geq 3$ which are related to the Picard group, the projective linear group $PSL(\mathbb{Z}[i])$ over the Gaussian integers $\mathbb{Z}[i]$. For $p = 7$ this gives the universal regular polytope $\{\{7,3\}_8, \{3,3\}\}$ with group $PSL_2(7^2) \ltimes C_2$ of order $7^2(7^4-1)$, whose facet is Klein's map $\{3,7\}_8$ of genus 3 ([10, 44]). For many other interesting examples with facets of small genus see also [24, 25].

For higher ranks, the locally projective case has received some attention. For instance, McMullen [23] constructs the regular polytopes $\mathcal{P}$ of rank $n \geq 5$ and type $\{3, 4, 3^{n-3}\}$ which are universal with respect to having their 4-faces isomorphic to the projective polytope $\{3,4,3\}_6$ (which is constructed from the 24-cell $\{3,4,3\}$ by identifying antipodal points); that is, $A(\mathcal{P})$ is the Coxeter group $[3,4,3^{n-3}] = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, factored out by the single extra relation

$$(\rho_0\rho_1\rho_2\rho_3)^6 = 1 .$$

In particular, $A(\mathcal{P}) \cong S_4 \wr S_{n-1}$ of order $24^{n-1}(n-1)!$. These polytopes are locally projective in the more general sense of the term that each minimal non-spherical section is a projective polytopes (which here is $\{3,4,3\}_6$). Similarly there are locally projective regular polytopes of types $\{3^k, 4, 3, 3, 4, 3^l\}$ all of whose non-spherical 4-sections of types $\{3,4,3\}$, $\{4,3,3\}$ and $\{3,3,4\}$ are isomorphic to $\{3,4,3\}_6$, $\{4,3,3\}_4$ and $\{3,3,4\}_4$, respectively.

We also mention two examples of mixed toroidal-projective type, the polytopes $\{\{6,3\}_{(s,s)}, \{3,5\}_5\}$ of rank 4 with hemi-icosahedral vertex-figures and with $s \geq 2$, and the polytopes $\{\{4,3,4\}_{(2^k,0^{3-k})}, \{3,4,3\}_6\}$ of rank 5 with $k = 1,2,3$. Using a suitable hermitian form as in Section 6.2 we find that the first polytope is infinite if $s \geq 3$; an application of the Coxeter-Todd coset enumeration algorithm suggests that it is also infinite if $s = 2$ (but there is no proof yet). The group of the second polytope is $C_2^{m_k} \ltimes [3,4,3]_6$, of order $576 \cdot 2^{m_k}$, with $m_k = 3, 5$ or 8 if $k = 1, 2$ or 3, respectively ([30]).

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FACE NUMBERS AN

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Abstract. The first part of t
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1. Numbers of Faces

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$f_2 \leq 2f_0 - 4.$

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