BULL. AUSTRAL. MATH. SOC. 10E05, 52A10, 52A43
VOL. 35 (1987) 441-454

ON THE AREA OF PLANAR CONVEX SETS CONTAINING MANY LATTICE POINTS

P.R. SCOTT

A classical theorem of van der Corput gives a bound for the volume of a symmetric convex set in terms of the number of lattice points it contains. This theorem is here generalized and extended for a large class of non-symmetric sets in the plane.

1. Introduction

Let Λ be a lattice in the plane, generated by vectors v_1, v_2, and having lattice determinant $d(\Lambda) = |\det(v_1, v_2)|$. Let K be an open convex set containing the origin O, and having area $A(K)$.

In 1936, van der Corput [2] showed that if K is symmetric about the origin O, and contains as well as O, at most p distinct pairs of non-zero lattice points, then $A(K) \le 4(p + 1)$. If we set $c = 2p + 1$ in this result, so that c denotes the total number of lattice points in K, we obtain for symmetric sets, $A(K) \le 2c + 2$. Arkinstall [1] sought to extend this result to non-symmetric sets in the following way.

Let a chord of K which is bisected by the origin O be called

Received 10 June 1986.

a *chord of symmetry* of K , and let $s(K)$ denote the number of such chords. Then Arkinstall shows:

If $s(K)$ is even or infinite, then $A(K)/d(\Lambda) \le 2c + 2$

If $s(K) > 1$ and $c \le 4$, then $A(K)/d(\Lambda) \le (2c + 2) + 1/(2c)$

If $s(K) > 3$ and $c \le 4$, then $A(K)/d(\Lambda) \le 2c + 2$.

The inequalities are best possible, but the proofs of the last two are long and involve much case-splitting. Further, it seems likely that the restriction $c \le 4$ is unnecessary.

In the present paper, we set aside Arkinstall's symmetry conditions, and show that the above inequalities hold for a large class of non-symmetric sets. It is hoped that this might be a useful step in establishing Arkinstall's result without restriction.

Let $k > 0$. As in [5] we say that K is *$k\Lambda$-bounded* if some translate of K is contained in some fundamental parallelogram of $k\Lambda$, but no translate is contained in any fundamental parallelogram of $(k - \varepsilon)\Lambda\ (\varepsilon > 0)$. We shall prove:

THEOREM 1. *Let* $k = 2c + 2$. *If* K *is* $k\Lambda$*-bounded, and the* c *lattice points in* K *are collinear, then*

$$A(K)/d(\Lambda) \le 2c + 2 + 1/(2c) .$$

It appears to be difficult to establish a similar result when the lattice points in K are not collinear. Instead therefore we proceed as follows. Let P be the fundamental parallelogram of Λ , centred at the origin O , and with side directions determined by the lattice vectors v_1, v_2 . Let Π denote the closed convex polygon (perhaps degenerate) obtained by taking the convex hull of the lattice points in K . We now define a new polygon Π^* to be the vector sum $\Pi + 5P$ of Π and a five-fold enlargement of P about the origin. This has the effect of surrounding Π by a generous border of width $\frac{5}{2}v_i$ in the v_i-direction, $(i = 1, 2)$.

We shall

THEOREM 2.

(a) $A(K)/d(\Lambda)$

(b) *if the la*

$A(K)/d(\Lambda)$

Both of these i

We observ

and having $c >$

implies it. Ho

thus the rectar

distorted sligh

condition of th

Since A

Π^* are invari

establish the

$d(\Lambda) = 1$.

The tria

cannot be impr

condition need

illustrated se

boundary lines

We shall prove:

THEOREM 2. *Suppose* $c > 1$ *and* $K \subset \Pi^*$. *Then*

(a) $A(K)/d(\Lambda) \leq 2c + \frac{5}{2}$;

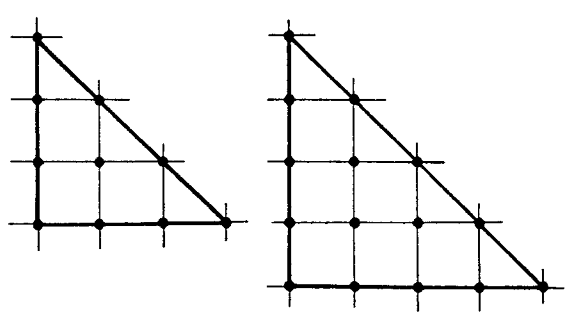
(b) *if the lattice points in* K *are not collinear, then* $A(K)/d(\Lambda) \leq 2c + 2$.

Both of these inequalities are best possible.

We observe that for sets K containing collinear lattice points, and having $c > 1$, Theorem 1 is stronger than Theorem 2(a), and in fact implies it. However, the result of Theorem 2 fails for sets having $c = 1$: thus the rectangle with sides along the lines $y = 0$, $y = 1$, $y = \pm\frac{5}{2}$, distorted slightly to contain the origin, satisfies the boundedness condition of the theorem, but has area arbitrarily close to 5.

Since $A(K)/d(\Lambda)$, c, and the definitions of $k\Lambda$-boundedness and Π^* are invariant under affine transformation, it will be sufficient to establish the theorems when Λ is the integral lattice, in which case $d(\Lambda) = 1$.

The triangles in Figure 1 show that the inequalities of the theorems cannot be improved. We also see from Figure 2 that some boundedness condition needs to be placed on the set K . For the area of each of the illustrated sets can be made as large as we please by taking the sloping boundary lines close to horizontal.



$c = 1 \quad A = 4\frac{1}{2} \qquad c = 3 \quad A = 8$

Figure 1.

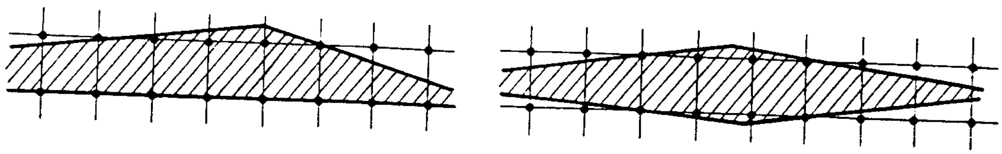


Figure 2.

We might mention that other bounds are known for $A(K)$; for example Nosarzewska [3] showed that $A(K) < c + \frac{1}{2}P(K)$, where $P(K)$ is the perimeter of K . However, there is no obvious way to relate this to Arkinstall's result.

Finally we note that for $c = 1$, the result of Theorem 1 is established in [5] . We shall henceforth assume that $c \geq 2$.

2. Setting up the problem

We define $t = t(K)$ to be the smallest integer such that all the lattice points in K lie on $t(K)$ parallel lattice lines. Thus for example, if the lattice points in K are collinear, $t(K) = 1$.

Let Π denote the convex lattice polygon (possibly degenerate) obtained by taking the convex hull of the lattice points contained in K . We say that polygon Π' is *equivalent* to Π if Π' can be obtained from Π using only integral unimodular transformations, and lattice translations.

The following lemma sets up a useful 'standard form' for Π' , together with some characterising properties.

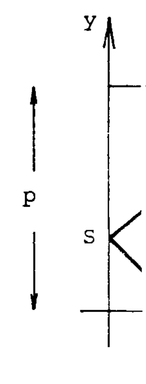
LEMMA 1. *There exists a positive integer* $r(K)$, *such that* Π *is equivalent to a convex lattice polygon* Π' *whose lattice points lie precisely on the lines* $y = 1, 2, \ldots, r(K)$.

Proof. If $t(K) = 1$, Π degenerates to a lattice line segment. This is clearly equivalent to a lattice line segment lying along the y-axis and satisfying the requirements of the lemma. (In this case, we shall specify

this position, rather thar

If $t(K) = 2$, we ca
lattice points in K to t
ing parallel segment will
we form a lattice triangle
and a lattice point of Π
no further lattice points
base on the y-axis has le
the altitude of τ cannot
unimodular shear having th
the second segment lie on

Suppose now that $t($
contained in, and has at l
rectangle, as in Figure 3,
Let $P,Q,R,S,$ be such ver
the projection of R on t



We now obtain an equ
in the following way. Usi
fixes the line $y = 1$, we
in fact decrease p' to a
that a shear "perpendicula
simply rotate the polygon

this position, rather than having Π lying along the line $y = 1$.)

If $t(K) = 2$, we can map the segment containing a maximal number of lattice points in K to the y-axis as above. We assert that the remaining parallel segment will now lie on one of the lines $x = \pm 1$. For, if we form a lattice triangle τ from two adjacent points on the y-axis, and a lattice point of Π not on the axis, then since $\tau \subset K$, τ can have no further lattice points on its boundary or in its interior. Since its base on the y-axis has length 1, and by Pick's theorem [4] it has area $\frac{1}{2}$, the altitude of τ cannot exceed 1 . Finally, a suitable integral unimodular shear having the y-axis as axis ensures that the points of the second segment lie on a subset of the lines $y = 1, 2, \ldots, r(K)$.

Suppose now that $t(K) \geq 3$. Then after suitable translation, Π is contained in, and has at least one vertex on each side of a $p \times p'$ rectangle, as in Figure 3, where we may assume that $p' \geq p \geq t(K) - 1 \geq 2$. Let $P, Q, R, S,$ be such vertices of Π , as in Figure 3, and let R' be the projection of R on the opposite side.

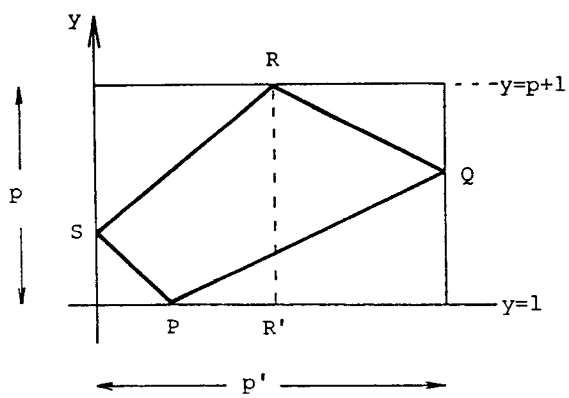


Figure 3.

We now obtain an equivalent polygon Π' , (perhaps identical to Π), in the following way. Using an integral unimodular shear, σ , which fixes the line $y = 1$, we make $|PR'| \leq p/2$. Such a transformation may in fact decrease p' to a value less than p . If this happens, noting that a shear "perpendicular to σ" leaves $|PR'|$ and p' unchanged, we simply rotate the polygon through a quarter turn about a suitable lattice point, and repeat the process. There can be at most a finite number of such rotations, since at each step the positive integer $p + p'$ is reduced by at least 1. Note that an adjacent pair of the points P, Q, R, S may coincide at a vertex of the rectangle.

We thus obtain an equivalent lattice polygon Π' contained in a $p \times p'$ rectangle (compare Figure 3) with $p' \geq p \geq 2$, and $|PR'| \leq p/2$. We now show that every line $y = k (1 \leq k \leq p + 1)$ contains at least one point of Π'. (This will clearly establish our lemma, with $r(K) = p$.)

Since $p \geq 2$ and $PQRS$ is convex, it will be sufficient to show that the lines $y = 2, y = p$ each contain a lattice point; by symmetry it will be sufficient to consider the line $y = p$.

If R does not lie at a vertex of the rectangle, then the intercept of $PQRS$ on the line $y = p$ is smallest when Q, S lie at the base vertices of the rectangle. Since $p' \geq p$, the (closed) intercept has length at least 1, and so contains a lattice point. If R coincides with S at the vertex $(0, p + 1)$ (say) of the rectangle, vertices P, Q must lie on the opposite two sides. Since $|PR'| \leq p/2$ and $p' \geq p$, $\angle PRQ$ contains the ray from R through the point $T(p, 1)$. But line RT has equation $x + y = p + 1$. We deduce that line $y = p$ contains the point $(1, p)$ of Π.

We proceed to establish our theorems for a general set K' whose associated lattice polygon Π' satisfies Lemma 1. This is sufficient, since the statements of the theorems are invariant under integral unimodular transformation, and lattice translation. For simplicity of notation we henceforth omit the prime.

3. Proof of Theorem 2

Let c_i denote the number of lattice points in K lying on $y = i (1 \leq i \leq r)$; then $\sum_{i=1}^{r} c_i = c$. On each line $y = i (1 \leq i \leq r)$, there exist lattice points P'_i, P_i such that $P'_i P_i$ is the segment of length $c_i + 1$ containing the c_i points of K on $y = i$ in its interior. By Lemma 1, $c_i \neq 0 \ (1 \leq i \leq r)$.

We now partition K into convex subsets with the lines $y = i + \frac{1}{2} (1 \leq i \leq r - 1)$, obtaining r subsets $K_1, K_2, \ldots, K_r$, with K_i containing just the lattice points on $y = i$. Clearly

$$A(K) = \sum_{i=1}^{r} A(K_i) .$$

By the convexity of K, for $2 \leq i \leq r - 1$, each K_i is contained in a trapezium bounded by the lines $y = i \pm \frac{1}{2}$, and lines through the points P_i', P_i . Such a trapezium has area $c_i + 1$. Hence

$$(*) \qquad A(K_i) \leq c_i + 1 (2 \leq i \leq r - 1)$$

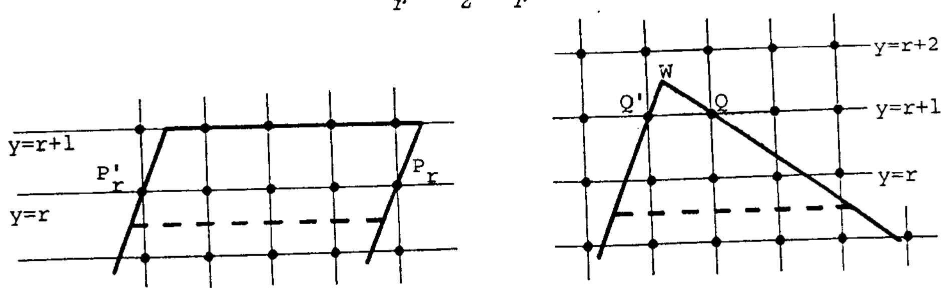
It remains to find bounds for $A(K_1)$ and $A(K_r)$.

LEMMA 2. $A(K_1) + A(K_r) \leq \frac{3}{2} (c_1 + c_r + 2) + \frac{1}{2}$, *and this result is best possible.*

Proof. We first find an upper bound for $A(K_r)$ for certain sets K_r .

Suppose that K_r is bounded by straight lines through P_r', P_r which are parallel, or meet in a point of the half-plane $y > r$. If K_r is also bounded by the line $y = r + 1$, then K_r is contained in a parallelogram of length $c_r + 1$ and height $\frac{3}{2}$ (Figure 4(a)). Hence

$$A(K_r) \leq \frac{3}{2} (c_r + 1)$$



(a) (b)

Figure 4.

Assume now that K_r extends beyond the line $y = r + 1$; by our boundedness condition on K, K_r does not extend beyond $y = r + \frac{5}{2}$. Let Q', Q be adjacent points on the line $y = r + 1$ such that K intercepts the segment $Q'Q$ (Figure4(b)). Since K is convex, there exists a point W such that K_r is bounded by $Q'W$, QW ; thus K_r lies within the triangle Δ determined by $Q'W$, QW and the line $y = r - \frac{1}{2}$. Let $Q'W$, QW cut off a segment of length t on the line $y = r$. Then Δ has base length $\frac{1}{2}(3t - 1)$ and altitude $(3t - 1)/2(t - 1)$. Hence

$$A(K_r) \le A(\Delta) = \frac{1}{2}.\frac{3t - 1}{2}.\frac{3t - 1}{2(t - 1)}$$

$$= \frac{(3t - 1)^2}{8(t - 1)}$$

For $t > 1$, this rational function of t assumes a minimal value of 3 at $t = \frac{5}{3}$. We consider several ranges of t .

(a) For $\frac{5}{3} \le t \le 2$, the y-coordinate w of W satisfies $r + 2 \le w \le r + \frac{5}{2}$, so Δ is not affected by the boundedness condition on K . Substituting $t = 2$ in the above formula,

$$A(K_r) \le 3\tfrac{1}{8} \le \frac{3}{2}(c_r + 1) + \frac{1}{8}$$

since $c_r \ge 1$.

(b) For $1 < t \le \frac{5}{3}$, triangle Δ becomes truncated by the horizontal line $y = r + \frac{5}{2}$, and K_r is bounded by the trapezium with sides along $Q'W$, QW and the lines $y = r + \frac{5}{2}$, $y = r - \frac{1}{2}$. The area of this trapezium is clearly constant for all t in this range, and so

$$A(K_r) \le 3 \le \frac{3}{2}(c_r + 1)$$

since $c_r \ge 1$.

(c) For $t > 2$, K is again contained in the triangle Δ , and we observe that

$$\begin{aligned} A(K_r) \le A(\Delta) &= \tfrac{3}{8}(3t + 1) + 1/2(t - 1) \\ &\le \tfrac{9}{8}t + \tfrac{3}{8} + \tfrac{1}{2} \\ &\le \tfrac{9}{8}(c_r + 1) + \tfrac{7}{8} \\ &= \tfrac{9}{8}c_r + 2 \\ &\le \tfrac{3}{2}(c_r + 1) \end{aligned}$$

since $c_r \ge 2$. Because $t > 2$, this condition is always satisfied.

Hence in each case,

$$A(K_r) \le \frac{3}{2}(c_r + 1) + \frac{1}{8}$$

We note that if similarly, $A(K_1) \le \frac{3}{2}(c_r + 1) + \frac{1}{8}$, then the inequality of the lemma is satisfied.

Suppose now that the lines through P_r', P_r meet in a point V in the halfplane $y < 1$. It is clear that a set K_r of maximal area will be bounded by the line $y = r + 1$. Hence a set K_r of maximal area will be contained in a trapezium T_r formed by the lines $V P_r'$, $V P_r$, $y = r + 1$, and $y = r - \frac{1}{2}$.

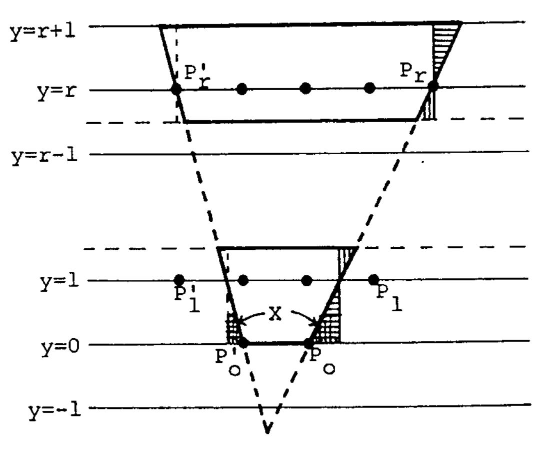


Figure 5

But since K is convex, the lines $V P_r'$, $V P_r$ will also bound K_1. Let K_1^+ denote the intersection of K_1 with the upper halfplane $\{(x,y) \mid y > 0\}$ and K_1^- the intersection with $\{(x,y) \mid y < 0\}$. As we intend applying our previous argument about K_r to the set K_1, we henceforth use t to denote the length of the segment of the line $y = 1$ cut off by $V P_r'$, $V P_r$.

Now if K_1^+ and K_r are contained in corresponding trapezia T_1, T_r as in Figure 5, let R_1, R_r denote the corresponding rectangles, obtained by replacing the sloping sides by vertical segments as shown.

For $t > 2$, V lies above the line $y = -1$, and $A(K_1^-) < \frac{1}{2}$. A geometric addition and subtraction of congruent triangles shows that

$$\begin{aligned} A(K_1) + A(K_r) &\le A(K_1^+) + A(K_r) + A(K_1^-) \\ &< A(R_1) + A(R_r) + \tfrac{1}{2} \\ &\le \tfrac{3}{2}(c_1 + 1) + \tfrac{3}{2}(c_r + 1) + \tfrac{1}{2} \\ &= \tfrac{3}{2}(c_1 + c_2 + 2) + \tfrac{1}{2} \end{aligned}$$

as required by the lemma.

(It may happen of course that the sloping sides of the lower trapezium meet, not at V, but at some point with greater y-coordinate. It is easily checked that the above addition and subtraction argument holds *a fortiori* in this case.)

For $t \le 2$,

$$\begin{aligned} A(K_1) + A(K_r) &= A(R_1) + A(K_1^-) + A(R_r) \\ &= A(K_1) + A(X) + A(R_r) \end{aligned}$$

where X is made up of the two cross-hatched regions in Figure 5.

From our previous argument with K_r,

$$A(K_1) \leq \frac{3}{2}(c_1 + 1) + \frac{1}{8} .$$

Also, regarding X as a split trapezium, having altitude $\frac{1}{2}$ and (combined) parallel side-lengths $t - 1$ and $t - \frac{1}{2}(t + 1)$, the area of X is given by

$$\begin{aligned} A(X) &= \frac{1}{2} . \frac{1}{2} (t - 1) + (t - \frac{1}{2}(t + 1)) \\ &= \frac{3}{8}(t - 1) \\ &\leq \frac{3}{8} \end{aligned}$$

since $t \leq 2$.

Hence

$$\begin{aligned} A(K_1) + A(K_r) &\leq \{\frac{3}{2}(c_1 + 1) + \frac{1}{8}\} + \frac{3}{8} + \frac{3}{2}(c_r + 1) \\ &= \frac{3}{2}(c_1 + c_r + 2) + \frac{1}{2} \end{aligned}$$

as required.

We note that the equality is required here for the large triangle in Figure 1. This completes the proof of the lemma.

We are now in a position to establish Theorem 2. Assuming $r \geq 2$, we have from (*) and Lemma 2.

$$\begin{aligned} \sum_{i=1}^{r} A(K_i) &\leq \frac{3}{2}(c_1 + c_r + 2) + \frac{1}{2} + \sum_{i=2}^{r-1}(c_i + 1) \\ &= c + \frac{1}{2}(c_1 + c_r) + r + \frac{3}{2} \\ &= c + \frac{1}{2}(c_1 + c_r + r - 2) + \frac{1}{2}(r + 1) + 2 \\ &\leq \frac{3}{2}c + \frac{1}{2}(r + 1) + 2 \end{aligned}$$

(noting that $c_1 + c_r + (r - 2) \leq c$) .

Now if the lattice points in K are not all collinear, then $r + 1 \leq c$, and

$$A(K) = \sum_{1}^{r} A(K_i) \leq 2c + 2 .$$

On the other hand, if the lattice points in K are collinear, then $r = c$, and

$$A(K) \leq 2c + 2 + \frac{1}{2}$$

This completes the proof of the theorem.

4. Proof of Theorem 1

As in Lemma 1, we shall assume that the lattice points of K lie along the y-axis, and that $c > 1$. We first symmetrize K about the x-axis to obtain a corresponding set K^* ; clearly $A(K^*) = A(K)$. In fact we seek a set K^* for which $A(K^*)$ is maximal; K^* will be a certain polygonal set, with its bounding lines determined by the lattice point contraints on K . Since K contains just c lattice points on the y-axis, we may therefore assume that K^* is bounded by lines $PV, P'V$ through the points $P(0,\frac{1}{2}(c+1))$, $P'(0, -\frac{1}{2}(c+1))$, where by symmetry, $V(v,0)$ lies on the x-axis. By reflecting K (K^*) in the y-axis if necessary, we may assume that V lies on the positive x-axis (possibly 'at infinity'). It is now clear that K^* will be bounded by the line $x = -1$.

There are several critical cases to consider, depending on the position of V .

(a) If $v > (2c+1)/c$, then the maximal set K^* is a trapezium T , with parallel sides along $x = \pm 1$, and sides through P, P' . In this case

$$A(K^*) = A(T) = 2c + 2$$

(b) If $2 \leq v \leq (2c+1)/c$, then K^* is an isosceles triangle, Δ , with base on the line $x = -1$ and sides through P, P' meeting in vertex V . (This constraint on v ensures that the line $x = 1$ intercepts K^* in a segment of length at most 1, corresponding to the fact that K contains no lattice points on the line $x = 1$.) It is easily checked that

Δ has maximal area when $v = (2c + 2)/c$; in this case we obtain

$$A(K^*) = 2c + 2 + 1/(2c) .$$

(c) If $1 < v < 2$, then K^* is an isosceles triangle with its base angles truncated by the horizontal lines $y = \pm(c + 1)$, such lines appearing as a result of the boundedness condition on K . An easy calculation shows that the area assumed for such truncated triangles does not exceed the area of the limiting figure, a $(2c + 2) \times 1$ rectangle, obtained as v approaches 1. Thus here,

$$A(K^*) < 2c + 2 .$$

This completes the proof of Theorem 1.

5. Comments

Although the results of the theorems are best possible, some refinements can probably be made. For example, for sets K with $t(K) \geq 3$, it seems likely that no boundedness condition need be imposed on K , although such conditions are necessary for this proof.

We also note that there are some analogous problems to be considered, corresponding to the n-dimensional form of van der Corput's theorem for symmetric sets of volume V :

$$V(K)/d(\Lambda) \leq 2^{n-1}(c + 1)$$

References

[1] J. Arkinstall, *Generalizations of Minkowski's theorem in the plane*, (Ph.D. thesis, University of Adelaide, South Australia, 1982).

[2] J.G. van der Corput, "Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen", *Acta Arithmetica* 2 (1936), 145-146.

[3] M. Nosarzewska, "Évaluation de la différence entre l'aire d'une region plane convexe et la nombre des points aux coordonnées entières couverte par elle", *Colloq. Math.* 1 (1947), 305-311.

[4] G. Pick, "Geometrisches zur Zahlenlahre", *Sitzungsber Lotos Prag.* (2) 19 (1900), 311-319.

[5] P.R. Scott, "An analogue of Minkowski's theorem in the plane", *J. London Math. Soc.* 8 (1974), 647-651.

Department of Mathematics,
University of Adelaide,
Adelaide,
South Australia,
Australia.