FORCING COLORATIONS
AND THE STRONG PERFECT GRAPH CONJECTURE

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Abstract: We give various reformulations of the Strong Perfect Graph Conjecture, based on a study of forced coloring procedures, uniquely colorable subgraphs and $\omega - 1$-cliques in minimal imperfect graphs.

1. Introduction

If G is a graph $\omega = \omega(G)$ denotes the cardinality of a maximum clique and $\alpha = \alpha(G)$ denotes the cardinality of a maximum stable set. A k-stable set or k-clique ($k \in \mathbb{N}$) will mean a clique or stable set of size k. $\chi = \chi(G)$ is the chromatic number of G. Subgraph means induced subgraph in this paper.

Let us replace $\{x\}$ by x throughout the paper. Paths an circuits go through every vertex at most once. They will be considered to be subgraphs or edge-sets. The vertex-set of the graph G will be denoted by $V(G)$, the edge-set by $E(G)$.

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every subgraph H of G, otherwise it is called *imperfect*. Polyhedra of the form $P = \{x \geq 0 : Ax \leq 1\}$, where A is a $0-1$ matrix have integer vertices if and only if the rows of A are the characteristic vectors of cliques, and the vertices of P are the characteristic vectors of stable sets of a perfect graph. (This is a straightforward consequence of Lovász's Perfect Graph Theorem (1972 a) and Fulkerson's theorem on antiblocking (1970,1971), as it was observed by Chvátal (1975).) Thus Perfect Graphs provide a pleasant reformulation of a wide range of integer programs with advantageous properties with respect to Optimization.

While the Perfect Graph Theorem has become the material of undergraduate textbooks, the interest has decreased toward the structure of perfect or minimal imperfect graphs in general since Fulkerson's Lovász's and Padberg's work (1970, 1972 a,b, 1974). A graph is called *minimal imperfect* if it is not perfect, but all its subgraphs are perfect.

A co-NP characterization of perfectness immediately follows from Lovász (1972 b), but perfect graphs are not known to belong to NP, and their recognition seems to be very difficult.

However, the direction of co-NP characterizations is not closed either. The strongest coNP charaterization would be given by the following conjecture of Berge (1961), (1962):

A graph isomorphic to an odd circuit of length at least five is called an *odd hole* , and the complement of such a graph is called an *odd antihole* .

Berge's Strong Perfect Graph conjecture (SPGC): *If a graph is minimal imperfect, then it is an odd hole, or an odd antihole.*

Since a co-NP characterization exists already, there could be doubts about whether this conjecture is interesting enough. However, its investigation may be justified by the closed relation it could have with the recognition of perfect graphs. It is of course a nice structural property interesting for its own sake.

Since the proof of the Strong Perfect Graph Conjecture occured to be difficult, researchers of the field have turned toward proving it for subclasses of graphs, and developped algorithms for recognizing subclasses of perfect graphs. A relatively small number of papers deals with the Strong Perfect Graph Conjecture in general.

Let us summarize one direction of general results, and on the way we define some notions. In the sequal we shall suppose that the reader is familiar with these notions and statements, and we shall use them without reference.

Lovász's theorem on partitionability(1972b), (1984):
Minimal imperfect graphs have $\alpha\omega + 1$ vertices;
It follows then by an easy computation, that
deleting an arbitrary vertex in a minimal imperfect graph, the remaining graph can be partitioned into α ω-cliques, and also into ω α-stable-sets.

Graphs G with $\alpha\omega + 1$ vertices and the property that deleting an arbitrary vertex the remaining graph can be partitioned into α ω-cliques, and also into ω α-stables are called *partitionable*, or (α, ω) graphs.

Since partitionable graphs are easily seen to be imperfect, the above theorem of Lovász provides a NP-characterization of imperfectness. It also implies Lovász's Perfect Graph Theorem stating that
A graph is perfect if and only if its complement is perfect.

Padberg's corollaries(1974): Let G be an (α, ω) graph. Let us mention four properties proved by Padberg which will be the most important for us:

a.) *For every ω-clique K there exists a unique α-stable set disjoint from it, let us denote it by $S(K)$; for every α-stable set S, there exists a unique ω-clique disjoint from it, denote it by $K(S)$.*

b.) *For every $v \in V(G)$, $G-v$ has one unique partition into ω-cliques, and one unique partition into α-stable sets. The former partition is the clique-partition of $G-v$, its elements are the clique classes of $G-v$, whereas the latter partition is the coloration of $G-v$, its elements are the color classes of $G-v$.*

c.) *Every $v \in V(G)$ is contained in α α-stable sets, namely*
$\{S(K) : K$ is a clique-class of $G-v$ $\}$, and ω ω-cliques, namely
$\{K(S) : S$ is a color-class of $G-v$ $\}$.

In this paper the representation of a set with a vector will always mean its characteristic vector as a vertex-set.

d.) *The set of ω-cliques of a minimal imperfect graph is linearly independent.*

In the sequal we shall suppose that the reader is familiar with these statements, and we shall use them without reference.

Chvátal's small transversals:

Chvátal (1976), (1984a), (1984b) has made a crucial remark:

In a minimal imperfect graph G there is no set of vertices whose cardinality is $\alpha+\omega-1$ and intersects every α-stable set and every ω-clique.

This is an immediate consequence just of the perfectness of subgraphs with $(\alpha-1)(\omega-1)+1$ vertices: such a subgraph must contain either an ω-clique or an α-stable set.

A set whose cardinality is $\alpha+\omega-1$ and intersects every α-stable set and every ω-clique will be called a *small transversal*.

The author does not know about any argument which would use other properties of minimal imperfect graphs than partitionability, and the perfectness of subgraphs on $\alpha'\omega'+1$ vertices, where either $\alpha' \leq 3$ or $omega' \leq 3$ or $\alpha' = \alpha - 1$ or $omega' = \omega - 1$.

Characterization of unique colorability

A graph which has one unique partition into $\omega = \omega(G)$ stable sets, is called *uniquely colorable.* Deleting a vertex from a minimal imperfect graph we get a uniquely colorable perfect graph (Padberg). The significance of uniquely colorable perfect graphs, and their relation to the Strong Perfect Graph Conjecture was first recognized by A. Tucker (1984). He stated conjectures about the relation of what he called "sequential colorings" and unique colorability, and pointed out the role such a relation could play in a proof of the Strong Perfect Graph Conjecture.

This approach was further developped in Fonlupt, Sebő (1990), where a good characterization of uniquely colorable perfect graphs is given, and the following separation of the

SPGC into two parts is exhibited:

Problem 1. Prove the existence of "combinatorial forcings" in uniquely colorable perfect graphs, and more particularly in graphs of the form $G - v$, where G is minimal imperfect, and $v \in V(G)$.

Problem 2. Prove that a minimal imperfect graph with "enough combinatorial forcings" is an odd hole or antihole.

The notions in quotation marks should be defined so that applying the result of Problem 1 to all graphs $G - x$ ($x \in V(G)$), where G is the minimal imperfect graph of Problem 2, one gets a sufficient number of forcings to deduce that G is an odd hole or an odd antihole, with the help of an already existing solution of Problem 2.

Let us explain now that both Problem 1 and Problem 2 are solved in a certain sense, and the "only" gap to fill in is between two different meanings of the expression "combinatorial forcing": the one *provided* by any existing solution of Problem 1, and the one *required* by any solution given to Problem 2.

The following solution of Problem 1 is an easy consequence of the above mentioned polyhedral characterization of clique matrices of perfect graphs:
A perfect graph G on n vertices has one unique partition into $\omega = \omega(G)$ stable sets, if and only if it has $n - \omega + 1$ linearly independent ω-cliques.
This linear algebraic relation can easily be translated into a combinatorial one, which, however, involves only a weaker forcing procedure than the condition under which Problem 2 can be solve at present (Theorem 1.2).

Thus the more particular the forcings in Problem 1 are, or the weaker the condition of Problem 2 is, the closer we are to the SPGC.

The main goal of this work is to make a step forward in Problem 2.

Let G be an arbitrary graph. Let us say that xKy ($x \neq y \in V(G), K \subseteq V(G)$) is a *forcing* (to color x and y to the same color in every ω-coloration), if both $K \cup x$ and $K \cup y$ are ω-cliques. We shall say that x and y are *forced*, if there exist forcings $x_1K_1x_2$, $x_2K_2x_3$, $\ldots$, $x_{t-1}K_{t-1}x_t$, $x = x_1$, $y = x_t$. Indeed, *if $\chi(G) = \omega(G)$, and x, y are forced, then clearly, x and y must have the same color in every ω-coloration of G.*

Problem 1 seems to be more difficult: we do not even know about the existence of just one forcing in a minimal imperfect graph. A first result in this direction has been reached by J. Fonlupt (1992), who has proved that a uniquely 3-colorable perfect graph there exists a forcing.

Forcings of the complement of a graph will be called *co-forcings*, and pairs of vertices forced in the complement will be called *co-forced*.

S.E. Markossian, G.S. Gasparian and A.S. Markossian's result (1986), (1992) is a breakthrough in this direction:

If in G two *adjacent* vertices are forced, then obviously, $\chi(G) > \omega(G)$, in particular G is not perfect. But does G contain a hole or an antihole ?

In a hole, any two adjacent vertices are co-forced, and any two vertices at distance two are forced. Markossian and Gasparian and Markossian (1986), (1992) prove that this statement can be reversed:

Theorem 1.1 *If a minimal imperfect graph G and its complement both have two forced vertices which are adjacent, then G is a hole or antihole.*

The key-result of the present work is the following generalization:

Theorem 1.2 *If a minimal imperfect graph G has two forced vertices which are adjacent, then G is a hole or antihole.*

For a proof of the SPGC it suffices now to prove that a minimal imperfect graph or its complement have adjacent forced vertices.

We prove Theorem 1.2 in Section 2. This provides a new, short proof of Theorem 1.1 itself. (In Markossian and Gasparian and Markossian's solution an ordered sequence of α different forcings, and ω different co-forcings is essential, their proof is quite involved, and uses an exhaustive case checking; the proof in Section 2 makes use of one forcing and two co-forcings only, and exploits these in a simple way.)

The other pole of this paper is a study of the $\omega - 1$-cliques of a minimal imperfect graph, with the goal of understanding forcings better, and getting closer to the conjectures below. The main results about $\omega - 1$-cliques are sketched in Section 4. We could still not prove *the existence of at least one forcing in a minimal imperfect graph, which would be a breakthrough we think*, but an interesting structure seems to come up: we shall call some restricted kind of $\omega - 1$-cliques *intervals*, and our goal is to prove that intervals arise as the intersection of ω-cliques. (The name "interval" comes from the fact that specializing them to "webs" they are exactly the $\omega - 1$-cliques forming intervals, and the properties we shall prove also remind intervals.) This turns out to be a rich notion: in Section 4 we work out the relations between vertices, intervals and ω-cliques. Our main tool for this work will be Theorem 1.2. We shall also prove and then use a new relation between unique colorability and forcings (that is some results on Problem 1), which may be interesting for its own sake:

Theorem 1.3 *Let G be minimal imperfect, and suppose K is an ω-clique, S is an α-stable, and $K \cap S = \emptyset$. If $G - K \cup S$ and its complement are uniquely colorable, then G is* *a hole or antihole.*

A proof of this theorem can be found at the end of Section 3.

The results of the paper lead us into a new range of simple reformulations of the Perfect Graph Conjecture, which we summarize in Section 5.

Let us finish this introduction with two conjectures, which imply together the Strong Perfect Graph conjecture, and which are the main motivation for the present research. The first is a weakening of a conjecture of Tucker (1984), see also Fonlupt, Sebő (1990). The second would sharpen the results of the paper.

Conjecture 1 *If G is a uniquely colorable perfect graph such that $\alpha(G)\omega(G) = V(G)$ and the set of its ω-cliques is linearly independent, then it has a forcing.*

Conjecture 2 *If G is a partitionable graph which has both a forcing and a co-forcing, then it also has a small transversal.*

2. Critical edges and the Perfect Graph Conjecture

An edge $xy \in E(G)$ is called *critical*, if $\alpha(G - xy) = \alpha + 1$. This means exactly that there exists S, $|S| = \alpha - 1$ such that xSy is a co-forcing.

If the pair (x, y) is a critical edge of the complement we shall call it a critical co-edge. (Then there exists then a unique $\omega - 1$-clique K so that xKy is a forcing.)

Throughout this section we suppose that G is an (α, ω) *partitionable graph*.

Markossian, Gasparian and Markossian (1986), (1992) made several simple but important observations about critical edges. Let us state two of them:

(2.1) *If $x_0x_1, \ldots, x_{k-1}x_k$ are critical edges, $k < \omega$, then $\{x_0, \ldots, x_k\}$ is a clique.*

Indeed, let $x_{i-1}S_ix_i$ be the coforcings corresponding to these critical edges ($i = 1, \ldots, k$). Then $\{x_0, \ldots, x_k\} \cup S_1 \cup \ldots \cup S_k$ is a proper subset of the vertices of G, because $k + 1 + k(\alpha - 1) = k\alpha + 1 < \omega\alpha + 1$. But a proper subset of an (α, ω)-graph has a partition into at most α cliques, and co-forced vertices must be in the same clique-class, whence they are adjacent, as claimed.

(2.2) *If xSy is a co-forcing, then there exists a unique ω-clique K_x containing x and not containing y, and $K_x = K(S \cup y)$.*

Proof. Clearly, an ω-clique K_x containing x is disjoint from S. If in addition $y \notin K_x$, then $K_x \cap (S \cup y) = \emptyset$.

Q.E.D.

Let us first prove that Theorem 1.2 is a special case of the following Theorem 2.1 . Indeed, if there are co-forced non-adjacent vertices, then it is easy to see from (2.1) that there exists a path $(x_0, \ldots, x_\omega)$, where $x_{i-1}x_i$ ($i = 1, \ldots, \omega$) are critical edges. Applying (2.1) again, we immediately get that x_0Kx_ω is a forcing, where $K = \{x_1, \ldots, x_{\omega-1}\}$. This forcing and the two critical edges x_0x_1, $x_{\omega-1}x_\omega$ satisfy the condition of the theorem below.

THEOREM 2.1 *Suppose G is an (α, ω) graph, and it has no small transversal. If there exists v_1, $v_2 \in V(G)$ and an $\omega - 1$-clique K such that v_1Kv_2 is a forcing, moreover there exist u_1, $u_2 \in K$ (not necessarily distinct), such that u_1v_1, u_2v_2 are critical edges, then G is a hole or an antihole.*

Proof. Let $u_1S_1v_1$, $u_2S_2v_2$ be the two co-forcings corresponding to the two critical edges of the theorem. If $u_1 = u_2$, then $\omega = 2$, because if $\omega \geq 3$, then by (2.1) $\{v_0, v_1, v_2\}$ is a clique, in contradiction with $v_1v_2 \notin E(G)$. G is an odd hole then.

Suppose now $u_1 \neq u_2$, and let U_1 be the color class of u_1 in the coloration of $G - u_2$, and let U_2 be the color class of u_2 in the coloration of $G - u_1$. $U_1 \setminus \{u_1\}$, $U_2 \setminus \{u_2\}$, S_1, S_2 are pairwise distinct, because $u_1 \notin K(U_1)$, but $u_2 \in K(U_1)$, whence by (2.2) $v_2 \in K(U_1)$, and since $v_1v_2 \notin E(G)$, we also have $v_1 \notin K(U_1)$: $K(U_1) \cap \{u_1, v_1, u_2, v_2\} = \{u_2, v_2\}$. Symetrically, $K(U_2) \cap \{u_1, v_1, u_2, v_2\} = \{u_1, v_1\}$. On the other hand, by (2.1), $K(S_1 \cup \{v_1\}) = K \cup \{v_2\}$, and therefore $K(S_1 \cup \{v_1\}) \cap \{u_1, v_1, u_2, v_2\} = \{u_1, u_2, v_2\}$ and symmetrically $K(S_2 \cup \{v_2\}) \cap \{u_1, v_1, u_2, v_2\} = \{u_2, u_1, v_1\}$. Thus, with a glance at the coloration of $G - u_2$ we actually see that U_1, S_1 and S_2 are pairwise disjoint, and and with a glance at the coloration of $G - u_1$ we see that so are U_2, S_1 and S_2 as well (see the Figure).

Let $\mathcal{S}$ be the family of α-stable sets containing u_1, different from $S_1 \cup u_1$, and of U_2. We have $|\mathcal{S}| = \alpha - 2$.

Case 1 There exists $S \in \mathcal{S}$ such that $S \setminus u_1$ is not a subset of $S_1 \cup S_2$.

Let then $s \in (S \setminus u_1) \setminus (S_1 \cup S_2)$, and let Q be the clique class of $G - s$ for which $u_2 \in Q$. Equivalently, $s \in S(Q)$, $u_2 \in Q$. Then we also have $v_2 \in Q$, because if not, then by (2.2) $S(Q) = S_2 \cup v_2$, and therefore $s \notin S(Q)$ in contradiction with the choice of Q.

Let $T := (S \setminus u_1) \cup (Q \setminus v_2) \cup v_1$.

$Q \cap S \neq \emptyset$, because otherwise S would be a color class of $G - u_2$, and since $u_1 \in S$ this means $S = U_1$, contradicting the choice of S.

$|T| = \alpha + \omega - 2$ follows.

We first show that *every ω-clique R different from $K(S)$ has a non-empty intersection with T*. Indeed, since $R \cap S \neq \emptyset$, this is obvious if $u_1 \notin R$; if $u_1 \in R$, then either $v_1 \in R$ and we are done, or $v_1 \notin R$, and then by (2.2) $R = K(S_1 \cup v_1) = K \cup v_2$. But then $u_2 \in R \cap T$, because $u_2 \in K$, and $u_2 \in Q \setminus v_2 \subseteq T$.

Second, we show that *every α-stable set U different from $S_2 \cup v_2$ has a non-empty intersection with T*. Indeed, if $U = S(Q)$, then this is true, because $s \in S(Q) \cap T$. If $U \neq S(Q)$, then $(Q \setminus v_2) \cup v_1$ intersects U, except if U is the unique stable set (see (2.2)) containing v_2 and not containing v_1. But this stable set is just $S_2 \cup v_2 \neq U$ by assumption.

Since $S \neq S_2 \cup v_2$ (because $u_1 \in S$, whence $v_2 \notin S$), $K(S)$ and $S_2 \cup v_2$ are not disjoint. Adding their intersection to T we get a small transversal, a contradiction. Thus Case 1 cannot hold.

But then Case 1 can also not hold if we interchange the indices 1 and 2. The only case that remains:

Case 2 Every α-stable set which contains u_1 or u_2 and which is different from U_1 and U_2 is a subset of $S_1 \cup S_2 \cup \{u_1, u_2\}$.

Let H be the graph induced by $S_1 \cup S_2 \cup \{u_1, u_2\}$. By the assumption, $S_1 \cup S_2 \cup \{u_1, u_2\}$ contains all the α-stable sets containing u_1 except one, and the same is true for u_2. Thus

H has at least $2(\alpha - 1)$ α-stable sets.

We show that this is possible only if $\alpha = 2$. Let H be the graph induced by $S_1 \cup S_2$. Since H is bipartite, it is perfect, whence it can be partitioned to α cliques, that is, there exists a perfect matching $M = \{x_1y_1, \ldots, x_\alpha y_\alpha\}$ in H. (We implicitly used here the Perfect Graph Theorem, or König's theorem about matching's in bipartite graphs. To use the latter note that $\alpha(H) = \alpha$ implies that the minimum cardinality of a transversal is also α.)

Let $\chi_i := \chi_{\{x_i, y_i\}}$ $(i = 1, \ldots, \alpha)$. (χ_i is 1 in x_i and y_i, and 0 otherwise.) Since every α-stable set of H contains exactly one of x_i and y_i, the vectors $\chi_1 - \chi_i$ $(i = 2, \ldots, \alpha)$ are all orthogonal to the characteristic vectors of α-stable sets of H. Thus the linear rank of the α-stable sets of H is at most $2\alpha - (\alpha - 1) = \alpha + 1$. Since the set of all α-stable sets of G is linearly independent (Padberg), we get that

H has at most $\alpha + 1$ α-stable sets.

Comparing this with the trivial lower bound above, we get that $2\alpha - 2 \le \alpha + 1$, that is $\alpha \le 3$.

Suppose $\alpha = 3$. Let S, S', S" be the three α-stable sets containing u_1, where S' is a color class of $G - u_2$, and $S'' = S_1 \cup u_1$. Let $T := (S \cup K \cup v_1) \setminus u_1$. $|T| = \alpha + \omega - 2$.

Since by assumption $S \subseteq S_1 \cup S_2 \cup \{u_1, u_2\}$, we have $S \cap S_2 \ne \emptyset$. $S \cap S_1 \ne \emptyset$ can also be supposed: if $S \cap S_1 = \emptyset$, then $|S \cap S_2| = 2$, and $u_1S_2u_2$ is a co-forcing; but then $u_1S_1u_2$ is not a co-forcing (because of (2.2) and the unicity of the α-stable set disjoint from a given ω-clique), and interchanging the indices 1 and 2 we have $S \cap S_1 \ne \emptyset$, $S \cap S_2 \ne \emptyset$.

It follows that T intersects every α-stable set except possibly S'. (Note that $S(K \cup v_1) = S_2 \cup v_2$.) It also intersects every ω-clique except $K(S)$. (The unique clique containing u_1 and not containing v_1 is $K \cup v_2$.)

Since $S' \ne S$, S' and $K(S)$ are not disjoint. Adding their intersection to T we get a small transversal. This contradiction proves $\alpha \le 2$. Thus G is an antihole.

Q.E.D.

3. Unique colorability and critical edges

Although the main stream of the paper is to study Problem 2, in this Section we wish to give some particular answers to Problem 1 . A general answer can be found in Fonlupt and Sebő (1990), but that cannot be sticked together with Theorem 1.2 to prove SPGC. We would like to provide forcings in uniquely colorable graphs. For some particular graphs we will succeed. These graphs will play a role in the following two sections.

Let us first mention a general relation between the rank $r(G)$ of the set of ω-cliques of a graph, perfectness and unique colorability, which is the trivial implication of a characterization of perfectness and unique colorability in Fonlupt, Sebő (1990).

(3.1) *Let G be arbitrary. If $\chi(G) = \omega(G)$, then $r(G) \leq n - \omega + 1$, and if in addition equality holds, then G is uniquely colorable.*

Indeed, let $\chi_1, \dots, \chi_\omega$ be the characteristic vectors of the color classes in an ω-coloration. Then the vectors $\chi_1 - \chi_2, \dots, \chi_1 - \chi_\omega$ are all orthogonal vectors to the ω-cliques (like in the proof of Theorem 2.1, Case 2), and they are linearly independent. If there exists another coloration, then let $\chi_{\omega+1}$ be a color class in this coloration, which is different from χ_i $(i = 1, \dots, \omega)$. It can be easily seen that $\chi_1 - \chi\omega + 1$ is linearly independent from $\{\chi_1 - \chi_2, \dots, \chi_1 - \chi_\omega\}$, and it is also orthogonal to the ω-cliques.

The following statement is a key observation for proving theorems 1.3 and 4.1:

(3.2) *If G is partitionable and K is an ω-clique such that $G - K$ is uniquely colorable, then*

a.) The critical edges of G whose both endpoint is in K form a spanning tree of K.

b.) If in addition there exists a critical edge as, $a \in K$, $s \notin K$, then the critical edges of $K \cup s$ form a path through all vertices of $K \cup s$.

c.) One of the endpoints of this path is s, let us denote the other by t. $s(K \setminus t)t$ is a forcing.

The heart of the proof is the following lemma:

Lemma: *Under the conditions of (3.2) the graph whose vertex set is K and whose edges are the critical edges induced by K, is connected.*

Proof. Let $\{X_1, X_2\}$ be an arbitrary partition of K, and $x_1 \in X_1$ $x_2 \in X_2$. The coloration of $G - x_1$ consists of S, of $|X_1| - 1$ color classes intersecting X_1, and $|X_2|$ color classes intersecting X_2. Similarly, the coloration of $G - x_2$ consists of S, of $|X_2| - 1$ color classes intersecting X_2, and $|X_1|$ color classes intersecting X_1. Thus, since by assumption the restriction of these two colorations to $G - K$ is the same, there exists an $\alpha - 1$-stable A, and $a_1 \in X_1$, $a_2 \in X_2$ such that $A \cup a_2$ is a color class of $G - x_1$, and $A \cup a_1$ is a color class of $G - x_2$. Hence $a_1 a_2$ is a critical edge.

We have proved that there is a critical edge between the classes of any partition of K, which proves the lemma.

Q.E.D.

It can be easily proved in various ways that the graph in the lemma cannot contain a circuit. A lemma of Giles, Trotter and Tucker (1984), see also Tucker (1984)) means exactly that the critical edges form a forest in general. Their proof is tricky but simple, and the special case we neeed is even easier. We use their idea in the following proof:

Proof of (3.2). We first prove (3.2)a : Suppose indirectly that C is a circuit consisting of only critical edges, $V(C) \subseteq K$. Let $v_1, \ldots, v_k$ be the vertices of this circuit so that $v_1 S_1 v_2, \ldots, v_{n-1} S_{n-1} v_n, v_n S_n v_1$ are co-forcings. It follows from (2.2) that $S_1 \cup v_2$ is a color-class of $G - v_1$. Replace $S_1 \cup v_2$ by $S_1 \cup v_1$ in the partition into α-stable sets (coloration) of $G - v_1$. We get a coloration of $G - v_2$. In the same way as before, $S_2 \cup v_3$ is a color class in this coloration.

In the general step we have a coloration of $G - v_i$, where $S_i \cup v_{i+1}$ is a color class, and replacing $S_i \cup v_{i+1}$ by $S_i \cup v_i$, we get a coloration of $G - v_{i+1}$ $(i = 1, \ldots, k)$. If $i = k$, then $i + 1 = 1$, and finally we get another coloration of $G - v_1$. This coloration cannot be the same as the one we started with, because that would mean that we permuted entire color classes. (K cannot contain a color class, because it is a clique.)

Thus the critical edges induced by K form a forest. By the Lemma they form a tree, and a.) is proved.

To prove "b." use (2.1) and the fact that in a tree which is not a path there is no Hamiltonian path: if the statement was not true, s would be joined to any other vertex of K with a path containing at most ω vertices; then, according to (2.1), s is adjacent to every vertex of K, and $K \cup s$ is an $\omega + 1$-clique, a contradiction.

Q.E.D.

Note that the extremities of the path provided by (3.2)b are non-adjacent co-forced vertices. (If they were adjacent, $K \cup s$ would form an $\omega + 1$-clique, see (2.1).)

Let us prove now Theorem 1.3:

Proof of Theorem 1.3 . The conditions of (3.2) are satisfied for both G and its complement $\bar{G}$. Apply first (3.2)a. both to G with the clique K, and to $\bar{G}$ with S.

Let $s \in S$ be a vertex of degree one in the tree formed by the critical edges of $\bar{G}$: the critical co-edges induced by $S \setminus s$ also form a tree.

Since S is a stable set, $s \notin Q$ for every forcing of the form $s_1 Q s_2$, $s_1, s_2 \in S$. Thus such forcings remain forcings in $G - s$. Hence the critical co-edges induced by $S \setminus s$ will be the same in $G - s$ as in G: they form a tree. But forced pairs of vertices have the same color in the perfect graph $G - s$, implying that $S \setminus s$ is the subset of a color-class A of $G - s$. Since A is also an α-stable, $A \setminus S$ consists of one vertex, let us denote it by a. as is a critical edge, and by (2.2) K is the unique clique containing a and not containing s: we got in particular that $a \in K$.

We can apply (3.2) b.) now: the critical edges induced by $K \cup s$ form a Hamiltonian path of $K \cup s$, where the endpoints of this path are s and t. Thus s and t are co-forced.

By (3.2)c.) $s(K \setminus t)t$ is a forcing, in particular s and t are non-adjacent.

We proved that G has two non-adjacent co-forced vertices. Since the conditions are "self-complementary", we also have two adjacent forced vertices. Thus Theorem 1.1 can be applied.

Q.E.D.

4. $\omega - 1$-cliques in minimal imperfect graphs

In this section we are going to study $\omega - 1$-cliques with the goal of getting closer to forcings. Therefore we are not interested in arbitrary $\omega - 1$-cliques, only in those which can participate in forcings, that is, might arise as the intersection of two ω-cliques. Unfortunately we cannot prove the existence of such $\omega - 1$-cliques, that is the existence of a forcing (we think this would be a decisive breakthrough concerning Problem 1, and the SPGC). Since we want to have existing objects, we define a more general notion, which, if the Perfect Graph Conjecture is true is what we want; it defines existing objects; it is restrictive enough to provide an interesting structure.

We suppose all over this section that G is partitionable.

We shall say that an $\omega - 1$ element subset of an ω-clique is an *interval*, if it is a class in a partition into ($\alpha - 1$ pieces of) $\omega - 1$-cliques of $G - K \cup S(K)$, for some ω-clique K.

If Q is an ω-clique and $I \subseteq Q$ is an interval, then $x \in V(G)$ with $\{x\} = Q \setminus I$ will be called an *extremity* of I or of Q. The same terminology is used for the intervals of the complement $\bar{G}$ of G. (That is an $\alpha - 1$ element subset of an α-stable will be called interval, if it is a color class of a graph of the form $G - K \cup S(K)$.)

It is easy to check that *$K \setminus x$ is an interval (K is an ω-clique, $x \in K$), if and only if there exists an α-stable set S with $S \cap K = \{x\}$, and $S \cap S(K) \neq \emptyset$.* This will be used without reference.

It will turn out that some *local structural properties of intervals and of their extremities are equivalent to the SPGC.*

(4.1) *Every $\omega - 1$-clique is contained in at most two ω-cliques.*

Indeed, *if K is an $\omega - 1$-clique, and $x \in K$ is arbitrary, then in the coloration of $G - x$ there are exactly two color classes disjoint from K.* Denote these two α-stables by S_1 and S_2. No other clique than $K(S_1)$ or $K(S_2)$ can contain K:
if S is a color class of $G - x$ but $S \neq S_1$, $S \neq S_2$, then $S \cap K \neq \emptyset$, and $K(S)$ is disjoint from this set, whence it does not contain K; if S is not in the coloration of $G - x$ at all, then $x \notin K(S)$, whence it does not contain K again. (4.1) is proved.

Note that in exactly the same way, every $\omega - i$-clique is contained in at most $i + 1$ ω-cliques.

(4.2) *Every ω-clique has at least two extremities.*

Indeed, let K be an arbitrary ω-clique. Choose $v \in V(G) \setminus (K \cup S(K)$. In the coloration of $G - v$ $S(K)$ is not a color class, and consequently there are at least two color-classes intersecting $S(K)$. These intersect K in different points, which are all extremities of intervals.

The following statement is also an easy exercise, we leave it to the reader.

(4.3) *Every vertex of G is the extremity of at least two ω-cliques.*

(4.1) and (4.2) imply that *there exist at least n different intervals.* (Furthermore, every ω-clique can be "represented" by a different interval contained in it ...)

For holes, antiholes (and more generally for "webs") equality holds in these statements. Conversely:

Theorem 4.1 *If G has an ω-clique with exactly two extremities, and there is no small transversal in G, then G is a hole or an antihole.*

Sketch of the Proof. Let K be an ω-clique with exactly two extremities a_1, a_2. This means exactly, (according to the equivalent definition of intervals noted above), that all α-stable sets T, $T \cap S(K) \neq \emptyset$, $T \neq S(K)$, contain a_1 or a_2. In particular, the number of such α-stable sets is at most $2(\alpha - 1)$. (At least one α stable set containing a_1 is disjoint from $S(K)$: the color class of a_1 in $G - a_2$.) Since the α-stable sets are linearly independent, we get: $r(\bar{G} - S(K)) \geq n - 2\alpha + 1$.

Applying (3.1) to $\bar{G} - S(K)$ ($n - \alpha$ replaces n, α replaces ω) *we get equality here*, and we also get that $\bar{G} - S(K)$ is uniquely colorable.

With some work one can also prove that there exists a critical edge from a_1 to $S(K)$. Thus, the conditions of (3.2)b are satisfied, and it follows by (3.2)c that *G has adjacent forced vertices.*

Now, *the condition of Theorem 1.2 is satisfied*, and we are done.

Q.E.D.

Note the similarity between the proof of Theorem 1.3 and 4.1. However, the condition here is not self-complementary, so we have only adjacent forced vertices, and Theorem 1.2 had to be used.

Theorem 4.2 *If G has a vertex which is the extremity of exactly two ω-cliques, and there is no small transversal, then G is a hole or an antihole.*

Note the kind of polarity relation between (4.2) and (4.3) and between Theorem 4.1 and 4.2. (This can be given a precise sense: Tucker has noted that the intersection graph of a partitionable graph is also partitionable.) However, we cannot prove one statement from the other, and Theorem 4.1 seems to be much more difficult: we use Theorem 1.2 in an essential way in the proof, whereas this is not necessary in the (here omitted) proof of

Theorem 4.2 . The reason why we cannot prove one of these statements from the other is that we do not know whether the polarity relation keeps small transversals.

5. Reformulations of the Strong Perfect Graph Conjecture

The goal of this section is to collect new reformulations of the SPGC which follow from the results of the paper.

We distinguish two levels of reformulations: to prove the SPGC from the second level statements, Theorem 1.2 seems to be essential, whereas the first level statements imply SPGC more easily.

Anyone of the following statements is equivalent to the Strong Perfect Graph Conjecture:

1st level

If G is minimal imperfect, then for every ω-clique K, $G - K$ is uniquely colorable;

If G is minimal imperfect, then the number of intervals is $|V(G)|$;

If G is minimal imperfect, then the set of intervals is linearly independent;

If G is minimal imperfect, then every interval has two extremities;

If G is minimal imperfect, then every ω-clique has two extremities;

If G is minimal imperfect, then every vertex is the extremity of two intervals;

If G is minimal imperfect, then there exists a vertex which is the extremity of two intervals;

2nd level

If G is minimal imperfect, then there exists an ω-clique K, such that $G - (K \cup S(K))$ and its complement are uniquely colorable;

If G is minimal imperfect, then there exist two adjacent forced vertices (or more generally, two co-forcings and one forcing like in Theorem 2.1) in G or $\bar{G}$;

If G is minimal imperfect, then there exists a forcing xKy such that $G - (K \cup \{x, y\})$ is uniquely colorable;

If G is minimal imperfect, then there exists a forcing xKy such that every clique intersecting K also intersects x or y.

If G is minimal imperfect, then there exists an interval which has exactly two extremities.

Can any of these be proved for a subclass for which the SPGC is open ?

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