Standard Pairs and Group Relaxations in Integer Programming

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Abstract

The main result of this paper is a non-Buchberger algorithm for constructing initial ideals and Gröbner bases of toric ideals, based on the connections between toric ideals and integer programming. The tools used are those of standard pair decompositions of standard monomials of a toric initial ideal, localizations of such ideals at their associated primes and group relaxations of integer programs. We give an algorithm for constructing standard pair decompositions, provide degree bounds for certain elements in the reduced Gröbner bases of toric ideals, and derive bounds on the arithmetic degree of initial ideals of monomial curves. We also exhibit new results for the localizations of initial ideals arising from toric ideals of codimension two.

1 Introduction and preliminaries

In this paper we study the initial ideals of a *toric ideal*. For any monomial ideal M, it is known that the *standard pair decomposition* [STV] of its standard monomials is closely related to the associated primes of M and hence to its *arithmetic degree* [BM]. When M is the initial ideal of a toric ideal I_A with respect to a cost vector c, its standard monomials are in bijection with the optimal solutions to the family of *integer programs* minimize$\{c \cdot x : Ax = b, x \in \mathbf{N}^n\}$, as b varies. The arithmetic degree of M is a complexity measure for these integer programs and gives an upper bound on the Castelnuovo-Mumford *regularity* of the toric ideal. The associated primes of M form a subposet in the face lattice of the *regular triangulation*, Δ_c, of the columns of A induced by c, and the localizations of M at its associated primes correspond to *group relaxations* of the above integer programs. Using these connections, we obtain a non-Buchberger algorithm for constructing initial ideals and Gröbner bases of toric ideals. The algorithm inputs the regular triangulation Δ_c and builds M by constructing its successive localizations by making repeated calls to an optimization oracle. We also give an algorithm to construct standard pair decompositions, derive bounds for the arithmetic degree of one dimensional toric initial ideals and give new results for localizations of initial ideals when the toric ideal is of codimension two. We first describe the setup.

Let $A = [a_1, \ldots, a_n] \in \mathbf{Z}^{d \times n}$ be a matrix of rank d where a_j denotes the jth column of A. The *toric ideal* of A, denoted I_A, is the d-dimensional prime ideal that is the kernel of the homomorphism $k[x_1, \ldots, x_n] \rightarrow k[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ such that $x_j \mapsto t^{a_j}$. The name comes from the fact that the affine variety $\mathcal{V}(I_A)$ is a *toric variety* in k^n [Ful]. Let $\mathbf{N}$ denote the set of non-negative integers. The ideal I_A is generated by *binomials* of the form $x^{u^+} - x^{u^-}$ where $u = u^+ - u^-$, $u^+, u^- \in \mathbf{N}^n$ lies in the $(n-d)$-dimensional lattice $ker_{\mathbf{Z}}(A) := ker(A) \cap \mathbf{Z}^n = \{u \in \mathbf{Z}^n : Au = 0\}$ and hence its Gröbner bases [AL], [Buc], [CLO] also consist of such binomials. We assume that $ker_{\mathbf{Z}}(A) \cap \mathbf{N}^n = \{0\}$ which implies that I_A is homogeneous with respect to some positive grading. Toric ideals play a crucial role in integer programming [CT], [Stu2] which is the context of this paper.

Consider the integer program $IP_{A,c}(b) := minimize\{c \cdot x \; : \; Ax = b, x \in \mathbf{N}^n\}$ where the *right hand side* vector b lies in the monoid $\mathbf{N}(A) := \{Ax : x \in \mathbf{N}^n\}$ and the *cost vector* $c \in \mathbf{R}^n$. A vector $x \in \mathbf{N}^n$ such that $Ax = b$ is called a *feasible solution* to $IP_{A,c}(b)$ and the polyhedron $P_b^I :=$ convex hull $\{x \in \mathbf{N}^n : Ax = b\}$, is called the *$b$-fiber* of A. Clearly, $IP_{A,c}(b)$ is *feasible* (i.e., $P_b^I \cap \mathbf{N}^n \neq \emptyset$) if and only if $b \in \mathbf{N}(A)$. A feasible solution to $IP_{A,c}(b)$ at which the functional $c \cdot x$ is minimized is called an *optimal* solution to $IP_{A,c}(b)$. Since $ker(A) \cap \mathbf{R}^n_{\geq 0} = \{0\}$, P_b^I is a polytope for all $b \in \mathbf{N}(A)$. Hence every cost vector $c \in \mathbf{R}^n$ has a bounded optimum over a fiber P_b^I. We say that c is a *generic* cost vector for A if the optimal solution to $IP_{A,c}(b)$ is a unique vertex of P_b^I for each $b \in \mathbf{N}(A)$. A cost vector c is generic for A if and only if the initial ideal of I_A with respect to c, denoted as $in_c(I_A)$, is a monomial ideal in $k[\mathbf{x}] := k[x_1, \ldots, x_n]$. We will always assume that c is generic for A. Let $IP_{A,c}$ denote the family of all integer programs $IP_{A,c}(b)$ obtained by varying $b \in \mathbf{N}(A)$ and let $\mathcal{G}_c$ be the reduced Gröbner basis of the toric ideal I_A with respect to c. It can be shown that if A satisfies the assumptions above and c is generic for A, then $\mathcal{G}_c$ is well defined regardless of whether c induces a term order on $k[\mathbf{x}]$ or not. We fix the above setting and assumptions on A, b and c in the rest of this paper. Conti and Traverso [CT] showed that $\mathcal{G}_c$ solves all programs in $IP_{A,c}$ as follows: Given a feasible solution α to the integer program $IP_{A,c}(b)$, the optimal solution β to $IP_{A,c}(b)$ is the exponent vector of the *normal form* of x^α with respect to $\mathcal{G}_c$. The reduced Gröbner basis $\mathcal{G}_c$ is a *test set* for the family $IP_{A,c}$ [Tho] in the sense of [BJ], [CGST], [Gra], [Sca] and [Sch].

Proposition 1.1 *[Tho] (i) The exponent vectors of the monomials in the initial ideal $in_c(I_A)$ are precisely the non-optimal solutions to the programs in $IP_{A,c}$.*
(ii) The monoid $\mathbf{N}(A)$ is in bijection with the set of exponent vectors of standard monomials of $in_c(I_A)$, denoted $\mathcal{S}_c$, via $b \leftrightarrow \beta_b$ where β_b is the unique optimal solution to $IP_{A,c}(b)$.

Let $\mathcal{A} := \{a_1, \ldots, a_n\}$ be the vector configuration in $\mathbf{Z}^d$ consisting of the columns of A. For a subset $\sigma \subseteq \mathcal{A}$, we let $pos(\sigma)$ denote the cone generated by σ. A *triangulation* Δ of $\mathcal{A}$ is a collection of subsets of $\mathcal{A}$ such that $\{pos(\sigma) : \sigma \in \Delta\}$ is a set of cones in a simplicial fan whose support is $pos(\mathcal{A})$. The simplicial nature allows us to write Δ_c as the set of its maximal simplices. The elements of Δ are called the *faces* or *cells* of Δ. For convenience we identify $\mathcal{A}$ with the set of indices $[n] := \{1, \ldots, n\}$ and any subset of $\mathcal{A}$ by the corresponding subset $\sigma \subseteq [n]$. A generic cost vector c induces the *regular triangulation* Δ_c of $\mathcal{A}$ [BFS],[GKZ] as follows: σ is a face of Δ_c if there exists a vector $y \in \mathbf{R}^d$ such that $a_j \cdot y = c_j$ whenever $j \in \sigma$ and $a_j \cdot y < c_j$ otherwise. The *Stanley-Reisner* ideal of Δ_c is the square-free monomial ideal $\langle x_{i_1} \cdots x_{i_r} : \{i_1, \ldots, i_r\}$ is a minimal *non-face* of $\Delta_c \rangle$ where $\tau \subseteq [n]$ is a minimal non-face of Δ_c if τ is not in Δ_c but every proper subset of τ is in Δ_c.

Theorem 1.2 *[Stu1] The radical of the initial ideal $in_c(I_A)$ is the Stanley-Reisner ideal of the regular triangulation Δ_c.*

This paper is organized as follows. In Section 2 we describe two known coverings of the standard monomials of a monomial ideal M: the standard pair decomposition and the *Stanley decomposition* [Sta]. It was shown in [STV] that the former decomposition is related to the associated primes of M and provides a combinatorial formula for the arithmetic degree of M. We show that the cardinality of a minimal Stanley decomposition is an upper bound to the arithmetic degree of M. We also give an algorithm to compute standard pair decompositions. The associated primes of $\mathrm{in}_c(I_A)$ are monomial ideals generated by subsets of $\{x_1, \ldots, x_n\}$. The set of all associated primes of $\mathrm{in}_c(I_A)$, denoted $\mathrm{Ass}(\mathrm{in}_c(I_A))$, forms a partially ordered set (poset) with respect to inclusion: for $P, Q \in \mathrm{Ass}(\mathrm{in}_c(I_A))$, we write $P \leq Q$ if $P \supseteq Q$. Hence, the minimal primes of $\mathrm{in}_c(I_A)$ are the maximal elements of $\mathrm{Ass}(\mathrm{in}_c(I_A))$. The poset $\mathrm{Ass}(\mathrm{in}_c(I_A))$ is contained (often strictly) in the face lattice of the regular triangulation Δ_c. We provide a family of toric initial ideals for which the two posets coincide. In Section 3 we describe the connections between localizations of $in_c(I_A)$ at its associated primes and group relaxations [Gom2] of programs in $IP_{A,c}$ along the lines of [SWZ]. In our sense, a group relaxation of $IP_{A,c}(b)$ is the *lattice program* obtained by relaxing non-negativity restrictions on any subset of variables whose indices form a cell in Δ_c. Such relaxations were studied in [Wol]. Degree bounds for certain Gröbner basis elements are derived. We also obtain bounds on the arithmetic degree of $in_c(I_A)$ in the special case when A is a $1 \times n$ matrix. In Section 4 we consider initial ideals arising from toric ideals of codimension two. An associated prime of such an initial ideal has codimension at most three. We show that such an initial ideal is uniquely determined by its localizations at minimal primes, a property that fails in higher codimension. We then examine the behaviour of localizations of such initial ideals at minimal primes as c is varied while keeping the radical of the initial ideal fixed. In Section 5 we describe the non-Buchberger algorithm to compute $in_c(I_A)$ and hence $\mathcal{G}_c$, using Δ_c as input. This algorithm is based on the results in Sections 2 and 3. We conclude with computational results for this algorithm.

2 Decompositions of standard monomials, associated primes and arithmetic degree

We first recall the standard pair decomposition [STV] of (the standard monomials of) an arbitrary monomial ideal M which allows a combinatorial description of the arithmetic degree [BM] of M. For $m \in \mathbf{N}^n$, let the *support* of m be denoted by $supp(m) := \{i : m_i \neq 0\}$ and let $supp(x^m) := supp(m)$.

Definition 2.1 *For a monomial $x^m \in k[\mathbf{x}]$ and $\sigma \subseteq [n]$, we say that (x^m, σ) is an* **admissible pair** *of a monomial ideal M if (i) $supp(m) \cap \sigma = \emptyset$ and (ii) every monomial in $x^m \cdot k[\mathbf{x}_\sigma] := x^m \cdot k[x_j : j \in \sigma]$ is a standard monomial of M. A monomial in $x^m \cdot k[\mathbf{x}_\sigma]$ is said to be covered by the standard pair (x^m, σ).*

There is a natural partial order on the set of all admissible pairs of M given by $(x^m, \sigma) \leq (x^{m'}, \sigma')$ if and only if x^m divides $x^{m'}$ and $supp(x^{m'}/x^m) \cup \sigma' \subseteq \sigma$.

Definition 2.2 *An admissible pair (x^m, σ) of M is called a* **standard pair** *of M if it is a minimal element in the poset of all admissible pairs with respect to the above partial order.*

The standard pairs of M induce a unique covering of the set of standard monomials of M which we refer to as the standard pair decomposition of M. It was shown in [STV] that the arithmetic degree of M, denoted *arithdeg(M)*, equals the number of standard pairs of M. This paper also gives a bound for $arithdeg(M)$ in terms of the degrees of the minimal generators and dimension of M. Algebraically, $arithdeg(M) = \sum mult(P)deg(P)$ where the sum is over all homogeneous prime ideals P in $k[\mathbf{x}]$, $mult(P)$ is the length of the largest ideal of finite length in the ring $k[\mathbf{x}]_P/Mk[\mathbf{x}]_P$, and $deg(P)$ is the geometric degree of the variety of P. The integer $mult(P)$ is positive if and only if P is an associated prime of M, and since in our situation M is a monomial ideal, every associated prime of M is of the form $p_\sigma := \langle x_j : j \notin \sigma \rangle$ for some $\sigma \subseteq [n]$ which implies that $deg(p_\sigma) = 1$.

Theorem 2.3 *[STV] Given a monomial ideal M and $p_\sigma = \langle x_j : j \notin \sigma \rangle$ for some $\sigma \subseteq [n]$:*
(i) p_σ is an associated prime of M if and only if M has a standard pair of the form $(\cdot, \sigma)$. If p_σ is an associated prime of M then $mult(p_\sigma)$ equals the number of standard pairs of M of the form $(\cdot, \sigma)$. Hence, $arithdeg(M) = \sum mult(p_\sigma)$, where the sum is over all associated primes of M.
(ii) p_σ is a minimal prime of M if and only if $(1, \sigma)$ is a standard pair of M.

Example 2.4 *The standard pairs of a monomial ideal.*
The ideal $M = \langle z^{11}, xz, x^4y^7, x^5y^5, x^6y^3, x^7y, x^8 \rangle \subset k[x, y, z]$ has 14 standard pairs with $\sigma = \{2\}$: $\{(z^{10}, \sigma), \ldots, (z, \sigma), (1, \sigma), (x, \sigma), (x^2, \sigma), (x^3, \sigma)\}$ and 16 standard pairs of the form $(x^i y^j, \emptyset)$ where $i = 4, \ldots, 7$ and $j = 0, \ldots, (14 - 2i)$. Hence $arithdeg(M) = 30$ and the associated primes of M are $p_{\{2\}} = \langle x, z \rangle$ and $p_\emptyset = \langle x, y, z \rangle$. Figure 1 shows the standard pair decomposition of M. □

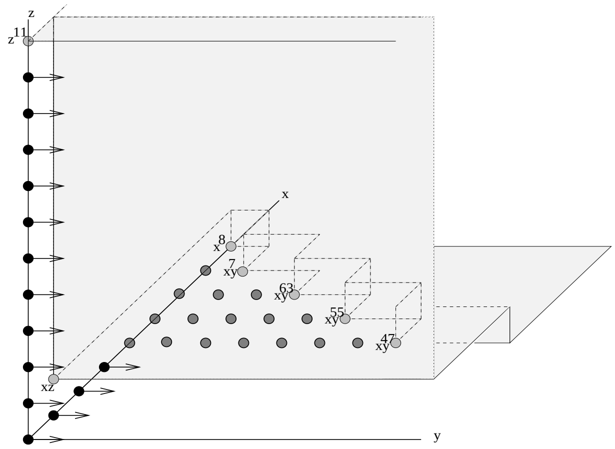

Figure 1: Standard pair decomposition.

We first provide an algorithm to compute the standard pair decomposition of a monomial ideal.

Algorithm 2.5 *How to compute the standard pairs of a monomial ideal* $M \subset k[x_1, \ldots, x_n]$.

INPUT: The minimal generators of M.
OUTPUT: The set S of standard pairs of M.
Let $k := 0$, $S := \emptyset$, and $M_0 := M$.
While $M_k \neq \langle 1 \rangle$ do

- $w := 1$, $\sigma := \{1, \ldots, n\}$, and $\tilde{M} := M_k$.
- While $\tilde{M} \neq \{0\}$ do
  1. Let x^m be a minimal generator of $\tilde{M}$ and choose $x_i,\ i \in \operatorname{supp}(x^m)$.
  2. $w := w \cdot x^m/x_i$, and $\sigma := \sigma \backslash \{i\}$.
  3. $\tilde{M} := (\tilde{M} : x^m/x_i) \cap k[x_j : j \in \sigma]$.
- If $w = x^u$, let $w' := \prod_{i \in supp(x^u) \cap \sigma^c} x^{u_i}$.
- If $(x^v, \tau) \not\leq (w', \sigma)$ for every $(x^v, \tau) \in S$, then $S := S \cup \{(w', \sigma)\}$.
- $M_{k+1} := M_k + \langle w \rangle$ and $k := k + 1$.

Proof of correctness: We first show that each time the inner while loop terminates, $M_k : w = p_\sigma$. Suppose $\sigma^c = \{1, 2, \ldots, r\}$, and as the algorithm passed through the inner while loop it detected x_i's in this order, i.e., first x_1, then $x_2, \ldots$, and finally x_r. Similarly, suppose $\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_r$ are the ideals the loop computed, and $x^{m_1}, x^{m_2}, \ldots, x^{m_r}$ are the respective minimal generators it picked. Then $w = (x^{m_1}/x_1)(x^{m_2}/x_2) \cdots (x^{m_r}/x_r)$. First we show that wx_i is in M_k for all $i = 1, \ldots, r$. Clearly, $(x^{m_i}/x_i)x_i$ is in $\tilde{M}_i$. But since $\tilde{M}_i = (\tilde{M}_{i-1} : x^{m_{i-1}}/x_{i-1}) \cap k[x_i, \ldots, x_n]$, $(x^{m_{i-1}}/x_{i-1})(x^{m_i}/x_i)x_i$

is in $\tilde{M}_{i-1}$. Continuing in this fashion we see $(x^{m_1}/x_1)\cdots(x^{m_{i-1}}/x_{i-1})(x^{m_i}/x_i)x_i$ is in $\tilde{M}_1$, which is M_k. This shows $wx_i \in M_k$ for all i. When the inner while loop terminates $0 = \tilde{M}_{r+1} = (\tilde{M}_r : x^{m_r}/x_r) \cap k[x_{r+1}, \ldots, x_n]$. Hence, no monomial in $x_{r+1}, \ldots, x_n$ multiplies w into M_k. Hence $M_k : w = p_\sigma$. Also clearly, (w', σ) is a standard pair of M_k. A standard pair of M_k is either a standard pair of M or covered by a standard pair of M. So we only need to show that if (w', σ) is covered by a standard pair of M, then it is covered by one in S. Suppose (x^v, τ) is a standard pair of M which covers (w', σ). If (x^v, τ) was a standard pair of M_k, that would be a contradiction since a standard pair cannot cover another standard pair. Therefore in the course of the algorithm a generator that was added to M_k has "destroyed" (x^v, τ). This happens when that standard pair is added to S. $\square$

There is a general decomposition scheme for any finitely generated k-algebra called a Stanley decomposition [Sta] [SW] which we now compare with the standard pair decomposition described above. The k-algebra of interest here is $k[\mathbf{x}]/M$ which as a k-vector space is spanned by the standard monomials of M.

Definition 2.6 *Let M be a monomial ideal in $k[\mathbf{x}]$. A* **Stanley decomposition** *of the standard monomials of M is a set of (not necessarily admissible) pairs (x^α, σ) such that these pairs partition the standard monomials, i.e., $x^\alpha \cdot k[\mathbf{x}_\sigma] \cap x^\beta \cdot k[\mathbf{x}_\tau] = \emptyset$ for two pairs (x^α, σ) and (x^β, τ). Equivalently, a Stanley decomposition is a collection of pairs (x^{α_i}, σ_i), $i = 1, \ldots, s$ such that $k[\mathbf{x}]/M \simeq \bigoplus_{i=1}^s x^{\alpha_i} \cdot k[\mathbf{x}_{\sigma_i}]$. A Stanley decomposition with minimal number of pairs is called a* **minimal** *Stanley decomposition.*

Example 2.7 *A Stanley decomposition of the standard monomials of Example 2.4:*
Using the algorithm described in Lemma 2.4 of [SW] we find that the following is a Stanley decomposition of the standard monomials of M (see Figure 2):

$$\bigcup_{i=1}^{10}(z^i, \{2\}) \ \cup\ \bigcup_{j=0}^{3}(y^7x^j, \{2\}) \ \cup\ \bigcup_{j=0}^{7}(x^j, \emptyset) \ \cup\ \bigcup_{\substack{j=0\\ i=1,2}}^{6}(y^ix^j, \emptyset) \ \cup\ \bigcup_{\substack{j=0\\ i=3,4}}^{5}(y^ix^j, \emptyset) \ \cup\ \bigcup_{\substack{j=0\\ i=5,6}}^{4}(y^ix^j, \emptyset).$$

This Stanley decomposition needs a total of 14 pairs of the form $(\cdot, \{2\})$ and 44 pairs of the form $(\cdot, \emptyset)$. Notice that the covering given in Figure 1 is also a Stanley decomposition. $\square$

Proposition 2.8 *The arithmetic degree of a monomial ideal M is at most the cardinality of a minimal Stanley decomposition.*

Proof: Let (x^α, σ) be a pair in a given Stanley decomposition of M and let $x^{\alpha'} = \prod_{i \in supp(x^\alpha) \cap \sigma^c} x^{\alpha_i}$. Then $(x^{\alpha'}, \sigma)$ is an admissible pair and we can find a standard pair (x^β, τ) such that $x^\alpha \cdot k[\mathbf{x}_\sigma] \subseteq x^{\alpha'} \cdot k[\mathbf{x}_\sigma] \subseteq x^\beta \cdot k[\mathbf{x}_\tau]$. Since standard pairs found this way cover the standard monomials of M, we conclude that arithdeg(M) is at most the cardinality of a minimal Stanley decomposition. $\square$

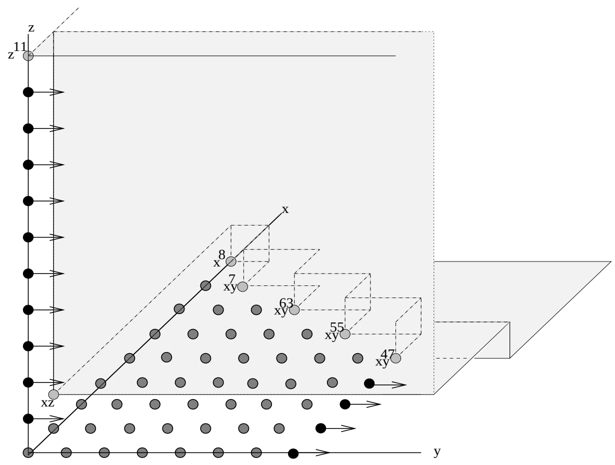

Figure 2: A Stanley decomposition of the standard monomials of M in Example 2.4.

Remark 2.9 *The cardinality of a minimal Stanley decomposition can be strictly larger than the arithmetic degree.*
Let $M = \langle x_1x_3, x_1x_4, x_2x_3, x_2x_4 \rangle \subset k[x_1, x_2, x_3, x_4]$. This ideal has two standard pairs, namely, $(1, \{1,2\})$ and $(1, \{3,4\})$. The only monomial covered by both standard pairs is the monomial 1. In order to avoid this in any Stanley decomposition one needs to introduce a new pair. For instance, $(1, \{1,2\}$, $(x_3, \{3,4\})$ and $(x_4, \{4\})$ constitute a minimal Stanley decomposition. We thank Bernd Sturmfels for pointing out this example to us.

In the rest of this section we focus our attention on monomial ideals that arise as initial ideals of toric ideals. For such ideals, the results in Theorem 2.3 can be augmented as follows.

Theorem 2.10 *([Stu2], Chapter 8) Let $in_c(I_A)$ be the initial ideal of the toric ideal I_A with respect to the generic cost vector c.*
(a) If p_σ is an associated prime of $in_c(I_A)$, then σ is a cell in the regular triangulation Δ_c.
(b) The prime ideal p_σ is a minimal prime of $in_c(I_A)$ if and only if σ is a maximal cell of Δ_c. In this case, $mult(p_\sigma) = vol(\sigma)$, where $vol(\sigma)$ is the normalized volume of σ in Δ_c.

The *normalized volume* of a maximal cell $\sigma \in \Delta_c$ is the quotient of the determinant of A_σ, the submatrix of A whose columns are indexed by σ, by the g.c.d. of the determinants of all A_τ such that τ is a maximal cell in Δ_c. By the above theorem, $arithdeg(in_c(I_A)) = \sum_{\sigma \in \Delta_c} mult(p_\sigma)$. We note that the converse to Theorem 2.10 (a) is false. If τ is a cell of Δ_c that is not of maximal dimension, then p_τ may or may not be an associated (embedded) prime of $in_c(I_A)$.

Example 2.11 Consider $A := \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and the lexicographic term order $a \succ b \succ c \succ d$ where a, b, c, d are the variables associated to the four columns of A. Then $in_{\succ}(I_A) = \langle bd, ad, ac \rangle$ and written in full, $\Delta_{\succ} = \{\{1,2\},\{2,3\},\{3,4\},\{1\},\{2\},\{3\},\{4\},\emptyset\}$. However, this initial ideal has no embedded primes. Its minimal primes are $p_{\{1,2\}} = \langle c, d \rangle$, $p_{\{2,3\}} = \langle a, d \rangle$ and $p_{\{3,4\}} = \langle a, b \rangle$ and each has multiplicity one which is the normalized volume of each maximal face in $\Delta_{\succ}$. $\square$

We now construct a family of matrices and term orders for which p_σ is an associated prime of $in_c(I_A)$ for *each* $\sigma \in \Delta_c$. Consider the matrix

$$A_q := \begin{bmatrix} q & q-1 & q-2 & \cdots & (q+1)/2 & (q-1)/2 & \cdots & 2 & 1 & 0 \\ 0 & 1 & 2 & \cdots & (q-1)/2 & (q+1)/2 & \cdots & q-2 & q-1 & q \end{bmatrix}$$

where $q \in \mathbf{N}$, q odd, $q \geq 5$. The toric variety $\mathcal{V}(I_{A_q})$ is the rational normal curve of degree q in $\mathbf{P}^q$. For ease of exposition, we label the variables associated with the columns of A_q as $x_0, x_1, x_2, \ldots, x_{(q-1)/2}, y_{(q-1)/2}, \ldots, y_2, y_1, y_0$. Consider the lexicographic ordering $\succ$ defined "cyclically" as follows: $x_{(q-1)/2} \succ y_1 \succ x_{(q-3)/2} \succ y_2 \succ x_{(q-5)/2} \succ y_3 \succ \cdots \succ x_1 \succ y_{(q-1)/2} \succ x_0 \succ y_0$. We index the vector configuration given by A_q similarly: $\mathcal{A}_q := \{0, 1, 2, \ldots, (q-1)/2, ((q-1)/2)', \ldots, 2', 1', 0'\}$. Then the regular triangulation $\Delta_{\succ}$ consists of the unique maximal face $\sigma = \{0, 0'\}$ and all its subfaces. Theorem 2.10 (b) implies that $mult(p_{\{0,0'\}}) = q$.

Proposition 2.12 *For each face τ of the regular triangulation $\Delta_{\succ}$ of $\mathcal{A}_q$ defined as above, the ideal p_τ is an associated prime of $in_{\succ}(I_{A_q})$.*

Proof: By Theorem 2.10, it suffices to show that for each lower dimensional face $\tau_1 = \{0\}, \tau_2 = \{0'\}, \tau_0 = \emptyset$ in $\Delta_{\succ}$, there is a standard pair for $in_{\succ}(I_{A_q})$ of the form $(\cdot, \tau)$.
(i) $(x_1, \{0\})$ *is a standard pair of* $in_{\succ}(I_{A_q})$: The exponent vector of $x_0^r x_1$ is the unique lattice point in the $((q-1)+rq, 1)^t$-fiber of A_q for all values of $r \in \mathbf{N}$. Hence all such monomials are standard with respect to $in_{\succ}(I_{A_q})$ showing that $(x_1, \{0\})$ is an admissible pair for $in_{\succ}(I_{A_q})$. The binomial $x_1 y_0 - y_{(q-1)/2}^2 \in I_{A_q}$ has initial term $x_1 y_0$ with respect to $\succ$ and hence $(x_1, \{0, 0'\})$ is not admissible for $in_{\succ}(I_{A_q})$. Hence, $(x_1, \{0\})$ is a standard pair of $in_{\succ}(I_{A_q})$.
(ii) $(y_1, \{0'\})$ *is a standard pair of* $in_{\succ}(I_{A_q})$: As above, the fiber of A_q containing the exponent vector of any monomial $y_1 y_0^r$, $r \in \mathbf{N}$ contains no other lattice point which shows that $(y_1, \{0'\})$ is admissible for $in_{\succ}(I_{A_q})$. This is a standard pair for $in_{\succ}(I_{A_q})$ since the binomial $y_1 x_0 - y_3 x_2 \in I_{A_q}$ has initial term $y_1 x_0$ with respect to $\succ$.
(iii) $(x_{(q-1)/2}, \emptyset))$ *is a standard pair of* $in_{\succ}(I_{A_q})$: Consider $x_{(q-1)/2} x_0 - x_{\lfloor (q-1)/4 \rfloor} x_{\lceil (q-1)/4 \rceil}$ and $x_{(q-1)/2} y_0 - y_{\lfloor (q+1)/4 \rfloor} y_{\lceil (q+1)/4 \rceil}$ in I_{A_q}. The leading term of each binomial with respect to $\succ$ is the first term. Hence neither $(x_{(q-1)/2}, \{0\})$ nor $(x_{(q-1)/2}, \{0'\})$ are admissible for $in_{\succ}(I_{A_q})$. Moreover, the fiber containing the exponent vector of $x_{(q-1)/2}$ has no other lattice points which shows that $(x_{(q-1)/2}, \emptyset)$ is a standard pair of $in_{\succ}(I_{A_q})$. $\square$

Example 2.13 Consider $A_9 = \begin{bmatrix} 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$. The variables associated with the columns of A_9 are $x_0, x_1, \ldots, x_4, y_4, \ldots, y_0$. The standard pairs of I_{A_9} are given below.

- $\sigma = \{0, 0'\}$: $(y_4^8, \sigma), (y_4^7, \sigma), (y_4^6, \sigma), (y_4^5, \sigma), (y_4^4, \sigma), (y_4^3, \sigma), (y_4^2, \sigma), (y_4, \sigma), (1, \sigma)$.
- $\tau_1 = \{0\}$: $(x_1^4, \tau_1), (x_1^3 y_4^2, \tau_1), (x_1^3 y_4, \tau_1), (x_1^3, \tau_1), (x_1^2 y_4^4, \tau_1), (x_1^2 y_4^3, \tau_1), (x_1^2 y_4^2, \tau_1), (x_1^2 y_4, \tau_1),$ $(x_1^2, \tau_1), (x_1 y_4^6, \tau_1), (x_1 y_4^5, \tau_1), (x_1 y_4^4, \tau_1), (x_1 y_4^3, \tau_1), (x_1 y_4^2, \tau_1), (x_1 y_4, \tau_1), (x_1, \tau_1)$.
- $\tau_2 = \{0'\}$: $(y_2 y_3, \tau_2), (y_2, \tau_2), (y_3^2 y_4^2, \tau_2), (y_3^2 y_4, \tau_2), (y_3^2, \tau_2), (y_3 y_4^5, \tau_2), (y_3 y_4^4, \tau_2),$ $(y_3 y_4^3, \tau_2), (y_3 y_4^2, \tau_2), (y_3 y_4, \tau_2), (y_3, \tau_2), (y_1, \tau_2)$.
- $\tau_0 = \emptyset$: $(x_2, \tau_0), (x_3, \tau_0), (x_2^2, \tau_0), (x_2 y_3, \tau_0), (x_2 x_1^2, \tau_0), (x_2 x_1, \tau_0), (y_3^2 x_1, \tau_0),$ $(y_3 x_1^2 y_4, \tau_0), (y_3 x_1^2, \tau_0), (y_3 x_1 y_4^3, \tau_0), (y_3 x_1 y_4^2, \tau_0), (y_3 x_1 y_4, \tau_0), (y_3 x_1, \tau_0), (x_4, \tau_0)$.

Thus $mult(p_\sigma) = 9$, $mult(p_{\tau_1}) = 12$, $mult(p_{\tau_2}) = 16$, $mult(p_{\tau_0}) = 14$ and $arithdeg(in_\succ(I_{A_9})) = 51$.

3 Group relaxations in integer programming and localization

The standard pair decomposition of $in_c(I_A)$ reduces the problem of solving integer programs in $IP_{A,c}$ to solving systems of linear equations: if β is the optimal solution to the program $IP_{A,c}(b)$, then the monomial x^β is covered by some standard pair (x^u, σ) and hence $x^\beta = x^u \cdot \mathbf{x}_\sigma^{\beta_\sigma}$ for some exponent vector β_σ. The exponent vector β_σ is the unique solution to the linear system $A_\sigma x_\sigma = b - A_{supp(u)} u$. Therefore, in order to solve $IP_{A,c}(b)$ it suffices to solve the arithdeg($in_c(I_A)$)-many linear systems of equations as above and picking as β_σ any solution that is both integral and non-negative. Hence $arithdeg(in_c(I_A))$ is a measure of the complexity of $IP_{A,c}$, and in Section 5 we will see that it also measures the complexity of computing the reduced Gröbner basis $\mathcal{G}_c$ of I_A.

In this section we connect a certain relaxation technique in integer programming, called *group relaxation* [Gom2], to localizations of initial ideals of toric ideals. This connection was first presented in [SWZ] and we will follow the treatment there. We start with a generalization of integer programs.

Definition 3.1 *Suppose $\mathcal{L}$ is a sublattice of $\mathbf{Z}^n$, $w \in \mathbf{R}^n$ and $v \in \mathbf{N}^n$. The* **lattice program** *$Lat_{\mathcal{L},w}(v)$ defined by this data is*

$$minimize \;\; w \cdot u : \; u \equiv v \; mod \, \mathcal{L}, \;\; u \in \mathbf{N}^n.$$

Integer programs are lattice programs: If v is a solution to $IP_{A,c}(b)$, then $IP_{A,c}(b)$ is the program:

$$\text{minimize} \;\; c \cdot x : \; x \equiv v \text{ mod } \ker_{\mathbf{Z}}(A), \;\; x \in \mathbf{N}^n.$$

Gröbner basis methods for integer programs can be extended to solve lattice programs. Given the lattice $\mathcal{L}$ and a cost vector w, we first construct the *lattice ideal* $I_\mathcal{L} = \langle \mathbf{x}^\alpha - \mathbf{x}^\beta : \alpha - \beta \in \mathcal{L}, \alpha, \beta \in$

$\mathbf{N}^n\rangle$. We then compute the reduced Gröbner basis of $I_{\mathcal{L}}$ with respect to a refined term order $\succ_w$, denoted as $\mathcal{G}_{\succ_w}$. For a particular lattice program $\mathrm{Lat}_{\mathcal{L},w}(v)$, the optimal solution is the exponent vector of the normal form of x^v with respect to $\mathcal{G}_{\succ_w}$.

Let $\sigma \subseteq [n]$ and $\pi_\sigma : \mathbf{Z}^n \longrightarrow \mathbf{Z}^{n-|\sigma|}$ be the coordinate projection map where the coordinates indexed by σ are eliminated. In the rest of this section $\mathcal{L} := ker_{\mathbf{Z}}(A)$ and $\mathcal{L}_\sigma := \pi_\sigma(\ker_{\mathbf{Z}}(A))$.

Lemma 3.2 *The map $\pi_\sigma : \mathcal{L} \longrightarrow \mathcal{L}_\sigma$ is an isomorphism if $rank(A_\sigma) = |\sigma|$.*

Definition 3.3 *Consider the integer program $IP_{A,c}(b)$ and a feasible solution v to this program. Let τ be a face of Δ_c and σ be any maximal face of Δ_c containing τ. Then the* **group relaxation** *of $IP_{A,c}(b)$ with respect to τ is the lattice program $Lat_{\mathcal{L}_\tau,\tilde{c}_\tau}(\pi_\tau(v))$ where $\tilde{c}_\tau := \pi_\tau(c - c_\sigma A_\sigma^{-1} A)$.*

Group relaxations are always defined with respect to a face τ of Δ_c. This guarantees that $rank(A_\tau) = |\tau|$. The next theorem justifies the word "relaxation" in the above definition.

Theorem 3.4 *Suppose $u' \in \mathbf{N}^{n-|\tau|}$ is the optimal solution to the group relaxation of $IP_{A,c}(b)$ with respect to τ. Then there exists a unique $u \in \mathbf{Z}^n$ such that $A(u-v) = 0$ for any feasible solution v to $IP_{A,c}(b)$ and $\pi_\tau(u) = u'$. If $u \geq 0$ then it is the optimal solution to $IP_{A,c}(b)$.*

Proof: Since π_τ is an isomorphism of $\mathcal{L}$ and $\mathcal{L}_\tau$, $u' - \pi_\tau(v)$ uniquely lifts to $(u-v) \in \mathcal{L}$. This proves the first claim. If $u \in \mathbf{N}^n$, then it is a feasible solution to $\mathrm{IP}_{A,c}(b)$. We need to show that it is actually optimal. Suppose w is another feasible solution with $c \cdot (u-w) > 0$. This implies that $(c - c_\sigma A_\sigma^{-1} A) \cdot (u-w) > 0$, and hence $\tilde{c}_\tau \cdot (u' - \pi_\tau(w)) > 0$. This contradicts the fact that u' is the optimal solution for the group relaxation. □

Example 3.5 *Solving integer programs via group relaxations.*

For $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 3 & 1 & 0 \\ 0 & 2 & 3 & 4 & 1 \end{bmatrix}$ and $c = [10,\ 2,\ 1,\ 1,\ 10]$, we get $\Delta_c = \{\{1,2,4\},\{1,4,5\},\{2,3,4\}\}$. Take $\sigma = \{1,2,4\}$. Then $\tilde{c}_\sigma = [1/12,\ 13/12]$, and $\mathcal{G}_{\tilde{c}_\sigma}(\mathcal{L}_\sigma) = \{\underline{x_5} - x_3, \underline{x_3^{12}} - 1\}$. For $b^T = [16,\ 36,\ 39]$ and the feasible solution $v = (0,\ 2,\ 9,\ 1,\ 4)$ of $IP_{A,c}(b)$, the remainder of $\mathbf{x}^{\pi_\sigma(v)} = x_3^9 x_5^4$ with respect to $\mathcal{G}_{\tilde{c}_\sigma}(\mathcal{L}_\sigma)$ is x_3 whose exponent vector is $(1,\ 0)$. This lifts to the optimal solution $(3,\ 6,\ 1,\ 6,\ 0)$ of the original integer program. However, for $b^T = [9,\ 13,\ 16]$ and the initial solution $v = (1,\ 0,\ 4,\ 0,\ 4)$ of $IP_{A,c}(b)$, the normal form of $\mathbf{x}^{\pi_\sigma(v)} = x_3^4 x_5^4$ is x_3^8 which lifts to $(5,\ -4,\ 8,\ 0,\ 0)$. In this case, the above group relaxation does not solve $IP_{A,c}(b)$. □

Proposition 3.6 *Under the ring homomorphism*

$$\hat{\pi}_\sigma : \ k[x_1,\ldots,x_n] \longrightarrow k[x_i : i \notin \sigma] \quad \textit{where} \quad \hat{\pi}_\sigma(x_i) = \begin{cases} 1 & \textit{if } i \in \sigma \\ x_i & \textit{if } i \notin \sigma \end{cases},$$

the image of the toric ideal I_A is the lattice ideal $I_{\mathcal{L}_\sigma}$. Moreover, $\hat{\pi}_\sigma(\mathcal{G}_c(I_A))$ is a Gröbner basis of $I_{\mathcal{L}_\sigma}$ with respect to the cost vector $\tilde{c}_\sigma$. This Gröbner basis is not reduced in general.

Proof: The first claim is a consequence of Lemma 3.2. The second follows from the fact that for $u-v \in \mathcal{L}$, $c \cdot (u-v) > 0$ if and only if $\tilde{c}_\sigma \cdot (\pi_\sigma(u) - \pi_\sigma(v)) > 0$. $\square$

Using Proposition 3.6 we will identify the initial ideal $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ with the localization of $\mathrm{in}_c(I_A)$ at the prime p_σ, i.e. $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}) = (\mathrm{in}_c(I_A)k[\mathbf{x}]_{p_\sigma}) \cap k[x_i : i \notin \sigma]$. This allows us to characterize which integer programs are solved by the group relaxation with respect to τ (see also [Stu2],Chapter 12).

Theorem 3.7 *The integer program $IP_{A,c}(b)$ is solved by the group relaxation with respect to $\tau \in \Delta_c$ if and only if the monomial x^β, where β is the optimal solution to $IP_{A,c}(b)$, is covered by a standard pair (x^α, τ') of $in_c(I_A)$ for some $\tau' \supseteq \tau$.*

Proof: The monomial x^β is standard with respect to $\mathrm{in}_c(I_A)$. Suppose it is covered by a standard pair as above. Then $x^\beta = x^\alpha x^{\beta_{\tau'}}$ for some $x^{\beta_{\tau'}}$ with $\mathrm{supp}(x^{\beta_{\tau'}}) \subseteq \tau'$. Since $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ is the localization of $\mathrm{in}_c(I_A)$ at p_τ the pair $(x^\alpha, \tau' - \tau)$ is a standard pair of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. Hence the optimal solution to the group relaxation will be $\alpha + \pi_\tau(\beta_{\tau'})$ which lifts to β. Conversely if the integer program is solved by this group relaxation then $\pi_\tau(\beta)$ must be a standard monomial of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. Hence $(\pi_\tau(\beta), \tau)$ is an admissible pair of $\mathrm{in}_c(I_A)$, and this means x^β is covered by a standard pair of the desired form. $\square$

¿From a practical point of view it is best if an integer program is solved by a group relaxation with respect to a maximal face τ of Δ_c. In this case the prime p_τ is a minimal prime of $\mathrm{in}_c(I_A)$, and the localized initial ideal $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ is zero-dimensional. Hence it has finitely many standard monomials and the Gröbner bases of zero-dimensional ideals are easier to compute than those of higher dimensional ideals.

Proposition 3.8 *If τ is a maximal face of Δ_c, then the number of standard monomials of $in_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ is $|det(A_\tau)|$. Therefore the degree of such a standard monomial is at most $|det(A_\tau)| - 1$.*

Proof: Theorem 2.10 implies that there are exactly $\mathrm{vol}(\tau)$-many standard monomials for $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$, and $\mathrm{vol}(\tau)$ is equal to $|\det(A_\tau)|$. The second statement follows immediately from this. $\square$

Corollary 3.9 *Suppose I_A is homogeneous with respect to the total degree grading and $g = x^\alpha - x^\beta$ is an element of the reduced Gröbner basis $\mathcal{G}_c$. If $\hat{\pi}_\tau(g)$ is also a reduced Gröbner basis element in $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ for a maximal face τ of Δ_c, then $deg(g) \leq (dim(I_A)+1)(deg(I_A))^2$.*

Proof: Let $v = \pi_\tau(\alpha - \beta)$. From the previous proposition it follows that $|v|_1 \leq 2\deg(I_A)$ using the fact that $\deg(I_A) \geq |\det(A_\tau)|$. Since $A_\tau(\alpha-\beta)_\tau = -A_{[n]\setminus\tau} v$, using Cramer's rule we get the bound that $|(\alpha-\beta)_\tau|_1 \leq 2(\dim(I_A))(\deg(I_A))^2$. In other words, $|\alpha - \beta|_1 \leq 2(\dim(I_A)+1)(\deg(I_A))^2$, and hence $\deg(g) \leq (\dim(I_A)+1)(\deg(I_A))^2$. $\square$

We can use the above results to derive an upper bound on the arithmetic degree of $in_c(I_A)$ when A is a $1 \times n$ matrix. Without loss of generality we assume that the greatest common divisor of the entries in A is one. The following theorem can also be deduced from [Gom1].

Theorem 3.10 *Let $A = [a_1, a_2, \ldots, a_n]$ where $0 < a_1 < a_2 < \ldots < a_n$ and g.c.d.$(a_1, \ldots, a_n) = 1$. Then for $c \in \mathbf{R}^n$, arithdeg$(in_c(I_A)) \leq \begin{cases} a_n(a_j - 1) & \text{whenever } \Delta_c = \{j\},\ j < n \\ a_{n-1}(a_n - 1) & \text{whenever } \Delta_c = \{n\} \end{cases}$*

The bijection given in Proposition 1.1 (ii) between the set of standard monomials of $\text{in}_c(I_A)$ and the set of optimal solutions to the integer programs in $IP_{A,c}$ allows an interpretation of the standard pair decomposition of $\text{in}_c(I_A)$ as a covering of $\mathbf{N}(A)$ as follows. For a standard pair (x^m, σ) of $\text{in}_c(I_A)$, the monomials in $x^m \cdot k[x_j : j \in \sigma]$ are in bijection with the elements of the translated monoid $A_{supp(m)} \cdot m + \mathbf{N}(\{a_j : j \in \sigma\})$ contained in $\mathbf{N}(A)$, where $\mathbf{N}(\{a_j : j \in \sigma\}) = \{\sum_{j \in \sigma} n_j a_j : n_j \in \mathbf{N}\}$. For a matrix A as in Theorem 3.10, the point configuration $\mathcal{A}$ has n distinct regular triangulations, each of the form $\Delta = \{j\}$ for $j = 1, \ldots, n$. For a cost vector $c \in \mathbf{R}^n$, if $\Delta_c = \{j\}$ then $\text{in}_c(I_A)$ has a_j standard pairs of the form $(\cdot, \{j\})$ and possibly more of the form $(\cdot, \emptyset)$. Further, if $(x^m, \cdot)$ and $(x^{m'}, \cdot)$ are two different standard pairs of $\text{in}_c(I_A)$ then $A_{supp(m)} \cdot m \neq A_{supp(m')} \cdot m'$. Let $b_0 := \max\{A \cdot m : (x^m, \{j\})$ is a standard pair of $\text{in}_c(I_A)$ $\}$. Then b_0 is an upper bound to the arithmetic degree of $\text{in}_c(I_A)$.

Proof of Theorem 3.10: Let $c \in \mathbf{R}^n$ be a cost vector such that $\Delta_c = \{j\}$, $j \in \{1, 2, \ldots, n\}$ and let M_j be the localization of the initial ideal $\text{in}_c(I_A)$ at the prime ideal $p_{\{j\}}$. By Proposition 3.8 all standard monomials of M_j have total degree at most $a_j - 1$ and hence their exponent vectors lie in the simplex in $\mathbf{R}^{n-1}$ with vertices $\{(a_j - 1)e_k, k \in [n] \backslash \{j\}\} \cup \{0\}$. This implies that b_0 is at most $a_n(a_j - 1)$ when $j < n$ and $a_{n-1}(a_n - 1)$ when $j = n$. □

Corollary 3.11 *The arithmetic degree of an initial ideal of I_A where A is as in Theorem 3.10 is bounded above by $(a_n - 1)^2$.*

Proof: For $j < n$, $a_j \leq a_{n-1} \leq a_n - 1$. Therefore, $a_n(a_j - 1) \leq a_n(a_n - 2) < a_n^2 - 2a_n + 1 = (a_n - 1)^2$. □

4 Toric ideals of codimension two

We now focus our attention on initial ideals that arise from toric ideals of codimension two, i.e., A is a $d \times (d+2)$ matrix of rank d. Such toric ideals have many special properties that fail in higher codimension (see [PS]). In this section we derive new results for the localizations of these initial ideals at their associated primes. We first invoke a result for general toric initial ideals which provides a lower bound for the dimension of an associated prime of $in_c(I_A)$.

Theorem 4.1 *[HT] Let A be a $d \times n$ matrix of rank d and $in_c(I_A)$ be its initial ideal with respect to a generic cost vector $c \in \mathbf{R}^n$. If p_σ is an associated prime of $in_c(I_A)$ for some σ in the regular triangulation Δ_c, then $n + 1 - 2^{n-d} \leq |\sigma| \leq d$.*

Corollary 4.2 *If I_A is a toric ideal of codimension two, then whenever p_σ is an associated prime of $in_c(I_A)$ for some $\sigma \in \Delta_c$, $d - 1 \leq |\sigma| \leq d$.*

Therefore, in codimension two, all minimal primes of $in_c(I_A)$ have dimension d and codimension $n - d = 2$ while all embedded primes (if there are any) have dimension $d - 1$ and codimension $n - (d - 1) = n - d + 1 = 2 + 1 = 3$. Hence, there are only two "levels" of non-trivial localizations for such initial ideals. We will be concerned with the localizations of $in_c(I_A)$ at its minimal primes.

Suppose c_1 and c_2 are two distinct cost vectors such that $\mathrm{in}_{c_1}(I_A) \neq \mathrm{in}_{c_2}(I_A)$ but $\Delta_{c_1} = \Delta_{c_2}$. Then both initial ideals have the same minimal primes although the corresponding primary components of the two initial ideals may differ. However, it is possible that even though $\mathrm{in}_{c_1}(I_A) \neq \mathrm{in}_{c_2}(I_A)$, the localizations of the two initial ideals at each minimal prime coincide.

Example 4.3 Consider $A = [1, 2, 3, 4]$ of corank three, $c_1 = (7, 5, 0, 1)$ and $c_2 = (7, 11, 0, 1)$. Here $\mathrm{in}_{c_1}(I_A) = \langle bd, bc^2, ad, ac, ab, d^3, b^2, a^2\rangle \neq \mathrm{in}_{c_2}(I_A) = \langle bd, bc, ad^2, ac, ab, d^3, b^2, a^2\rangle$ although $\Delta_{c_1} = \Delta_{c_2} = \{\{3\}\}$. However, both ideals when localized at $p_{\{3\}}$ give $\langle a, b, d^3\rangle \subset k[a, b, d]$. □

Theorem 4.4 *Let I_A be a toric ideal of codimension two, and let c_1 and c_2 be two distinct cost vectors with $\Delta_{c_1} = \Delta_{c_2} =: \Delta$. Then $in_{c_1}(I_A) = in_{c_2}(I_A)$ if and only if $in_{\tilde{c_1}_\sigma}(I_{\mathcal{L}_\sigma}) = in_{\tilde{c_2}_\sigma}(I_{\mathcal{L}_\sigma})$ for every maximal face σ of the regular triangulation Δ.*

In order to prove the above theorem we need to look at a different way of encoding b-fibers and group relaxations. See [SST] for an alternate proof. Suppose $B \in \mathbf{Z}^{n\times(n-d)}$ is a matrix whose columns form a basis for $\mathcal{L} := ker_{\mathbf{Z}}(A)$. Given a b-fiber P_b^I = convex hull $\{x \in \mathbf{N}^n : Ax = b\}$ and any lattice vector u in this fiber, the lattice points in P_b^I are in bijection with the lattice points in the polytope $P_u := \{z \in \mathbf{R}^{n-d} : Bz \leq u\}$. This bijection is given by $z \in P_u \cap \mathbf{Z}^{n-d} \leftrightarrow u - Bz$. If u and v are two different lattice points of P_b^I then the corresponding polytopes P_u and P_v are lattice translates of each other. Besides P_u we introduce two related polyhedra: for a vector $c \in \mathbf{R}^n$ we define $P_{u,c} := \{z \in \mathbf{R}^{n-d} : Bz \leq u,\ (cB)z \geq 0\}$, and for the same vector c and a subset τ of $[n]$ we let $P_{u,c}^\tau := \{z \in \mathbf{R}^{n-d} : B^\tau z \leq \pi_\tau(u),\ (cB)z \geq 0\}$, which is obtained from $P_{u,c}$ by omitting the inequalities indexed by τ. The following result characterizes the standard pairs of $\mathrm{in}_c(I_A)$ using $P_{u,c}^\tau$. For the details of the proof we refer to [HT].

Theorem 4.5 *Let $u \in \mathbf{N}^n$ and $\tau \subseteq [n]$ such that $supp(u) \cap \tau = \emptyset$. Then the pair (x^u, τ) is a standard pair of $in_c(I_A)$ if and only if the origin is the only lattice point in $P_{u,c}^\tau$, and for $\tau_i := \tau \cup \{i\}$ where $i \notin \tau$, $P_{u,c}^{\tau_i}$ contains some non-zero lattice point.*

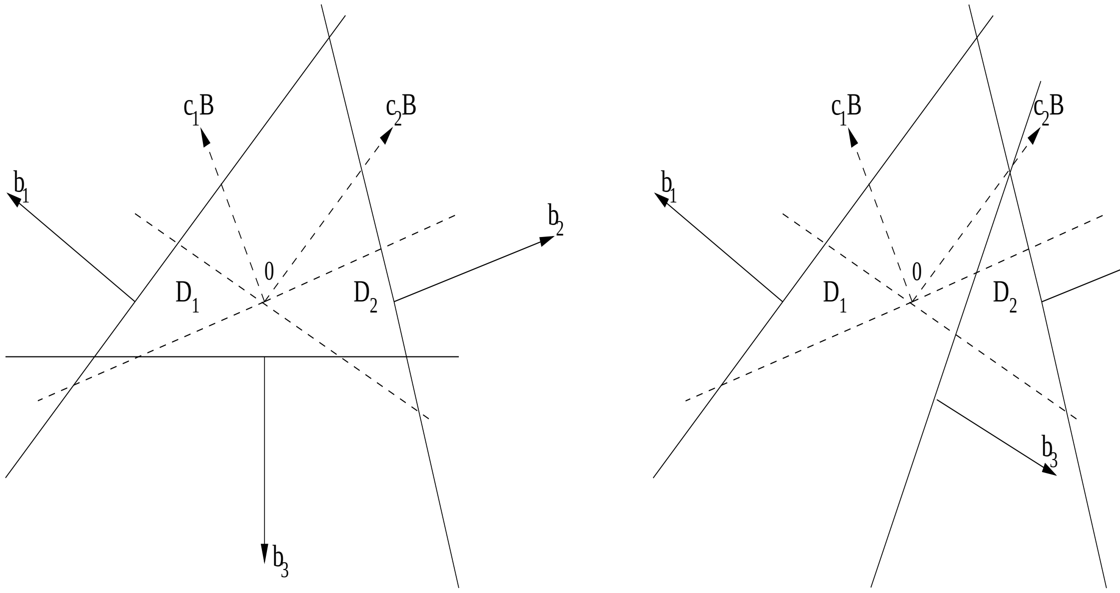

Figure 3: Two different P^{τ}_{u,c_1} and P^{τ}_{u,c_2} for the proof of Theorem 4.4.

Proof of Theorem 4.4: One direction is trivial. For the other direction suppose $\mathrm{in}_{\tilde{c_1}_\sigma}(I_{\mathcal{L}_\sigma}) = \mathrm{in}_{\tilde{c_2}_\sigma}(I_{\mathcal{L}_\sigma})$ for every maximal face σ of Δ, but $\mathrm{in}_{c_1}(I_A) \neq \mathrm{in}_{c_2}(I_A)$. By Corollary 4.2 there exists a $(d-1)$ dimensional face τ of Δ such that $\mathrm{in}_{\tilde{c_1}_\tau}(I_{\mathcal{L}_\tau}) \neq \mathrm{in}_{\tilde{c_2}_\tau}(I_{\mathcal{L}_\tau})$. Therefore, there is a standard pair $(x^u, \emptyset)$ of $\mathrm{in}_{\tilde{c_1}_\tau}(I_{\mathcal{L}_\tau})$ such that x^u is not a standard monomial of $\mathrm{in}_{\tilde{c_2}_\tau}(I_{\mathcal{L}_\tau})$. Using the polyhedra introduced above this means the following: Let $B \in \mathbf{Z}^{(d+2)\times 2}$ be as above. Without loss of generality we assume that the rows of B^τ are b_1, b_2, b_3, and $\sigma = \{2, 3, \ldots, d+2\}$ is a maximal face of Δ containing τ. By Theorem 4.5, P^{τ}_{u,c_1} does not contain any non-zero lattice points and all the inequalities are *essential* - i.e., the removal of any inequality will introduce a new lattice point, whereas P^{τ}_{u,c_2} contains a non-zero lattice point z^*. Let D_1 be the triangle $\{z \in \mathbf{R}^2 : b_1 z \leq u_1, (c_1 B)z \geq 0, (c_2 B)z \leq 0\}$ and D_2 be the triangle $\{z \in \mathbf{R}^2 : b_2 z \leq u_2, (c_1 B)z \leq 0, (c_2 B)z \geq 0\}$. Now the lattice point z^* which is in P^{τ}_{u,c_2} but not in P^{τ}_{u,c_1} must be in D_2. The constraint $b_3 z \leq u_3$ cuts through D_2 or D_1, or both. If it cuts only through D_2 and if $b_2 z^* = \hat{u}_2$, then $P^{\sigma}_{(u_1,\hat{u}_2),c_2}$ will contain the same lattice point but $P^{\sigma}_{(u_1,\hat{u}_2),c_1}$ will be lattice point free (Figure 3, second diagram). Using Theorem 4.5 we conclude that $(x_1^{u_1} x_2^{\hat{u}_2}, \emptyset)$ is a standard pair for $\mathrm{in}_{\tilde{c_1}_\sigma}(I_{\mathcal{L}_\sigma})$, but $x_1^{u_1} x_2^{\hat{u}_2}$ is not a standard monomial of $\mathrm{in}_{\tilde{c_2}_\sigma}(I_{\mathcal{L}_\sigma})$. This contradicts the hypothesis that $\mathrm{in}_{\tilde{c_1}_\sigma}(I_{\mathcal{L}_\sigma}) = \mathrm{in}_{\tilde{c_2}_\sigma}(I_{\mathcal{L}_\sigma})$. If the constraint $b_3 z \leq u_3$ cuts through D_1 (see Figure 3, first diagram), then for $v = (0, u_2, u_3)$ the polytope P^{τ}_{v,c_1} will not contain a non-zero lattice point, but P^{τ}_{v,c_2} will contain one. However for both polytopes the inequality $b_3 z \leq u_3$ is not essential. Hence, $(x_2^{u_2}, \emptyset)$ is a standard pair of $\mathrm{in}_{\tilde{c_1}_\sigma}(I_{\mathcal{L}_\sigma})$, but $x_2^{u_2}$ is not a standard monomial of $\mathrm{in}_{\tilde{c_2}_\sigma}(I_{\mathcal{L}_\sigma})$ - a contradiction. $\square$

In the rest of this section we examine the behaviour of the zero-dimensional ideals $\hat{\pi}_\sigma(in_c(I_A))$ (I_A is of codimension two) for a fixed maximal simplex σ in $\Delta := \Delta_c$, as c varies. We first recall some results for a general $d \times n$ matrix A satisfying the assumptions of Section 1. For a fixed regular triangulation Δ of $\mathcal{A}$, the set $\{c' \in \mathbf{R}^n : \Delta_{c'} = \Delta\}$ is an n-dimensional polyhedral cone in $\mathbf{R}^n$ called the *secondary cone* of Δ [BFS],[GKZ]. It has the rowspan of A as its lineality space and hence we will refer to its image in $ker(A) \simeq \mathbf{R}^n/rowspan(A) \simeq \mathbf{R}^{n-d}$ as the secondary cone of Δ, denoted $\mathcal{C}_\Delta$. For the initial ideal $in_c(I_A)$, the set $\{c' \in \mathbf{R}^n : in_{c'}(I_A) = in_c(I_A)\}$ is an n-dimensional polyhedral cone in $\mathbf{R}^n$ called the *Gröbner cone* of $in_c(I_A)$ [MR]. It also has $rowspan(A)$ as lineality space and again we refer to its image in $ker(A)$, as the Gröbner cone of $in_c(I_A)$, denoted $\mathcal{K}_c$. The Gröbner cones of all initial ideals $in_c(I_A)$ with $\Delta_c = \Delta$ form a *polyhedral complex* whose support is $\mathcal{C}_\Delta$ [Stu1]. Recall that the matrix $B \in \mathbf{Z}^{n \times n-d}$ introduced earlier was chosen such that its columns form a $\mathbf{Z}$-basis for $\mathcal{L} = ker_{\mathbf{Z}}(A)$ and hence a basis for $ker(A)$. Hence for each $u \in ker(A)$, there is a unique vector $\lambda_u \in \mathbf{R}^{n-d}$ such that $B\lambda_u = u$. Let $H^u := \{x \in \mathbf{R}^{n-d} : \lambda_u x = 0\}$. For $u \in ker_{\mathbf{Z}}(A)$ and $\lambda_c \in \mathbf{R}^{n-d}$ if $\lambda_u \cdot \lambda_c > 0$, then $in_c(x^{u^+} - x^{u^-}) = x^{u^+}$ and if $\lambda_u \cdot \lambda_c < 0$ then $in_c(x^{u^+} - x^{u^-}) = x^{u^-}$. We say that $x^{u^+} - x^{u^-}$ is a *valid binomial* for a cone K in $\mathbf{R}^{n-d}$ if H^u is a *supporting hyperplane* of K, i.e., $H^u \cap int(K) = \emptyset$. The binomial $x^{u^+} - x^{u^-}$ is called a *facet binomial* of K if H^u is the linear span of a facet of K and for every $\lambda_\omega \in int(K)$, $in_\omega(x^{u^+} - x^{u^-}) = x^{u^+}$. A vector $u \in ker_{\mathbf{Z}}(A)$ (or equivalently $x^{u^+} - x^{u^-} \in I_A$) is a *circuit* of I_A if $supp(u)$ is minimal with respect to inclusion and the g.c.d. of the components of u is one. Every circuit of I_A is an element of some reduced Gröbner basis of I_A. The facet binomials of a Gröbner cone of I_A are elements in the corresponding reduced Gröbner basis while the facet binomials of a secondary cone are circuits of I_A [ST].

Lemma 4.6 *The binomial $x^p - x^q \in I_A$ is invalid for the secondary cone $\mathcal{C}_\Delta$ if and only if* **both** *$supp(p)$ and $supp(q)$ contain non-faces of Δ.*

Proof: The binomial $x^p - x^q$ is invalid for $\mathcal{C}_\Delta$ if and only if there exists a non-zero $\lambda_\omega \in \mathbf{R}^{n-d}$ such that $\lambda_\omega \in H^{p-q} \cap int(\mathcal{C}_\Delta)$. Hence we can choose λ_{ω_1} and λ_{ω_2} in $int(\mathcal{C}_\Delta)$, arbitrarily close to λ_ω and on opposite sides of H^{p-q} such that, $in_{\omega_1}(x^p - x^q) = x^p$ and $in_{\omega_2}(x^p - x^q) = x^q$. However, both $in_{\omega_1}(I_A)$ and $in_{\omega_2}(I_A)$ are supported on Δ and since $x^p \in in_{\omega_1}(I_A)$ and $x^q \in in_{\omega_2}(I_A)$, by Theorem 1.2, both $supp(p)$ and $supp(q)$ contain minimal non-faces of Δ. $\square$

When A is of corank two, $\mathcal{C}_\Delta$ is a two dimensional cone in $\mathbf{R}^2$. Let $x^\alpha - x^\beta$ and $x^\gamma - x^\delta$ in I_A be the two facet binomials of $\mathcal{C}_\Delta$. Then $supp(\alpha)$ and $supp(\gamma)$ are minimal non-faces of Δ while $supp(\beta)$ and $supp(\delta)$ are faces. Let the Gröbner cones refining $\mathcal{C}_\Delta$ be labeled $\mathcal{K}_1, \ldots, \mathcal{K}_t$ such that $\mathcal{K}_l$ is the Gröbner cone of $in_{c_l}(I_A)$, $x^\alpha - x^\beta$ is a facet binomial of $\mathcal{K}_1$, and $\mathcal{K}_l$ is adjacent to $\mathcal{K}_{l+1}$ for $l = 1, \ldots, t-1$. Recall that $\Delta_{c_1} = \cdots = \Delta_{c_t} = \Delta$. We denote the common facet binomial of the Gröbner cones $\mathcal{K}_l$ and $\mathcal{K}_{l+1}$ as $x^{p^{l+1}} - x^{q^l}$. Each such binomial is invalid for $\mathcal{C}_\Delta$.

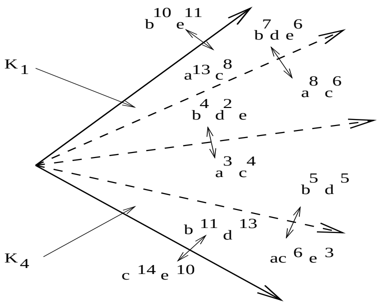

Figure 4: The secondary cone $\mathcal{C}_\Delta$ and the four Gröbner cones contained in it.

We now fix a maximal simplex $\sigma \in \Delta$ and consider $\hat{\pi}_\sigma(in_\omega(I_A))$ as λ_ω sweeps from one facet of $\mathcal{C}_\Delta$ to the other. To avoid the trivial case, we assume that $\mathcal{C}_\Delta$ supports at least two Gröbner cones, i.e., $t \geq 2$. Since A is of corank two, we can assume that $[n] \backslash \sigma = \{i, j\}$. Then among the generators of $\hat{\pi}_\sigma(in_\omega(I_A)) \subset k[x_i, x_j]$ there are two of the form x_i^* and x_j^*. We call these the *single variable generators* of $\hat{\pi}_\sigma(in_\omega(I_A))$. Our goal is to prove (Theorem 4.11) that as λ_ω sweeps through $\mathcal{C}_\Delta$, the degree of one single variable generator of $\hat{\pi}_\sigma(in_\omega(I_A))$ monotonically increases while the degree of the other monotonically decreases. The following example illustrates this phenomenon.

Example 4.7 Consider the matrix $A := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 2 & 0 \\ 0 & 1 & 4 & 5 & 2 \end{bmatrix}$ and its regular triangulation $\Delta = \{\{1, 2, 5\}, \{2, 3, 5\}, \{3, 4, 5\}\}$. We associate the variables a, b, c, d and e with the columns of A. The above triangulation supports four distinct initial ideals of I_A which we list below along with a cost vector that induces it:

- $M_1 := \langle b^4d^2e, a^5c^2d, b^5d^5, b^7de^6, a^2bd^3, a^7d^4, a^{13}c^8 \rangle$, $c_1 = (17, 21, 0, 0, 0)$
- $M_2 := \langle b^4d^2e, a^5c^2d, a^2bd^3, b^5d^5, a^7d^4, a^8c^6 \rangle$, $c_2 = (1, 1, 0, 0, 0)$
- $M_3 := \langle a^5c^2d, a^2bd^3, b^5d^5, a^3c^4, a^7d^4 \rangle$, $c_3 = (14, 5, 0, 0, 0)$
- $M_4 := \langle a^5c^2d, a^2bd^3, ab^6d^8, b^{11}d^{13}, a^3c^4, a^7d^4, ac^6e^3 \rangle$, $c_4 = (16, 1, 0, 0, 0)$

The facet binomials of $\mathcal{C}_\Delta$ are $a^{13}c^8 - b^{10}e^{11}$ and $b^{11}d^{13} - c^{14}e^{10}$. This secondary cone and its Gröbner cones are shown in Figure 4. Figure 5 shows the localizations of M_1, M_2, M_3, M_4, in that order, at the minimal prime $p_{\{2,3,5\}}$. Notice that the degree of the single variable generator d^* increases monotonically from left to right while the degree of a^* decreases. $\square$

The following proposition holds for matrices of corank two and no higher.

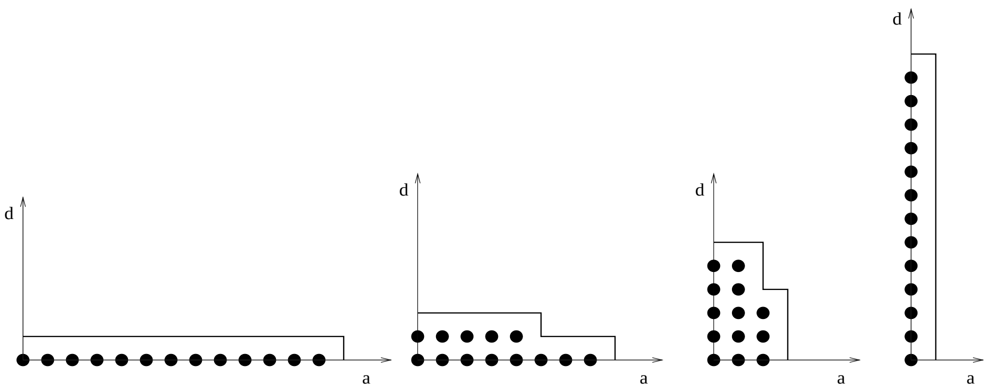

Figure 5: Localizations of the four initial ideals supported on Δ at the minimal prime $p_{\{2,3,5\}}$.

Proposition 4.8 *Let $\mathcal{C}_\Delta$ be a secondary cone of a toric ideal I_A of codimension two with facet binomials $x^\alpha - x^\beta$ and $x^\gamma - x^\delta$ such that $\mathcal{C}_\Delta$ supports at least two Gröbner cones. Then*
(i) $supp(\alpha) \neq supp(\gamma)$,
(ii) $x^p - x^q \in I_A$ is invalid for $\mathcal{C}_\Delta$ if and only if $supp(\alpha) \subseteq supp(p)$ and $supp(\gamma) \subseteq supp(q)$, and
(iii) $supp(\alpha) \cap supp(\gamma) = \emptyset$.

Proof: (i) Suppose $supp(\alpha) = supp(\gamma)$. Let $\mathcal{C}_{\Delta'}$ be the secondary cone adjacent to $\mathcal{C}_\Delta$ such that $\mathcal{C}_\Delta$ and $\mathcal{C}_{\Delta'}$ share the facet binomial $x^\gamma - x^\delta$. Then $supp(\gamma) = supp(\alpha)$ is a face of Δ'. Let $\Delta' = \Delta_\omega$ for $\omega \in \mathbf{R}^n$. If $x^\alpha - x^\beta$ is a valid binomial for $\mathcal{C}_{\Delta'}$, then $in_\omega(x^\alpha - x^\beta) = x^\alpha$ which contradicts Theorem 1.2. If $x^\alpha - x^\beta$ is invalid for $\mathcal{C}_{\Delta'}$, then we get a contradiction to Lemma 4.6.
(ii) The *if* direction follows from Lemma 4.6. Suppose $x^p - x^q \in I_A$ is invalid for $\mathcal{C}_\Delta$. Then $p-q$ is a non-negative linear combination of $\alpha - \beta$ and $\delta - \gamma$ which implies that $supp(p) \subseteq supp(\alpha) \cup supp(\delta)$ and $supp(q) \subseteq supp(\beta) \cup supp(\gamma)$. However, since $supp(\gamma) \cap supp(\delta) = \emptyset$ and $supp(\gamma) \not\subseteq supp(\alpha)$ it follows that $supp(\gamma) \not\subseteq supp(p)$. ($supp(\gamma) \not\subset supp(\alpha)$ since they are both minimal nonfaces of Δ and $supp(\gamma) \neq supp(\alpha)$ by (i).) Similarly, $supp(\alpha) \not\subseteq supp(q)$. If $supp(\alpha) \subseteq supp(p)$ and $supp(\gamma) \subseteq supp(q)$, we are done. Else, one of the terms, say x^p supports a non-face μ, $supp(\mu) \neq supp(\alpha)$, $supp(\gamma)$. Since $supp(\mu)$ is a minimal non-face of Δ there exists a circuit $x^\mu - x^\nu \in I_A$ that is a valid binomial for $\mathcal{C}_\Delta$ such that $supp(\nu)$ is a face of Δ. We can assume without loss of generality that $\lambda_{\mu-\nu}$ comes before $\lambda_{\alpha-\beta}$ which comes before λ_{p-q} in clockwise order. Then $supp(\mu) \subseteq supp(p)$ implies that the three coplanar vectors $\mu - \nu, \alpha - \beta$ and $p - q$ all lie in the intersection of the halfspaces $x_j \geq 0$ for all $j \in supp(\mu)$. Therefore, $supp(\mu) \subseteq supp(\alpha)$. But since both are minimal nonfaces of Δ this implies that $supp(\mu) = supp(\alpha)$ which is a contradiction.
(iii) Since $t \geq 2$, for each $x^{p_{l+1}} - x^{q_l}$, $l \in \{1, \ldots, t-1\}$, $supp(\alpha) \subseteq supp(p_{l+1})$ and $supp(\gamma) \subseteq supp(q_l)$. Since $supp(p_{l+1}) \cap supp(q_l) = \emptyset$, $supp(\alpha) \cap supp(\gamma) = \emptyset$. $\square$

Lemma 4.9 *Let σ be a fixed maximal simplex in $\Delta = \Delta_c$ with $[n]\backslash\sigma = \{i, j\}$ and $x^\alpha - x^\beta$ and $x^\gamma - x^\delta$ be the facet binomials of the secondary cone $\mathcal{C}_\Delta$. Then either x_i divides x^α and x_j divides x^γ or x_j divides x^α and x_i divides x^γ. Neither monomial is divisible by $x_i x_j$.*

Proof: If neither x_i nor x_j divides x^α, then $\hat{\pi}_\sigma(in_c(I_A)) = \langle 1 \rangle$ which is a contradiction. Similarly for x^γ. Hence at least one of x_i or x_j divides both x^α and x^δ. However, since $supp(\alpha) \cap supp(\gamma) = \emptyset$, exactly one of x_i or x_j divides x^α and the other variable divides x^γ. $\square$

Corollary 4.10 *Suppose x_i divides x^α and x_j divides x^γ. Then the common facet binomial $x^{p_{l+1}} - x^{q_l}$ of the Gröbner cones $\mathcal{K}_l$ and $\mathcal{K}_{l+1}$ contained in $\mathcal{C}_\Delta$ has the following properties:*
(i) $supp(\alpha) \subseteq supp(p^{l+1})$ and $supp(\gamma) \subseteq supp(q^l)$.
(ii) x_i divides $x^{p^{l+1}}$ and x_j divides x^{q^l}. Neither monomial is divisible by $x_i x_j$.
(iii) $\hat{\pi}_\sigma(x^{p^{l+1}} - x^{q^l}) = x_i^{p_i^{(l+1)}} - x_j^{q_j^l}$.

Theorem 4.11 *For each of the Gröbner cones $\mathcal{K}_l$, $l = 1, \ldots, t$ contained in the secondary cone $\mathcal{C}_\Delta$, let $x_i^{i_l}$ and $x_j^{j_l}$ denote the single variable generators of the localization $\hat{\pi}_\sigma(in_{c_l}(I_A))$. Then either $i_1 \geq i_2 \geq \cdots \geq i_t$ and $j_1 \leq j_2 \leq \cdots \leq j_t$ or $i_1 \leq i_2 \leq \cdots \leq i_t$ and $j_1 \geq j_2 \geq \cdots \geq j_t$*

Proof: Suppose x_i divides x^α and x_j divides x^γ. Therefore $i_1 \leq \pi_\sigma(\alpha) = \alpha_i$. The minimal generator $x_i^{i_1}$ comes from a valid binomial of $\mathcal{K}_1$. If this binomial is $x^p - x^q$ and is invalid for $\mathcal{C}_\Delta$, then only x^q belongs to $in_{c_1}(I_A)$ but x_i does not divide x^q. Therefore, the minimal generator $x_i^{i_1}$ comes from a valid binomial of $\mathcal{C}_\Delta$. However, this binomial is valid for all Gröbner cones supported on Δ which implies that $i_1 \geq i_l$ for all $l = 2, \ldots, t$. Suppose $i_1 = i_2 = \cdots = i_l > i_{l+1}$. Then this jump is caused by $x^{p^{l+1}}$ and hence $\pi_\sigma(p^{l+1}) = i_{l+1}$. However, all cost vectors $c_{l+1}, \ldots, c_t$ pick $x^{p^{l+1}}$ as the initial term in the binomial $x^{p^{l+1}} - x^{q^l}$ which is valid for all cones $\mathcal{K}_{l+1}, \ldots, \mathcal{K}_t$. This implies that $x^{p^{l+1}}$ lies in each of the initial ideals $in_{c_{l+1}}(I_A), \ldots, in_{c_t}(I_A)$. Therefore, $i_{l+1} \geq i_k$ for all $k \in \{l+2, \ldots, t\}$. By a similar argument, starting from the other facet of $\mathcal{C}_\Delta$, we can show that $j_1 \leq j_2 \leq \cdots \leq j_t$. $\square$

5 A non-Buchberger algorithm for toric Gröbner bases

We now present the non-Buchberger algorithm to compute the reduced Gröbner basis $\mathcal{G}_c$ of the toric ideal I_A. The basic idea is to build $\text{in}_c(I_A)$ from the initial ideals $\text{in}_{\tilde{c}}(I_{\mathcal{L}_\tau})$ where p_τ is an associated prime of $\text{in}_c(I_A)$. By Theorem 2.10, $Ass(in_c(I_A))$, the poset of all associated primes of $in_c(I_A)$ is a subposet of the regular triangulation Δ_c and both posets have the same maximal elements. We assume that Δ_c is given. The construction starts at the top level of this poset, and for each maximal simplex σ of Δ_c constructs a minimal generating set (and hence a Gröbner basis) for $\text{in}_{\tilde{c}}(I_{\mathcal{L}_\sigma})$. It then moves down the successive levels of the face lattice of Δ_c and at each lower dimensional face τ of Δ_c computes $\text{in}_{\tilde{c}}(I_{\mathcal{L}_\tau})$ if p_τ is an embedded prime of $in_c(I_A)$. This amounts to computing all standard pairs of $in_c(I_A)$ of the form $(\cdot, \tau)$.

5.1 Computing the localized initial ideal for each maximal face of Δ_c

The algorithm starts by computing a minimal generating set for the localized initial ideal corresponding to each maximal face σ of Δ_c. The following fact is used in the algorithm.

Proposition 5.1 *For a generic cost vector c, $\tilde{c}_\sigma > 0$ for each maximal face σ of Δ_c.*

Proof: The maximal faces of Δ_c are in bijection with the *optimal bases* of the linear programming relaxation of $\mathrm{IP}_{A,c}(b)$ (see Theorem 1.1 in [ST] and [WW]): minimize$\{c \cdot x \,:\, Ax = b,\, x \geq 0\}$, as b varies. The optimality criterion for linear programming [Sch] implies that for a maximal face σ, $\tilde{c}_\sigma \geq 0$. For generic c, $\tilde{c}_\sigma$ is strictly positive. $\square$

For each maximal face σ, $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ is a zero-dimensional ideal, and hence for each $i \notin \sigma$, $x_i^{m_i}$ is a minimal generator of $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ for some m_i. Moreover, by Proposition 3.8 the degree of every minimal generator is at most $|\det(A_\sigma)|$. So for every monomial $x^\alpha \in k[x_i : i \notin \sigma]$ such that $|\alpha|_1 \leq |\det(A_\sigma)|$ we determine $T_\alpha := \{x^\beta \,:\, x^\alpha - x^\beta \in I_{\mathcal{L}_\sigma} \text{and } \mathbf{x}^\alpha \succ_{\tilde{c}_\sigma} x^\beta\}$. Since $\tilde{c}_\sigma$ is strictly positive, T_α is finite and it is empty if and only if x^α is a standard monomial.

Algorithm 5.2 *A non-Buchberger algorithm to compute Mingen($in_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$), the minimal generators of $in_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$, and $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$, the reduced Gröbner basis of $I_{\mathcal{L}_\sigma}$ for a maximal face σ of Δ_c when c is a generic cost vector.*

INPUT: $A \in \mathbf{Z}^{d \times n}$, a generic cost vector $c \in \mathbf{R}^n$, and a maximal face σ of Δ_c.
OUTPUT: $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}))$ and $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$.
1. Let Gen $:= \emptyset$. For every $x^\alpha \in k[x_i : i \notin \sigma]$ with $|\alpha|_1 \leq |\det(A_\sigma)|$ compute T_α. If $T_\alpha = \emptyset$ then x^α is a standard monomial, else Gen $:=$ Gen $\cup \{x^\alpha\}$.
2. Let $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}))$ be the minimal set of monomials in Gen with respect to divisibility.
3. Let $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}) := \emptyset$. For every $x^\alpha \in \mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}))$, let x^β be the minimal element of T_α with respect to $\succ_{\tilde{c}_\sigma}$, and let $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}) := \mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}) \cup \{x^\alpha - x^\beta\}$.
4. Output $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma}))$ and $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$.

Example 5.3 In this example we consider the matrix A and cost vector c from Example 3.5. We use Algorithm 5.2 to compute $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ for each maximal face of Δ_c. There are three maximal faces: $\sigma_1 = \{1,2,4\}$, $\sigma_2 = \{1,4,5\}$, $\sigma_3 = \{2,3,4\}$ with $\tilde{c}_{\sigma_1} = [1/12, 13/12]$, $\tilde{c}_{\sigma_2} = [13/4, 9/4]$, $\tilde{c}_{\sigma_3} = [1,2]$ respectively. Since $mult(p_{\sigma_1}) = vol(\sigma_1) = 12$, $\mathrm{in}_{\tilde{c}_{\sigma_1}}(I_{\mathcal{L}_{\sigma_1}})$ has 12 standard monomials, and we see that for x_3^i, $i = 0, \ldots, 11$, $T_{(i,0)} = \emptyset$. Hence, all these monomials are standard. We compute $T_{(12,0)} = \{1\}$ and $T_{(0,1)} = \{x_3\}$, and hence $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_{\sigma_1}}(I_{\mathcal{L}_{\sigma_1}})) = \{x_3^{12}, x_5\}$ and $\mathcal{G}_{\tilde{c}_{\sigma_1}}(I_{\mathcal{L}_{\sigma_1}}) = \{x_3^{12} - 1, x_5 - x_3\}$. Since $|\det(A_{\sigma_2})| = 4$, $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_{\sigma_2}}(I_{\mathcal{L}_{\sigma_2}})) \subset k[x_2, x_3]$ has four standard monomials. We compute that $T_{(1,0)} = \{x_3\}$, $T_{(0,1)} = T_{(0,2)} = T_{(0,3)} = \emptyset$ and $T_{(0,4)} = \{1\}$. This means that

$\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_{\sigma_2}}(I_{\mathcal{L}_{\sigma_2}})) = \{x_2, x_3^4\}$ and $\mathcal{G}_{\tilde{c}_{\sigma_2}}(I_{\mathcal{L}_{\sigma_2}}) = \{x_2 - x_3, x_3^4 - 1\}$. Since $|\det(A_{\sigma_3})| = 1$, we already know that $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_{\sigma_3}}(I_{\mathcal{L}_{\sigma_3}})) = \{x_1, x_5\}$. We see that $T_{(1,0)} = \{1\}$ and $T_{(0,1)} = \{x_1, 1\}$. So $\mathcal{G}_{\tilde{c}_{\sigma_3}}(I_{\mathcal{L}_{\sigma_3}}) = \{x_1 - 1, x_5 - 1\}$. We will see that although $x_5 - x_1$ is not a reduced Gröbner basis element here, it will "lift up" to $x_5x_2 - x_1x_3$ which is a reduced Gröbner basis element of I_A. This is the reason why we need to compute and store the entire T_α for every minimal generator x^α of the localized initial ideal instead of just storing the corresponding Gröbner basis element. □

5.2 Computing localized initial ideals for lower dimensional faces of Δ_c

Now we assume that for all superfaces σ of τ with $\dim(\tau) = k = \dim(\sigma) - 1$ we have computed the minimal generators of $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ along with T_α for each such minimal generator. For $k = d-1$, these assumptions hold since we can use Algorithm 5.2 for maximal faces. Let $\sigma_1, \ldots, \sigma_s$ be the $(k+1)$-dimensional faces of Δ_c containing τ. Let Min_j be the set of minimal generators of $\mathrm{in}_{\tilde{c}_{\sigma_j}}(I_{\mathcal{L}_{\sigma_j}}) \subset k[x_i : i \notin \sigma_j]$ and let M_j be the monomial ideal generated by Min_j in $k[x_i : i \notin \tau]$.

Theorem 5.4 *The ideal $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ is contained in $M = \bigcap_{\sigma_j \supset \tau} M_j$. Moreover, the standard pairs of M is the union of the standard pairs of M_j.*

Proof: A monomial x^α is a standard monomial of M if and only if it is a standard monomial of some M_j. Also, any standard monomial of an M_j is also a standard monomial of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. This proves the first claim. For the second claim, observe that a standard pair of any M_j is an admissible pair for M, and hence is also admissible for $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. But since this pair is actually a standard pair for $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ it is a standard pair for M as well. For the converse note that M does not have standard pairs of the form $(x^\alpha, \emptyset)$ since this would mean that $\langle x_i : i \notin \tau \rangle$ is an embedded prime of M. But this is not true since this prime is not an embedded prime of any M_j. So if (x^α, σ) is a standard pair of M then it is also a standard pair of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. Since $\sigma \neq \emptyset$, the same pair is a standard pair for some M_j. □

This theorem implies that the only standard monomials of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ which are not standard monomials of M are precisely those monomials x^α where $(x^\alpha, \emptyset)$ is a standard pair of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. So we just need to identify these "new" standard monomials. A minimal generator x^α of M_j equals $x^{\pi_{\sigma_j}(\bar{\alpha})}$ where $x^{\bar{\alpha}}$ is a minimal generator of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. So first for each $x^\beta \in T_\alpha$ where $x^\alpha \in \mathrm{Min}_j$ we compute the unique binomial $x^{\bar{\alpha}} - x^{\bar{\beta}} \in I_{\mathcal{L}_\tau}$ such that $x^{\pi_{\sigma_j}(\bar{\alpha})} - x^{\pi_{\sigma_j}(\bar{\beta})} = x^\alpha - x^\beta$. We define $\mathrm{lift}(x^\alpha)$ to be that $x^{\bar{\alpha}}$ where the coordinate of $\bar{\alpha}$ which corresponds to $\sigma_j \backslash \tau$ is minimum. We collect the union of all $\mathrm{lift}(x^\alpha)$ where $x^\alpha \in \bigcup_{j=1}^{s} \mathrm{Min}_j$ in Gen. Clearly Gen gives a subset of the minimal generators of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. But there could be more generators and we need to find them.

If the monomials in Gen generate M, then we are done: by Theorem 5.4 there are no standard pairs of the form $(x^\alpha, \emptyset)$ and $M = \mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. If that is not the case, then there is at least one generator x^α of M which is not in Gen. So we compute T_α for this monomial; x^α is a standard monomial of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ (which corresponds to a standard pair $(\mathbf{x}^\alpha, \emptyset)$) if and only if $T_\alpha = \emptyset$. In this case we update M to be a slightly smaller monomial ideal $M := \langle (\mathrm{Mingen}(M) \backslash \{x^\alpha\}) \cup \bigcup_{i \notin \tau} \{x^{\alpha + e_i}\} \rangle$ where e_i is the i-th unit vector. Note that the updated M still contains $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. If $T_\alpha \neq \emptyset$ then $x^\alpha \in \mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ and we update Gen to be the minimal generating set of $\langle \mathrm{Gen} \cup \{x^\alpha\} \rangle$. We repeat the same for the updated M and Gen and we stop when $\langle \mathrm{Gen} \rangle = M$. In this case $M = \mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. Once again, using the T_α computed for the minimal generators of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ we can also output the reduced Gröbner basis of $I_{\mathcal{L}_\tau}$. We summarize this in the following algorithm.

Algorithm 5.5 *How to compute Mingen($in_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$) and $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$, for a non-maximal face τ of Δ_c.*

INPUT: A k-dimensional face τ of Δ_c; Min_j, the minimal generators of $\mathrm{in}_{\tilde{c}_{\sigma_j}}(I_{\mathcal{L}_{\sigma_j}})$ for each $(k+1)$-dimensional face σ_j with $\sigma_j \supset \tau$, and T_α for every $x^\alpha \in \mathrm{Min}_j$.
OUTPUT: $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau}))$ and $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$.
1. Let $M_j := \langle \mathrm{Min}_j \rangle \subset k[x_i : i \notin \tau]$ and compute the minimal generators $\mathrm{Mingen}(M = \bigcap_j M_j)$.
2. For $x^\alpha \in \bigcup_j \mathrm{Min}_j$ compute $\mathrm{lift}(x^\alpha)$ (see the discussion above) and collect these in Gen.
3. While $\mathrm{Mingen}(M) \neq \mathrm{Gen}$ do
• Pick $x^\alpha \in \mathrm{Mingen}(M) \backslash \mathrm{Gen}$ and compute T_α.
• If $T_\alpha = \emptyset$ then let $M := \langle (\mathrm{Mingen}(M) \backslash \{x^\alpha\}) \cup \bigcup_{i \notin \tau} \{x^{\alpha+e_i}\} \rangle$. If $T_\alpha \neq \emptyset$ then update Gen to be the minimal generating set of $\langle \mathrm{Gen} \cup \{x^\alpha\} \rangle$.
4. Let $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau}) := \emptyset$. For each $x^\alpha \in \mathrm{Gen}$ find the minimal monomial x^β with respect to $\succ_{\tilde{c}_\tau}$ in T_α and set $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau}) := \mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau}) \cup (x^\alpha - x^\beta)$.
5. Output $\mathrm{Mingen}(\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})) = \mathrm{Gen}$ and $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$.

Example 5.6 *Example 5.3 continued.* We will compute $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ for the lower-dimensional faces of Δ_c. We start with $\tau = \{4, 5\}$ which is contained in the unique three-dimensional face σ_2 of Δ_c; therefore $M = \langle x_2, x_3^4 \rangle \in k[x_1, x_2, x_3]$. In order to compute $\mathrm{lift}(x_2)$ we compute the unique binomial whose image under $\hat{\pi}_{\sigma_2}$ is $x_2 - x_3$. This binomial is $x_2 - x_3 x_1$, and hence $\mathrm{lift}(x_2) = x_2$. This actually says x_2 does not lift up. Similarly, we compute that $\hat{\pi}_{\sigma_2}(x_3^4 - x_1^7) = x_3^4 - 1$ and hence $\mathrm{lift}(x_3^4) = x_3^4$. We conclude that $M = \mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. For $\tau = \{2, 4\}$ we have σ_1 and σ_3 containing τ. Therefore, $M = M_3 \cap M_1 = \langle x_1, x_5 \rangle \cap \langle x_3^{12}, x_5 \rangle = \langle x_1 x_3^{12}, x_5 \rangle$. We calculate that $\mathrm{lift}(x_1) = x_1 x_3^{12}$ since $x_1 - 1$ lifts up to $x_1 x_3^{12} - 1$. As a minimal generator of M_3 to find $\mathrm{lift}(x_5)$ we need to consider $x_5 - 1$ and $x_5 - x_3$. Since $\hat{\pi}_{\sigma_3}(x_5 x_3^{11} - 1) = x_5 - 1$ and $\hat{\pi}_{\sigma_1}(x_5 - x_1 x_3) = x_5 - x_3$ we conclude $\mathrm{lift}(x_5) = x_5$. For the generators of M_1, we also see that $\mathrm{lift}(x_5) = x_5$, and $\mathrm{lift}(x_3^{12}) = x_1 x_3^{12}$. Thus once again $M = \mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$. Now let's look at $\tau = \{1, 4\}$ which is contained in σ_1 and σ_2, so $M = M_1 \cap M_2 = \langle x_3^{12}, x_5 \rangle \cap \langle x_2, x_3^4 \rangle = \langle x_3^{12}, x_3^4 x_5, x_2 x_5 \rangle$. We compute $\mathrm{lift}(x_3^{12}) = x_3^{12}$, $\mathrm{lift}(x_5) = x_5 x_2$, $\mathrm{lift}(x_2) = x_2 x_5$ and $\mathrm{lift}(x_3^4) = x_3^4 x_5^8$; therefore $\langle \mathrm{Gen} \rangle = \langle x_3^{12}, x_3^4 x_5^8, x_2 x_5 \rangle \neq M$. We actually deduce right away that $x_3^4 x_5^i, i = 1, \ldots, 7$ have to be "new" standard monomials of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$, since we know that $x_3^4 x_5^8$ is a minimal generator for the same ideal. So we replace M with $\langle x_3^{12}, x_3^4 x_5^8, x_3^4 x_2 x_5, x_3^5 x_5, x_2 x_5 \rangle =$

$\langle x_3^{12}, x_3^4x_5^8, x_3^5x_5, x_2x_5\rangle$. Since T_α corresponding to $x_3^5x_5$ is empty, we see that $x_3^5x_5$ is also standard for $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ and we further replace M by $\langle x_3^{12}, x_3^4x_5^8, x_3^5x_5^2, x_3^6x_5, x_2x_5\rangle$. After five iterations we see that $x_3^5x_5^i, i = 1, \ldots, 6$ are all "new" standard monomials, but $T_\alpha = \{x_2\}$ for $x_3^5x_5^7$ (since $x_3^5x_5^7 - x_2 \in I_{\mathcal{L}_\tau}$ and $x_3^5x_5^7 \succ_{\tilde{c}_\tau} x_2$). Therefore M becomes the ideal $\langle x_3^{12}, x_3^4x_5^8, x_3^5x_5^7, x_3^6x_5, x_2x_5\rangle$ and Gen $= \{x_3^{12}, x_3^4x_5^8, x_3^5x_5^7, x_2x_5\}$. Now we can pick $x_3^6x_5$ and repeat. Eventually we see that $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau}) = \langle\mathrm{Gen}\rangle$ where Gen $= \{x_3^{12}, x_3^4x_5^8, x_3^5x_5^7, x_3^6x_5^6, x_3^7x_5^5, x_3^8x_5^4, x_3^9x_5^3, x_3^{10}x_5^2, x_3^{11}x_5, x_2x_5\}$. □

Now we can combine Algorithm 5.2 and Algorithm 5.5 into an algorithm which computes $\mathcal{G}_c(I_A)$.

Algorithm 5.7 *A non-Buchberger algorithm to compute $\mathcal{G}_c(I_A)$ when c is a generic cost vector.*

INPUT: $A \in \mathbf{Z}^{d\times n}$, a generic cost vector $c \in \mathbf{R}^n$, and the regular triangulation Δ_c.
OUTPUT: $\mathcal{G}_c(I_A)$.
1. For each maximal face σ of Δ_c compute $\mathcal{G}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ along with T_α for each minimal generator of $\mathrm{in}_{\tilde{c}_\sigma}(I_{\mathcal{L}_\sigma})$ using Algorithm 5.2.
2. For $k = d-1$ to 0 do: For every k-dimensional face τ of Δ_c compute $\mathcal{G}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ along with T_α for each minimal generator of $\mathrm{in}_{\tilde{c}_\tau}(I_{\mathcal{L}_\tau})$ using Algorithm 5.5.
3. Output $\mathcal{G}_c(I_A) = \mathcal{G}_{\tilde{c}_\emptyset}(I_{\mathcal{L}_\emptyset})$.

As stated above, the algorithm needs to check every face of Δ_c for a possible associated prime of $in_c(I_A)$. The following result for toric initial ideals provides a much stronger stopping criterion for our algorithm. Since this theorem is proved using very different techniques from those in this paper we do not include its proof.

Theorem 5.8 *[HT] Let τ be a face of the regular triangulation Δ_c and $\sigma_1, \ldots, \sigma_t \in \Delta_c$ be such that τ is a facet of each σ_j for $j = 1, \ldots, t$. If p_{σ_j} is not an embedded prime of $in_c(I_A)$ for $j = 1, \ldots, t$ then neither is p_τ.*

This result says that if p_τ is an embedded prime of $in_c(I_A)$, then there exists a chain of associated primes of $\mathrm{in}_c(I_A)$ $p_\tau = p_{\sigma_0} \supset p_{\sigma_1} \supset \cdots \supset p_{\sigma_k} = p_\sigma$ where p_σ is a minimal prime of $in_c(I_A)$ and $|\sigma_i| = |\sigma_{i-1}| + 1$ for $i = 1, \ldots, k$. Theorem 5.8 fails for general monomial ideals as shown below.

Example 5.9 Consider the family of $(r-1)$-dimensional monomial ideals
$M_{(r,p)} := \langle(\prod_{i=1}^r x_i)x_j^{p-1} \,:\, j = 1, \ldots, r\rangle$ in $k[x_1, \ldots, x_r]$ where $p \geq 2$. The minimal primes of $M_{(r,p)}$ are all the principal ideals of the form $\langle x_i\rangle$ each occurring with multiplicity one. The maximal ideal $\langle x_1, \ldots, x_r\rangle$ is the only embedded prime of $M_{(r,p)}$ and its multiplicity is $(p-1)^r$. □

Computational experience

We have done a first implementation of Algorithm 5.7 to test its computational efficiency. It is a rough implementation which could be improved in the future. In the table below we provide the timings we obtained for randomly generated test instances. The matrix A and the cost vector c have entries between 0 and 20. We assume that we know the regular triangulation Δ_c, as well as a basis for $\ker(A) \cap \mathbf{Z}^n$ (both of these computations are polynomial-time computations). When computing the initial ideals and the reduced Gröbner bases of $I_{\mathcal{L}_\sigma}$ when σ is a maximal simplex of Δ_c we did not use Algorithm 5.2 but just used the Buchberger's algorithm. This speeds up the overall algorithm greatly.

The first column of the table gives the dimension of the matrix A. The second column is the number of maximal simplices in Δ_c. The third column is the number of Gröbner basis elements for the toric ideal I_A, and the last column is the running time (in seconds). All the times are obtained on an UltraSparc 1.

Problem Size	$\lvert\Delta_c\rvert$	# GB el.	Time
4×6	9	35	21
4×7	13	78	29
4×8	24	169	44
5×7	10	55	23
5×8	19	143	55
5×9	27	198	72
5×10	32	257	112
6×9	26	172	88
6×10	37	213	105
6×11	33	220	148

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