Primary Decomposition of Lattice Basis Ideals

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Abstract

We study the primary decomposition of lattice basis ideals. These ideals are binomial ideals with generators given by the elements of a basis of a saturated integer lattice. We show that the minimal primes of such an ideal are completely determined by the sign pattern of the basis elements, while the embedded primes are not. We examine the ideal generated by the 2×2 adjacent minors of a generic $m \times n$ matrix. In particular we determine all minimal primes in the $3 \times n$ case. We also present faster ways of computing a generating set for the associated toric ideal from a lattice basis ideal.

1 Introduction

A lattice $\mathcal{L} \subset \mathbf{Z}^n$ is called *saturated* if for $u \in \mathbf{Z}^n$ and $s \in \mathbf{Z}$, $su \in \mathcal{L}$ implies $u \in \mathcal{L}$. This is equivalent to saying there exists a $d \times n$ integral matrix A with $\text{rank}(A) = d = n - \text{rank}(\mathcal{L})$ and $\mathcal{L} = \ker(A) \cap \mathbf{Z}^n$. We will assume throughout that $\mathcal{L}$ is saturated and $\mathcal{L} \cap \mathbf{N}^n = \{0\}$, i.e., there are no positive vectors in $\mathcal{L}$. The *toric ideal* of $\mathcal{L}$ in the polynomial ring $S = k[x_1, \ldots, x_n]$ is the ideal

$$I_{\mathcal{L}} = \langle \mathbf{x}^\alpha - \mathbf{x}^\beta : \ \alpha, \beta \in \mathbf{N}^n \text{ and } \alpha - \beta \in \mathcal{L} \rangle,$$

which, since $\mathcal{L}$ is saturated, is a prime ideal. We fix a lattice basis $\{u_1, \ldots, u_r\} \subset \mathbf{Z}^n$ for the saturated integer lattice $\mathcal{L}$. We let $f_{u_i} = \mathbf{x}^{(u_i)_+} - \mathbf{x}^{(u_i)_-}$ and $J = \langle f_{u_1}, \ldots, f_{u_r} \rangle$. In general J is strictly contained in $I_{\mathcal{L}}$, but in the Laurent polynomial ring $k[\mathbf{x}^{\pm 1}] = k[x_i^{\pm 1} : i = 1, \ldots, n]$, they are equal.

Proposition 1.1 *(cf. Corollary 2.5 in [5], Lemma 2.2 in [6])* $Jk[\mathbf{x}^{\pm 1}] = I_{\mathcal{L}} k[\mathbf{x}^{\pm 1}]$ *and hence* $J : (\prod_{i=1}^n x_i)^\infty = I_{\mathcal{L}}$.

One can decide whether $J = I_{\mathcal{L}}$ by just looking at the sign pattern of the basis vectors $u_1, \ldots, u_r$. For this we need first the following definition.

Definition 1.2 *Suppose M is an $r \times n$ matrix. Then M is called a* **mixed** *matrix if every row of M has both a positive and a negative entry; otherwise it is called* **unmixed**. *The matrix M is called* **dominating** *if it is mixed, but no $t \times t$ submatrix of M is mixed for $1 \leq t \leq min(r, n)$ (see [7]).*

Theorem 1.3 *(cf. Theorem 2.9 in [7]) Let M be the $r \times n$ matrix with rows $u_1, \ldots, u_r$. Then M is dominating if and only if $J = I_{\mathcal{L}}$*

In this paper we will look at the case when $J \neq I_{\mathcal{L}}$. One way of measuring the "distance" of J to being the toric ideal $I_{\mathcal{L}}$ is to consider a primary decomposition of J. In Section 2, we will show that the minimal primes of J are characterized only by the sign pattern of the elements $\{u_1, \ldots, u_r\}$ in the lattice basis. This is an extension of Theorem 1.3 and the work of Fischer, Morris and Shapiro ([6], [7], [8], [9]). In Section 3, we will show that the sign pattern of the generators is not the only factor determining the embedded primes. In this case we do not have a complete characterization, but we will illustrate the general form of these associated primes in terms of subsets of variables and sublattices of $\mathcal{L}$. As a family of examples we will look at the lattice basis ideal generated by the adjacent 2×2 minors of a generic $m \times n$ matrix. These ideals have applications in computational statistics, and they were studied by Diaconis, Eisenbud and Sturmfels [2]. They determined all minimal primes of such an ideal when $m = 2$, and furthermore they gave a minimal primary decomposition when $m = n = 4$. In this case there are 15 minimal and 17 embedded primes. We will determine all minimal primes of such an ideal when $m = 3$, and furthermore we will give a minimal primary decomposition when $m = 3$ and $n = 5$. In this case there are 10 minimal primes and 9 embedded primes. In the last section, we will determine a set of variables of cardinality no more than $\lfloor n/2 \rfloor$ such that every minimal prime (except $I_{\mathcal{L}}$) of J contains at least one variable from this set. This leads to a faster *successive* Gröbner basis algorithm for computing a generating set for the toric ideal $I_{\mathcal{L}}$ from the generators of J.

2 Minimal primes

In this section we will characterize the minimal primes of a lattice basis ideal. Since $\{u_1, \ldots, u_r\}$ generates a saturated lattice, so does any non-empty subset of this set. Thus as an easy consequence of Proposition 1.1 we have:

Corollary 2.1 *$I_{\mathcal{L}}$ is a primary component in every minimal primary decomposition of J. Furthermore, $I_{\mathcal{L}}$ is a minimal prime and every minimal prime P of J other than $I_{\mathcal{L}}$ has the form $P = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_t}, I_{\mathcal{L}'} \rangle$ where $t \geq 1$ and $\mathcal{L}'$ is the lattice generated by the basis vectors $u_{j_1}, \ldots, u_{j_k}$ such that $supp(u_{j_p}) \cap \{i_1, \ldots, i_t\} = \emptyset$ for $p = 1, \ldots, k$. Thus P is determined by the set of variables contained in it.*

Without loss of generality we can assume that a minimal prime P of J which is not $I_{\mathcal{L}}$ contains the first t variables, i.e., $P = \langle x_1, \ldots, x_t, I_{\mathcal{L}'} \rangle$ where $\mathcal{L}'$ is the lattice generated by

$u_{s+1}, \ldots, u_r$. In other words, the matrix M has the form

$$M = \left[\begin{array}{c|c} N & B \\ \hline 0 & D \end{array} \right], \tag{1}$$

where N is an $s \times t$ matrix with no row of zeros and D is the $(r-s) \times (n-t)$ matrix whose rows come from $u_{s+1}, \ldots, u_r$. Conversely, given a decomposition of M as in (1) where N is mixed, the ideal generated by the variables corresponding to the columns of N together with the toric ideal of the lattice generated by rows of D is a prime ideal containing J. Our next results describes the minimal primes of J in terms of these decompositions.

Definition 2.2 *A matrix N is called* **irreducible** *if*

1. *N is a mixed $s \times t$ matrix where $t \leq s$, and*
2. *one cannot bring N into the following form after permuting its rows and columns,*

$$N = \left[\begin{array}{c|c} N' & B' \\ \hline 0 & D' \end{array} \right], \tag{2}$$

where N' is a mixed $s' \times t'$ matrix with $t' \leq s'$ and D' is $(s-s') \times (t-t')$ with $t-t' > s-s'$.

Remark 2.3 *The second condition in the above definition includes also the degenerate case when D' is $0 \times (t-t')$.*

Example 2.4 Let

$$N_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, N_2 = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \end{array} \right], N_3 = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{array} \right].$$

Then N_1 is irreducible, while N_2 and N_3 are not.

$\square$

Theorem 2.5 *Let P be a prime induced by a decomposition as in (1). Then P is a minimal prime of J if and only if N is irreducible.*

Proof: Suppose P is a minimal prime. Clearly N is a mixed matrix, since otherwise P has to contain variables other than $x_1, \ldots, x_t$. First we show that $t \leq s$. Suppose not. If N is a dominating matrix then by Corollary 2.7 of [7] every linear combination of the rows of N is a mixed vector. This means that every binomial in the toric ideal $I_{\mathcal{L}^*}$ associated with the lattice $\mathcal{L}^*$ generated by the first s rows of M is strictly contained in P. This further implies that P strictly contains the toric ideal $I_{\mathcal{L}}$, and hence cannot be a minimal prime. So N is not dominating, and therefore it contains a $k \times k$ mixed submatrix. Now let T be a mixed

$s' \times t'$ submatrix of N with $s' \geq t'$ such that T is maximal with respect to this property. Then we can assume that N has the following decomposition

$$N = \left[\begin{array}{c|c} T & S \\ \hline 0 & R \end{array}\right].$$

If R is not degenerate (see Remark 2.3), then it must be a dominating matrix since otherwise we can construct a bigger mixed matrix containing T. But then it follows again from Corollary 2.7 of [7] that P contains the prime ideal $\langle x_1, \ldots, x_{t'}, I_{\mathcal{L}^*}\rangle$ where now $\mathcal{L}^*$ is the lattice generated by $u_{s'+1}, \ldots, u_r$. Since J is contained in this prime ideal, P cannot be a minimal prime over J. Thus $N = [T, S]$ where T is $s \times t'$ with $t' < t$. But then once again we conclude that P contains the prime ideal $\langle x_1, \ldots, x_{t'}, I_{\mathcal{L}^*}\rangle$ where $\mathcal{L}^*$ is generated by $u_{s+1}, \ldots, u_r$. This shows that N has to be an $s \times t$ mixed matrix with $t \leq s$.

Now assume that after permuting its rows and columns N can be put into a form as in (2) where N' is a maximal offender and has size $s' \times t'$. If $s' = s$, then the prime ideal $\langle x_1, \ldots, x_{t'}, I_{\mathcal{L}^*}\rangle$ would contain J where $t' < t$ and where $I_{\mathcal{L}^*}$ is as above. Hence P could not be a minimal prime over J. Otherwise, the maximality of N' implies that D' is dominating. Hence as before the prime ideal $\langle x_1, \ldots, x_{t'}, I_{\mathcal{L}^*}\rangle$ where $\mathcal{L}^*$ is the lattice generated by $u_{s'+1}, \ldots, u_r$ would be contained in P, again a contradiction.

Conversely, suppose there is a decomposition of M as in (1) where N is irreducible. Suppose P is not a minimal prime of J. Then there exists a minimal prime Q of J, such that $Q \subset P$. This means $Q = \langle x_1, \ldots, x_{t'}, I_{\mathcal{L}^*}\rangle$ with $t' < t$ and $\mathcal{L}^*$ the lattice generated by $u_{s'+1}, \ldots, u_r$. Then Q induces a decomposition

$$N = \left[\begin{array}{c|c} N' & B' \\ \hline 0 & D' \end{array}\right],$$

where N' is an $s' \times t'$ matrix. Since N is irreducible, D' is a mixed matrix with at least as many rows as columns. Now, if the rows of D' are linearly dependent, then there exists a linear combination u of the rows $u_{s'+1}, \ldots, u_s$ such that the support of u is in $\{t+1, \ldots, n\}$. Clearly f_u is in Q but not in P, contradicting the fact that P contains Q. If the rows of D' are linearly independent, then there exists a linear combination of the rows $u_{s'+1}, \ldots, u_s$ such that the resulting integral vector v has nonnegative coordinates in $t'+1, \ldots, t$. But then the corresponding binomial f_v is in Q but not in P, once again contradicting the fact that $P \supset Q$. This shows that P is a minimal prime. $\square$

Remark 2.6 *We will call the irreducible matrix N in a decomposition of M corresponding to a minimal prime P the* **variable matrix** *of P. If N_1 and N_2 are two variable matrices such that by permuting rows and columns of M we get a submatrix*

$$N = \left[\begin{array}{c|c} N_1 & 0 \\ \hline 0 & N_2 \end{array}\right],$$

then N is irreducible and hence it is the variable matrix of a different minimal prime of J.

Example 2.7 *The minimal primes of a lattice basis ideal of a lattice $\mathcal{L}$ with rank($\mathcal{L}$) = 3.*

Let J be the lattice basis ideal whose generators correspond to the rows of the following matrix:

$$M = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & -2 \\ -3 & 0 & 0 & 1 & -1 & 3 \\ 1 & 1 & 0 & -1 & 1 & -2 \end{bmatrix}.$$

Other than the toric ideal $I_{\mathcal{L}} = J : (x_1x_2x_3x_4x_5x_6)^\infty = \langle x_1x_6 - x_3^2, x_2x_6 - x_1^2, x_1x_4x_6 - x_2^2x_5, x_1x_2x_5 - x_4x_6^2\rangle$, there are four other minimal primes. We list them together with the decomposition of the matrix M:

$$\begin{bmatrix} 1 & -1 & * & * & * & * \\ -1 & 1 & * & * & * & * \\ 0 & 0 & 1 & -1 & 2 & -2 \end{bmatrix} P_1 = \langle x_4, x_5, x_1x_3^2 - x_2x_6^2\rangle, \begin{bmatrix} 1 & -2 & * & * & * & * \\ -3 & 3 & * & * & * & * \\ 1 & -2 & * & * & * & * \end{bmatrix} P_2 = \langle x_1, x_6\rangle,$$

$$\begin{bmatrix} 2 & 0 & -2 & * & * & * \\ 0 & -1 & 3 & * & * & * \\ 0 & -1 & -2 & * & * & * \end{bmatrix} P_3 = \langle x_3, x_5, x_6\rangle, \begin{bmatrix} 1 & -1 & 0 & * & * & * \\ -3 & 0 & 1 & * & * & * \\ 1 & 1 & -1 & * & * & * \end{bmatrix} P_4 = \langle x_1, x_2, x_4\rangle.$$

In fact, one can check with `Macaulay 2` that $\text{rad}(J) = I_{\mathcal{L}} \cap P_1 \cap P_2 \cap P_3 \cap P_4$. □

The minimal primes in the above example are nice because three of them consist of only variables. This will be the case for all minimal primes (other than $I_{\mathcal{L}}$) when the lattice basis ideal is *generic*, i.e., every variable appears in each generator f_{u_i} of J. Our terminology is motivated by the generic lattice ideals studied by Peeva and Sturmfels [11].

Corollary 2.8 *If the generators of J have full support, i.e. supp(u_i) = $\{1, \ldots, n\}, i = 1, \ldots, r$, then the non-toric minimal primes of J are generated only by subsets of variables.*

Proof: This follows from Theorem 2.5 and the fact that if there are no zeros in matrix M then the minimal primes will not have an extra "toric" part $I_{\mathcal{L}'}$. □

Corollary 2.9 *Let J be a lattice basis ideal that is associated to a saturated lattice $\mathcal{L}$ with rank($\mathcal{L}$) = r. Then a minimal prime of J contains at most r variables.*

Proof: Suppose P is a minimal prime which induces a decomposition of M as in (1). Since M has r rows and the variable matrix N has at least as many rows as its columns, P can contain at most r variables. □

Example 2.10 *Homogeneous primitive partition identities.*

A homogeneous partition identity with parts at most n is an identity of integers of the form

$$a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots b_k,$$

with $0 < a_i, b_j \leq n$. Such an identity is called *primitive* if there is no proper subidentity

$$a_{i_1} + a_{i_2} + \cdots + a_{i_r} = b_{j_1} + b_{j_2} + \cdots b_{j_r}.$$

Homogeneous primitive partition identities with parts at most n can be calculated by computing any reduced Gröbner basis of the toric ideal associated with $\Lambda(A)$, the Lawrence lifting [14] of

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n \end{bmatrix}.$$

This Gröbner basis has applications in computational statistics (see [3]). In order to get a Gröbner basis for this toric ideal we can start with the following lattice basis:

$$\begin{bmatrix} 1, & -2, & 1, & 0, & \cdots & 0, & -1, & 2, & -1, & 0, & \cdots & 0 \\ 2, & -3, & 0, & 1, & \cdots & 0, & -2, & 3, & 0, & -1, & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n-2), & -(n-3), & 0, & 0, & \cdots & 1, & -(n-2), & (n-3), & 0, & 0, & \cdots & -1 \end{bmatrix}$$

The following has also been shown by Bigatti, La Scala and Robbiano ([1], Corollary 2.10).

Corollary 2.11 *Let $J \subset k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ideal generated by $f_{u_1}, \ldots, f_{u_{n-2}}$ corresponding to the rows of the above matrix. The minimal primes of J are the associated toric ideal $I_{\mathcal{L}}$ and $P_1 = \langle x_1, x_2 \rangle$, $P_2 = \langle x_1, y_1 \rangle$, $P_3 = \langle x_2, y_2 \rangle$, and $P_4 = \langle y_1, y_2 \rangle$. Hence $V(J : (x_1 y_2)^\infty) = V(J : (x_2 y_1)^\infty) = V(I_{\mathcal{L}})$.*

Proof: By Theorem 2.4, $P_1, \ldots, P_4$ are minimal primes. In other possible decompositions of the above matrix as in (1), the variable matrix N is either:

1. the submatrix consisting of column 2 or column $n+1$ together with the identity matrix of the columns 3 through n, or
2. the submatrix consisting of column 1 or column $n+2$ together with the negative identity matrix of the columns $n+3$ through $2n$, or
3. a $k \times k$ identity matrix as a submatrix of the identity matrix of the columns 3 through n together with the corresponding $k \times k$ negative identity matrix as a submatrix of the negative identity matrix of the columns $n+3$ through $2n$ where $1 \leq k \leq n-2$.

But in all of these cases the matrix N has more columns than rows, hence the corresponding prime cannot be a minimal prime of J. $\square$

3 Embedded primes and adjacent minors

In this section we first show that the embedded primes of a lattice basis ideal do not depend on the sign pattern of the generators. Then we will describe the general form of these embedded primes. Lastly, we will examine the lattice basis ideal generated by the 2×2 adjacent minors of a generic $m \times n$ matrix. We will describe all minimal primes in the $3 \times n$ case. We will also give a minimal primary decomposition when the generic matrix is 3×5.

Example 3.1 *The embedded primes of a lattice basis ideal do not depend on the sign pattern of the generators.*

Consider the following matrices

$$M_1 = \begin{bmatrix} 2 & -1 & -2 & 0 & 0 \\ 1 & -2 & 0 & -3 & 0 \\ 2 & -1 & 0 & 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 3 & -2 & -1 & 0 & 0 \\ 2 & -1 & 0 & -5 & 0 \\ 2 & -2 & 0 & 0 & -3 \end{bmatrix},$$

with the saturated lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ generated by the rows of M_1 and M_2 respectively. Then the lattice basis ideal J_1 corresponding to M_1 is radical and $J_1 = I_{\mathcal{L}_1} \cap \langle x_1, x_2 \rangle$. But although M_1 and M_2 have the same sign pattern, J_2 picks up an embedded prime. In fact, $J_2 = I_{\mathcal{L}_2} \cap \langle x_1^2, x_2 \rangle \cap \langle x_1^2, x_2^2, x_4^5 \rangle$. □

Following the notation in [5] we define for each vector of positive integers $s = (s_1, \ldots, s_n)$ and each subset $\mathcal{E}$ of $\{1, 2, \ldots, n\}$

$$J_{\mathcal{E}}^{(s)} := (J + (\{x_i^{s_i}\}_{i \notin \mathcal{E}})) : (\prod_{i \in \mathcal{E}} x_i)^{\infty}. \qquad (3)$$

Theorem 6.2 in [5] states that if the integers s_i are chosen sufficiently large, then

$$J = \bigcap_{\mathcal{E} \subseteq \{1,\ldots,n\}} J_{\mathcal{E}}^{(s)}.$$

Thus an irredundant primary decomposition of J can be obtained from primary decompositions of the $J_{\mathcal{E}}^{(s)}$ by deleting redundant components. Corollary 2.6 in [5] implies that any associated prime P of J has the form $\langle \{x_i\}_{i \notin \mathcal{E}}, I_{\mathcal{L}'} \rangle$ for some $\mathcal{E} \subseteq \{1, \ldots, n\}$ and a saturated lattice $\mathcal{L}'$ of vectors with support in $\mathcal{E}$. The next proposition gives more information on $I_{\mathcal{L}'}$.

Proposition 3.2 *Let J be a lattice basis ideal for the saturated lattice $\mathcal{L} \subset \mathbf{Z}^n$. Then an embedded prime P of J has the form $\langle \{x_i\}_{i \notin \mathcal{E}}, I_{\mathcal{L}'} \rangle$ for some $\mathcal{E} \subseteq \{1, \ldots, n\}$, and $\mathcal{L}'$ is a saturated sublattice of $\mathcal{L} \cap \mathbf{Z}^{\mathcal{E}}$.*

Proof: Suppose $b = \mathbf{x}^{\alpha} - \mathbf{x}^{\beta}$ is in $I_{\mathcal{L}'}$. From (3) we see that there exists a polynomial f and and an integer k such that $bf(\prod_{i \in \mathcal{E}} x_i)^k$ is in $J + (\{x_i^{s_i}\}_{i \notin \mathcal{E}})$. We can assume that $f \in k[x_i : i \in \mathcal{E}]$. With respect to an elimination term order where $x_i \succ x_j$ for all $i \notin \mathcal{E}$ and $j \in \mathcal{E}$, the normal form of $f(\prod_{i \in \mathcal{E}} x_i)^k$ will be a polynomial $g \in k[x_i : i \in \mathcal{E}]$. Suppose

$g = g_1 + \cdots + g_p$ where the g_j are the terms of g. Then Theorem 5.2(a) of [5] implies that there exists a positive integer r such that $(\mathbf{x}^{r\alpha} - \mathbf{x}^{r\beta})g_j$ is in $J + (\{x_i^{s_i}\}_{i \notin \mathcal{E}})$ for all j. Since no term of $(\mathbf{x}^{r\alpha} - \mathbf{x}^{r\beta})g_j$ is in $J + (\{x_i^{s_i}\}_{i \notin \mathcal{E}})$ (otherwise P has to contain a variable x_i where $i \in \mathcal{E}$), Proposition 1.10 of [5] implies that $(\mathbf{x}^{r\alpha} - \mathbf{x}^{r\beta})g_j$ is in J. This shows that $r(\alpha - \beta)$, and hence, $(\alpha - \beta)$ is in $\mathcal{L}$. We conclude that $\mathcal{L}'$ is a saturated sublattice of $\mathcal{L} \cap \mathbf{Z}^{\mathcal{E}}$. □

We now examine the minimal primes in the case where the lattice basis ideal is generated by the 2×2 adjacent minors of the generic $m\times n$ matrix of variables $X = (x_{ij})$. A 2×2 adjacent minor of X is the binomial $x_{ij}x_{i+1,j+1} - x_{i+1,j}x_{i,j+1}$, for $1 \le i \le m-1$ and $1 \le j \le n-1$. We denote this lattice basis ideal by J_{mn}. The corresponding toric ideal is the determinental ideal generated by all 2×2 minors of X (Proposition 5.4 in [13]). In [1], the set of all minimal primes was given for the ideals J_{2n}. Here we determine the minimal primes over J_{3n}.

Let $M(X)$ be the relation matrix whose rows correspond to the adjacent minors of X. Thus $M(X)$ is a $(m-1)(n-1)\times mn$ matrix where each row has four non-zero entries. We index the columns of $M(X)$ by the variables x_{ij} in X. Furthermore, we use the lexicographic ordering on the columns, i.e., $[i,j] < [k,l]$ if $i < k$, or if $i = k$ and $j < l$. The columns $[1,1],[1,n],[m,1],[m,n]$ contain exactly one non-zero entry. The columns $[1,j]$ and $[n,j]$, $j = 2\ldots,n-1$, contain two non-zero entries. All other columns contain four non-zero entries. As an example, if X is 3×5, then $M(X)$ is

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}.$$

Our first lemma shows that the "corner" variables of X are not in any minimal primes.

Lemma 3.3 *The variables $x_{11}, x_{1n}, x_{m1}, x_{mn}$ are not in any minimal primes over J_{mn}.*

Proof: By symmetry it suffices to do the proof for x_{11}. Suppose to the contrary that x_{11} is in a minimal prime P over J_{mn}. This minimal prime ideal corresponds to a decomposition of the matrix M as in Theorem 2.5. Furthermore, the column $[1,1]$ necessarily intersects the variable submatrix N in this decomposition. This column has one non-zero entry which is a 1, and we can assume that it is in the first row of N. If column [2,2] intersects N, then the first row of N contains another 1. Hence the first column of N could be deleted while leaving the remaining submatrix of N still mixed. This would contradict the irreduciblity of N. Furthermore, the first row of N must be mixed. Thus either column $[1,2]$ or $[2,1]$ intersects N. In the first case the only other non-zero entry in this column is a 1 in row 2. As we have seen N cannot intersect column $[2,2]$, thus N must intersect column $[1,3]$. Continuing in this fashion N must intersect columns $[1,2]$ through $[1,k]$. If $k < n$, then N

intersects column $[2,k]$. Hence deleting columns $[1,1]$ through $[1,k-1]$ and rows 1 through $k-2$ still leaves a mixed submatrix of N - a contradiction. On the other hand, if $k=n$, the deletion of columns $[1,1]$ through $[1,n]$ and rows 1 through $n-1$ from N also leaves a mixed submatrix of N. This again is a contradiction to Theorem 2.5.

Now we can return to the case where the first row of N intersects column $[2,1]$ of $M(X)$. Then N must intersect row n which contains a 1 in the $n+1$ position. If N intersects column $[3,2]$, which contains another 1 in row n, then columns $[1,1],[2,1]$ and row 1 could be deleted from N leaving a mixed matrix behind - also a contradiction with the irreducibility of N. Since N must have a negative entry in row n it must intersect either column $[2,2]$ or column $[3,1]$. However, as previously noted, N cannot intersect column $[2,2]$. Therefore N intersects column $[3,1]$. If $m=3$, then we have a contradiction, since columns $[1,1],[2,1],[3,1]$ and rows 1 and n could be deleted leaving a mixed matrix. If $m>3$, then N must intersect row $2n$, which contains a 1 in column $[3,1]$. Row $2n$ must also contain a negative entry which cannot be from column $[3,2]$. Therefore N must intersect column $[4,1]$. Continuing in this fashion N must intersect rows $1,n,2n,\ldots,(m-2)n$ and columns $[1,1],[2,1],[3,1],\ldots,[(m-1),1]$. This set contains more columns than rows, yet deleting them from N leaves a mixed matrix, a contradiction with Theorem 2.5. Thus the result is proved. $\square$

Next we describe the minimal primes over J_{3n}. As we saw in Corollary 2.1 every non-toric minimal prime ideal P has the form $\langle x_{i_1},x_{i_2},\ldots,x_{i_t},I_{\mathcal{L}'}\rangle$ where $t\geq 1$ and $\mathcal{L}'$ is the lattice generated by the vectors $u_{j_1},\ldots,u_{j_k}$ such that $\mathrm{supp}(u_{j_p})\cap\{i_1,\ldots,i_t\}=\emptyset$ for $p=1,\ldots,k$. Thus the problem comes down to deciding what variable sets can be in a minimal prime. We will show that a minimal prime P will contain all the variables in the columns indexed by $S=\{i_1,\ldots,i_k\}\subset\{2,\ldots,n-1\}$ where $0=i_0<i_1<i_2<\ldots<i_k<i_{k+1}=n+1$ are non-consecutive integers (we assume that X has two imaginary columns, namely column 0 and column $n+1$). Furthermore we will prove that the minimal prime P associated to S will contain all the variables in the non-adjacent columns of X indexed by S such that

1. if two columns i_j and i_{j+1} $j=0,\ldots,k$ are separated by more than one column (i.e. $i_{j+1}>i_j+2$) then P either contains
   i) all variables on the second row of X between columns i_j and i_{j+1} or it contains
   ii) all 2×2 minors of the columns $i_j+1,\ldots i_{j+1}-1$,
2. for $i_j>i_{j-1}+2$ and $i_{j+1}>i_j+2$ we do not allow case i) happen for the columns $i_{j-1}+1,\ldots,i_j-1$ and $i_j+1,\ldots,i_{j+1}-1$ at the same time.

Note that when $S=\emptyset$ we obtain either the toric ideal $I_{\mathcal{L}}$ or the ideal generated by the variables $\{x_{21},\ldots,x_{2n}\}$. Both of these ideals are minimal primes of J_{3n}. The three bottom rows in the figure on page 14 are the minimal primes of J_{35}. We invite the reader to compare these with the description we just gave. It was shown in [2] that the number of minimal primes over J_{2n} is the n-th Fibonacci number f_n. A recursive formula for the number $\phi(n)$ of minimal primes $\phi(n)$ over J_{3n} can be deduced from the above description. We define $\phi(0)=\phi(1)=1$. Clearly $\phi(2)=2,\phi(3)=3$ and $\phi(4)=6$. Then $\phi(n)=\phi(n-1)+\phi(n-2)+\sum_{i=0}^{n-5}\phi(i)$. It is not difficult to see that $\phi(n)$ grows much faster than f_n. To give an idea we include some values of $\phi(n)$ and f_n:

n	5	10	15	20	25
f_n	5	55	610	6765	75025
$\phi(n)$	10	182	3296	59661	1079925

Theorem 3.4 *All prime ideals of the above form are minimal over J_{3n}.*

Proof: We will use induction on the number of columns in X to show that the above ideals are derived from a decomposition of $M(X)$ as in Theorem 2.5. When $n = 3$, then $J_{32} = I_{\mathcal{L}} \cap \langle x_{21}, x_{22} \rangle$ as shown in [2]. The result can also be deduced from the fact that

$$M(X) = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Clearly the only irreducible submatrix is the 2×2 matrix containing columns $[2, 1]$ and $[2, 2]$.

Let P be a prime ideal of the above type and suppose that P contains the i th column of X, but not the k th column for any $k < i$. We first assume that P does not contain the variables $x_{21}, \ldots, x_{2,i-i}$. Let N be the 4×3 mixed submatrix of $M(X)$ consisting of the columns $[1, i], [2, i], [3, i]$ and whose rows correspond to the four equations that contain the variable x_{2i}. Then N is irreducible and all the other rows of M have no support in the columns of N. Thus we have a decomposition of $M(X)$. Furthermore, the rows corresponding to the adjacent minors of the first $i - 1$ columns of X do not intersect with N. Thus we can view these rows as being contained in what is referred to as the submatrix D in Theorem 2.5. Additionally, the support of these rows do not intersect with the support of the any of the other rows of D. Thus M can be assumed to have the form

$$\begin{bmatrix} N & * & * \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix}$$

where D_1 and D_2 are the relation matrices corresponding to the adjacent minors of the first $i - 1$ columns and last $n - i$ columns of X respectively. Let X' denote the matrix obtained from X by deleting the first i columns. Then P induces a prime ideal of the above type over X'. By induction on the number of columns in the variable matrix, this is a minial prime over J_{3n-i}. By Remark 2.6, this induces a decomposition of $M(X)$ which corresponds to a minimal prime over J_{3n}. Clearly this prime ideal is P.

Now assume that P contains the variables $x_{21}, \ldots, x_{2,i-1}$. Then one checks that the submatrix N of M consisting of the $2i$ rows corresponding to the adjacent minors containing the variables $x_{21}, x_{22}, \ldots, x_{2i}$ and the $i + 2$ columns $[2, 1], \ldots, [2, i], [1, i], [3, i]$ is irreducible. Now proceed as we did in the first case and we have the result. $\square$

Remark 3.5 Let P be a minimal prime over J_{3n} and suppose that $x_{1i} \in P$ for some i, $2 \le i \le n-1$. Since P is prime and contains all the 2×2 adjacent minors of the variable matrix, either P contains both $x_{1,i-1}$ and $x_{1,i+1}$ or it contains x_{2i}. A symmetric argument applies if $x_{3i} \in P$. Also note, if $x_{2i} \in P$, then either x_{3i} and x_{1i} are in P or $x_{2,i-1}$ and $x_{2,i+1}$ are in P. Furthermore, since P is a minimal prime over J_{3n}, if it contains $x_{2,i-1}$, x_{2i} and $x_{2,i+1}$, then it cannot contain either x_{1i} or x_{3i}.

Next show that all minimal primes over J_{3n} have the indicated form.

Theorem 3.6 *Every minimal prime ideal over J_{3n} besides the toric ideal and the ideal $\langle x_{21}, \ldots, x_{2n} \rangle$ is an ideal of type 1 or type 2.*

Proof: Let P be a minimal prime ideal over J_{3n} other than the toric ideal. Then P must contain a variable. Let N denote the irreducible variable submatrix of M associated to P, whose existence is guaranteed by Theorem 2.5. Let x_{ij} be the first variable under the lexicographic ordering that is in P. First suppose that $i = 1$. Then by Lemma 3.3 this cannot be the variable x_{11}. Since $x_{1,j-1}$ is not in P, by Remark 3.5 x_{2j} must be in P. Hence x_{3j} must also be in P. We also claim that P cannot contain all three variables from the $j+1$ th column of X. If it did, then N would interesect the columns $[1,j], [2,j], [3,j], [1,j+1], [2,j+1], [3,j+1]$. and the rows corresponding to the six equations that contain the variables x_{2j} and x_{2j+1}. Hence N would contain the submatrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Observe that if columns 2, 4 and 6, and rows 3 and 6 of the above matrix are deleted, then the remaining matrix is still mixed. But this contradicts the fact that N is irreducible.

If $x_{1,j+1}$ is in P, then either $x_{2,j+1}$ or $x_{1,j+2}$ is in P. However, if the former is in P, then so is $x_{3,j+1}$, which we have already seen is not possible. Thus we can assume that $x_{1,j+2} \in P$. By repeating this process, since $x_{1n} \notin P$, it follows that for some $k, j+1 < k < n$, P contains the variables $x_{1j+1}, \ldots, x_{1k}, x_{2k}, x_{3k}$. Then the columns $[1, j+1], \ldots, [1, k-1]$ and the rows corresponding to rows $j+1, \ldots, k-2$ of the original matrix $M(X)$ must intersect N. However their deletion (if $k = j+2$, don't remove any rows) leaves a submatrix N' of N that is still mixed. Since we have removed more columns than rows this is a contradiction to N being irreducible. Thus $x_{1,j+1}$ is not in P and by symmetry neither is $x_{3,j+1}$.

We also claim that P cannot contain any variables of the form x_{3k} where $k < j$. For if it did, then as we saw in Remark 3.5, P would either have to contain all the variables $x_{3l}, l < j$, which is impossible by Lemma 3.3, or contain the t th column of the variable matrix X for some $t < j$. But this contradicts our choice of x_{1j} as being the first variable in P. This proves the claim.

It is possible that $x_{2k} \in P$ for some $k < j$. However, by Remark 3.5, P would have to contain all variables $x_{2t}, t < j$. Also by Remark 3.5, if P contains any more variables, then the next one it contains under the lexicographic order would have to be of the form x_{1t} with $t > j$. Proceeding as before, we see that the variable set of P corresponds to one of the variable sets of a prime ideal of type 1 or type 2. Since minimal prime ideals of lattice basis ideals are determined by the variables they contain, we are done. Finally, if P does not contain any variables of the form x_{1j}, then we have $P = \langle x_{21}, \ldots, x_{2n} \rangle$. $\square$

In general it is difficult to get a minimal primary decomposition of J_{mn}, but here we present a decomposition for J_{35}. Theorem 3.4 and 3.6 imply that J_{35} has 10 minimal primes:

$$I_{\mathcal{L}}, \quad A = \langle x_{21}, x_{22}, x_{23}, x_{24}, x_{25} \rangle,$$

$$B_1 = \langle x_{12}, x_{22}, x_{32}, x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{25} - x_{15}x_{23}, x_{13}x_{34} - x_{14}x_{33}, x_{13}x_{35} - x_{15}x_{33},$$

$$x_{14}x_{25} - x_{15}x_{24}, x_{14}x_{35} - x_{15}x_{34}, x_{23}x_{34} - x_{24}x_{33}, x_{23}x_{35} - x_{25}x_{33}, x_{24}x_{35} - x_{25}x_{34} \rangle,$$

$$B_2 = \langle x_{14}, x_{24}, x_{34}, x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{23} - x_{13}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{11}x_{33} - x_{13}x_{31},$$

$$x_{12}x_{23} - x_{13}x_{22}, x_{12}x_{33} - x_{13}x_{32}, x_{21}x_{32} - x_{22}x_{32}, x_{21}x_{33} - x_{23}x_{31}, x_{22}x_{33} - x_{23}x_{32} \rangle,$$

$$C_1 = \langle x_{12}, x_{22}, x_{32}, x_{23}, x_{24}, x_{25} \rangle, \quad C_2 = \langle x_{14}, x_{24}, x_{34}, x_{21}, x_{22}, x_{23} \rangle,$$

$$D = \langle x_{12}, x_{22}, x_{32}, x_{14}, x_{24}, x_{34} \rangle,$$

$$E = \langle x_{13}, x_{23}, x_{33}, x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{12}x_{23} - x_{13}x_{22},$$

$$x_{14}x_{25} - x_{15}x_{24}, x_{14}x_{35} - x_{15}x_{34}, x_{24}x_{35} - x_{25}x_{34} \rangle,$$

$$F1 = \langle x_{13}, x_{23}, x_{33}, x_{24}, x_{25}, x_{11}x_{22} - x_{12}x_{21}, x_{11}x_{32} - x_{12}x_{31}, x_{12}x_{23} - x_{13}x_{22} \rangle,$$

$$F2 = \langle x_{13}, x_{23}, x_{33}, x_{21}, x_{22}, x_{14}x_{25} - x_{15}x_{24}, x_{14}x_{35} - x_{15}x_{34}, x_{24}x_{35} - x_{25}x_{34} \rangle.$$

In other words, $\mathrm{rad}(J_{35}) = I_{\mathcal{L}} \cap A \cap B_1 \cap B_2 \cap C_1 \cap C_2 \cap D \cap E \cap F_1 \cap F_2$. We next describe the embedded primary components of J_{35}. The first primary ideal is

$$G_1 = \langle x_{12}, x_{22}, x_{32} \rangle + [(J^{[345]} + \langle x_{23}^2, x_{24}^2, x_{25}^2, x_{14}^2, x_{34}^2 \rangle) : x_{13}x_{15}x_{33}x_{35}],$$

where $J^{[345]}$ is the ideal generated by the adjacent 2×2 minors of the last three columns of X. The associated prime is the codimension 8 prime $\langle x_{12}, x_{22}, x_{32}, x_{23}, x_{24}, x_{25}, x_{14}, x_{34} \rangle$. By symmetry we have a similar primary component

$$G_2 = \langle x_{14}, x_{24}, x_{34} \rangle + [(J^{[123]} + \langle x_{21}^2, x_{22}^2, x_{23}^2, x_{12}^2, x_{32}^2 \rangle) : x_{11}x_{13}x_{31}x_{33}].$$

The next primary component is

$$H = (J_{35} + \langle x_{13}^2, x_{33}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{24}^2, x_{25}^2, (x_{11}x_{32} - x_{12}x_{31})^2, (x_{14}x_{35} - x_{15}x_{34})^2 \rangle) : \prod_{j \neq 3} x_{1j}x_{3j}.$$

The radical of H is $\langle x_{13}, x_{33}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{11}x_{32} - x_{12}x_{31}, x_{14}x_{35} - x_{15}x_{34}\rangle$. This ideal has codimension 9. The third type of primary component has also codimension 9:

$$K_1 = (J_{35} + \langle x_{12}^2, x_{32}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{24}^2, x_{25}^2, (x_{13}x_{34} - x_{14}x_{33})^2,$$
$$(x_{13}x_{35} - x_{15}x_{33})^2, (x_{14}x_{35} - x_{15}x_{34})^2\rangle) : \prod_{j\neq 2} x_{1j}x_{3j}.$$

$\mathrm{rad}(K_1)$ is $\langle x_{12}, x_{32}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{13}x_{34} - x_{14}x_{33}, x_{13}x_{35} - x_{15}x_{33}, x_{14}x_{35} - x_{15}x_{34}\rangle$, and of course we have a symmetric primary component:

$$K_2 = (J_{35} + \langle x_{14}^2, x_{34}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{24}^2, x_{25}^2, (x_{11}x_{32} - x_{12}x_{31})^2,$$
$$(x_{11}x_{33} - x_{13}x_{31})^2, (x_{12}x_{33} - x_{13}x_{32})^2\rangle) : \prod_{j\neq 4} x_{1j}x_{3j}.$$

The last codimension 9 primary component is

$$L = (J_{35} + \langle x_{12}^2, x_{32}^2, x_{14}^2, x_{34}^2, x_{21}^2, x_{22}^2, x_{23}^2, x_{24}^2, x_{25}^2\rangle) : x_{11}x_{13}x_{15}x_{31}x_{33}x_{35}$$

with the monomial associated prime $\langle x_{12}, x_{32}, x_{14}, x_{34}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}\rangle$. There is one type of codimension 10 primary component

$$R_1 = (J_{35} + \langle x_{12}^2, x_{13}^2, x_{22}^2, x_{23}^2, x_{32}^2, x_{33}^2, x_{21}^3, x_{24}^3, x_{25}^3, (x_{14}x_{35} - x_{15}x_{34})^2\rangle) : x_{11}x_{14}x_{15}x_{31}x_{34}x_{35},$$

where $\mathrm{rad}(R_1) = \langle x_{12}, x_{13}, x_{22}, x_{23}, x_{32}, x_{33}, x_{21}, x_{24}, x_{25}, x_{14}x_{35} - x_{15}x_{34}\rangle$. The symmetric primary component is

$$R_2 = (J_{35} + \langle x_{13}^2, x_{14}^2, x_{23}^2, x_{24}^2, x_{33}^2, x_{34}^2, x_{21}^3, x_{22}^3, x_{25}^3, (x_{11}x_{33} - x_{13}x_{31})^2\rangle) : x_{11}x_{12}x_{15}x_{31}x_{32}x_{35}.$$

The most embedded primary component is the codimension 11 ideal

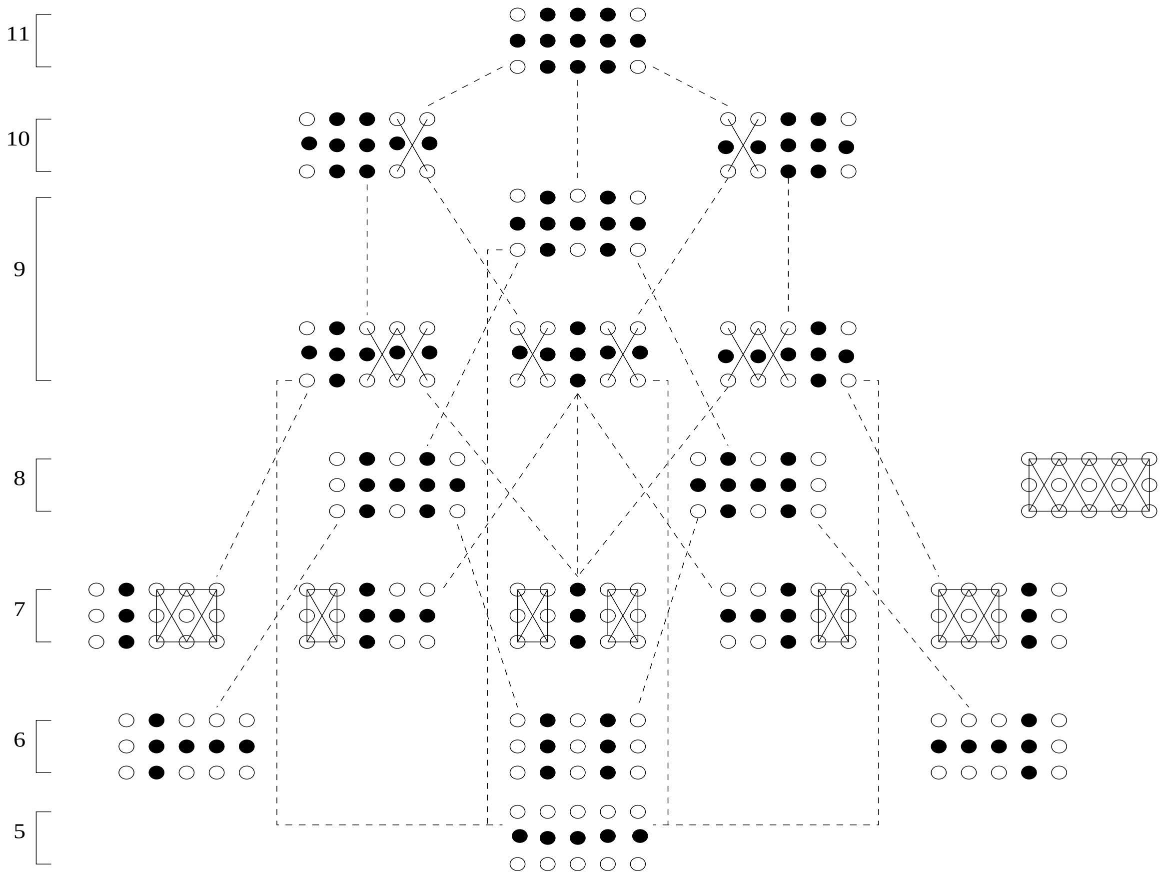
$$U = (J_{35} + \langle x_{12}^4, x_{13}^4, x_{14}^4, x_{32}^4, x_{33}^4, x_{34}^4, x_{21}^4, x_{22}^4, x_{23}^4, x_{24}^4, x_{25}^4\rangle) : (x_{11}x_{15}x_{31}x_{35})^3(x_{11}x_{35} - x_{15}x_{31}).$$

The associated prime is the monomial prime $\langle x_{12}, x_{13}, x_{14}, x_{32}, x_{33}, x_{34}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}\rangle$. We note that one can check that all the ideals above are primary by either using Theorem 1.1 in [4] or Algorithm 9.4 in [5]. We summarize this in the following.

Proposition 3.7 *The ideal J_{35} has 19 associated primes, 10 minimal and 9 embedded. A minimal primary decomposition is given by*

$$J_{35} = \mathit{rad}(J_{35}) \cap G_1 \cap G_2 \cap H \cap K_1 \cap K_2 \cap L \cap R_1 \cap R_2 \cap U.$$

The next figure is the partially ordered set of the associated primes of J_{35} ordered by inclusion. The numbers on the left are the codimensions of the primes on the corresponding level. The black dots represent the variables which are contained in the corresponding prime. For instance the 3×5 array of dots at the top is the radical of U (see above). The "crossed" white dots in a box represent the ideal generated by all 2×2 minors of the submatrix in that box. For instance, the isolated codimension 8 prime is the toric ideal $I_{\mathcal{L}}$. The crossed white dots not in a box represent the ideal generated by all 2×2 minors of the submatrix consisting of variables the crosses touch. The three bottom rows of the figure together with the toric ideal are the minimal primes of J_{35}. We invite the reader to compare with Theorem 3.4 and Theorem 3.6.


11
10
9
8
7
6
5

4 Computing a generating set for the toric ideal

Proposition 1.1 and the fact that $J : (\prod_{i=1}^{n} x_i)^\infty = (\cdots(J : x_1^\infty) : x_2^\infty)\cdots : x_n^\infty)$ gives an algorithm which computes a generating set of $I_\mathcal{L}$ starting from the generators of J by doing n Gröbner bases computations (cf. Algorithm 1 in [10]). The following theorem shows that we need at most $\lfloor n/2 \rfloor$ Gröbner bases computations.

Theorem 4.1 *There exists variables $x_{i_1}, \ldots, x_{i_k}$ such that $I_\mathcal{L} = J : (\prod_{j=1}^{k} x_{i_j})^\infty$ where $k \leq \lfloor n/2 \rfloor$.*

Proof: We construct a subset $T = \{x_{i_1}, \ldots, x_{i_k}\}$ of the variables such that for every associated prime P of J we have $P \cap T \neq \emptyset$. We start with $T = \emptyset$. We can assume that the first k_1 entries of the first row of M are nonzero and the rest of u_1's entries are zero. Furthermore we can also assume that the first s_1 entries of u_1 are positive where $s_1 \leq \lfloor k_1/2 \rfloor$. We set $T = T \cup \{x_1, \ldots, x_{s_1}\}$. Then we delete the first row and the first k_1 columns of M. If the resulting matrix have all unmixed rows we stop, if not, we can assume that the top row is mixed. We can further assume that the first k_2 entries of this row are nonzero and the first $s_2 \leq \lfloor k_2/2 \rfloor$ of them are positive; and we set $T = T \cup \{x_{k_1+1}, \ldots, x_{k_1+s_2}\}$ and we repeat. When we are done, T will have $s_1 + s_2 + \cdots + s_p \leq \lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor + \cdots + \lfloor k_p/2 \rfloor \leq \lfloor n/2 \rfloor$ variables. So M will be in the form

$$\left[\begin{array}{c|c} B & 0 \\ \hline C & D \end{array}\right]$$

where B is an $p \times (\sum_{j=1}^{p} k_j)$ matrix of the form

$$\begin{bmatrix} + & \cdots & + & - & \cdots & - & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \cdots & * & * & \cdots & * & + & \cdots & + & - & \cdots & - & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & & \vdots & & & \vdots & & & \ddots & & & \ddots & & & \ddots & \\ * & \cdots & * & * & \cdots & * & * & \cdots & * & * & \cdots & * & * & \cdots & * & + & \cdots & + & - & \cdots & - \end{bmatrix}$$

and D is a matrix whose rows are unmixed. Now suppose x is a variable contained in an associated prime of J. If $x \in T$ then we are done. So suppose that $x = x_i \notin T$ and it corresponds to a column of B. This means that x_i corresponds to an entry in the final block of negative entries in a row u_j of B. Then P must contain yet another variable $x_{i'}$ which corresponds to one of the positive entries of row j. If $x_{i'}$ is in T, we are done. If not, since $i' < i$, we repeat this finitely many times and eventually find a variable in $T \cap P$. If x corresponds to a column of D, then we know that P must contain another variable corresponding to one of the columns of B because each row of D is unmixed. This brings us back to the previous case. □

Bigatti, La Sacala and Robbiano [1] report an efficient implementation for computing generators of toric ideals starting from lattice basis ideals using successive saturations with respect to variables. They observe that in most cases the number of variables needed is far smaller than $n/2$. Here we provide an example where precisely $n/2$ variables are needed.

Suppose M is a $2 \times n$ matrix whose rows generate a saturated lattice of rank two, where the entries in the first $n/2$ columns are all positive and the entries in the last $n/2$ columns are all negative (we assume that n is even). Then Theorem 2.5 implies that the non-toric minimal primes of the lattice basis ideal corresponding to M are of the form $\langle x_i, x_j \rangle$ where $1 \leq i \leq n/2$ and $n/2 < j \leq n$. We need a subset of variables so that every such minimal prime contains at least one variable from that set. Obviously such a subset contains at least $n/2$ variables.

Recall that the saturated lattices we are considering are the lattices of integral vectors in the kernel of a $d \times n$ matrix A. The following corollary will be useful for computing a generating set for $I_{\mathcal{L}}$ when d is small compared to n.

Corollary 4.2 *Let A be a $d \times n$ matrix where $\mathcal{L} = ker(A) \cap \mathbf{Z}^n$. Then there exists a lattice basis of $\mathcal{L}$ and a subset of variables $x_{i_1}, \ldots, x_{i_k}$ with $k = d+1$ such that $I_{\mathcal{L}} = J : (\prod_{j=1}^{k} x_{i_j})^\infty$.*

Proof: Suppose M is a matrix whose rows form a basis of $\mathcal{L}$. By applying the Hermite normal form algorithm (Section 5.3 in [12]) to M we get a new basis matrix $M' = [C, D]$ where D is a $(n-d) \times (n-d)$ lower triangular matrix in which every entry on and below the main diagonal is positive. We take J to be the lattice basis ideal corresponding to M'. The first row of M' has support in $\{1, \ldots, d+1\}$, and so we can choose $x_1, \ldots, x_{d+1}$ as a sufficient set of variables such that $I_{\mathcal{L}} = J : (\prod_{j=1}^{d+1} x_j)^\infty$. This follows from the proof of Theorem 4.1. $\square$

Remark 4.3 *Notice that if the first row of M' in the above proof has support equal to $\{1, \ldots, d+1\}$, then we can choose a smaller set of variables $x_{i_1}, \ldots, x_{i_k}$ with $k \leq \lfloor (d+1)/2 \rfloor$.*

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