Polytopes of Partitions of Numbers

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ABSTRACT

We study the vertices and facets of the polytopes of partitions of numbers. The partition polytope P_n is the convex hull of the set of incidence vectors of all partitions $n = x_1 + 2x_2 + ... + nx_n$. We show that the sequence $P_1, P_2, ..., P_n, ...$ can be treated as an embedded chain. Dynamics of behavior of the vertices of P_n, as n increases, is established. Some sufficient and some necessary conditions for a point of P_n to be its vertex are proved. Representation of the partition polytope as a polytope on a partial algebra – which is a generalization of the group polyhedron in the group theoretic approach to the integer linear programming – allows to prove subadditive characterization of the non-trivial facets of P_n. These facets $\sum_{i=1}^{n} p_i x_i \geq p_0$ correspond to extreme rays of the cone of subadditive functions $p:\{1,2,...,n\} \to \mathbb{R}$ with additional requirements $p_0 = p_n$ and $p_i + p_{n-i} = p_n$, $1 \leq i < n$. The trivial facets are explicitly indicated. We also show how all vertices and facets of the polytopes of constrained partitions – in which some numbers are forbidden to participate – can be obtained from those of the polytope P_n. All vertices and facets of P_n for $n \leq 8$ and $n \leq 6$, respectively, are presented.

1. INTRODUCTION

Any representation of a positive integer number *n* as a sum of positive integers

$$n = n_1 + n_2 + \ldots + n_k,\ n_i \in \mathbb{Z}, n_i > 0,\ i = 1, \ldots, k,$$

is called a partition of *n*. For centuries the partitions of numbers were a subject of thorough investigations [1]. In this paper the set of unordered partitions of *n* is studied from the polyhedral point of view. Each partition is associated with its incidence vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$; the component x_i, $i = 1, \ldots, n$, is the number of times the item *i* appears in the partition. The object of our interest is the polytope $P_n \subset \mathbb{R}^n$, which is the convex hull of the set

$$T_n = \left\{ x \in \mathbb{Z}^n \mid x_1 + 2x_2 + \ldots + nx_n = n,\ x_i \in \mathbb{Z},\ x_i \geq 0,\ i = 1, \ldots, n \right\} \tag{1}$$

of incidence vectors of all unordered partitions of *n*: $P_n = convT_n$. We call P_n the polytope of unordered partitions of *n*. This definition guarantees that T_n is the set of all integer points of P_n. So one can study P_n in effort to describe the set of unordered partitions of *n*.

In section 2 of the paper a relation between the partition polytopes for different numbers *n* is established. It is shown that the polytope P_n is of dimension $n-1$ and that the sequence $P_1, P_2, P_3, \ldots$ can be treated as an embedded chain. Dynamics of behavior of the vertices of P_n, as *n* increases, is established. Some sufficient and some necessary conditions for a point to be a vertex of P_n are proposed. All vertices of P_n, for $n \leq 8$, are established with the aid of these conditions. They are listed in Appendix 1.

In section 3 the faces of maximal dimension of the partition polytope are described. Since $dim\, P_n = n-1$, an inequality

$$\sum_{i=1}^{n} p_i x_i \geq p_0 \tag{2}$$

defines a facet of P_n if it is valid for P_n and is satisfied as equality by some $n-1$ affinely independent points of P_n. According to another definition, (2) is a facet if it is valid for P_n and cannot be expressed as a sum of two other valid inequalities, unless each is a positive multiple of (2) plus a scalar multiple of equation (1). We divide all facets into two classes. The facets of the first class, which we call trivial, are explicitly listed: they are all coordinate hyperplanes of $\mathbb{R}^n$, except $x_1 = 0$. As to the non-trivial facets, we prove their subadditive characterization, which allows us to finally describe them as certain solutions of a system of equations and inequalities.

The algebraic technique of subadditive characterization of the facets of polyhedra on algebraic structures was originally proposed in the group theoretic approach to integer linear programming [4, 5]. According to this approach, a relaxation of the original integer linear programming problem is reduced to a linear minimization problem over the master group polyhedron

$$P(G, g_0) = conv\,\{x = (x(g),\ g \in G,\ g \neq 0) \mid \sum_{g \in G, g \neq 0} g x(g) = g_0,\ x(g) \in \mathbb{Z},\ x(g) \geq 0\,\} \tag{3}$$

of all solutions of an equation on a finite abelian group G with some $g_0 \in G$ as the right-hand side. More precisely, this reduction produces a group polyhedron $P(H, g_0)$, $H \subseteq G$, that is (3) with $g \in G$ substituted by $g \in H$. Details of this reduction can be easily found [7, 11]. The hierarchy of valid, subadditive and minimal inequalities related to this polyhedron was constructed [5, 6]. An inequality

$$\sum_{g \in G, g \neq 0} p(g) x(g) \geq p(g_0) \tag{4}$$

is called valid for $P(G, g_0)$ if it is satisfied by all the points of $P(G, g_0)$. A valid inequality (4) is called minimal if any other inequality $\sum_{g \in G, g \neq 0} r(g) x(g) \geq r(g_0)$ satisfying $r(g_0) \geq p(g_0)$ and $r(g) \leq p(g)$, $g \in G$, where at least one of these constraints is strict, is not valid for $P(G, g_0)$.

Definition of subadditive inequalities is based on the notion of a subadditive function. Let $\hat{+}$ denote the addition operation in *G*. A function p: $p: G \to \mathbb{R}$ is called $\hat{+}$-subadditive if $p(g_1 \hat{+} g_2) \leq p(g_1) + p(g_2)$, for all g_1, $g_2 \in G$. An inequality (4) is called $\hat{+}$-subadditive if $p(g)$, $g \in G$, are the values of some subadditive function on *G*. Subadditive functions on *G*, as well as subadditive inequalities for the polyhedron $P(G, g_0)$, form polyhedral cones. Hence one can talk about their extreme rays. Subadditive characterization of the polyhedron $P(G, g_0)$ in the group theoretic approach asserts that its non-trivial facets are exactly those extreme subadditive inequalities that are minimal [5]. This description was extended to polyhedra on certain Abelian semigroups and additive systems (finite sets closed in respect to one everywhere defined binary algebraic operation) [2, 3, 8]. The author generalized these results for the case of polyhedra on partial algebras [12]. The notion of partial algebra is referred to as it is defined in [9]. An algebra is an arbitrary nonempty set together with some algebraic operations of arbitrary arity defined on it. In a partial algebra operations can be defined on the basic set only partially.

We show that the partition polytope P_n can be represented as a polytope on a partial algebra with one operation. So, essentially, subadditive characterization of the non-trivial facets of P_n follows from [12]. Since this work is not easily available we reproduce its main results here, though in the form applicable to our case. Some theorems and proofs are close to those in [8], but the main results are substantially new. Such are Theorem 10 that describes the trivial facets and Theorem 7 that describes the minimal valid inequalities and is crucial for the final description of the non-trivial facets in Theorem 13. The list of all non-trivial facets of P_n, for $n \leq 6$, obtained by the use of Theorem 13 is presented in Appendix 2.

In section 4 the polytopes of constrained partitions, in which some numbers are forbidden to appear, are considered. We show that, similar to the case of group polyhedra, these polytopes are just certain cuts of the master partition polytope P_n and that their facets are provided by the facets of P_n.

2. POLYTOPES OF PARTITIONS AND THEIR VERTICES

Theorem 1. Affine dimension of the polytope P_n is equal to $n-1$**.**

Proof. One can easily check that P_1 is the point x_1=1 in $\mathbb{R}$, and P_2 is the closed line segment in $\mathbb{R}^2$ with the endpoints (2, 0) and (0, 1) corresponding to partitions 2 = 1+1 and 2 = 2. So the theorem is true for P_1 and P_2: $dim\, P_1$ = 0 and $dim\, P_2$ = 1. For a given $n > 1$, the point $e = (0^{n\text{-}1},\ 1)$, with coordinates $e_i =\ 0,\ 1 \le i \le n-1,\ e_n = 1$, is a single vertex of P_n with $x_n > 0$. Hence P_n is the convex hull of e and all integer points of the set $Q_{n-1} = \{x \in \mathbb{R}_+^{n-1} \mid \sum_{i=1}^{n-1} ix_i = n\}$.

Equation $\sum_{i=1}^{n-1} ix_i = n$ is equivalent to $x_1' + \sum_{i=2}^{n-1} ix_i = n-1$, where $x_1' = x_1 - 1$. Therefore, Q_{n-1} translated by -1 along the axis x_1 contains P_{n-1}. By induction on n, we have $dim\, P_n$ = $dim\, P_{n-1}$ +1, which proves the theorem.

As can be seen from the proof, the polytope P_n is a pyramid with the point $(0^{n\text{-}1},\ 1)$ as the apex. The base of the pyramid lies in the hyperplane $x_n = 0$ and contains the polytope P_{n-1} translated by 1 along the axis x_1 and embedded into $\mathbb{R}^n$. If we identify P_{n-1} with its image under the translation $\varphi_1:\ (x_1, x_2, \dots, x_{n-1}) \mapsto (x_1 + 1, x_2, \dots, x_{n-1}, 0)$ we can consider P_{n-1} to be a part of P_n. With this convention, the partition polytopes constitute an embedded chain

$$P_1 \subset P_2 \subset \dots \subset P_n \subset \dots.$$

The vertices $x = (x_1, x_2, \dots, x_n)$ of P_n with $x_1 > 0$ and $x_n = 0$ are inherited from P_{n-1}. Indeed, $\varphi_1^{-1}(x)$ is a vertex of P_{n-1} since it cannot be a convex combination of any

$y^1, y^2, \dots, y^m \in T_{n-1}$ unless the same is true for x and $\varphi_1(y^1), \varphi_1(y^2), \dots, \varphi_1(y^m) \in T_n$. A similar relation holds for the vertices of P_n with the first coordinate $x_i > 0$, for $2 \le i \le \left[\frac{n}{2}\right]$: they are inherited by P_n from P_{n-i} via translation $\varphi_i : \mathbb{R}^{n-i} \to \mathbb{R}^n$: $(x_1, x_2, \dots, x_{n-i}) \mapsto (x_1, x_2, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{n-i}, 0^i)$. Since every $x \in T_n$, except $(0^{n-1}, 1)$, has $x_i > 0$ for some i, $1 \le i \le \left[\frac{n}{2}\right]$, we obtain

Theorem 2. All vertices of P_n, except $(0^{n-1}, 1)$, are the φ_i-images of vertices of some preceding polytopes P_{n-i}, $i = 1, 2, \dots, \left[\frac{n}{2}\right]$: if $x \ne (0^{n-1}, 1)$ is a vertex of P_n and $i = \min j$, for which $x_j > 0$, then $x = \varphi_i(y)$ for some vertex y of P_{n-i}.

On the other hand, some vertices of P_{n-1} do not remain vertices of P_n since they are captured by the convex hull of some other vertices. Such is the vertex (1, 1, 0) of P_3: $\varphi_1(1,1,0) =$ (2, 1, 0, 0) is the half-sum of (0, 2, 0, 0) and (4, 0, 0, 0) and is not a vertex of P_4. The corollary below shows that finally this is the destiny of almost all vertices of the partition polytopes and elucidates how soon this happens.

Next two theorems give two sufficient and two necessary conditions for a point $x \in T_n$ to be a vertex of P_n. These conditions proved to be rather strong: they were successfully used to check all partitions of n, up to $n = 8$, for being vertices of P_n, see Appendix 1.

Theorem 3. (i) Let $\{i_1, i_2, \dots, i_k\}$ be a set of integers, $1 \le i_j \le n$, $j = 1, 2, \dots, k$, such that the equation $i_1 x_1 + i_2 x_2 + \dots + i_k x_k = n$, $x_j \in \mathbb{Z}_+$, has one or two solutions. Then for each solution $a_1, a_2, \dots, a_k$, the point $x = (x_1, x_2, \dots, x_n)$, with $x_i = a_i$, for $i \in \{i_1, i_2, \dots, i_k\}$, and $x_i = 0$, for $i \notin \{i_1, i_2, \dots, i_k\}$, is a vertex of P_n.

(ii) Let $1 = i_1 < i_2 < \ldots < i_k \le n$ be an increasing sequence of integers. Define $n_k = n$, $x_{i_k} = \left[\frac{n_k}{i_k}\right]$; $n_{k-1} = n_k - x_{i_k} i_k$, $x_{i_{k-1}} = \left[\frac{n_{k-1}}{i_{k-1}}\right]$; … ; $n_1 = n_2 - x_{i_2} i_2$, $x_1 = x_{i_1} = \left[\frac{n_1}{i_1}\right] = n_1$; and $x_i = 0$, for $i \ne i_1, i_2, \ldots, i_k$. Then, $x = (x_1, x_2, \ldots, x_n)$ is a vertex of P_n.

Proof. To prove (i), it is sufficient to notice that if an integer point $x \in P_n$ is not a vertex then there are at least two other integer points $y^1, y^2 \in P_n$ such that $y_i^1 = y_i^2 = 0$ whenever $x_i = 0$. To prove (ii), suppose x is not a vertex of P_n. Then, $x = \lambda_1 y^1 + \lambda_2 y^2 + \ldots + \lambda_m y^m$ for some integer points $y^1, y^2, \ldots, y^m \in P_n$ with $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$. Then, for all $j = 1, 2, \ldots, m$, subsequently hold $y_{i_k}^j = x_{i_k}$, $y_{i_{k-1}}^j = x_{i_{k-1}}$, … , $y_{i_1}^j = x_{i_1}$. So all $y^1, y^2, \ldots, y^m = x$, and x is a vertex.

Theorem 4. Every vertex x of P_n satisfies the following relations:

(i) $ix_i < k$ for all i and k such that $1 \le i < k \le n$, i divides k, and $x_k > 0$,

(ii) $ix_i < m - k$ for all triples of indices i, k, m such that $k < m$, i divides $m - k$, and $x_k, x_m > 0$.

Proof. To prove (i), note that if $x_i \ge \frac{k}{i}$ then x is the half-sum of points y^1 and y^2 with coordinates $y_i^1 = x_i - \frac{k}{i}$, $y_k^1 = x_k + 1$, $y_i^2 = x_i + \frac{k}{i}$, $y_k^2 = x_k - 1$, and $y_j^1 = y_j^2 = x_j$, for all $j \ne i, k$, and both y^1 and y^2 belong to P_n.

For (ii), if $x_i \ge \frac{m-k}{i}$ then x is the half-sum of points y^1 and y^2 with coordinates $y_i^1 = x_i + \frac{m-k}{i}$, $y_k^1 = x_k + 1$, $y_m^1 = x_m - 1$, $y_i^2 = x_i - \frac{m-k}{i}$, $y_k^2 = x_k - 1$, $y_m^2 = x_m + 1$, and $y_j^1 = y_j^2 = x_j$, for all $j \ne i, k, m$, and both y^1 and y^2 belong to P_n.

Corollary. All vertices of P_n, except $(n,0^{n\text{-}1})$, do not remain vertices of P_{2n}, and P_{2n} is the first polytope, for which this happens.

Proof. Let $x=(x_1,x_2,\ ...\ ,\ x_n)\neq(n,0^{n-1})$ be a vertex of P_n. Then, $x_k > 0$ for some $k > 1$, and $\varphi_1^n(x) = (x_1+n,x_2,\ldots,x_n,0^n)$ violates necessary condition (i) of Theorem 4. Hence $(n,0^{n-1})$ is the only vertex of P_n that is still a vertex of P_{2n}. To conclude the proof, note that $\varphi_1^{n-1}(0^{n-1},1)=(n-1,0^{n-2},1,0^{n-1})$ is a vertex of P_{2n-1}.

3. FACETS OF PARTITION POLYTOPES

Let us consider the partial algebra $\mathcal{N}=\left\langle N,\hat{+}\right\rangle$ with the basic set $N = \{1, 2, ... , n\}$ and partial operation $\hat{+}$ on N defined by

$$i \mathbin{\hat{+}} j = \{i + j, \quad \text{if} \quad i + j \leq n; \quad \text{and} \quad \textit{not defined}, \quad \text{if} \quad i + j > n\}, \quad i, j \in N.$$

Successively applying operation $\hat{+}$ to the elements of N and already built subexpressions, one can recursively construct a variety of formal expressions E on $\mathcal{N}$, such as $E = (((1\mathbin{\hat{+}}5)\mathbin{\hat{+}}4)\mathbin{\hat{+}}(3\mathbin{\hat{+}}1))$. Each formal expression E can be associated with its incidence vector $t(E) \in \mathbb{R}^n$, with the components t_i equal to the number of times $i \in N$ occurs in E. Continuing the example above for $n = 7$, we have $t(E) = (2, 0, 1, 1, 1, 0, 0)$. Some formal expressions can be successfully evaluated, finally yielding certain elements $v(E) \in N$. For the others, evaluation stumbles at one of the steps on an indefiniteness. This is the case in our example: $(1\mathbin{\hat{+}}5)\mathbin{\hat{+}}4 = 6\mathbin{\hat{+}}4$ is undefined for $n = 7$. Both computability of an expression E and the value of $v(E)$ depend only on the incidence vector $t(E)$, hence we can regard operation $\hat{+}$ as commutative and associative.

Let $T(\mathcal{N},n)$ be the set of the incidence vectors of those expressions E, for which $v(E) = n$. We define the polyhedron $P(\mathcal{N},n)$ on the partial algebra $\mathcal{N}$ as the convex hull of $T(\mathcal{N},n)$. It is obvious that $T(\mathcal{N},n)=T_n$ and $P(\mathcal{N},n) = P_n$.

Henceforth, we denote the inequality (2) by the $(n+1)$-dimensional vector $(p_0; p) = (p_0; p_i, i \in N)$. As for the group case, an inequality $(p_0; p)$ is called valid for the polytope P_n if it is valid for all $t \in T(\mathcal{N}, n)$. All inequalities valid for P_n form a cone in $\mathbb{R}^{n+1}$, which we denote by $V(P_n)$.

A function $p: N \to \mathbb{R}$ is called $\hat{+}$-subadditive if $p(i \hat{+} j) \leq p(i) + p(j)$, for all $i, j \in N$ such that $i \hat{+} j$ is defined. In other words, p is a $\hat{+}$-subadditive function if

$$p(i + j) \leq p(i) + p(j), \quad i, j \in N, \quad i + j \leq n. \tag{5}$$

In the following we simply call such functions subadditive and write p_i instead of $p(i)$. Subadditive functions on N form a cone in $\mathbb{R}^n$, denote it by $S(N)$.

Lemma 1. If $p \in S(N)$ and E is an expression on $\mathcal{N}$ with the incidence vector $t = t(E)$ and the value $v(E) = m \in N$, then $\sum_{i=1}^{n} p_i t_i \geq p_m$.

Proof. The statement follows from subadditivity of p: $\sum_{i=1}^{n} p_i t_i \geq \sum_{i=1}^{n} p(it_i) \geq p(\sum_{i=1}^{n} it_i) = p_m$.

Lemma 1 implies that for each subadditive function p and each $p_0 \leq p_n$, an inequality $(p_0; p)$ is valid for P_n. We call such inequalities subadditive. Subadditive inequalities form a cone in $\mathbb{R}^{n+1}$, denote it by $S(P_n)$. Next theorem is an immediate consequence of Lemma 1.

Theorem 5. $S(P_n) \subseteq V(P_n)$.

As for the group case, we call an inequality $(p_0; p) \in V(P_n)$ minimal valid inequality if its coefficients p_i cannot be decreased and the right-hand side p_0 increased without violating its validity for P_n. Let $M(P_n)$ be the set of all minimal valid inequalities for P_n.

Lemma 2. If $(p_0; p) \in M(P_n)$ and E is an expression on $\mathcal{N}$ with the incidence vector $t = t(E)$ and the value $v(E) = m \in N$, then $\sum_{i=1}^{n} p_i t_i \geq p_m$.

Proof. On the contrary, suppose that for some minimal valid inequality $(p_0; p)$ there exists an expression E such that $\sum_{i=1}^{n} p_i t_i < p_m$. Define a new inequality $(p_0; q)$ by setting $q_i = p_i$, for $i \neq m$, and $q_m = \sum_{i=1}^{n} p_i t_i$. If we show that $(p_0; q)$ is a valid inequality, this would contradict minimality of $(p_0; p)$ and complete the proof. Suppose the opposite: $(p_0; q)$ is not valid, i.e. there exists an incidence vector $u \in T_n$, for which $\sum_{i=1}^{n} q_i u_i < p_0$. Then, $u_m \geq 1$ since q differs from p only in the m-th component. Let us take an expression corresponding to the incidence vector u and substitute each item m in it by the expression E. We will obtain a new expression with some incidence vector $w \in T_n$. Let N_t be the set of those indices i, for which $t_i > 0$. The coordinates of w are: $w_i = u_i$, for $i \neq m$ and $i \notin N_t$; $w_i = u_m t_i + u_i$, for $i \in N_t$; $w_m = 0$. The following calculation shows that $\sum_{i=1}^{n} p_i w_i < p_0$:

$$p_0 > \sum_{i=1}^{n} q_i u_i = \sum_{i \in N - N_t,\, i \neq m} q_i u_i + q_m u_m + \sum_{i \in N_t} q_i u_i = \sum_{i \in N - N_t,\, i \neq m} p_i u_i + u_m \sum_{i \in N_t} p_i t_i + \sum_{i \in N_t} p_i u_i =$$

$$= \sum_{i \in N - N_t,\, i \neq m} p_i u_i + \sum_{i \in N_t} p_i (u_m t_i + u_i) = \sum_{i \in N - N_t,\, i \neq m} p_i w_i + \sum_{i \in N_t} p_i w_i = \sum_{i=1}^{n} p_i w_i .$$

However, this contradicts validity of $(p_0; p)$. Therefore, inequality $(p_0; q)$ is valid and lemma is proved.

Lemma 2 implies the following theorem.

Theorem 6. $M(P_n) \subseteq S(P_n)$.

Theorem 7. An inequality $(p_0; p)$, valid for P_n, is minimal if and only if it satisfies the following conditions:

(i) $p_0 = p_n$,

(ii) $p_i + p_{n-i} = p_n$, $1 \le i \le \left[\dfrac{n}{2}\right]$.

Proof. At first, let $(p_0; p)$ be a minimal valid inequality for P_n. Its validity for the point $(0^{n-1},1) \in T_n$ implies $p_0 \le p_n$. However, by Lemma 2, the inequality $(p_n; p)$ is valid. Therefore, $p_0 < p_n$ could not be the case, as it would contradict minimality of $(p_0; p)$, and (i) is proved.

To prove (ii), notice that, by Theorem 6, minimality of $(p_0; p)$ implies its subadditivity. In particular, $p_i + p_{n-i} \ge p_n$, for all $i < n$. Suppose $p_k + p_{n-k} > p_n$ for some $k < n$. Then either (1) $k \ne \dfrac{n}{2}$, and we can assume that $k > \dfrac{n}{2}$, or (2) $k = \dfrac{n}{2}$ for an even n. We show further that in each case an inequality $(p_0;q) \in V(P_n)$ can be constructed in such way that $q_i \le p_i$, for all i, and some $q_i < p_i$. This will contradict to minimality of $(p_0; p)$ and prove condition (ii).

Consider the case (1) first. Define a function q by setting $q_k = p_n - p_{n-k}$ and $q_i = p_i$, $i \ne k$. Suppose $(p_0;q) \notin V(P_n)$. Then $\sum_{i=1}^{n} q_i t_i < p_0$ for some $t \in T_n$. Since q and p differ only in their k-th components and $t_k > 1$ is impossible, then $t_k = 1$. Let E be an expression with the incidence vector t. Then $E = E_1 \hat{+} k$, where E_1 is some expression with the value $\sum_{i \ne k}^{n} i t_i = n - k$.

By Lemma 1 and condition (i), $p_0 > \sum_{i=1}^{n} q_i t_i = \sum_{i \ne k} q_i t_i + q_k = \sum_{i \ne k} p_i t_i + p_n - p_{n-k} \ge p_{n-k} + p_n - p_{n-k} = p_n = p_0$, which is absurd.

In the case (2) we define q by $q_k = \frac{p_n}{2}$ and $q_i = p_i$, $i \neq k$. Now, if $\sum_{i=1}^{n} q_i t_i < p_0$ for some $t \in T_n$, then either $t_k = 1$ or $t_k = 2$. If $t_k = 1$ then we obtain a contradiction in the same way as in case (1): $p_0 > \sum_{i=1}^{n} q_i t_i = \sum_{i \neq k} q_i t_i + q_k = \sum_{i \neq k} p_i t_i + \frac{p_n}{2} \geq p_k + \frac{p_n}{2} > p_n = p_0$. If $t_k = 2$, we again have $p_0 > \sum_{i=1}^{n} q_i t_i = 2q_k = p_n = p_0$. So, $(p_0; q) \in V(P_n)$ in each possible case, and (ii) is proved.

Now we have to prove that if a valid inequality $(p_n; p)$ satisfies (ii) then it is minimal. Suppose the opposite: $(p_n; p) \notin M(P_n)$. Then there exists a valid inequality $(r_0; r)$ such that $r_0 \geq p_n$ and $r_i \leq p_i$, $i \in N$, where at least one constraint is strict. In the case $r_0 > p_n$ we have $r_1 + r_{n-1} \leq p_1 + p_{n-1} = p_n < r_0$. If $r_0 = p_n$, then $r_k < p_k$ for some $k \leq n$, and we have either $r_k + r_{n-k} < p_k + p_{n-k} = p_n = r_0$ for some $k < n$, or $r_n < p_n = r_0$ for $k = n$. In all cases we obtain a contradiction with validity of the inequality $(r_0; r)$ for the incidence vector t, with $t_k = t_{n-k} = 1$ and all other coordinates zero, or for the incidence vector $(0^{n-1}, 1)$. This ends the proof.

Let us define an equality $\sum_{i=1}^{n} p_i x_i = p_0$ to be a valid equality for P_n if it holds for all $t \in T_n$. Without loss of strictness we can use the same notation $(p_0; p)$ for a valid equality. Denote the set of all equalities valid for P_n by $W(P_n)$. Obviously, $W(P_n) \subseteq V(P_n)$. In fact, the inclusion is more strict.

Theorem 8. $W(P_n) \subseteq M(P_n)$.

Proof. We know that P_n lies in the hyperplane $x_1 + 2x_2 + ... + nx_n = n$ and $\dim P_n = n - 1$. Hence any valid equality $(p_0; p)$ defines the same hyperplane, i.e. $(p_0; p) = \lambda(n; 1, 2, ..., n)$, $\lambda \neq 0$. Since $(n; 1, 2, ..., n)$ satisfies conditions (i) and (ii) of Theorem 7, $(p_0; p) \in M(P_n)$.

Thus, we have the following chain of inclusions:

$$W(P_n) \subseteq M(P_n) \subseteq S(P_n) \subseteq V(P_n). \tag{6}$$

Recall some basic facts from the polyhedral theory [10]. For arbitrary cone $K \subset \mathbb{R}^k$, denote by *lin.space* K the maximal linear space contained in K. A cone K is said to be a pointed cone if *lin.space* K is zero. A point $x \in K$ is said to define an extreme ray of a pointed cone K, if the equality $x = \frac{1}{2}(x^1 + x^2)$, for some x^1, $x^2 \in K$, implies $x^i = \lambda_i x$, $\lambda_i > 0$, $i = 1, 2$; in fact, $\frac{1}{2}$ can be omitted here. Any pointed polyhedral cone K has a finite set of extreme rays, which we denote by *Ext* K. If a cone K is not pointed then it can be factorized by *lin.space* K, i.e. two points $v_1, v_2 \in K$ can be considered as different if and only if $v_1 - v_2 \notin$ *lin.space* K. The general situation is that the factor-cone K by *lin.space* K is a pointed cone, and the original cone K is generated by nonnegative combinations of the extreme rays of the factor-cone plus linear combinations of a basis of *lin.space* K [10, 8]. Extreme rays of the factor-cone are defined by the points $x \in K$ such that an equality $x = \frac{1}{2}(x^1 + x^2)$, for x^1, $x^2 \in K$, implies $x^i = \lambda_i x + l^i$, for some $\lambda_i \geq 0$ and $l^i \in$ *lin.space* K, $i = 1,2$. If we set $K = V(P_n)$ then, according to the second definition of a facet, $(p_0; p)$ defines a facet of P_n if and only if it is an extreme ray of the factor-cone $V(P_n)$ by *lin.space* $V(P_n)$.

Theorem 9. The cones $V(P_n)$ and $S(P_n)$ have the common maximal linear space:

$$lin.space\ V(P_n) = lin.space\ S(P_n) = W(P_n).$$

Proof. Equality $lin.space V(P_n) = W(P_n)$ is obvious. The rest of the statement follows from (6).

Let V_n, S_n and M_n be, respectively, the pointed factor-cones of $V(P_n)$ and $S(P_n)$ and the factor-set of $M(P_n)$ by $W(P_n)$. Inclusions

$$M_n \subseteq S_n \subseteq V_n \tag{7}$$

follow from (6).

The inequalities $\lambda(-1;0^n)$ and $\lambda(0;e^i)$, $i \in N$, where $\lambda > 0$ and e^i is the vector with components $e_i^i = 1$ and $e_j^i = 0$, for $j \neq i$, are trivially valid for P_n. We call them trivial valid inequalities. Next theorem shows that almost all inequalities of the second type are the facets of P_n. We call them trivial facets.

Theorem 10. The inequalities $x_i \geq 0$, $2 \leq i \leq n$, are facets of the polytope P_n, for $n \geq 2$.

Proof. Let us fix $n \geq 2$, and i, $2 \leq i \leq n$. Since $dim\, P_n = n-1$, the facets of P_n have dimension $n-2$ and contain $n-1$ affine-independent points of P_n. If we find such points in the hyperplane $x_i = 0$ theorem will be proved.

As was shown in the first part, the intersection of the polytope P_i with the hyperplane $x_i = 0$ contains translated polytope P_{i-1}, whose dimension is $i-2$. Let us take $i-1$ affine-independent points $t^j = (t_1^j, t_2^j, \dots, t_{i-1}^j)$, $j = 1,2,...,i-1$, of P_{i-1}, including the vertex $(i-1,\ 0^{i-2})$. Then, $i-1$ points $\varphi_1^{n-i+1}(t^j) = (t_1^j + n - i + 1, t_2^j, \dots, t_{i-1}^j, 0^{n-i+1})$ are affine-independent, belong to P_n and have last $n-i+1$ coordinates zero.

Every pass from P_{k-1} to P_k, $i < k \leq n$, is accompanied by emergence of a new point – the vertex $u^k = (0^{k\text{-}1},1)$ of P_k, which lies in the hyperplane $x_i = 0$ of $\mathbb{R}^k$. These points provide the rest $n-i$ points $\varphi_1^{n-k}(u^k) = (n-k, 0^{k-2}, \dots, 1, 0^{n-k}) \in P_n$, $i < k \leq n$. Indeed, all the n points $\varphi_1^{n-i+1}(t^j)$, $j = 1,2,\dots,i-1$, and $\varphi_1^{n-k}(u^k)$, $i < k \leq n$, belong to P_n and to the hyperplane $x_i = 0$, and are affine-independent. Theorem is proved.

To finish with the trivial valid inequalities, note that $(-1;0^n)$ is not a facet since the corresponding hyperplane does not contain any $t \in T_n$. Neither is the hyperplane $x_1 = 0$ since for any incidence vector t, $t_1 = 0$ implies $t_{n-1} = 0$, and the n–2 components remained are not sufficient to construct n–1 affine-independent points of P_n.

Now we are on the last lap to prove the subadditive characterization of the non-trivial facets.

Theorem 11. Every nontrivial valid inequality, which is extreme in V_n, is a minimal valid inequality.

Proof. Suppose that some extreme in V_n valid inequality $(p_0; p)$ is not minimal. Then, there exist $\delta_0 \geq 0$ and $\delta_i \leq 0$, $i \in N$, such that not all of them are equal to zero and $(p_0 + \delta_0; p + \delta)$ is a valid inequality. Then, $(p_0 - \delta_0; p - \delta)$ is also a valid inequality and $(p_0; p) = \frac{1}{2}(p_0 + \delta_0; p + \delta) + \frac{1}{2}(p_0 - \delta_0; p - \delta)$. Extremality of $(p_0; p)$ implies $(p_0 - \delta_0; p - \delta) = \lambda(p_0; p) + l$, for some $\lambda \geq 0$ and $l \in W(P_n)$, which is equivalent to $(1-\lambda)(p_0; p) = (\delta_0; \delta) + l$. The fact that not all δ_0 and δ_i are equal to zero implies $\lambda \neq 1$, hence $(p_0; p) = \frac{1}{1-\lambda}(\delta_0; \delta) + \frac{1}{1-\lambda} l$. Assumption $\lambda < 1$ contradicts to validity of $(p_0; p)$, since $\delta_0 \geq 0$, $\delta_i \leq 0$, $i \in N$, and all $t \in T_n$ are nonnegative. Therefore, $\lambda > 1$ and $(p_0; p) = \frac{-\delta_0}{1-\lambda}(-1, 0^n) + \sum_{i=1}^{n} \frac{\delta_i}{1-\lambda}(0, e^i) + \frac{l}{1-\lambda}$, which is a representation of $(p_0; p)$ as a nonnegative combination of the trivial valid inequalities. Therefore, $(p_0; p)$ can be extreme only in the case that it is one of these trivial valid inequalities. Theorem is proved.

Lemma 3. If a minimal valid for P_n inequality $(p_n; p)$ is a sum of two valid inequalities $(r_0^1; r^1)$ and $(r_0^2; r^2)$, then both $(r_0^1; r^1)$ and $(r_0^2; r^2)$ are minimal valid inequalities.

Proof. Suppose, on the contrary, that for example $(r_0^1; r^1)$ is not minimal. Then there exists a valid inequality $(r_0^3; r^3)$ satisfying $r_0^3 \geq r_0^1$ and $r_i^3 \leq r_i^1$, for all *i*, and at least one of these conditions is strict. Inequality $(r_0^3 + r_0^2;\ r^3 + r^2)$ is valid, but $p_n = r_0^1 + r_0^2 \leq r_0^3 + r_0^2$ and $p_i = r_i^1 + r_i^2 \geq r_i^3 + r_i^2$, for all *i*. Since one of the restrictions is strict this contradicts to minimality of $(p_0; p)$ and proves lemma.

Theorem 12. The set of non-trivial extreme valid inequalities for the polytope P_n is the set of minimal valid inequalities extreme in the cone of subadditive inequalities S_n:

$$Ext(V_n) = Ext(S_n) \cap M_n .$$

Proof. Let $(p_0; p)$ be a non-trivial inequality extreme in V_n. By Theorem 11, $(p_0; p) \in M_n$ and, by (7), $(p_0; p) \in S_n$. Together with inclusion $S_n \subseteq V_n$ this implies that $(p_0; p)$ is extreme in S_n. Conversely, let $(p_0; p) \in Ext(S_n) \cap M_n$ and suppose that $(p_0; p)$ is not extreme in V_n. Then it can be expressed as a half-sum of two valid inequalities: $(p_0; p) = \frac{1}{2}(r_0^1; r^1) + \frac{1}{2}(r_0^2; r^2)$. It follows from Lemma 3 and (7) that both $(r_0^1; r^1)$ and $(r_0^2; r^2)$ belong to M_n and, therefore, to S_n, which contradicts extremality of $(p_0; p)$ in S_n and completes the proof.

So we proved that every non-trivial facet of P_n is generated by an extreme ray of the factor-cone of subadditive functions by the line $\lambda(1, 2, \dots, n)$, $\lambda \in \mathbb{R}$. Let us call a subadditivity inequality (5) active for a subadditive function *p* if *p* turns it into equality. The factor-cone is of dimension *n*–1, hence for any its extreme ray there exist some *n*–2 linearly independent active inequalities (5). Theorems 12 and 7 indicate that for each non-trivial facet the minimality conditions (ii) give a part of order *n*/2 of linearly independent active inequalities. We cannot say how to augment this subsystem to obtain a system of n–2 linearly independent active inequalities that provides a non-trivial facet, but we can specify the facets a little more.

Lemma 4. Every non-trivial facet $(p_n; p)$ of P_n is equivalent in the factor-cone V_n to some facet $(q_n; q)$ with $q_i \geq 0,\ 1 \leq i \leq n$, and at least one $q_j = 0$, for j not dividing n, $2 \leq j \leq n-1$.

Proof. Let $(p_n; p)$ be a non-trivial facet and $m \in N$ be an index such that $\dfrac{p_m}{m} = \min\limits_{i \in N} \dfrac{p_i}{i}$. Then the inequality $(p_n; p) - \dfrac{p_m}{m}(n;1,2,\ldots,n)$ can serve as the facet $(q_n; q)$. Indeed, $(q_n; q)$ is equivalent to $(p_n; p)$, $q_m = 0$ and inequalities $q_i \geq 0$ hold. Since $(p_n; p) \in Ext(S_n)$, $(p_n; p)$ is not equivalent to $(n;1,2,\ldots,n)$ and, by Lemma 1, $m \neq 1$; thus $q_1 > 0$. By Theorem 7, minimality of $(p_n; p)$, and therefore of $(q_n; q)$, implies $q_n = q_1 + q_{n-1} > 0$. Inequalities $q_j > 0$, for j dividing n, follow from subadditivity of $(q_n; q)$, $q_n > 0$ and Lemma 1.

Next theorem summarizes all that we know about the non-trivial facets of P_n for $n > 2$. It was successfully used to construct all facets of P_n for small n, see Appendix 2.

Theorem 13. An inequality $(p_n; p)$ is a non-trivial facet of the partition polytope P_n if and only if its coefficient vector p turns into equalities n–2 linearly independent rows of the system

$$p_i + p_{n-i} = p_n, \quad 1 \leq i \leq \left[\frac{n}{2}\right],$$

$$p_i + p_j \geq p_{i+j}, \quad 1 \leq i, j < n, \quad i + j \leq n$$

and is non-collinear to the vector $(1,2,\ldots,n)$. The facets can be supposed to have nonnegative coefficients with some $p_j = 0$, for j not dividing n, and $p_i > 0$, for all i dividing n.

4. POLYTOPES OF CONSTRAINED PARTITIONS

Let M be a subset of N, $|M| = m$, and consider the polytope $P_n(M)$ of the incidence vectors of the partitions of n

$$\sum_{i \in M} i x_i = n,\ \ x_i \in \mathbb{Z},\ \ x_i \geq 0\,, \tag{8}$$

in which only the numbers $i \in M$ are allowed to appear. Partitions of this kind are often studied [1]. We show further that the theory developed in the previous part can be applied to this case. This extension follows from the relation between the master group polyhedron (3) and the particular polyhedra defined by (3) but with summation by $g \in H,\ H \subset G$. In fact, the parallel between the partition polytopes and the group polyhedra is unexpectedly so straight that we could make only slight changes in the Gomory's reasoning [5].

Let $E(M)$ be the $(n - m)$-dimensional subspace in $\mathbb{R}^n$, in which $x_i = 0$ for all $i \notin M$.

Theorem 14. $P_n(M)$ is a face of P_n and is equal to $P_n \cap E(M)$.

Proof. We prove first that $P_n(M) = P_n \cap E(M)$. Any point $t \in P_n(M)$ lies in $E(M)$. Since t satisfies (8), it satisfies (1) and belongs to P_n. Hence $P_n(M) \subseteq P_n \cap E(M)$. Conversely, let a point t belong to $P_n \cap E(M)$. Since $t \in P_n$, it is a convex combination of some vertices t^i of P_n: $t = \sum_i \lambda_i t^i$, with $\lambda_i \geq 0$. Since $t \in E(M)$, its j-th coordinate $t_j = 0$, for $j \notin M$. So the same is true for the j-th coordinates of each t^i, and $t^i \in E(M)$. However, since all t^i satisfy (1) and lie in $E(M)$ they satisfy (8), and each $t^i \in P_n(M)$. Thus, t is a convex combination of the vertices of $P_n(M)$ and belongs to $P_n(M)$. So, $P_n \cap E(M) \subseteq P_n(M)$ and, in fact, the equality holds.

Now recall that the inequalities $x_i \geq 0$, $i \notin M$, $i \neq 1$, are facets of P_n. Furthermore, since the hyperplane $x_1 = 0$ contains the vertex $(0^{n-1},1)$ and $x_1 \geq 0$ is valid for P_n, $x_1 = 0$ defines a face of P_n, though it is not a facet. So $P_n(M)$ is the intersection of some facets and/or a face of P_n and, in its turn, is a face of P_n. Theorem is proved.

The next theorem clarifies connection between the vertices and facets of the partition polytope P_n and those of the polytopes of constrained partitions $P_n(M)$.

Theorem 15. (i) Every vertex of $P_n(M)$ is a vertex of P_n. A vertex of P_n is a vertex of $P_n(M)$ if and only if it belongs to the subspace $E(M)$.

(ii) An inequality $(q_0;q)$ with q an m-vector provides a facet of $P_n(M)$ if and only if there exists a facet $(p_0;p)$ of P_n with $p_i = q_i$, for all $i \in M$, and $p_0 = q_0$.

Proof. Both statements follow from the fact that $P_n(M)$ is a face of P_n. All vertices of a face of a polytope are those vertices of the polytope that belong to this face. The facets of a face of a polytope are given by some facets of the polytope.

Theorem 15 states that each facet of $P_n(M)$ can be obtained by taking some facet $(p_0;p)$ of P_n and simply omitting the components p_i, for $i \notin M$. After this is done for all facets of P_n, all facets of $P_n(M)$ will be obtained plus some valid but superfluous inequalities.

5. CONCLUSION

Studying the set of partitions of numbers as a polytope allowed to clear up its general structure. Each polytope P_n is a pyramid. Its base and apex are located in adjacent layers of the integer lattice, hence P_n has no strictly interior points. Each P_n contains in its base translated polytopes of partitions of all numbers lesser than n. Due to emergence of new vertices, subsequent polytopes of the sequence $P_1, P_2, \dots, P_n, \dots$ gradually almost completely capture the preceding polytopes. We proposed rather strong sufficient and necessary conditions for a partition to be a vertex of the polytope. Though we cannot answer the question which vertices of the preceding polytopes still remain the vertices of P_n, and which of them and when cease to be. Another problem of interest is to estimate how the number of vertices of P_n grows in comparison to the total number of partitions.

While the vertices of P_n form a kind of basis in the set of all partitions, the facets of P_n can be used, for example, to solve optimization problems on partitions. In principle, they can be found by the cutting plane methods of the integer linear programming theory, but our aim was to obtain them beforehand. The algebraic approach used enabled us to connect the non-trivial facets with extreme rays of certain subcone of subadditive functions relative to the partial addition on the set {1, 2, ... , *n*}. The general problem of obtaining description of extreme rays of the cones of various subadditive functions appears to be of great importance, some results on this account were obtained in [13]. This problem is far from being easy but the auxiliary minimality conditions make it much simpler in our particular case.

The results of the paper can be used for computer calculation of the vertices and facets of the partition polytopes. Additional information on the relations between the coordinates of the vertices and the coefficients of the facets would be helpful.

APPENDIX 1. **Vertices of the polytopes of partitions P_n for $n \leq 8$.**

The table below demonstrates embeddings $P_1 \subset P_2 \subset ... \subset P_8$. The columns $x_1, x_2, ..., x_8$ contain all integer points of P_8. The parts of the table surrounded by the bold lines serve to provide the lists of integer points of preceding polytopes P_n, $n < 8$. The only thing to be done is to substitute the values in the x_1-column by those from the column x_1-for-P_n considered. In the last column we indicate for each point whether it is or is not a vertex of P_n, and confirm this by the relevant theorem.

For example the row 6 provides four points $(3,1,1,0^5) \in P_8$, $(2,1,1,0^4) \in P_7$, $(1,1,1,0^3) \in P_6$, $(0,1,1,0,0) \in P_5$. By Theorem 3, condition (ii) [or condition (i)], the last point is a vertex of P_n as it is induced by the sequence of indices 1, 2, 3. Condition (ii) of Theorem 4 implies that the other three points are not vertices of the corresponding polytopes.

x_1 for							P_n	Points in P_n								Vertex ?	
P_7	P_6	P_5	P_4	P_3	P_2	P_1		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8		
7	6	5	4	3	2	1	P_1	8	0	0	0	0	0	0	0	Yes, $n \geq 2$	Th. 3 (ii)
5	4	3	2	1	0		P_2	6	1	0	0	0	0	0	0	Yes, $2 \leq n \leq 3$ No, $n \geq 4$	Th. 3 (ii) Th. 4 (i)
4	3	2	1	0			P_3	5	0	1	0	0	0	0	0	Yes, $3 \leq n \leq 5$ No, $n \geq 6$	Th. 3 (ii) Th. 4 (i)
3	2	1	0					4	2	0	0	0	0	0	0	Yes, $4 \leq n \leq 5$ No, $n \geq 6$	Th. 3 (ii) Th. 4 (i)
3	2	1	0				P_4	4	0	0	1	0	0	0	0	Yes, $4 \leq n \leq 5$ No, $n \geq 8$	Th. 3 (ii) Th. 4 (i)
2	1	0						3	1	1	0	0	0	0	0	Yes, $n = 5$ No, $n \geq 6$	Th. 3 (ii) Th. 4 (ii)
2	1	0					P_5	3	0	0	0	1	0	0	0	Yes, $5 \leq n \leq 9$ No, $n \geq 10$	Th. 3 (ii) Th. 4 (i)
1	0							2	3	0	0	0	0	0	0	Yes, $n = 6, 7$ No, $n \geq 8$	Th. 3 (ii) Th. 4 (i)
1	0							2	1	0	1	0	0	0	0	Yes, $n = 6, 7$ No, $n \geq 8$	Th. 3 (ii) Th. 4 (i)
1	0							2	0	2	0	0	0	0	0	Yes, $6 \leq n \leq 8$ No, $n \geq 9$	Th. 3 (ii) Th. 4 (i)
1	0						P_6	2	0	0	0	0	1	0	0	Yes, $6 \leq n \leq 11$ No, $n \geq 12$	Th. 3 (ii) Th. 4 (i)
0								1	2	1	0	0	0	0	0	Yes, $n = 7$ No, $n \geq 8$	Th. 3 (i) Th. 4 (ii)
0								1	1	0	0	1	0	0	0	Yes, $n = 7, 8$ No, $n \geq 9$	Th. 3 (ii) Th. 4 (i)
0								1	0	1	1	0	0	0	0	Yes, $n = 7$ No, $n \geq 8$	Th. 3 (ii) Th. 4 (ii)
0							P_7	1	0	0	0	0	0	1	0	Yes, $7 \leq n \leq 13$ No, $n \geq 14$	Th. 3 (ii) Th. 4 (i)
								0	4	0	0	0	0	0	0	Yes, $n = 8, 9$ No, $n \geq 10$	Th. 3 (ii) Th. 4 (i)
								0	2	0	1	0	0	0	0	No, $n \geq 8$	Th. 4 (i)
								0	1	2	0	0	0	0	0	Yes, $n = 8, 9$	Th. 3 (ii)
								0	1	0	0	0	1	0	0	Yes, $n = 8, 9$	Th. 3 (ii)
								0	0	1	0	1	0	0	0	Yes, $8 \leq n \leq 10$	Th. 3 (ii)
								0	0	0	2	0	0	0	0	Yes, $8 \leq n \leq 11$	Th. 3 (ii)
							P_8	0	0	0	0	0	0	0	1	Yes, $8 \leq n \leq 15$	Th. 3 (ii)

APPENDIX 2. **Non-trivial facets of the polytopes of partitions P_n for $n \le 6$.**

P_n	Non-trivial facets	Active inequalities	Vertices on facet
P_1 = point (1)	No	No	No
P_2 = segment, endpoints: (2, 0), (0,1)	No	No	No
P_3 = triangle, vertices: (3, 0, 0), (1, 1, 0), (0, 0, 1)	$x_1 + x_3 = 1$	$p_1 + p_2 \ge p_3$	(1, 1, 0) (0, 0, 1)
P_4 = pyramid, vertices: (0, 0, 0, 1), (4, 0, 0, 0), (0, 2, 0, 0), (1, 0, 1, 0)	$2x_1 + x_2 + 2x_4 = 2$	$p_1 + p_3 \ge p_4$ $2p_2 \ge p_4$	(1, 0, 1, 0) (0, 2, 0, 0) (0, 0, 0, 1)
P_5	$x_1 + 2x_2 + x_4 + 2x_5 = 2$	$p_1 + p_4 \ge p_5$ $p_2 + p_3 \ge p_5$ $2p_1 \ge p_2$	(1, 0, 0, 1, 0) (0, 1, 1, 0, 0) (2, 0, 1, 0, 0) (0, 0, 0, 0, 1)
	$x_1 + x_3 + x_5 = 1$	$p_1 + p_4 \ge p_5$ $p_2 + p_3 \ge p_5$ $p_1 + p_2 \ge p_3$	(1, 0, 0, 1, 0) (0, 1, 1, 0, 0) (1, 2, 0, 0, 0) (0, 0, 0, 0, 1)
P_6	$x_1 + 2x_2 + x_3 + x_5 + 2x_6 = 2$	$p_1 + p_5 \ge p_6$ $p_2 + p_4 \ge p_6$ $2p_3 \ge p_6$ $2p_1 \ge p_2$	(1, 0, 0, 0, 1, 0) (0, 1, 0, 1, 0, 0) (0, 0, 2, 0, 0, 0) (0, 0, 0, 0, 0, 1) (2, 0, 0, 1, 0, 0)
	$6x_1 + 2x_2 + 3x_3 + 4x_4 + 6x_6 = 6$	$p_1 + p_5 \ge p_6$ $p_2 + p_4 \ge p_6$ $2p_3 \ge p_6$ $2p_2 \ge p_4$	(1, 0, 0, 0, 1, 0) (0, 1, 0, 1, 0, 0) (0, 0, 2, 0, 0, 0) (0, 0, 0, 0, 0, 1) (0, 3, 0, 0, 0, 0)

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