A Lower Bound for the Simplexity of the N-Cube via Hyperbolic Volumes

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ABSTRACT

Let $T(n)$ denote the number of n-simplices in a minimum cardinality decomposition of the n-cube into n-simplices. For $n \geq 1$ we show that $T(n) \geq H(n)$, where $H(n)$ is the ratio of the hyperbolic volume of the ideal cube to the ideal regular simplex. $H(n) \geq \frac{1}{2} \cdot 6^{n/2}(n+1)^{-\frac{n+1}{2}} n!$. Also $\lim_{n\to\infty} \sqrt{n}[H(n)]^{1/n} \approx 0.9281$. Explicit bounds for $T(n)$ are tabulated for $n \leq 10$, and we mention some other results on hyperbolic volumes.

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1. Introduction

The problem of determining the minimal triangulation of the n-cube originally arose in connection with minimizing the number of pivot steps in simplicial algorithms for finding approximate fixed points (see Scarf [22], Mara [14]), and seems an intrinsically interesting combinatorial problem. A triangulation of a polytope $\mathbf{P}$ in $\mathbf{R^n}$ is a *vertex-triangulation* if all the vertices of the simplices in the triangulation are vertices of $\mathbf{P}$ and it is *face to face* if the intersection of any two simplices is a lower dimensional simplex j-face of both of them. Let $T_{\rm v}(n)$, $T_{\rm f}(n)$ and $T_{\rm vf}(n)$ denote respectively the minimal number of simplices in a vertex triangulation, face to face triangulation, and face to face vertex triangulation of the n-cube. Let $T(n)$ denote the minimal number of simplices in a *general* decomposition of the n-cube into simplices, with extra vertices possibly being used, and with the triangulation not necessarily being face to face. The additional subscript "c" will be added to the set to denote the demand that the triangulation be "corner cutting," i.e. all 2^{n-1} of the simplices "cutting off" the corners of $[0,1]^n$ with even coordinate sum by hyperplanes through their n neighbor vertices, must be used. Clearly $T(n) \le T_{\rm v}(n) \le T_{\rm vf}(n) \le T_{\rm cvf}(n)$; it seems plausible that extra vertices and non-face to face triangulations don't help, so that $T(n) = T_{\rm vf}(n)$, but this has not been proved. Guy [6] calls one of $\{T(n), T_{\rm v}(n), T_{\rm f}(n), T_{\rm vf}(n)\}$ the *simplexity* of the n-cube, but it is not clear which one he meant.

The best known lower and upper bounds for small values of n are listed in table 1. Most of them come from exhaustive searches, computer aided constructions, and/or bounds obtained from enormous linear or integer programs [12] [9].

By considering simplices in which the n coordinate values are ordered according to fixed permutations (or just by induction on n), one may easily see that $T_{\rm vf}(n) \le n!$. Haiman [8] showed $T_{\rm vf}(kn) \le [T_{\rm vf}(n)/n!]^k (kn)!$, enabling one triangulation of an n-cube with cardinality below $n!$ to be turned into an infinite sequence of such triangulations. Thus $T_{\mathrm{vf}}(7) \leq 1493$ [12] implies $T_{\mathrm{vf}}(7n) \leq (1493/7!)^n (7n)! < .840463^{7n} (7n)!$, which is the best known upper bound asymptotically.

A simple volume argument in Euclidean space [8] shows $T(n) \geq E(n) = 2^n (n+1)^{-(n+1)/2} n!$ (and see also [19]).

Hughes [9] was able to get a lower bound on $T_{\mathrm{v}}(n)$ for small n by solving enormous linear or integer programs, in which the variables corresponded to exterior j-face tuples, with computer aid. These methods are remarkably powerful for small n, actually getting the apparent right answers when $n \leq 7$, and (as table 1 shows) yielding bounds at least as good as the present paper's for $n \leq 11$, i.e. all n for which Hughes's computer was able to calculate his bound. But because Hughes's lower bound is for $T_{\mathrm{v}}(n)$, not $T(n)$, it really does *not* ever dominate our bounds. Also, Hughes defined a quantity "$b(n)$" which he knew was larger than his lower bound, and whose asymptotic behavior could be analysed. The result is

$$b(n) \leq n^{n/2} (\frac{2}{e})^n \exp(\sqrt{en}) n^{3/4} 2 \exp(-\frac{e+11}{4})(1 + O(n^{-1/2}));$$

i.e. asymptotically Hughes's bound is only better than the trivial Euclidean volume bound $E(n)$ by a subexponential factor $\exp(\sqrt{en(1+o(1))})$ at most.

This paper obtains exponentially improved lower bounds for $T(n)$ using a volume argument in hyperbolic geometry. To state our main result, let $\mathbf{H^n}$ denote n-dimensional hyperbolic space and let $\mathrm{Hvol}_n(\cdot)$ denote hyperbolic volume.

Theorem 1.1. *One has*

$$T(n) \geq H(n) = \frac{\mathrm{Hvol}_n(\textit{regular ideal } n\textit{-cube in } \mathbf{H^n})}{\mathrm{Hvol}_n(\textit{regular ideal simplex in } \mathbf{H^n})} \ . \tag{1.1}$$

$$H(n) \geq \frac{1}{2} \cdot 6^{n/2} (n+1)^{-\frac{n+1}{2}} n!, \quad n \geq 1. \tag{1.2}$$

$$\lim_{n \to \infty} \left[\frac{H(n)}{E(n)} \right]^{1/n} = A \approx 1.261522510 \ . \tag{1.3}$$

Actually in (1.1) one could use any hyperbolic polytope that has the combinatorial type of an n-cube. The bound (1.2) improves on the Sallee, Hughes, and Euclidean volume lower bounds by a exponential factor roughly of order $\left(\frac{3}{2}\right)^{n/2}$ as $n \to \infty$; and the theorem, as is shown by (1.3), actually improves on it by the slightly larger exponential factor A^n, since $A > \sqrt{3/2} \approx 1.224744871$.

Such hyperbolic volume bounds were first used by Sleator, Thurston and Tarjan [23] on a related problem.

There remains a huge gap between our lower and the Haiman-Hughes-Anderson upper bound for $T(n)$ as $n \to \infty$.

our bound	known results	sources
$\lceil H(1) \rceil = 1$	$T(1) = T_{\rm vf}(1) = T_{\rm cvf}(1) = 1$	
$\lceil H(2) \rceil = 2$	$T(2) = T_{\rm vf}(2) = T_{\rm cvf}(2) = 2$	
$\lceil H(3) \rceil = 5$	$T(3) = T_{\rm vf}(3) = T_{\rm cvf}(3) = 5$	[12]
$\lceil H(4) \rceil = 15$	$T_{\rm v}(4) = T_{\rm vf}(4) = T_{\rm cvf}(4) = 16$	[12] [5]
$\lceil H(5) \rceil = 48$	$T_{\rm v}(5) = T_{\rm vf}(5) = T_{\rm cvf}(5) = 67$	[12] [13] [19] [3] [4]
$\lceil H(6) \rceil = 174$	$270 \leq T_{\rm v}(6) \leq T_{\rm vf}(6) = 308 < T_{\rm cvf}(6) = 324$	[12] [10] [11] [21]
$\lceil H(7) \rceil = 681$	$1175 \leq T_{\rm v}(7) \leq T_{\rm vf}(7) = 1493 < T_{\rm cvf}(7) = 1820$	[12]
$\lceil H(8) \rceil = 2863$	$5522 \leq T_{\rm v}(8) \leq T_{\rm vf}(8) \leq 13136$	[9] [3]
$\lceil H(9) \rceil = 12811$	$26593 \leq T_{\rm v}(9) \leq T_{\rm vf}(9) \leq 105341$	[9] [3]
$\lceil H(10) \rceil = 60574$	$131269 \leq T_{\rm v}(10) \leq T_{\rm vf}(10) \leq 928780$	[9] [3]
$\lceil H(11) \rceil = 300956$	$665272 \leq T_{\rm v}(11)$	[9]
$\lceil H(12) \rceil = 1564340$	$T_{\rm vf}(12) \leq 87654336$	[8] & $T_{\rm vf}(6) = 308$

Table 1: Bounds for $T(n)$, $T_{\rm v}(n)$, $T_{\rm vf}(n)$, and $T_{\rm cvf}(n)$. $H(n)$ is our hyperbolic volume lower bound on $T(n)$.

2. Hyperbolic Geometry

Before proving Theorem 1.1 we recall basic facts about hyperbolic geometry, see Beardon [1], Milnor [15], Haagerup and Munkholm [7], Thurston [26].

The "*projective model*" of $\mathbf{H^n}$ is the interior of the unit ball $\mathbf{B}_n$ of $\mathbf{R^n}$ with the Riemannian metric

$$\mathrm{d}s^2 = \frac{1}{1-r^2}\sum_{i=1}^{n} \mathrm{d}x_i^2 + \frac{1}{(1-r^2)^2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathrm{d}x_i \mathrm{d}x_j \ .$$

The geodesics are straight lines, but angles between straight lines are usually not the same as Euclidean geometry angles. The *volume form* for $\mathbf{H^n}$ is

$$\mathrm{d}V = (1-r^2)^{-\frac{n+1}{2}} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \ . \tag{2.1}$$

In this model a *polytope* in $\mathbf{H^n}$ is an ordinary Euclidean convex polytope in the unit ball $\mathbf{B}_n$. It is an *ideal polytope* if all its vertices lie on the boundary $\mathbf{S}^{n-1} = \{x : \|\mathbf{x}\| = \sum_{i=1}^n x_i^2 = 1\}$. A *regular polytope* $\mathbf{P}$ in $\mathbf{H^n}$ is a polytope such that the group G of hyperbolic isometries σ with $\sigma(\mathbf{P}) = \mathbf{P}$ is transitive on the facets of $\mathbf{P}$. It is known that an ideal polytope in $\mathbf{H^n}$ that is a regular polytope viewed in Euclidean geometry on $\mathbf{B}_n$ is a regular ideal polytope.

A key property of hyperbolic space $\mathbf{H^n}$ is that the volume of any polytope $\mathbf{P}$ of fixed combinatorial type is bounded above by a finite constant, depending on n and the combinatorial type of $\mathbf{P}$. For example, no triangle in $\mathbf{H^2}$ can have hyperbolic area exceeding π.

Proof of Theorem 1.1 (except for (1.3)): Regard the n-cube $\blacksquare_n$ as embedded in the projective model of $\mathbf{H^n}$. We use the following basic result.

Theorem 2.1. (Haagerup and Munkholm [7]) *The simplex of maximum volume in* $\mathbf{H^n}$ *is a regular ideal simplex.*

Given a triangulation of the n-cube by simplices $\{\mathbf{U}^{(i)} : 1 \leq i \leq T(n)\}$. Theorem 2.1 gives

$$\mathrm{Hvol}(\blacksquare_n) = \sum_{c=1}^{T(n)} \mathrm{Hvol}(\mathbf{U}^{(i)}) \leq T(n)\mathrm{Hvol}(\widetilde{\blacktriangle}_n)$$

where $\widetilde{\blacktriangle}_n$ denotes an ideal regular simplex. (We will use a tilde "$\widetilde{x}$" to denote the version of "x" scaled to be inscribed in $\mathbf{B}_n$.) Hence

$$T(n) \geq \frac{\mathrm{Hvol}(\blacksquare_n)}{\mathrm{Hvol}(\widetilde{\blacktriangle}_n)} \tag{2.2}$$

and (1.1) follows.

To obtain (1.2) we observe that the Euclidean volume of an ideal regular simplex inscribed in $\mathbf{B}_n$ is

$$\mathrm{Vol}(\widetilde{\blacktriangle}_n) = \frac{\sqrt{n+1}}{n!}\left(1+\frac{1}{n}\right)^{n/2} \tag{2.3}$$

From the volume form

$$\mathrm{Hvol}(\widetilde{\blacktriangle}_n) = \int_{\widetilde{\blacktriangle}_n} (1-r^2)^{-(\frac{n+1}{2})} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

and Haagerup and Munkholm [7] proved (p. 11) that

$$\frac{\mathrm{Hvol}(\widetilde{\blacktriangle}_n)}{\mathrm{Vol}(\widetilde{\blacktriangle}_n)} < \left(\frac{n}{n-1}\right)^{\frac{n+1}{2}} \ , \tag{2.4}$$

where $\mathrm{Vol}(\cdot)$ denotes Euclidean volume. For the regular ideal n-cube

$$\widetilde{\blacksquare}_n := \{\mathbf{x} \in \mathbf{B}_n : \text{ all } |x_i|^2 \leq \frac{1}{n}\}$$

one has

$$\mathrm{Vol}(\widetilde{\blacksquare}_n) = \left(\frac{4}{n}\right)^{n/2} \ . \tag{2.5}$$

For any concave-$\cup$ function $f(t)$

$$\frac{1}{\mathrm{Vol}(\widetilde{\blacksquare}_n)} \int_{\widetilde{\blacksquare}_n} f(r^2) \mathrm{d}x_1 \ldots \mathrm{d}x_n \geq f\left(\frac{1}{3}\right) \ . \tag{2.6}$$

This holds because equality holds in (2.6) for all linear functions, and in particular for the tangent line to $f(r)$ at $r = \frac{1}{\sqrt{3}}$, a fact easily deduced from

$$\frac{1}{\mathrm{Vol}(\widetilde{\blacksquare}_n)} \int_{\widetilde{\blacksquare}_n} r^2 \mathrm{d}x_1 \ldots \mathrm{d}x_n = \frac{1}{3} \ .$$

On choosing $f(t) = (1-t)^{-\frac{n+1}{2}}$ in (2.6) we obtain

$$\frac{\mathrm{Hvol}(\widetilde{\blacksquare}_n)}{\mathrm{Vol}(\widetilde{\blacksquare}_n)} \geq \left(\frac{3}{2}\right)^{\frac{n+1}{2}} \ . \tag{2.7}$$

Choosing $\blacksquare_n = \widetilde{\blacksquare}_n$ in (2.2) and using (2.3)–(2.7) yields

$$\begin{aligned} T(n) \ &\geq \ \frac{\left(\frac{3}{2}\right)^{\frac{n+1}{2}} \left(\frac{4}{n}\right)^{n/2}}{\left(\frac{n}{n-1}\right)^{\frac{n+1}{2}} \frac{\sqrt{n+1}}{n!} \left(1+\frac{1}{n}\right)^{n/2}} \\ &\geq \ \frac{1}{2} 6^{n/2} (n+1)^{-\frac{n+1}{2}} n! \ , \end{aligned} \tag{2.8}$$

where the last step used $\left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{n}{n-1}\right)^{-\frac{n+1}{2}} \geq \left(\frac{3}{2e}\right)^{\frac{1}{2}} \left(\frac{n}{n-1}\right)^{-1} \geq \frac{1}{2}$, for $n \geq 4$. For $1 \leq n \leq 3$ our bound (1.2) may be verified by explicit computation. $\Box$.

3. The asymptotic hyperbolic volume of the ideal hyperbolic n-cube

We previously showed

$$\mathrm{Hvol}(\widetilde{\blacksquare}_n) = \int_{\widetilde{\blacksquare}_n} (1-r^2)^{-\frac{n+1}{2}} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ . \tag{3.1}$$

We will now prove that

$$\frac{\mathrm{Hvol}(\widetilde{\blacksquare}_n)}{\mathrm{Vol}(\widetilde{\blacksquare}_n)} = \frac{2\, n^{(n+1)/2}}{\Gamma([n+1]/2)} \int_0^\infty \mathrm{daw}(t)^n \mathrm{d}t \tag{3.2}$$

where

$$\mathrm{daw}(u) = e^{-u^2} \int_0^u e^{x^2} \mathrm{d}x \tag{3.3}$$

is "Dawson's integral." It then follows immediately by the "saddlepoint method" that

$$\lim_{n\to\infty} \left(\frac{\mathrm{Hvol}(\widetilde{\blacksquare}_n)}{\mathrm{Vol}(\widetilde{\blacksquare}_n)}\right)^{1/n} = A = \sqrt{2e}M \approx 1.261522510 \tag{3.4}$$

where $M = \max_{u>0} \operatorname{daw}(u) \approx .54104422463518169847$ (which occurs when $u \approx .92413887300459176701$).

Proof.

$$\frac{\operatorname{Hvol}(\widetilde{\blacksquare}_n)}{\operatorname{Vol}(\widetilde{\blacksquare}_n)} = 2^{-n} n^{n/2} \int_{\widetilde{\blacksquare}_n} (1-r^2)^{-(n+1)/2} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

$$= 2^{-n} n^{n/2} \int_{\widetilde{\blacksquare}_n} \frac{1}{\Gamma([n+1]/2)} \int_0^\infty s^{(n-1)/2} e^{(r^2-1)s} \mathrm{d}s \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

$$= 2^{-n} \frac{n^{n/2}}{\Gamma([n+1]/2)} \int_0^\infty \int_{\widetilde{\blacksquare}_n} s^{(n-1)/2} e^{(r^2-1)s} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \, \mathrm{d}s$$

$$= \frac{2^{-n} n^{n/2}}{\Gamma([n+1]/2)} \int_0^\infty s^{(n-1)/2} e^{-s} \int_{\widetilde{\blacksquare}_n} e^{r^2 s} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \, \mathrm{d}s$$

$$= \frac{n^{n/2}}{\Gamma([n+1]/2)} \int_0^\infty s^{-1/2} \operatorname{daw}(\sqrt{s/n})^n \mathrm{d}s$$

$$= \frac{2\, n^{(n+1)/2}}{\Gamma([n+1]/2)} \int_0^\infty \operatorname{daw}(t)^n \mathrm{d}t \qquad \square$$

For the simplex one has asymptotically

$$\frac{\operatorname{Hvol}(\widetilde{\blacktriangle}_n)}{\operatorname{vol}(\widetilde{\blacktriangle}_n)} = \sqrt{e}\left(1 + \frac{1}{4n} + O\left(\frac{1}{n^2}\right)\right) \tag{3.5}$$

as $n \to \infty$ by converting results of Rogers [17] to hyperbolic space and using the fact that the (Euclidean) dihedral angles of $\widetilde{\blacktriangle}_n$ are $\operatorname{arcsec}(n-1)$, $n \geq 2$. (Or this is proven directly in [24].)

Combining (3.4) and (3.5) leads to (1.3). $\square$

4. Discussion

The improvement on the Euclidean volume bound arose because the regular ideal n-cube has a much larger hyperbolic volume than its Euclidean counterpart, while the regular ideal n-simplex has only a constant increase in volume compared to its Euclidean counterpart.

It seems quite possible that even better lower bounds could be obtained by either (a) using the linear programming ideas of [9] [10] [11] [12] but with hyperbolic rather than euclidean volumes, or (b) considering, instead of the hyperbolic volume metric, instead other volume weighting functions[1].

The lower bounds of Table 1 were obtained from (1.1) using numerical integration estimates for $\operatorname{Hvol}(\widetilde{\blacktriangle}_n)$, $\operatorname{Hvol}(\widetilde{\blacksquare}_n)$ for $n \leq 10$. We state such bounds as volume ratios to Euclidean

dimension	$\frac{\mathrm{Hvol}(\widetilde{\blacksquare}_n)}{\mathrm{Vol}(\blacksquare_n)}$	$\frac{\mathrm{Hvol}(\widetilde{\blacktriangle}_n)}{\mathrm{Vol}(\blacktriangle_n)}$
2	3.14159265	2.41839915
3	3.29611955	1.97767173
4	3.921	1.84709829
5	4.819	1.78775
6	5.991	1.75474
7	7.488	1.73399
8	9.3874	1.719864
9	11.7882	1.709665
10	14.8194	1.701978

Table 2: Volume Ratios $\frac{\mathrm{Hvol}(\mathbf{U}_n)}{\mathrm{Vol}(\mathbf{U}_n)}$ for $\mathbf{U}_n = \widetilde{\blacksquare}_n$ or $\widetilde{\blacktriangle}_n$

volumes in Table 2. They are taken from Smith [24]. Note that $\mathrm{Vol}(\widetilde{\blacksquare}_n)$ is given by (2.5) and that the limiting value of $\frac{\mathrm{Hvol}(\widetilde{\blacktriangle}_n)}{\mathrm{vol}(\blacktriangle_n)}$ as $n \to \infty$ is $\sqrt{e} \approx 1.648721$ by (3.5).

In $\mathbf{H^3}$ the regular ideal cube $\widetilde{\blacksquare}_3$ can be triangulated by 5 ideal regular simplices $\widetilde{\blacktriangle}_3$ (an observation dating at least to Coxeter in 1935), showing that $T(3) = 5$ as claimed in table 1. This also proves that the regular ideal cube $\widetilde{\blacksquare}_3$ has the largest volume in $\mathbf{H^3}$ of any polyhedron of its combinatorial type. (Similarly $T(2) = 2$ and the ideal square is maximal.) Maximality is not known for $\widetilde{\blacksquare}_n$ for $n \geq 4$, although it is likely to be true. For $n \geq 4$ a decomposition of $\widetilde{\blacksquare}_n$ into ideal regular simplices is apparently impossible.

Acknowledgments and history.

Most of the results herein were in chapter II.A and II.B of my PhD thesis [24], but remained otherwise unpublished for 10 years. J.C.Lagarias suggested a reorganization and abridgement of that chapter, which the present writeup is based on. It is also shown (more precisely, the main term is shown; the second term is only conjectured) in [24] that the ideal regular n-octahedron has hyperbolic volume

$$\frac{\mathrm{Hvol}(\widetilde{\blacklozenge}_n)}{\mathrm{vol}(\blacklozenge_n)} = e\left(1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right)$$

I eventually hope to publish a larger work on noneuclidean volumes. My exact expression of A in terms of Dawson's integral was not in [24] since I only discovered it in about 1990. It (and

[1] But, the fact that very different Euclidean simplices whose vertices are cube vertices, e.g. the "cut off corner" and the "Hadamard matrix" simplices, are in fact isometric in $\mathbf{H^n}$, suggests that perhaps the "perfect balancing" induced by the symmetries of $\mathbf{H^n}$ causes the hyperbolic volume metric to indeed be *the* natural weighting function.

many of my results) were then rediscovered about 5 years later by Timothy H. Marshall of the University of Auckland, NZ. The proof presented here incorporates a simplification found by Marshall (my original proof had instead been in the "imaginary sphere model" of $\mathbf{H}^n$ and had been inspired by [18]).

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