# Lecture #4: Polynomial Functions

**Order polynomials**
and
**Ehrhart polynomials**

10:30 – 11:30 a.m.
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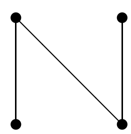
## Polynomial Functions

**Lemma:** Let $P$ be a poset with $n$ elements and, for $m \geq 0$, let $\Omega(P, m)$ be the number of order-preserving maps $P \to [m]$. Then

$$\sum_{m \geq 0} \Omega(P, m) x^m = \frac{\displaystyle\sum_{S \subseteq [n-1]} \beta(S) x^{|S|+1}}{(1-x)^{n+1}}.$$

**Example:** Let $P =$ . Then

$$\begin{aligned} \sum_{m \geq 0} \Omega(P, m) x^m &= \frac{x + 3x^2 + x^3}{(1-x)^5} \\ &= x + 8x^2 + 31x^3 + 85x^4 + \cdots \end{aligned}$$

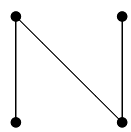
**Theorem:** Let $n \geq 0$ and $f : \mathbf{N} \to \mathbf{C}$. Then

$$\sum_{m \geq 0} f(m) x^m = \frac{p(x)}{(1-x)^{n+1}}$$

for some $p(x) \in \mathbf{C}[x]$ if and only if $f(m)$ is a polynomial function of $m$ of degree at most $n$ (exactly $n$ if and only if $p(1) \neq 0$).

## Stanley's Reciprocity Theorem for Order Polynomials

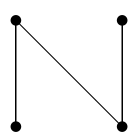
**Corollary:** The function $\Omega(P, m)$, $m \geq 0$, is a polynomial function of $m$.

**Example:** Let $P =$ . Then

$$\Omega(P, m) = \frac{1}{12}m + \frac{7}{24}m^2 + \frac{5}{12}m^3 + \frac{5}{24}m^4.$$

**Theorem:** Let $P$ be a poset with $n$ elements. Let $\Omega(P, m)$ be the number of order-preserving maps $P \to [m]$, and let $\overline{\Omega}(P, m)$ be the number of strict order-preserving maps $P \to [m]$. Then the polynomials $\Omega(P, m)$ and $\overline{\Omega}(P, m)$ satisfy

$$\overline{\Omega}(P, m) = (-1)^n \Omega(P, -m).$$

**Example:** Let $P =$ . Then

$$\overline{\Omega}(P, m) = -\frac{1}{12}m + \frac{7}{24}m^2 - \frac{5}{12}m^3 + \frac{5}{24}m^4.$$

Check that $\overline{\Omega}(P, 2) = 1$.

## Rational Generating Functions

**Theorem:** Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ with $\alpha_n \neq 0$. Then $\displaystyle\sum_{m \geq 0} f(m)x^m = \frac{p(x)}{q(x)}$, where $q(x) = 1 + \alpha_1 x + \cdots + \alpha_n x^n$ and $p(x)$ has degree less than $n$, if and only if

$$f(m+n) + \alpha_1 f(m+n-1) + \cdots + \alpha_n f(m) = 0,$$

for all $m \geq 0$, if and only if $\displaystyle f(m) = \sum_{i=1}^{k} p_i(m)\gamma_i^m$, where $1 + \alpha_1 x + \cdots + \alpha_n x^n = \prod_{i=1}^{k}(1 - \gamma_i x)^{n_i}$, and $p_i(m)$ has degree less than $n_i$.

**Proposition:** Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ with $\alpha_n \neq 0$. Suppose $f : \mathbf{Z} \to \mathbf{C}$ satisfies

$$f(m+n) + \alpha_1 f(m+n-1) + \cdots + \alpha_n f(m) = 0,$$

for all $m \in \mathbf{Z}$. So $F(x) = \sum_{m \geq 0} f(m)x^m$ is a rational function. Then

$$\sum_{m \geq 1} f(-m)x^m = -F(1/x).$$

## Proof of the Reciprocity Result

**Proof:** Let $F(x) = p(x)/q(x)$, where $q(x) = 1 + \alpha_1 x + \cdots + \alpha_n x^n$. Multiplication by $q(x)$ is a linear transformation on the $\mathbf{C}$-vector space of Laurent series. So the hypothesis on $f$,

$$q(x) \sum_{m \in \mathbf{Z}} f(m) x^m = 0,$$

implies that

$$\begin{aligned} q(x) \sum_{m \geq 1} f(-m) x^{-m} &= -q(x) \sum_{m \geq 0} f(m) x^m \\ &= -p(x). \end{aligned}$$

Substitution of $1/x$ for $x$ yields

$$\sum_{m \geq 1} f(-m) x^m = -\frac{p(1/x)}{q(1/x)} = -F(1/x).$$

## Ehrhart Polynomials

**Theorem:** Let $\mathcal{P}$ be a $d$-polytope in $\mathbf{R}^n$ with integer vertices. Let $i(\mathcal{P}, m)$ be the number of integer points in $m\mathcal{P}$, and let $\bar{i}(\mathcal{P}, m)$ be the number of integer points in the (relative) interior of $m\mathcal{P}$. Then $i(\mathcal{P}, m)$ and $\bar{i}(\mathcal{P}, m)$ are polynomial functions of $m$ of degree $d$ that satisfy $i(\mathcal{P}, 0) = 1$ and $\bar{i}(\mathcal{P}, m) = (-1)^d i(\mathcal{P}, -m)$.

**Example:** Let $\mathcal{P} = [0, 1]^n$ be the unit cube in $\mathbf{R}^n$, so $d = n$. Then the Ehrhart polynomials are $i(\mathcal{P}, m) = (m+1)^n$ and $\bar{i}(\mathcal{P}, m) = (m-1)^n$.

**Proposition:** Let $\mathcal{P}$ be an $n$-polytope in $\mathbf{R}^n$ with integer vertices. Then the leading coefficient of the Ehrhart polynomial $i(\mathcal{P}, m)$ is the volume of $\mathcal{P}$.

Use the fact that $i(\mathcal{P}, 0) = 1$ and reciprocity for Ehrhart polynomials to express the volume of $\mathcal{P}$ as a function of any $n$ of the numbers $i(\mathcal{P}, 1)$, $i(\mathcal{P}, 2)$,. . . , $\bar{i}(\mathcal{P}, 1)$, $\bar{i}(\mathcal{P}, 2)$, . . . .

## Pick's formula for $n = 2$, Reeve's for $n = 3$, and Macdonald's for $n \geq 4$

**Corollary:** If $\mathcal{P} \subset \mathbf{R}^2$ is a 2-polytope with integer vertices, then the volume $v(\mathcal{P})$ of $\mathcal{P}$ is

$$v(\mathcal{P}) = \frac{1}{2}\left(i(\mathcal{P},1) + \bar{i}(\mathcal{P},1) - 2\right).$$

**Proof:** Evaluate $i(\mathcal{P},m) = v(\mathcal{P})m^2 + ?m + 1$ at $m = 1$ and $m = -1$ to obtain the equations:

$$\begin{aligned} i(\mathcal{P},1) &= v(\mathcal{P}) + ? + 1, \text{ and} \\ \bar{i}(\mathcal{P},1) &= v(\mathcal{P}) - ? + 1. \end{aligned}$$

**Corollary:** If $\mathcal{P} \subset \mathbf{R}^3$ is a 3-polytope with integer vertices, then the volume $v(\mathcal{P})$ of $\mathcal{P}$ is

$$v(\mathcal{P}) = \frac{1}{6}\left(i(\mathcal{P},2) - 3i(\mathcal{P},1) - \bar{i}(\mathcal{P},1) + 3\right).$$

If $\mathcal{P} \subset \mathbf{R}^n$ is an n-polytope with integer vertices, then the volume $v(\mathcal{P})$ of $\mathcal{P}$ is

$$v(\mathcal{P}) = \frac{1}{n!}\left((-1)^n + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k}\, i(\mathcal{P},k)\right).$$

## References

I.G. Macdonald, The volume of a lattice polyhedron.

Richard Stanley, Enumerative Combinatorics.

Richard Stanley, Ordered Structures and Partitions.