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On the Hilbert function of a graded Cohen–Macaulay domain

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Abstract

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A condition is obtained on the Hilbert function of a graded Cohen–Macaulay domain $R = R_0 \oplus R_1 \oplus \cdots$ over a field $R_0 = K$ when R is integral over the subalgebra generated by R_1. A result of Eisenbud and Harris leads to a stronger condition when char $K = 0$ and R is generated as a K-algebra by R_1. An application is given to the Ehrhart polynomial of an integral convex polytope.

1. Introduction

By a *graded algebra* over a field K, we mean here a commutative K-algebra R with identity, together with a vector space direct sum decomposition $R = \coprod_{i \geq 0} R_i$, such that: (a) $R_i R_j \subseteq R_{i+j}$, (b) $R_0 = K$ (i.e., R is *connected*), and (c) R is finitely-generated as a K-algebra. R is *standard* if R is generated as a K-algebra by R_1, and *semistandard* if R is integral over the subalgebra $K[R_1]$ of R generated by R_1. The *Hilbert function* $H(R, \cdot)$ of R is defined by $H(R, i) = \dim_K R_i$, for $i \geq 0$, while the *Hilbert series* is given by

$$F(R, \lambda) = \sum_{i \geq 0} H(R, i) \lambda^i .$$

There has been considerable recent interest in the connections between the behavior of $H(R, i)$ and the structure of R. In particular, Hilbert functions of the following classes of standard graded algebras have been completely characterized: (a) arbitrary [11, Theorem 2.2] (essentially a result of Macaulay), (b) Cohen–

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Macaulay, or more generally, of fixed depth and Krull dimension [11, Corollaries 3.10 and 3.11] (again essentially due to Macaulay), (c) complete intersections [11, Corollary 3.4] (again Macaulay, and also independently, Gröbner), and (d) reduced (i.e., no nonzero nilpotents) [1]. Partial results have been achieved for Gorenstein rings [11, Theorem 4.1; 9]. One class of rings conspicuously absent from the above list is the (integral) domains. Some results in this direction are due to Roberts and Roitman [10]. In particular, they obtain [10, Theorem 4.5] a strong restriction on the Hilbert function of a standard graded domain of Krull dimension one, viz., if the function $\Delta H(R, i) := H(R, i) - H(R, i-1)$ starts to decrease strictly, then it strictly decreases until reaching 0. Moreover, they show [10, p. 103], based on an idea of A. Geramita, that for any $d \geq 0$ there does not exist a graded domain R of Krull dimension d and Hilbert series

$$F(R, \lambda) = \frac{1 + 2\lambda + \lambda^2 + \lambda^3}{(1-\lambda)^d}\,. \tag{1}$$

(They assume that R is standard, but their proof does not use this fact.) Moreover, there do exist reduced Cohen–Macaulay standard graded algebras R with this Hilbert series when $d \geq 1$.

Our main result (Theorem 2.1) will be a condition on the Hilbert function (or Hilbert series) of a semistandard Cohen–Macaulay domain R. We point out how further results follow from work of Eisenbud and Harris [2] related to Castelnuovo theory when R is standard and char $K = 0$. Finally in Section 4 we give an application to the Ehrhart polynomial of a convex polytope.

2. Semistandard Cohen–Macaulay domains

Let R be a semistandard graded K-algebra of Krull dimension d. Let $K[R_1]$ be the subalgebra of R generated by R_1, so $K[R_1]$ is a standard graded K-algebra. Since R is integral over $K[R_1]$ it follows that R is a finitely-generated $K[R_1]$-module. Hence by well-known properties of Hilbert series we have

$$F(R, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d}\,,$$

for certain integers $h_0, \ldots, h_s$ satisfying $\sum h_i \neq 0$ and $h_s \neq 0$. We call the vector $h(R) := (h_0, \ldots, h_s)$ the *h-vector* of R.

Theorem 2.1. *Suppose R is a semistandard graded Cohen–Macaulay domain with $h(R) = (h_0, \ldots, h_s)$. Then*

$$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i} \tag{2}$$

for all $0 \leq i \leq s$.

Proof. Let $\Omega(R)$ denote the canonical module of R (see [4]), which exists since R is Cohen–Macaulay. $\Omega(R)$ has the structure $\Omega(R) = \Omega(R)_0 \oplus \Omega(R)_1 \oplus \cdots$ of a finitely-generated graded R-module with Hilbert series

$$F(\Omega(R), \lambda) = \frac{h_s + h_{s-1}\lambda + \cdots + h_0\lambda^s}{(1-\lambda)^d}\,. \tag{3}$$

(See the proof of Theorem 4.4 of [11]. The integer q of [11, equation (12)] may be chosen arbitrarily by shifting the grading of $\Omega(R)$; we choose q so that (3) above is valid.) Pick an element $0 \neq u \in \Omega(R)_0$. Since R is a domain, $\Omega(R)$ is a torsion-free R-module. (In fact, $\Omega(R)$ is isomorphic to an ideal of R [4, Corollary 6.7].) Hence as R-modules we have $uR \cong R$.

We now use the following result from [8, Exercise 14(2) on p. 103] (in the special case $I = R_+$). Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded R-modules, with $R_+A \neq A$, $R_+B \neq B$, $R_+C \neq C$. (If A, B, C are finitely-generated, then these last conditions are equivalent to $A \neq 0$, $B \neq 0$, $C \neq 0$.) Assume depth $B >$ depth C. Then depth $A = 1 +$ depth C.

Apply this result to the exact sequence

$$0 \to uR \to \Omega(R) \to \Omega(R)/uR \to 0\,. \tag{4}$$

Since $R \neq 0$, we always have $uR \cong R \neq 0$ and $\Omega(R) \neq 0$. Thus if $\Omega(R)/uR \neq 0$, then

$$\operatorname{depth} uR = 1 + \operatorname{depth} \Omega(R)/uR\,.$$

Now depth $uR = d$ since $uR \cong R$ and R is Cohen–Macaulay. Hence either $\Omega(R) = uR$, or depth $\Omega(R)/uR = d - 1$. But since $\Omega(R)$ is isomorphic to a nonzero ideal of the domain R, it follows that $\dim \Omega(R)/uR < \dim R = d$. Therefore, we have

$$\Omega(R) = uR\,, \quad \text{or} \quad \dim \Omega(R)/uR = \operatorname{depth} \Omega(R)/uR = d - 1\,. \tag{5}$$

In the latter case we have that $\Omega(R)/uR$ is Cohen–Macaulay of Krull dimension $d - 1$.

Note. (5) can also be obtained from the long exact sequence of some depth-sensitive functor such as local cohomology (with respect to the ideal $R_+ = R_1 \oplus R_2 \oplus \cdots$ of R), applied to the short exact sequence (4).

If $\Omega(R) = uR$, then $\Omega(R) \cong R$ so R is Gorenstein. In this case we have $h_i = h_{s-i}$ [11, Theorem 4.1], so (2) holds with equality. Hence assume $\Omega(R)/uR \neq 0$. We may tensor the R-module $M = \Omega(R)/uR$ with an infinite extension field of K without altering the Cohen–Macaulay property, the Krull dimension, or the Hilbert series. Thus assume that K is infinite. Let $R' = R/(\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{x \in R : xM = 0\}$. Since K is infinite, the subalgebra $K[R_1']$ of R'

generated by R_1' has a homogeneous system of parameters (h.s.o.p.) $\theta_1, \ldots, \theta_{d-1}$ of degree one. Since R is integral over $K[R_1]$, it follows that $\theta_1, \ldots, \theta_{d-1}$ is an h.s.o.p. for R'. Any h.s.o.p. for $R/(\text{Ann } M)$ is an h.s.o.p. for M, so $\theta_1, \ldots, \theta_{d-1}$ is an h.s.o.p. for M.

Let $N = M/(\theta_1 M + \cdots + \theta_{d-1} M)$. Since M is Cohen–Macaulay we have [11, Corollary 3.2]

$$F(M, \lambda) = \frac{F(N, \lambda)}{\prod_{i=1}^{d-1} (1 - \lambda^{\deg \theta_i})} = \frac{F(N, \lambda)}{(1-\lambda)^{d-1}} .$$

Thus the polynomial $F(N, \lambda) = \sum k_i \lambda^i$ has nonnegative coefficients. But

$$\begin{aligned} F(M, \lambda) &= F(\Omega(R), \lambda) - F(uR, \lambda) \\ &= \frac{h_s + h_{s-1}\lambda + \cdots + h_0 \lambda^s}{(1-\lambda)^d} - \frac{h_0 + h_1 \lambda + \cdots + h_s \lambda^s}{(1-\lambda)^d} \\ &= \frac{k_0 + k_1 \lambda + \cdots + k_{s-1}\lambda^{s-1}}{(1-\lambda)^{d-1}} . \end{aligned}$$

An easy computation shows that

$$k_i = (h_s + h_{s-1} + \cdots h_{s-i}) - (h_0 + h_1 + \cdots + h_i) ,$$

and the proof follows. □

Note. The module $M = \Omega(R)/uR$ has the interesting property that it is a 'Gorenstein module' in the sense that $\Omega(M) \cong M$, where $\Omega(M)$ is the canonical module of M as defined, e.g., in [12, equation (15)].

3. Some further results

For the sake of completeness we mention the following easy and well-known result. Geometrically, it asserts when R is standard that an irreducible projective variety of dimension zero over an algebraically closed field consists of a single point.

Proposition 3.1. *Let R be a graded domain of Krull dimension one over an algebraically closed field K. Then R is isomorphic to the monoid algebra $K[\Gamma]$ of some (additive) submonoid Γ of $\mathbb{N} = \{0, 1, 2, \ldots\}$. In other words, R is isomorphic to a graded subalgebra of the polynomial ring $K[x]$ (with the standard grading $\deg x = 1$), i.e., a subalgebra generated (or spanned) by monomials. In particular, if R is semistandard, then $R \cong K[x]$.*

Proof. It clearly suffices to show that $H(R, i) = 0$ or 1 for every $i \geq 0$. Suppose $H(R, i) \geq 2$. Let $u, v \in R_i$ be linearly independent. Since $\dim R = 1$, u and v satisfy a nontrivial homogeneous polynomial equation $P(u, v) = 0$. Since K is algebraically closed, $P(u, v)$ factors into linear factors $\alpha u + \beta v$. Since R is a domain, at least one of these factors must be zero, contradicting the linear independence of u and v. □

Of course Proposition 3.1 fails for K nonalgebraically closed, e.g., $R = \mathbb{R}[x, y]/(x^2 + y^2)$.

Now assume R has Krull dimension at least two. If L is a purely transcendental extension field of L, then $R \otimes_K L$ will be a graded L-algebra which preserves such properties of K as being standard, semistandard, Cohen–Macaulay, and a domain, as well as the Hilbert function, depth, and Krull dimension. (For all these properties except being a domain, L can be any extension field of K.) Thus in the proof of Corollary 3.3 below it is valid to replace K by a purely transcendental extension field.

Insofar as Hilbert functions of standard graded domains R of Krull dimension at least two are concerned, Bertini's theorem from algebraic geometry (see [15, p. 68] and also [3, Chapter II, Theorem 8.18 and Remark 8.18.1]) tells us that we may assume $\dim R = 2$. For completeness we state a weak form of this result in the following algebraic form.

Proposition 3.2. *Let R be a standard graded domain of Krull dimension at least three over an infinite field K. Then there exists a parameter θ of degree one (i.e., $\theta \in R_1$ and $\dim R/\theta R = \dim R - 1$) such that if $S = R/\theta R$, then $S/H^0(S)$ is a domain. Here*

$$H^0(S) = \{x \in S : xS_+^n = 0 \text{ for some } n \geq 1\} ,$$

the 0th local cohomology module of S (with respect to the irrelevant ideal S_+). □

Corollary 3.3. *Let R be a standard Cohen–Macaulay graded domain of Krull dimension $d \geq 2$. Then the h-vector $h(R)$ is the h-vector of a standard Cohen–Macaulay graded domain of Krull dimension two.*

Proof. Extend the field K by a purely transcendental extension field if necessary. By Proposition 3.2 there is a regular sequence $\theta_1, \ldots, \theta_{d-2} \in R_1$ for which $R/(\theta_1 R + \cdots + \theta_{d-2} R)$ is a standard Cohen–Macaulay graded domain of Krull dimension two. But for any graded algebra A, if $\theta \in A_i$ is a non-zero-divisor, then $F(A/\theta A, \lambda) = (1 - \lambda^i) F(A, \lambda)$. Hence R and $R/(\theta_1 R + \cdots + \theta_{d-2} R)$ have the same h-vector, as desired. □

Finally we mention how a result of Eisenbud and Harris leads to some results related to Theorem 2.1 when R is standard and char $K = 0$.

Proposition 3.4. *Let R be a standard graded Cohen–Macaulay domain of Krull dimension $d \geq 2$ over a field K of characteristic 0. Let $h(R) = (h_0, h_1, \ldots, h_s)$, where $h_s \neq 0$. Let $m \geq 0$ and $n \geq 1$, with $m + n < s$. Then*

$$h_{m+1} + h_{m+2} + \cdots + h_{m+n} \geq h_1 + h_2 + \cdots + h_n .$$

Proof. The quantity $h_\Gamma(n)$ of [2, Chapter 3] is equal, in our notation, to $h_0 + \cdots + h_n$. Moreover, the degree d in [2] is our $h_0 + \cdots + h_s$. Corollary 3.5 of [2] asserts that

$$h_\Gamma(m+n) \geq \min(d, h_\Gamma(m) + h_\Gamma(n) - 1) ,$$

so in our notation,

$$h_0 + \cdots + h_{m+n} \geq \min(h_0 + \cdots + h_s, h_0 + \cdots + h_m + h_1 + \cdots + h_n) ,$$

since $h_0 = 1$. This is easily seen to be equivalent to the desired result. □

For instance, if $n = 1$ in Proposition 3.4, we obtain $h_1 \leq h_i$ for $1 \leq i \leq s - 1$. In particular, if R is Gorenstein (so $h_i = h_{s-i}$) and $s \leq 5$, then $h(R)$ is unimodal. It is not known whether $h(R)$ is unimodal for any standard Cohen–Macaulay (or Gorenstein) graded domain R (see [14, Conjecture 4(a)], [5, Conjecture 1.5]). If R is just assumed to be standard Gorenstein (but not a domain), then $h(R)$ need not be unimodal [11, p. 70]. If R is assumed to be a semistandard Gorenstein graded domain, then again $h(R)$ need not be unimodal, as shown by the example

$$R = K[y, x_1x_2y, x_1x_3y, x_2x_3y, x_1x_2x_3y^2]$$

(with the grading given by $\deg x_1^{a_1}x_2^{a_2}x_3^{a_3}y^b = b$), where $h(R) = (1, 0, 1)$. A related conjecture of Hibi [5, Conjecture 1.4] states that $h_0 \leq h_1 \leq \cdots \leq h_{[s/2]}$ and $h_i \leq h_{s-i}$ for all $0 \leq i \leq [s/2]$, when R is a standard Cohen–Macaulay graded domain. We also do not know whether Proposition 3.4 continues to hold for arbitrary fields K. It would be interesting to investigate to what extent the techniques of [2] can be used to obtain additional results about Hilbert functions of standard graded domains.

4. An example: The Ehrhart polynomial

In this section we will give a combinatorially interesting example of a semistandard Cohen–Macaulay graded domain. Let $\mathcal{P}$ be a d-dimensional convex polytope in $\mathbb{R}^n$ with integer vertices. Let $R_\mathcal{P}$ be the subalgebra of

$$K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, y]$$

generated by all monomials

$$x_1^{a_1} \cdots x_n^{a_n} y^b \quad \text{with } b \geq 1 \text{ and } \frac{1}{b}(a_1, \ldots, a_n) \in \mathcal{P} .$$

In fact, $R_\mathcal{P}$ as a K-vector space has a basis consisting of these monomials together with 1. Define a grading on $R_\mathcal{P}$ by setting $\deg x_1^{a_1} \cdots x_n^{a_n} y^b = b$. Thus the Hilbert function $H(R_\mathcal{P}, j)$ is equal to the number of points $\alpha \in \mathcal{P}$ satisfying $j\alpha \in \mathbb{Z}^n$, or in other words

$$H(R_\mathcal{P}, j) = \#(j\mathcal{P} \cap \mathbb{Z}^n) .$$

Then $H(R_\mathcal{P}, j)$ is a polynomial function of j of degree d, known as the *Ehrhart polynomial* of $\mathcal{P}$ and denoted $i(\mathcal{P}, j)$. For an introduction to Ehrhart polynomials, see [13, pp. 235–241].

Since $\deg H(R_\mathcal{P}, j) = d$ it follows that $\dim R_\mathcal{P} = d + 1$. Moreover, it is easy to see that $R_\mathcal{P}$ is normal, so by a theorem of Hochster [7] $R_\mathcal{P}$ is Cohen–Macaulay. Trivially $R_\mathcal{P}$ is a domain. Finally, the subalgebra $K[(R_\mathcal{P})_1]$ contains the monomials $x_1^{a_1} \cdots a_n^{a_n} y$ for which $(a_1, \ldots, a_n)$ is a vertex of $\mathcal{P}$. It then follows easily from the convexity of $\mathcal{P}$ that $R_\mathcal{P}$ is integral over $K[(R_\mathcal{P})_1]$. Hence $R_\mathcal{P}$ is semistandard. Thus from Theorem 2.1 we obtain the following proposition:

Proposition 4.1. *Let $\mathcal{P}$ be a convex d-polytope in $\mathbb{R}^n$ with integer vertices. Let $i(P, j)$ denote its Ehrhart polynomial, and write*

$$\sum_{j \geq 0} i(\mathcal{P}, j)\lambda^j = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^{d+1}} , \tag{6}$$

where $h_s \neq 0$. (Since $i(\mathcal{P}, j)$ is a polynomial for all *j we have $s \leq d$.) Then*

$$h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i}$$

for all $0 \leq i \leq s$. □

The algebra $R_\mathcal{P}$ need not be standard, e.g., when $\mathcal{P}$ is the simplex with vertices $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$. For this example $R_\mathcal{P}$ is just the ring R mentioned at the end of Section 3, so $h(R) = (h_0, \ldots, h_s) = (1, 0, 1)$.

In [6, Theorem 1] Hibi obtains the additional inequality

$$h_0 + h_1 + \cdots + h_{i+1} \geq h_d + h_{d-1} + \cdots + h_{d-i} ,$$

$0 \leq i \leq d$, where $(h_0, \ldots, h_s)$ is given by (6) (and where we set $h_{s+1} = h_{s+2} = \cdots = h_d = 0$). Such an inequality exists because one can describe an explicit ideal I of $R_\mathcal{P}$ for which $I \cong \Omega(R)$ and then apply an argument to $R_\mathcal{P}/I$ similar to what was done in the proof of Theorem 2.1 to $\Omega(R)/uR$.

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[illegible]rnal of Pure and Applied Algebra 73 (1991) 315
[illegible]rth-Holland

Author Index Volume 73 (1991)

(The issue number is given in front of the page numbers.)