LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS AND MAGIC LABELINGS OF GRAPHS

RICHARD P. STANLEY

1. Introduction. Let G be a finite graph allowing loops and multiple edges. Hence G is a *pseudograph* in the terminology of [10]. We shall denote the set of vertices of G by V, the set of edges by E, the number $|V|$ of vertices by p, and the number $|E|$ of edges by q. Also if an edge e is incident to a vertex v, we write $v \in e$. Any undefined graph-theoretical terminology used here may be found in [10]. A *magic labeling of G of index r* is an assignment $L : E \rightarrow \{0, 1, 2, \cdots\}$ of a nonnegative integer $L(e)$ to each edge e of G such that for each vertex v of G the sum of the labels of all edges incident to v is r (counting each loop at v once only). In other words,

$$(1) \qquad \sum_{e\,:\,v \in e} L(e) = r, \qquad \qquad \text{for all } v \in V.$$

For each edge e of G let z_e be an indeterminate and let z be an additional indeterminate. For each vertex v of G define the homogeneous linear form

$$(2) \qquad P_v = z - \sum_{e\,:\,v \in e} z_e, \qquad \qquad v \in V,$$

where the sum is over all e incident to v. Hence by (1) a magic labeling L of G corresponds to a solution of the system of equations

$$(3) \qquad P_v = 0, \qquad \qquad v \in V,$$

in nonnegative integers (the value of z is the index of L). Thus the theory of magic labelings can be put into the more general context of *linear homogeneous diophantine equations*. Many of our results will be given in this more general context and then specialized to magic labelings.

It may happen that there are edges e of G that are always labeled 0 in any magic labeling. If this is the case, then these edges may be ignored in so far as studying magic labelings is concerned; so we may assume without loss of generality that for any edge e of G there is a magic labeling L of G for which $L(e) > 0$. We then call G a *positive* pseudograph. If in a magic labeling L of G every edge receives a positive label, then we call L a *positive* magic labeling. If L_1 and L_2 are magic labelings, we define their *sum* $L = L_1 + L_2$ by $L(e) = L_1(e) + L_2(e)$ for every edge e of G. Clearly if L_1 and L_2 are of index r_1 and r_2, then L is magic of index $r_1 + r_2$. Now note that every positive pseudograph G possesses a positive magic labeling L, e.g., for each edge e of G let L_e be a magic labeling positive on e, and let $L = \sum L_e$.

In general, there appears to be no simple criterion for determining when a finite pseudograph is positive. However, the following condition, which is a special case of Stiemke's theorem [21], is sometimes useful. A finite pseudograph G is *not* positive if and only if there exists a labeling $K : V \to \mathbf{Z}$ of the vertices of G by integers such that $\sum_{v \in V} K(v) \leq 0$ and for all $e \in E$, $\sum_{v : v \in e} K(v) \geq 0$, with at least one of these $q + 1$ sums not equal to zero.

Let $H_G(r)$ be the number of magic labelings of G of index r, and let $\bar{H}_G(r)$ be the number of positive magic labelings of index r. Hence $\bar{H}_G(r) \leq H_G(r)$, $H_G(0) = 1$, $\bar{H}_G(0) = 0$ (unless G has no edges), and $H_G(1)$ is the number of 1-factors of G (counting loops as having degree 1). Our primary purpose is to prove Theorems 1.1, 1.2, and 1.3 below, which deal with the functions $H_G(r)$ and $\bar{H}_G(r)$. Our results, however, will be developed as generally as possible within the context of linear homogeneous diophantine equations.

THEOREM 1.1. *Let G be a finite positive pseudograph with at least one edge. Then there exist polynomials $P_G(r)$, $Q_G(r)$, $\bar{P}_G(r)$, and $\bar{Q}_G(r)$ such that for all nonnegative integers r, $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ and $\bar{H}_G(r) = \bar{P}_G(r) + (-1)^r \bar{Q}_G(r) - \delta_{0r}(-1)^m$, where δ_{0r} is the Kronecker delta and where $m = \deg P_G(r)$ (given explicitly by Proposition 5.2).*

Note that if G has no edges, then $H_G(r) = \delta_{0r}$, which is not of the form $P_G(r) + (-1)^r Q_G(r)$. The reason for this exception to Theorem 1.1 will become clear in the proof of Corollary 3.10.

THEOREM 1.2. *If in the previous theorem the graph obtained by removing all loops from G is bipartite, then $Q_G(r) = \bar{Q}_G(r) = 0$.*

By Theorem 1.1 we can evaluate $H_G(r)$ and $\bar{H}_G(r)$ when r is a negative integer. The next theorem tells us the significance of these numbers.

THEOREM 1.3. *With the hypotheses of Theorem 1.1, we have for all integers $r \neq 0$, $(-1)^m H_G(-r) = \bar{H}_G(r)$, where $m = \deg P_G(r)$.*

COROLLARY 1.4. *If G (as in Theorem 1.1) is regular of degree d (counting loops as having degree 1), then $H_G(-1) = H_G(-2) = \cdots = H_G(-d+1) = 0$ and $(-1)^m H_G(r) = H_G(-d-r)$.*

Proof of Corollary 1.4. If G is regular of degree d, then a one-to-one correspondence exists between magic labelings L of G of index r and positive magic labelings $\bar{L}$ of G of index $d + r$ by defining $\bar{L}(e) = 1 + L(e)$ for all $e \in E$. The proof now follows from Theorem 1.3.

The Anand–Dumir–Gupta conjecture. Let $H_n(r)$ be the number of $n \times n$ matrices of nonnegative integers summing to r in every row and column. Anand, Dumir, and Gupta [2] conjectured that $H_n(r)$ is a polynomial in r of degree $(n-1)^2$ satisfying $H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$ and $H_n(r) = (-1)^{n-1} H_n(-n-r)$. Equivalently, there exist integers c_i, $0 \leq i \leq \binom{n-1}{2}$,

depending only on n, such that

$$H_n(r) = \sum$$

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$\binom{r + n - 1 + i}{n - 1 + 2i}$, $0 \leq i \leq \binom{n - }{}$

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Let G be the complete bipartite partitioned into two classes $V_1 = \{v_1$ there is an edge between every v_i an L of G of index r an $n \times n$ matrix necting v_i and v'_i, then each A_{ij} is column sum of A is r. Conversely with every row and column sum r index r. Hence Theorem 1.2 and C Gupta conjecture except for the sta be given a simple *ad hoc* proof (see to this conjecture see [1], [15], [18], computed for $n \leq 6$ assuming the jecture. However, the sketched pr invalid.

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$$H_2^*(r) = \binom{r + 4}{4} +$$

and

$$H_3^*(r) = \binom{r + 9}{9} + 24\binom{r + 8}{9} + 156$$

$$+ 280\binom{r + 6}{9} .$$

More generally, if G is any finite let $H_G^*(r)$ be the number of *submagic* la of maps $E \to \{0, 1, 2, \cdots\}$ such that th or equal to r. Then $H_G^*(r) = H_{G'}(r)$, by adding a loop at each vertex (in addi

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depending only on n, such that

$$H_n(r) = \sum c_i \binom{r + n - 1 + i}{n - 1 + 2i}.$$

(The two forms of the conjecture are equivalent since the polynomials $\binom{r + n - 1 + i}{n - 1 + 2i}$, $0 \leq i \leq \binom{n-1}{2}$, form a basis over the integers for all polynomials $f(r)$ of degree $(n-1)^2$ mapping integers into integers and satisfying $f(-1) = f(-2) = \cdots = f(-n + 1) = 0$, $f(r) = (-1)^{n-1}f(-n - r)$.)

Let G be the complete bipartite graph K_{nn} so the vertices V of G may be partitioned into two classes $V_1 = \{v_1, \cdots, v_n\}$ and $V_2 = \{v'_1, \cdots, v'_n\}$ such that there is an edge between every v_i and v'_i. If we associate with a magic labeling L of G of index r an $n \times n$ matrix A, where A_{ij} is the label of the edge connecting v_i and v'_j, then each A_{ij} is a nonnegative integer and every row and column sum of A is r. Conversely an $n \times n$ matrix A of nonnegative integers with every row and column sum r corresponds to a magic labeling of K_{nn} of index r. Hence Theorem 1.2 and Corollary 1.4 reduce to the Anand–Dumir–Gupta conjecture except for the statement $\deg H_n(r) = (n - 1)^2$, which can be given a simple *ad hoc* proof (see Proposition 5.2). For some work relating to this conjecture see [1], [15], [18], and [19]. In particular, in [19], $H_n(r)$ is computed for $n \leq 6$ assuming the validity of the Anand–Dumir–Gupta conjecture. However, the sketched proof in [19] that $H_n(r)$ is a polynomial is invalid.

As a modification of the Anand–Dumir–Gupta conjecture, define $H_n^*(r)$ to be the number of $n \times n$ matrices of nonnegative integers with every row and column sum less than or equal to r. This corresponds to taking G to be K_{nn} with one loop adjoined to each vertex. Hence by Theorem 1.2, Corollary 1.4, and Proposition 5.2, $H_n^*(r)$ is a polynomial in r of degree n^2 satisfying $H_n^*(-1) = H_n^*(-2) = \cdots = H_n^*(-n) = 0$, $H_n^*(r) = (-1)^n H_n^*(-n - 1 - r)$. For instance,

$$H_2^*(r) = \binom{r + 4}{4} + 2\binom{r + 3}{4} + \binom{r + 2}{4}$$

and

$$H_3^*(r) = \binom{r + 9}{9} + 24\binom{r + 8}{9} + 156\binom{r + 7}{9}$$
$$+ 280\binom{r + 6}{9} + 156\binom{r + 5}{9} + 24\binom{r + 4}{9} + \binom{r + 3}{9}.$$

More generally, if G is any finite pseudograph (not necessarily positive), let $H_G^*(r)$ be the number of *submagic* labelings of G of index r, i.e., the number of maps $E \to \{0, 1, 2, \cdots\}$ such that the edge sums at each vertex are less than or equal to r. Then $H_G^*(r) = H_{G'}(r)$, where G' is the graph obtained from G by adding a loop at each vertex (in addition to whatever loops are already there).

Clearly G' is positive. Hence Theorems 1.1–1.3 may be applied to $H_G^*(r)$. Note that $H_G^*(r)$ is also equal to the number of ways of labeling the *vertices and edges* of G with nonnegative integers such that the sum of the label of any vertex together with the labels of all edges incident to that vertex is r. Some other aspects of such labelings were considered by Kotzig and Rosa [13].

The Carlitz conjecture. Let $S_n(r)$ be the number of symmetric $n \times n$ matrices of nonnegative integers summing to r in every row (and hence in every column). L. Carlitz [5] conjectured that $S_n(2r)$ and $S_n(2r + 1)$ are polynomial functions of the nonnegative integer r. Let G be the graph with n vertices $v_1, v_2, \cdots, v_n$ and exactly one edge between any two vertices (including one loop at each vertex). Associate with a magic labeling of G an $n \times n$ matrix A, where A_{ij} is the label of the edge between v_i and v_j. Then A is an $n \times n$ symmetric matrix of nonnegative integers with every row and column sum r, and conversely every such A corresponds to a magic labeling of G of index r. Hence Theorem 1.1 reduces to the Carlitz conjecture. We also obtain from Corollary 1.4 that $S_n(-1) = S_n(-2) = \cdots = S_n(-n + 1) = 0$, $S_n(r) = \pm S_n(-n - r)$. (The correct sign is (-1) to the power $\binom{n}{2}$, as shown in Section 5.)

Throughout this paper we use $\mathbf{N}$ to denote the nonnegative integers and $\mathbf{P}$ the positive integers, $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $\mathbf{P} = \{1, 2, 3, \cdots\}$. We shall call a solution in nonnegative integers to a system of equations such as (3) an $\mathbf{N}$-*solution*, while a solution in positive integers is a $\mathbf{P}$-*solution*. Similarly, a nonnegative integer combination of vectors $\alpha, \beta, \cdots$, i.e., a sum $a\alpha + b\beta + \cdots$, where $a, b, \cdots \in \mathbf{N}$, will be called an $\mathbf{N}$-*combination*, while a positive integer combination will be called a $\mathbf{P}$-*combination*.

2. The Hilbert syzygy theorem.

In this section we prove that for a finite pseudograph G (which we may assume is positive) there exist polynomials $P_G(r)$ and $Q_G(r)$ such that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for all but finitely many nonnegative integers r. The proof is based on the *Hilbert syzygy theorem*. A separate argument is needed to show that then $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for *all* nonnegative integers r (cf. Section 3). We shall also show in this section that if G minus its loops is bipartite, then $Q_G(r) = 0$.

Consider the following general situation. Let

$$P_1(z_1, \cdots, z_s) = 0$$
$$\vdots$$
$$P_p(z_1, \cdots, z_s) = 0 \tag{4}$$

be a system of p homogeneous linear equations with integer coefficients in the unknowns $z_1, \cdots, z_s$. Let $\alpha^1 = (\alpha_1^1, \cdots, \alpha_s^1), \cdots, \alpha^\sigma = (\alpha_1^\sigma, \cdots, \alpha_s^\sigma)$ be a finite set of nonzero $\mathbf{N}$-solutions to (4); so each α_i^j is a nonnegative integer, and for each j some $\alpha_i^j \neq 0$. All the $\mathbf{N}$-solutions to (4) form a commutative semigroup Φ with identity $0 = (0, 0, \cdots, 0)$ under the operation of componentwise addition. Let $X_1, \cdots, X_\sigma$ be new indeterminates corresponding to the solutions

$\alpha^1, \cdots, \alpha^\sigma$, and form the polynomi⣿ field. We impose on R the structu defining R_α to be the vector s] $X_1^{a_1} X_2^{a_2} \cdots X_\sigma^{a_\sigma}$ such that $a_1\alpha^1 + a_2\alpha$ so we have indeed defined a Φ-grad

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For a simple proof of Lemma 2.1

PROPOSITION 2.2. *The following*

(i) *The R-module M is finitely-ge*
(ii) *For every $\alpha \in \Phi$ there exists a $n\alpha$ is an $\mathbf{N}$-combination of α^1,*

Proof. (i) $\Rightarrow$ (ii). Assume M is has the form $\sum k_\alpha[\alpha]$, where the s where $k_\alpha \in K$, we may take the where $\beta^i \in \Phi$. Let $\gamma \in \Phi$. Now s integers n there must exist $j_n \in$ $a_{1n}, \cdots, a_{\sigma n}$ such that $[n\gamma] = X_1^{a_{1\cdots}}$

$$n\gamma = \text{[} \tag{5}$$

Since there are only finitely many d such that $j_n = k$ for infinitely many Lemma 2.1 or is easily proved directl are values $n_u < n_s$ such that $a_{in_u} \leq a$ $\sum_1^\sigma (a_{in_s} - a_{in_u})\alpha^i$ so (ii) holds.

(ii) $\Rightarrow$ (i). Assume that (ii) hold solutions to (4), finite in number by integers $n_1, n_2, \cdots, n_m$ such that We claim M is generated by all $\epsilon_i < n_i$. Let $\gamma \in \Phi$; so $\gamma = a_1\gamma^1 +$

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$\alpha^1, \cdots, \alpha^g$, and form the polynomial ring $R = K[X_1, \cdots, X_g]$, where K is any field. We impose on R the structure of a Φ-graded ring, $R = \sum_{\alpha \in \Phi} R_\alpha$, by defining R_α to be the vector space over K spanned by all monomials $X_1^{a_1} X_2^{a_2} \cdots X_g^{a_g}$ such that $a_1\alpha^1 + a_2\alpha^2 + \cdots + a_g\alpha^g = \alpha$. Clearly $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$; so we have indeed defined a Φ-grading.

Now define a Φ-graded R-module M as follows. M is a vector space over K with a basis consisting of all "symbols" $[\alpha]$, where α is an **N**-solution to (4). The action $R \times M \to M$ of R on M is uniquely defined by imposing the condition of K-linearity and the relations $X_i[\alpha] = [\alpha^i + \alpha]$, $\alpha \in \Phi$, $i = 1, 2, \cdots, g$. If we define M_α to be the (one-dimensional) vector space generated by $[\alpha]$, where $\alpha \in \Phi$, then it is easily seen that M becomes a Φ-graded R-module, i.e., $M = \sum_{\alpha \in \Phi} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$.

We need to know when M is finitely-generated. For this purpose we first require a lemma.

LEMMA 2.1 (Hilbert). *There are only finitely many nonzero* **N**-*solutions $\gamma \in \Phi$, called fundamental solutions, such that if $\gamma = \alpha + \beta$, where $\alpha, \beta \in \Phi$, then $\alpha = 0$ or $\beta = 0$. Every solution $\alpha \in \Phi$ is an* **N**-*combination of fundamental solutions.*

For a simple proof of Lemma 2.1 see, e.g., [7; §97].

PROPOSITION 2.2. *The following two conditions are equivalent.*

(i) *The R-module M is finitely-generated.*

(ii) *For every $\alpha \in \Phi$ there exists a positive integer n (depending on α) such that $n\alpha$ is an* **N**-*combination of $\alpha^1, \alpha^2, \cdots, \alpha^g$.*

Proof. (i) $\Rightarrow$ (ii). Assume M is finitely-generated. Since each element of M has the form $\sum k_\alpha[\alpha]$, where the sum ranges over finitely many $\alpha \in \Phi$ and where $k_\alpha \in K$, we may take the generators of M to be $[\beta^1], [\beta^2], \cdots, [\beta^m]$, where $\beta^i \in \Phi$. Let $\gamma \in \Phi$. Now since M is finitely-generated, for all positive integers n there must exist $j_n \in \{1, 2, \cdots, m\}$ and nonnegative integers $a_{1n}, \cdots, a_{gn}$ such that $[n\gamma] = X_1^{a_{1n}} \cdots X_g^{a_{gn}}[\beta^{j_n}]$ or, equivalently,

$$(5) \qquad n\gamma = \beta^{j_n} + \sum_1^g a_{in}\alpha^i.$$

Since there are only finitely many distinct j_n, there is some $k \in \{1, 2, \cdots, m\}$ such that $j_n = k$ for infinitely many values $n_1 < n_2 < \cdots$ of n. It follows from Lemma 2.1 or is easily proved directly by induction on g [7; §§152–153] that there are values $n_u < n_v$ such that $a_{in_u} \le a_{in_v}$ for $i = 1, 2, \cdots, g$. Hence $(n_v - n_u)\gamma = \sum_1^g (a_{in_v} - a_{in_u})\alpha^i$ so (ii) holds.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Let $\gamma^1, \gamma^2, \cdots, \gamma^m$ be the fundamental solutions to (4), finite in number by Lemma 2.1. Hence by (ii) there are positive integers $n_1, n_2, \cdots, n_m$ such that each $n_i\gamma^i$ is an **N**-combination of the α^i's. We claim M is generated by all $n_1 n_2 \cdots n_m$ symbols $[\sum \epsilon_i\gamma^i]$, where $0 \le \epsilon_i < n_i$. Let $\gamma \in \Phi$; so $\gamma = a_1\gamma^1 + a_2\gamma^2 + \cdots + a_m\gamma^m$, $a_i \in \mathbf{N}$. Define ϵ_i by

$a_i \equiv \epsilon_i \pmod{n_i}$ and $0 \le \epsilon_i < n_i$. Hence $\gamma = \sum_1^m n_i b_i \gamma^i + \sum_1^m \epsilon_i \gamma^i$ for some $b_i \in \mathbf{N}$. Since $n_i \gamma^i$ is an $\mathbf{N}$-combination of the α^i's, we have $\gamma = \sum_1^g c_i \alpha^i + \sum_1^m \epsilon_i \gamma^i$ for some $c_i \in \mathbf{N}$. Hence $[\gamma] = X_1^{c_1} X_2^{c_2} \cdots X_g^{c_g} [\sum_1^m \epsilon_i \gamma^i]$; so M is indeed finitely-generated.

Note. It is easily seen (though we have no need for this fact) that M is generated by one element, viz., $[0]$, if and only if $\alpha^1, \cdots, \alpha^g$ include all the fundamental solutions to (4).

We are now in a position to invoke the Hilbert syzygy theorem. For our purposes the following special case is convenient.

2.3. *Hilbert syzygy theorem.* Let Φ be the semigroup of $\mathbf{N}$-solutions to the system (4) of linear homogeneous equations with integer coefficients. Let $\alpha^1, \cdots, \alpha^g$ be nonzero elements of Φ satisfying condition (ii) of Proposition 2.2. Let $R = K[X_1, \cdots, X_g]$ be the Φ-graded ring constructed from $\alpha^1, \cdots, \alpha^g$ as before, and let M be the corresponding Φ-graded R-module. Then there exists an exact sequence

$$(6) \qquad 0 \to M^g \to M^{g-1} \to \cdots \to M^1 \to M \to 0$$

where the M^i are *free* finitely generated Φ-graded R-modules and all homomorphisms are of degree 0, i.e., if $M^i = \sum_{\alpha \in \Phi} M_\alpha^i$, then the image of M_α^i, $2 \le i \le g$, in (6) lies in M_α^{i-1}, while the image of M_α^1 lies in M_α.

Note. The statement that M^i is free means the following. M^i is free as an R-module (ignoring the grading), and M^i is Φ-graded. However, the grading on M^i may be "shifted". In other words, the homogeneous generators of M^i need not be of degree 0 but may have any degree $\alpha \in \Phi$.

In Hilbert's original proof [11; Theorem III] he considers only $\mathbf{N}$-graded polynomial rings $K[X_1, \cdots, X_g]$ and $\mathbf{N}$-graded R-modules, where also $\deg X_i = 1$. His proof, however, remains valid under the assumptions of Theorem 2.3. The much simpler proof given in [22; Chapter III, §13] is also valid under the assumptions of Theorem 2.3.

We now introduce the basic tool of *generating functions*. Given the system (4) with $\mathbf{N}$-solution set Φ, define the (formal) power series $F(x_1, \cdots, x_s)$ in the variables $x_1, \cdots, x_s$ by

$$(7) \qquad F(x_1, \cdots, x_s) = \sum_{\alpha \in \Phi} x_1^{\alpha_1} \cdots x_s^{\alpha_s}$$

where $\alpha = (\alpha_1, \cdots, \alpha_s)$. Hence F "lists" all elements of Φ; knowing F is equivalent to having a list of elements of Φ. We shall use the symbolic notation $\mathbf{x} = (x_1, \cdots, x_s)$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ so that (7) may be rewritten

$$(8) \qquad F(\mathbf{x}) = \sum_{\alpha \in \Phi} \mathbf{x}^\alpha.$$

In this section and the next we shall prove two fundamental results concerning the form of the generating function $F(\mathbf{x})$ (Theorems 2.5 and 3.9). We need for the first of these results one additional concept.

DEFINITION. A *completely fun* solution such that for all positive i $\beta = k\alpha$ and $\gamma = (n - k)\alpha$ for som

By taking $n = 1$ in the above de mental solution is a fundamental be true. For instance, the equatio tions, $\alpha = (2, 0, 1)$, $\beta = (0, 2, 1)$ completely fundamental since $2\gamma =$

LEMMA 2.4. *There are finitely* $\beta^2, \cdots, \beta^h$ *to* (4). *For every solution* *on* α) *such that* $n\alpha$ *is an* $\mathbf{N}$-combina

Proof. Since every completely follows from Lemma 2.1 that ther solutions.

Assume $\alpha \in \Phi$. If for some $r \in$ $r\beta^i$, then $n\alpha$ is an $\mathbf{N}$-combination c assume α is not a multiple of any ℓ of nonzero coordinates of α that f $\beta^1, \beta^2, \cdots, \beta^h$. This is clearly tru for some k satisfying $0 < k \le s$ (wi $\alpha' \in \Phi$ with less than k nonzero co an $\mathbf{N}$-combination of $\beta^1, \beta^2, \cdots, \beta'$

Suppose now that $\alpha \in \Phi$ has k not a multiple of any β^i. It is ea $\beta, \gamma \in \Phi$ such that neither β nor γ Since α, β, and γ are all $\mathbf{N}$-vectors a at most k nonzero coordinates.

Let p/q be the largest rational nu $q \in \mathbf{P}$) such that every coordina p/q exists. Moreover, $q\beta - p\alpha \ne$ multiple of α. Hence by definition with less nonzero coordinates than ℓ nates. Similarly define p'/q' with Hence $q'\gamma - p'\alpha$ has less than k no

Now from $n\alpha = \beta + \gamma$ we get $(qq'$ $p'\alpha)$. Since $q\beta - p\alpha$, $q'\gamma - p'\alpha$, a $q' > 0$, we have $qq'n - pq' - p'q \ge$ than k nonzero coordinates, by the such that $n_1(q\beta - p\alpha)$ and $n_2(q'\gamma -$ Hence $n_1 n_2 (qq'n - pq' - p'q)\alpha$ is a proof follows by induction.

We now come to the main result

$= \sum_1^m n_i b_i \gamma^i + \sum_1^m \epsilon_i \gamma^i$ for some
...e α^i's, we have $\gamma = \sum_1^\sigma c_i \alpha^i +$
$X_2^{c_2} \cdots X_\sigma^{c_\sigma}[\sum_1^m \epsilon_i \gamma^i]$; so M is

o need for this fact) that M is
y if $\alpha^1, \cdots, \alpha^\sigma$ include all the

lbert syzygy theorem. For our
t.

emigroup of N-solutions to the
with integer coefficients. Let
condition (ii) of Proposition 2.2.
g constructed from $\alpha^1, \cdots, \alpha^\sigma$
graded R-module. Then there

$\Gamma^1 \to M \to 0$

ded R-modules and all homo-
M_α^i, then the image of M_α^i,
f M_α^1 lies in M_α.
he following. M^i is free as an
;raded. However, the grading
iomogeneous generators of M^i
e $\alpha \in \Phi$.

] he considers only N-graded
ided R-modules, where also
lid under the assumptions of
[22; Chapter III, §13] is also

nctions. Given the system (4)
· series $F(x_1, \cdots, x_s)$ in the

$\cdots x_s^{\alpha_s}$

elements of Φ; knowing F is
hall use the symbolic notation
(7) may be rewritten

indamental results concerning
ems 2.5 and 3.9). We need

DEFINITION. A *completely fundamental solution* α to (4) is a nonzero N-solution such that for all positive integers n if $n\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi$, then $\beta = k\alpha$ and $\gamma = (n - k)\alpha$ for some $k \in \mathbf{N}$.

By taking $n = 1$ in the above definition, we see that every completely fundamental solution is a fundamental solution. The converse, however, need not be true. For instance, the equation $z_1 + z_2 = 2z_3$ has three fundamental solutions, $\alpha = (2, 0, 1)$, $\beta = (0, 2, 1)$, $\gamma = (1, 1, 1)$. However, only α and β are completely fundamental since $2\gamma = \alpha + \beta$.

LEMMA 2.4. *There are finitely many completely fundamental solutions* β^1, $\beta^2, \cdots, \beta^h$ *to (4). For every solution* $\alpha \in \Phi$ *there is a positive integer* n *(depending on* α*) such that* $n\alpha$ *is an* **N***-combination of* $\beta^1, \beta^2, \cdots, \beta^h$.

Proof. Since every completely fundamental solution is fundamental, it follows from Lemma 2.1 that there are finitely many completely fundamental solutions.

Assume $\alpha \in \Phi$. If for some $r \in \mathbf{P}$ and some $i = 1, 2, \cdots, h$ we have $\alpha = r\beta^i$, then $n\alpha$ is an **N**-combination of $\beta^1, \beta^2, \cdots, \beta^h$ for $n = 1$. Hence we may assume α is not a multiple of any β^i. We prove by induction on the number k of nonzero coordinates of α that for some $n > 0$, $n\alpha$ is an **N**-combination of $\beta^1, \beta^2, \cdots, \beta^h$. This is clearly true for $k = 0$, since then $\alpha = 0$. Assume that for some k satisfying $0 < k \le s$ (with s as in (4)) we have proved that for every $\alpha' \in \Phi$ with less than k nonzero coordinates there is an $n \in \mathbf{P}$ for which $n\alpha'$ is an **N**-combination of $\beta^1, \beta^2, \cdots, \beta^h$.

Suppose now that $\alpha \in \Phi$ has k nonzero coordinates. By assumption α is not a multiple of any β^i. It is easily seen that then there are an $n \in \mathbf{P}$ and $\beta, \gamma \in \Phi$ such that neither β nor γ are rational multiples of α and $n\alpha = \beta + \gamma$. Since α, β, and γ are all **N**-vectors and α has k nonzero coordinates, β and γ have at most k nonzero coordinates.

Let p/q be the largest rational number (reduced to lowest terms with $p \in \mathbf{N}$, $q \in \mathbf{P}$) such that every coordinate of $\beta - (p/q)\alpha$ is nonnegative. Clearly p/q exists. Moreover, $q\beta - p\alpha \ne 0$ since by assumption β is not a rational multiple of α. Hence by definition of p/q, $q\beta - p\alpha$ is a nonzero element of Φ with less nonzero coordinates than β; so $q\beta - p\alpha$ has less than k nonzero coordinates. Similarly define p'/q' with β replaced by γ in the definition of p/q. Hence $q'\gamma - p'\alpha$ has less than k nonzero coordinates.

Now from $n\alpha = \beta + \gamma$ we get $(qq'n - pq' - p'q)\alpha = q'(q\beta - p\alpha) + q(q'\gamma - p'\alpha)$. Since $q\beta - p\alpha$, $q'\gamma - p'\alpha$, and α are **N**-vectors with $q\beta - p\alpha \ne 0$ and $q' > 0$, we have $qq'n - pq' - p'q > 0$. Since $q\beta - p\alpha$ and $q'\gamma - p'\alpha$ have less than k nonzero coordinates, by the induction hypothesis there exist $n_1, n_2 \in \mathbf{P}$ such that $n_1(q\beta - p\alpha)$ and $n_2(q'\gamma - p'\alpha)$ are **N**-combinations of $\beta^1, \beta^2, \cdots, \beta^h$. Hence $n_1 n_2(qq'n - pq' - p'q)\alpha$ is an **N**-combination of $\beta^1, \beta^2, \cdots, \beta^h$, and the proof follows by induction.

We now come to the main result of this section.

THEOREM 2.5. *The generating function $F(\mathbf{x})$ defined by (8) is a rational function of the x_i's (in the algebra of formal power series), which when reduced to lowest terms has denominator*

$$D(\mathbf{x}) = \prod_\beta (1 - \mathbf{x}^\beta)$$

where β ranges over all completely fundamental solutions to (4).

Proof. It was known since about 1900 that $F(\mathbf{x})$ is a rational function (cf. Section 3); this will also come out of this proof. Let $\beta^1, \cdots, \beta^h$ be the completely fundamental solutions to (4), let $R = K[X_1, \cdots, X_h]$ be the Φ-graded polynomial ring corresponding to $\beta^1, \cdots, \beta^h$, and let M be the corresponding Φ-graded R-module. By Proposition 2.2 and Lemma 2.4, M is finitely-generated. Hence by Theorem 2.3 there is a Φ-graded exact sequence

$$(9) \qquad 0 \to M^h \to M^{h-1} \to \cdots \to M^1 \to M \to 0$$

where the M^i are free finitely-generated Φ-graded R-modules. If N is a Φ-graded R-module, let N_α denote the homogeneous component of N corresponding to $\alpha \in \Phi$. Thus N_α has the structure of a vector space over K. Since (9) is Φ-graded, i.e., the homomorphisms are of degree 0, the exact sequence (9) induces exact sequences

$$(10) \qquad 0 \to M^h_\alpha \to M^{h-1}_\alpha \to \cdots \to M^1_\alpha \to M_\alpha \to 0$$

for each $\alpha \in \Phi$, where the homomorphisms are linear transformations. Since M and each M^i are finitely-generated, the vector spaces appearing in (10) are finite-dimensional. Hence by a well-known property of such exact sequences [21; p. 233]

$$(11) \qquad \dim M_\alpha = \dim M^1_\alpha - \dim M^2_\alpha + \cdots + (-1)^{h-1} \dim M^h_\alpha .$$

Multiply (11) by $\mathbf{x}^\alpha$ and sum on all $\alpha \in \Phi$. Since $\dim M_\alpha = 1$ for all $\alpha \in \Phi$, we get

$$(12) \qquad F(\mathbf{x}) = F^1(\mathbf{x}) - F^2(\mathbf{x}) + \cdots + (-1)^{h-1} F^h(\mathbf{x})$$

where $F^i(\mathbf{x}) = \sum_{\alpha \in \Phi} (\dim M^i_\alpha) \mathbf{x}^\alpha$. ($F(\mathbf{x})$ may be thought of as the multivariable analogue of the *Poincaré series* of M corresponding to the function $\dim (\cdot)$.) Suppose M^i has free homogeneous generators $Y^i_1, \cdots, Y^i_k$ of degrees $\alpha^{i1}, \cdots, \alpha^{ik}$ respectively. Then a K-basis for M^i_α consists of all elements $X_1^{a_1} \cdots X_h^{a_h} Y^i_j$, where $\sum_1^h a_k \beta^k + \alpha^{ij} = \alpha$. Hence $F^i(\mathbf{x}) = \sum_{a_1=0}^\infty \cdots \sum_{a_h=0}^\infty \sum_{j=1}^k \mathbf{x}^t = (\sum_{j=1}^k \mathbf{x}^{\alpha^{ij}}) / \prod_{r=1}^h (1 - \mathbf{x}^{\beta^r})$, where $t = \sum_1^h a_r \beta^r + \alpha^{ij}$. Thus by (12) the denominator $D(\mathbf{x})$ of $F(\mathbf{x})$ divides $\prod (1 - \mathbf{x}^\beta)$, β completely fundamental.

It remains to prove that $D(\mathbf{x})$ cannot be a proper divisor of $\prod (1 - \mathbf{x}^\beta)$. If any factor $1 - \mathbf{x}^\beta$ were reducible, then $\beta = n\beta'$ for some $n > 1$, $\beta' \in \Phi$. But this contradicts the fact that β is completely fundamental (or even fundamental).

Hence we must prove that $F(\mathbf{x})$ (
γ ranges over a proper subset T (
any $n \in \mathbf{P}$ the term $\mathbf{x}^{n\beta}$ must .
Hence there is some term $\mathbf{x}^\delta$ in
such that $\delta + \sum_{\gamma \in T} a_\gamma \gamma = n\beta$, (
we have $\delta \in \Phi$. Thus by the
using the fact that β and each
$\sum a_\gamma \gamma = 0$ so $n\beta = \delta$. Hence
is nonzero for all $n \in \mathbf{P}$; so this
the proof.

One may regard the exact seq
holding among the $\mathbf{N}$-solutions t
related by $\alpha + \beta = \gamma + \delta$, the
syzygies are specified by the h
among syzygies of the first kin
specified by $M^3 \to M^2$, etc. Th
"chain of syzygies" terminates

Applications to magic labelin
labelings we need some infor
labelings, i.e., completely funda
based upon the following lemma

LEMMA 2.6. *Let X be an n >
row and column has the same s*

Lemma 2.6 is a well-known
Birkhoff [4] (see also [17; p. 56]
matrices consists of the permut

PROPOSITION 2.7. *Let G be
fundamental magic labeling of G
magic labeling of G, then $2L$ is a*

Proof. If L is any magic labe
by the vertices of G, where A_u
u and v. If L has index r, then
by Lemma 2.6, A is a sum of
symmetric, we have $A = \sum \pi$
Each $\pi + \pi^*$ is a symmetric m
L_r of index 2 as follows. For
let $e_1, e_2, \cdots, e_k$ be the edges co
$(l_1, \cdots, l_k)$, where $l_i = L(e_i)$.
$2l = \sum l^r$ of nonnegative integ
$l^r_1 + \cdots + l^r_k$. Now define th
$1 \leq i \leq k$. (In particular, if
$(\pi + \pi^*)_{uv}$.) We therefore ha

defined by (8) *is a rational ...eries), which when reduced to*

ons to (4).

$F(\mathbf{x})$ *is a rational function*

Let $\beta^1, \cdots, \beta^h$ be the com-
$_1, \cdots, X_h]$ be the Φ-graded
let M be the corresponding
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:quence

$\rightarrow M \rightarrow 0$

-modules. If N is a Φ-graded
ment of N corresponding to
space over K. Since (9) is
: 0, the exact sequence (9)

$\rightarrow M_\alpha \rightarrow 0$

near transformations. Since
spaces appearing in (10) are
rty of such exact sequences

$\cdot (-1)^{h-1} \dim M_\alpha^h$.

$\dim M_\alpha = 1$ for all $\alpha \in \Phi$,

$-1)^{h-1} F^h(\mathbf{x})$

be thought of as the multi-
rresponding to the function
ators $Y_1^i, \cdots, Y_k^i$ of degrees
M_α^i consists of all elements
e $F^i(\mathbf{x}) = \sum_{a_1=0}^\infty \cdots \sum_{a_k=0}^\infty$
$t = \sum_1^h a_r \beta^r + \alpha^{i}{}^i$. Thus
$1 - \mathbf{x}^\beta$), β completely funda-

per divisor of $\prod (1 - \mathbf{x}^\beta)$.
or some $n > 1, \beta' \in \Phi$. But
ental (or even fundamental).

Hence we must prove that $F(\mathbf{x})$ cannot have a denominator $\prod (1 - \mathbf{x}^\gamma)$, where γ ranges over a proper subset T of $\beta^1, \cdots, \beta^h$. Suppose $\beta = \beta^i \notin T$. Now for any $n \in \mathbf{P}$ the term $\mathbf{x}^{n\beta}$ must appear in the power series expansion of $F(\mathbf{x})$. Hence there is some term $\mathbf{x}^\delta$ in the numerator of $F(\mathbf{x})$ with nonzero coefficient such that $\delta + \sum_{\gamma \in T} a_\gamma \gamma = n\beta, a_\gamma \in \mathbf{N}$. Since δ is an $\mathbf{N}$-vector and $\gamma, n\beta \in \Phi$, we have $\delta \in \Phi$. Thus by the definition of completely fundamental solution, using the fact that β and each $\gamma \in T$ are completely fundamental, we have $\sum a_\gamma \gamma = 0$ so $n\beta = \delta$. Hence the coefficient of $\mathbf{x}^{n\beta}$ in the numerator of $F(\mathbf{x})$ is nonzero for all $n \in \mathbf{P}$; so this numerator is not a polynomial. This completes the proof.

One may regard the exact sequence (9) as specifying all relations (or *syzygies*) holding among the $\mathbf{N}$-solutions to (4). For instance, if $\alpha, \beta, \gamma, \delta$ are $\mathbf{N}$-solutions related by $\alpha + \beta = \gamma + \delta$, then they form a *syzygy of the first kind*. All such syzygies are specified by the homomorphism $M^2 \rightarrow M^1$. Similarly, relations among syzygies of the first kind are called *syzygies of the second kind* and are specified by $M^3 \rightarrow M^2$, etc. Thus the Hilbert syzygy theorem states that this "chain of syzygies" terminates within h steps.

Applications to magic labelings. In order to apply Theorem 2.5 to magic labelings we need some information about completely fundamental magic labelings, i.e., completely fundamental solutions to (3). This information is based upon the following lemma.

LEMMA 2.6. *Let X be an $n \times n$ matrix of nonnegative integers such that every row and column has the same sum. Then X is a sum of permutation matrices.*

Lemma 2.6 is a well-known simple consequence of the theorem of Garrett Birkhoff [4] (see also [17; p. 56]) that the convex hull of the doubly stochastic matrices consists of the permutation matrices.

PROPOSITION 2.7. *Let G be a finite pseudograph. Then every completely fundamental magic labeling of G has index 1 or 2. More precisely, if L is any magic labeling of G, then $2L$ is a sum of magic labelings of index 2.*

Proof. If L is any magic labeling of G, form the symmetric matrix A indexed by the vertices of G, where A_{uv} is the sum of the labels of all edges between u and v. If L has index r, then every row and column of A sums to r. Hence by Lemma 2.6, A is a sum of permutation matrices π, $A = \sum \pi$. Since A is symmetric, we have $A = \sum \pi^*$ (* denotes transpose) so $2A = \sum (\pi + \pi^*)$. Each $\pi + \pi^*$ is a symmetric matrix, and we associate with it a magic labeling L_π of index 2 as follows. For any pair of vertices u and v (including $u = v$) let $e_1, e_2, \cdots, e_k$ be the edges connecting u and v. Define the vector $l = l(u, v) = (l_1, \cdots, l_k)$, where $l_i = L(e_i)$. It is obvious that $2l$ can be written as a sum $2l = \sum l^\pi$ of nonnegative integer vectors $l^\pi = (l_1^\pi, \cdots, l_k^\pi)$ with $(\pi + \pi^*)_{uv} = l_1^\pi + \cdots + l_k^\pi$. Now define the magic labeling L_π by the condition $L_\pi(e_i) = l_i^\pi$, $1 \leq i \leq k$. (In particular, if only one edge e connects u to v, then $L_\pi(e) = (\pi + \pi^*)_{uv}$.) We therefore have $2L = \sum L_\pi$ as required.

A proof of Proposition 2.7 can also be based on Petersen's result [16] that a regular graph of even degree has a 2-factorization, but care must be given to the fact that we are taking loops to have degree 1 rather than degree 2 [12; p. 165, Satz 8].

The reader may be curious as to whether some analogue of Proposition 2.7 holds for fundamental magic labelings. Therefore we remark that it follows from the results of [12; Chapter XI, §3] that for $r \in \mathbf{P}$ there exists a finite pseudograph (or graph) with a fundamental magic labeling of index r if and only if $r = 2$ or r is odd.

COROLLARY 2.8. *Let G be a finite pseudograph. Then there exist polynomials $P_G(r)$ and $Q_G(r)$ such that for all but finitely many $r \in \mathbf{N}$, $H_G(r) = P_G(r) + (-1)^r Q_G(r)$.*

Proof. Let $e_1, e_2, \cdots, e_q$ be the edges of G, and define the formal power series

$$(13) \qquad F_G(x_1, x_2, \cdots, x_q, y) = \sum_L x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_q^{\alpha_q} y^r$$

where the sum ranges over all magic labelings $L(e_i) = \alpha_i$ of G and where r is the index of L. It follows that

$$(14) \qquad F_G(1, 1, \cdots, 1, y) = \sum_{r=0}^{\infty} H_G(r) y^r.$$

Now by Theorem 2.5 and Proposition 2.7, $F_G(x_1, \cdots, x_q, y)$ has a denominator of the form $\prod (1 - x_1^{\beta_1} x_2^{\beta_2} \cdots x_q^{\beta_q} y^s)$, where in each factor $s = 1$ or $s = 2$. Hence $F_G(1, 1, \cdots, 1, y)$ has a denominator of the form $(1 - y)^a (1 - y^2)^b$, $a, b \in \mathbf{N}$. By (14) this implies (e.g., by expanding $F_G(1, 1, \cdots, 1, y)$ by partial fractions) the desired result.

Observe that if for a given pseudograph G the completely fundamental magic labelings all have index one, then the proof of Corollary 2.8 would show that $F_G(1, 1, \cdots, 1, y)$ has a denominator of the form $(1 - y)^a$ so that $Q_G(r) = 0$. The next result tells us when this phenomenon occurs.

PROPOSITION 2.9. *Let G be a finite pseudograph. The following conditions are equivalent.*
  (i) *Every fundamental magic labeling of G has index one.*
  (ii) *Every completely fundamental magic labeling of G has index one.*
  (iii) *If G' is any spanning subgraph of G such that every connected component of G' is a loop, an edge, or a cycle (of length greater than or equal to 3), then every one of these cycles of length greater than or equal to 3 must have even length.*

Proof. (i) $\Rightarrow$ (ii). The proof is trivial, since a completely fundamental magic labeling is fundamental.

(ii) $\Rightarrow$ (i). Assume (ii). Let L be Proposition 2.7, $2L = \sum L'$, where assumption, each L' has index 1; so L' since for any L' appearing in the abo $e \in E$, we have that $L(e) - L'(e) \geq$ Since L is fundamental, we must hav

(ii) $\Rightarrow$ (iii). Suppose (ii) holds but of G such that every component of G greater than or equal to 3, with C a equal to 3. Define a magic labeling an edge of G', (b) $L(e) = 2$ if e is a loop if e is in a cycle of G' of length greater Since the cycle C does not possess a decompose nL into magic labelings o completely fundamental, a contradict

(iii) $\Rightarrow$ (ii). Suppose (iii) holds. Let G'' be the spanning subgraph o $L(e) = 2$, then e is either a loop or a sin components of G'' have $L(e) = 1$. Su (possibly void) with a loop at each possesses a spanning subgraph whos Hence by assumption every cycle in has even length. It is now easy to index one. Hence L is not comple (ii) follows.

COROLLARY 2.10. *If G is a finite (equivalent) conditions of Proposition 2 is 0. In particular, $Q_G(r) = 0$ if G m graph has no cycles of odd length.*

Example. Let G be any spanning [10; p. 46], where p is even. Then G of Proposition 2.9. Moreover, precis of W_p are positive (including W_p itse subgraphs (such as W_p when $p > 2$) a of Corollary 2.10 is stronger than the s G of W_p, p even, have the additior module M corresponding to the cor magic labelings is *free* on one genera

$$H_G(r) = \binom{r + m}{m}, \text{ where } m = \deg F$$

We mention that on the other extr with p odd. Any spanning subgrapl

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of G, and define the formal power

$$\_ x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_q^{\alpha_q} y^r$$

s $L(e_i) = \alpha_i$ of G and where r is

$$\sum_{r=0}^{\infty} H_G(r) y^r.$$

$c_1, \cdots, x_q, y)$ has a denominator
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(ii) ⇒ (i). Assume (ii). Let L be a fundamental magic labeling of G. By Proposition 2.7, $2L = \sum L'$, where each L' is completely fundamental. By assumption, each L' has index 1; so $L'(e) = 0$ or 1 for every edge $e \in E$. Hence since for any L' appearing in the above sum for $2L$, $2L(e) - L'(e) \geq 0$ for all $e \in E$, we have that $L(e) - L'(e) \geq 0$ also. Thus $L - L'$ is a magic labeling. Since L is fundamental, we must have $L = L'$, and so L has index one.

(ii) ⇒ (iii). Suppose (ii) holds but (iii) fails. Let G' be a spanning subgraph of G such that every component of G' is a loop, an edge, or a cycle of length greater than or equal to 3, with C a cycle in G' of odd length greater than or equal to 3. Define a magic labeling L of G as follows. (a) $L(e) = 0$ if e is not an edge of G', (b) $L(e) = 2$ if e is a loop or edge component of G', and (c) $L(e) = 1$ if e is in a cycle of G' of length greater than or equal to 3. Thus L has index two. Since the cycle C does not possess a magic labeling of index one, we cannot decompose nL into magic labelings of index one for any $n \in \mathbf{P}$. Hence L is completely fundamental, a contradiction.

(iii) ⇒ (ii). Suppose (iii) holds. Let L be a magic labeling of G of index 2. Let G'' be the spanning subgraph of G whose edges e satisfy $L(e) > 0$. If $L(e) = 2$, then e is either a loop or a single edge component of G''. The remaining components of G'' have $L(e) = 1$. Such a component must be a cycle or a path (possibly void) with a loop at each end. Any path with a loop at each end possesses a spanning subgraph whose components are loops or single edges. Hence by assumption every cycle in G'' of length greater than or equal to 3 has even length. It is now easy to decompose L into two magic labelings of index one. Hence L is not completely fundamental. By Proposition 2.7, (ii) follows.

COROLLARY 2.10. *If G is a finite pseudograph satisfying either of the three (equivalent) conditions of Proposition 2.9, then $Q_G(r)$ (as defined by Corollary 2.8) is 0. In particular, $Q_G(r) = 0$ if G minus its loops is bipartite, since a bipartite graph has no cycles of odd length.*

Example. Let G be any spanning subgraph of the *wheel* $W_p = C_{p-1} + K_1$ [10; p. 46], where p is even. Then G has no loops and satisfies the conditions of Proposition 2.9. Moreover, precisely 2^{p-1} of these spanning subgraphs G of W_p are positive (including W_p itself) and many of these positive spanning subgraphs (such as W_p when $p > 2$) are not bipartite. Hence the first sentence of Corollary 2.10 is stronger than the second sentence. The spanning subgraphs G of W_p, p even, have the additional interesting property that the graded module M corresponding to the completely fundamental (or fundamental) magic labelings is *free* on one generator [0]. It follows that $H_G(r) = \delta_{0r}$ or

$$H_G(r) = \binom{r + m}{m}, \text{ where } m = \deg P_G(r).$$

We mention that on the other extreme from W_p, p even, are the wheels W_p with p odd. Any spanning subgraph G of W_p, p odd, possesses no magic

labelings of odd index; so $P_G(r) = Q_G(r)$. This is true of any pseudograph with p odd and with no loops.

Example. This example shows that unfortunately the conditions of Proposition 2.9 are not necessary for $Q_G(r) = 0$. Let G_1 and G_2 be disjoint copies of the complete 4-graph K_4 with one edge removed. Let G consist of G_1, G_2, and an edge connecting a vertex of degree two in G_1 to a vertex of degree two in G_2. Then $H_G(r) = \binom{r+3}{3}$, but G has a completely fundamental magic labeling of index two.

Hilbert functions. Let G be a finite pseudograph and $L_1, \cdots, L_s$ a set of nonzero magic labelings satisfying condition (ii) of Proposition 2.2. Form the corresponding ring $R = K[X_1, \cdots, X_s]$ and the module M as before so M is finitely generated by Proposition 2.2. We have considered R and M to be graded by the semigroup Φ of all magic labelings, but it is possible to take less discriminating gradings. Define an $\mathbf{N}$-grading on R by letting each X_i be a homogeneous element whose degree is the index of L_i, and define an $\mathbf{N}$-grading on M by letting $[L]$ be a homogeneous element whose degree is the index of L for all $L \in \Phi$. Thus a K-basis for M_r, the r-th homogeneous component of M, consists of all symbols $[L]$ such that L is magic of index r. Hence $\dim_K M_r = H_G(r)$. Now $\dim_K M_r$ is by definition the *Hilbert function* of M; so we see that $H_G(r)$ is a Hilbert function.

If each X_i has degree one (the classical case considered by Hilbert), then Hilbert showed that $\dim_K M_r$ agrees with a polynomial for r sufficiently large, the *Hilbert polynomial* of M. This is in accordance with Corollary 2.8 and Corollary 2.10. Since we will prove that $H_G(r)$ is a polynomial for *all* $r \in \mathbf{N}$ (when each X_i has degree one), we see that $H_G(r)$ is actually the Hilbert polynomial of M. The easy generalization of Hilbert's result to the case where $\deg X_i$ is an arbitrary positive integer was first observed by Serre (cf. [3; Theorem 11.1]).

3. The Elliott–MacMahon algorithm. We now wish to prove that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for *all* $r \in \mathbf{N}$. We have shown that the generating function $F_G(1, 1, \cdots, 1, y)$ of (14), which we will abbreviate as $F_G(y) = F_G(1, 1, \cdots, 1, y)$, has the form $N(y)/(1 - y)^a(1 - y^2)^b$, where $N(y)$ is a polynomial in y and $a, b \in \mathbf{N}$. By well-known properties of rational functions, the statement that $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for *all* $r \in \mathbf{N}$ is then equivalent to $\deg F_G(y) < 0$, i.e., $\deg N(y) - a - 2b < 0$. (When $Q_G(r) = 0$ we may take $b = 0$, but this has no effect on the argument.)

We know from (2) and (3) that magic labelings of G correspond to $\mathbf{N}$-solutions of a system $P_v = 0$, $v \in V$, of homogeneous linear equations with integral coefficients. The algorithm of Elliott–MacMahon allows the explicit determination of the generating function $F(\mathbf{x})$ of (8) corresponding to a system (4). We shall complete our proofs by an analysis of this algorithm.

We proceed to describe the *Elliott–MacMahon algorithm* or the *EM-algorithm*

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for short. Elliott [6] defined the ⟂ MacMahon [14; §VIII] extended i interesting applications. In partic the Anand–Dumir–Gupta conjectu

$$H_3(r) = 3$$

Suppose we have a system (4) ⟂ coefficients. For each form $P_i = .$ and for each variable z_i associate a function"

$$(15) \qquad \mathfrak{F} = \prod_{i=1}^{} (1$$

where a_{ij} is the coefficient of z_j in of $\mathfrak{F}$ is formally expanded as a I Laurent expansions are formally terms which do not involve any $F(\mathbf{x})$ of (8). In MacMahon's no terms of $\mathfrak{F}$ free from λ_i's.

We compute $\Omega\mathfrak{F}$ as follows. Su the denominator of $\mathfrak{F}$ involve a var a positive power. Apply the ident

$$(16) \qquad \frac{1}{(1 - A)(1 - B)} = \frac{}{(1}$$

This breaks $\mathfrak{F}$ into three terms. I cedure whenever possible, i.e., wh variable λ_i with a positive expon negative exponent. Continue this is expressed in the form

$$(17) \qquad \mathfrak{F} = \sum \frac{}{(1 -}$$

where in each term $\pm 1/(1 - A)($ always a nonnegative exponent or on i).

To see that this process does ind⟂ indefinitely, define the *weight* of a occurring at any stage of the algori the exponents of all λ_i's appearing monomial A has some λ_i with a nor λ_i with a nonzero exponent of the ⟂ $(1 - B)$ is strictly greater than th

is true of any pseudograph

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and G_2 be disjoint copies of
Let G consist of G_1, G_2, and
a vertex of degree two in G_2.

fundamental magic labeling

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Proposition 2.2. Form the
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of G correspond to N-solutions
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responding to a system (4).
is algorithm.
algorithm or the *EM-algorithm*

for short. Elliott [6] defined the algorithm only for one equation $P = 0$, while MacMahon [14; §VIII] extended it to finitely many equations and gave several interesting applications. In particular, MacMahon [14; Paragraph 407] verified the Anand-Dumir-Gupta conjecture in the case $n = 3$ by showing

$$H_3(r) = 3\binom{r+3}{4} + \binom{r+2}{2}.$$

Suppose we have a system (4) of linear homogeneous equations with integer coefficients. For each form $P_i = P_i(z_1, \cdots, z_s)$ of (4) associate a variable λ_i, and for each variable z_j associate a new variable x_j. Form the "crude generating function"

(15)
$$\mathfrak{F} = \prod_{i=1}^{s} (1 - \lambda_1^{a_{1i}} \lambda_2^{a_{2i}} \cdots \lambda_p^{a_{pi}} x_i)^{-1}$$

where a_{ij} is the coefficient of z_j in P_i (so a_{ij} can be negative). If each factor of $\mathfrak{F}$ is formally expanded as a Laurent expansion in the λ_i's and if these s Laurent expansions are formally multiplied, then it is easy to see that the terms which do not involve any of the λ_i's are just the generating function $F(\mathbf{x})$ of (8). In MacMahon's notation $\Omega \mathfrak{F} = F(\mathbf{x})$, where $\Omega \mathfrak{F}$ denotes those terms of $\mathfrak{F}$ free from λ_i's.

We compute $\Omega \mathfrak{F}$ as follows. Suppose two factors $(1 - A)$ and $(1 - B)$ in the denominator of $\mathfrak{F}$ involve a variable λ_i, one to a negative power and one to a positive power. Apply the identity

(16)
$$\frac{1}{(1-A)(1-B)} = \frac{1}{(1-AB)} \left(\frac{1}{1-A} + \frac{1}{1-B} - 1 \right).$$

This breaks $\mathfrak{F}$ into three terms. For each of these terms apply the same procedure whenever possible, i.e., whenever in some term some factor contains a variable λ_i with a positive exponent and another factor contains λ_i with a negative exponent. Continue this process until no longer possible, i.e., until $\mathfrak{F}$ is expressed in the form

(17)
$$\mathfrak{F} = \sum \frac{\pm 1}{(1-A)(1-B)\cdots(1-D)}$$

where in each term $\pm 1/(1-A)(1-B) \cdots (1-D)$ each variable λ_i has always a nonnegative exponent or nonpositive exponent (the choice depending on i).

To see that this process does indeed terminate in (17), i.e., does not continue indefinitely, define the *weight* of a term $\pm 1/(1-A)(1-B) \cdots (1-D)$ occurring at any stage of the algorithm to be the sum of the absolute values of the exponents of all λ_i's appearing in this term. Now it is easily seen that if a monomial A has some λ_i with a nonzero exponent and another monomial B has λ_i with a nonzero exponent of the opposite sign, then the weight of $1/(1-A)(1-B)$ is strictly greater than the weight of $1/(1-AB)(1-A)$, $1/(1-$

$AB)(1 - B)$, and $1/(1 - AB)$. Hence after each step of the algorithm we have replaced a term with three terms of smaller weight. Clearly a term of weight 0 cannot be reduced any further—it is free from λ_i's. (Of course an irreducible term may also have positive weight.) Thus the algorithm has a tree-like structure with each node except endpoints having three successors and with no chain of length more than the weight of $\mathfrak{F}$ as expressed in (15). Such a tree must be finite; so the algorithm indeed terminates. A crude upper bound for the number of steps is $1 + 3 + 3^2 + \cdots + 3^{w-1}$, where w is the weight of $\mathfrak{F}$ as expressed in (15), i.e., w is the sum of the absolute values of the coefficients of the forms $P_1, \cdots, P_p$ of (4).

We can now read off the generating function $F(\mathbf{x})$ directly from (17). In any term $\pm 1/(1 - A)(1 - B) \cdots (1 - D)$ retain only those factors $1 - X$ which are free from the λ_i's. Thus $F(\mathbf{x})$ is expressed in the form

$$(18) \qquad F(\mathbf{x}) = \sum \frac{\pm 1}{(1 - X)(1 - Y) \cdots (1 - Z)}$$

where $X, Y, \cdots, Z$ are monomials $x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_s^{\gamma_s}$ and each $\gamma_i \in \mathbf{N}$.

The reader who may be mystified by our sketchy description will benefit considerably by examining the examples worked out by Elliott and MacMahon.

The following simple result will be useful in what follows.

LEMMA 3.1. *Suppose at any stage of the EM-algorithm applied to the system* (4) *some denominator has a factor* $1 - \lambda_1^{a_1} \cdots \lambda_p^{a_p} x_1^{\alpha_1} \cdots x_s^{\alpha_s}$. *Then for all* $i = 1, 2, \cdots, p$ *we have* $P_i(\alpha_1, \cdots, \alpha_s) = a_i$.

Proof. The proof is by induction on the steps of the EM-algorithm. At the beginning we have (15), and $P_i(0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0) = a_{ij}$ is clear. It suffices to show that if the lemma holds for the factors $1 - \lambda_1^{a_1} \cdots \lambda_p^{a_p} \mathbf{x}^\alpha$ and $1 - \lambda_1^{b_1} \cdots \lambda_p^{b_p} \mathbf{x}^\beta$, then it holds for $1 - \lambda_1^{a_1+b_1} \cdots \lambda_p^{a_p+b_p} \mathbf{x}^{\alpha+\beta}$. But this follows from the linearity property $P_i(\alpha + \beta) = P_i(\alpha) + P_i(\beta)$.

We now wish to analyze the EM-algorithm in further detail. We first require some preliminary discussion. In general, standard notation and terminology from combinatorial topology will be employed; confer, e.g., [8]. We shall also use, however, some terminology of our own, as follows.

By a *nonnegative integral polyhedral cone* or *NIP-cone*, for short, in $\mathbf{R}^s$ we mean a set $\mathcal{C}$ of nonnegative integer vectors in $\mathbf{R}^s$, including the origin, such that the convex hull $\bar{\mathcal{C}}$ of $\mathcal{C}$ is a polyhedral cone (in the usual sense) for which every integer vector in $\bar{\mathcal{C}}$ is in $\mathcal{C}$. Observe that the $\mathbf{N}$-solutions of a system (4) of linear homogeneous equations with integer coefficients form an NIP-cone in $\mathbf{R}^s$. $\mathcal{C}$ is said to be *t-dimensional* if the vector space over $\mathbf{R}$ spanned by $\bar{\mathcal{C}}$ has dimension t (equivalently, the rank of the free abelian group generated by $\mathcal{C}$ is t). A *boundary face* $\mathcal{C}'$ of $\mathcal{C}$ consists of the intersection of $\mathcal{C}$ with some boundary face $\bar{\mathcal{C}}'$ of $\bar{\mathcal{C}}$ of dimension greater than or equal to 0. Thus the origin is always a boundary face of $\mathcal{C}$, while the null set $\varnothing$ is not. If a boundary face $\bar{\mathcal{C}}'$ of $\bar{\mathcal{C}}$

is a d-dimensional cone, then $\mathcal{C}'$ of $\mathcal{C}$, denoted $\partial \mathcal{C}$, consists of the

A *lattice cone* $\mathcal{L}$ consists of all set $\alpha^1, \cdots, \alpha^t$ of linearly independent *integer vectors* for short) such that is in $\mathcal{L}$. We then write $\mathcal{L} = \langle \alpha^1,$ set $\{\alpha^1, \cdots, \alpha^t\}$ is uniquely determined vectors $\alpha^1, \cdots, \alpha^t$ to generate a there exist integer vectors $\alpha^{t+1}, \cdots$

$$(19)$$

For the absolute value of the ab fundamental parallelopiped spann on the geometry of numbers for lattice cone $\mathcal{L}$, then the boundary subsets of S.

If $\mathcal{C}$ is an NIP-cone, we define $\Delta = \{\mathcal{L}_i\}$ of lattice cones such that of $\mathcal{L}_i$ is in Δ, (b) if $\mathcal{L}_i$ and $\mathcal{L}_j$ and common boundary face $\mathcal{L}_k$ of $\mathcal{L}_i$ and hulls $\bar{\mathcal{L}}_i$ of the $\mathcal{L}_i$'s form a triangu

LEMMA 3.2. *Let* Γ *be a triangu homeomorphic to a solid* $(t - 1)$-*sp dimensional simplices in* Γ *not conta* $(-1)^{t-1} f_1^0 = 1$.

Proof. Let f_i be the total number total number of $(i - 1)$-simplices Let $\chi(\mathcal{P})$ (respectively $\chi(\partial \mathcal{P})$) deno $\partial \mathcal{P}$). Now $\mathcal{P} \neq \varnothing$ since $t \geq 1$. Th $(-1)^{t-1} f_t = 1$ and $\chi(\partial \mathcal{P}) = f_1' - f$ $f_t^0 - f_{t-1}^0 + \cdots + (-1)^{t-1} f_1^0 = ($

Remark. The assumption in L $\mathcal{P} \neq \varnothing$) is not merely a matter of Corollary 3.10 that the failure of explains the failure of Theorem 1.1

LEMMA 3.3. *Let* Γ *be a rectilin convex polytope* $\mathcal{P} \subseteq \mathbf{R}^s$. *For any* α S *in* Γ *containing* α *and* (b) *every bo*

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is a d-dimensional cone, then $\mathcal{C}'$ is a d-dimensional NIP-cone. The *boundary* of $\mathcal{C}$, denoted $\partial \mathcal{C}$, consists of the union of all boundary faces of $\mathcal{C}$.

A *lattice cone* $\mathcal{L}$ consists of all $\mathbf{N}$-combinations $a_1 \alpha^1 + \cdots + a_t \alpha^t$ of some set $\alpha^1, \cdots, \alpha^t$ of linearly independent vectors with integer coordinates (called *integer vectors* for short) such that every integer vector in the convex hull $\bar{\mathcal{L}}$ of $\mathcal{L}$ is in $\mathcal{L}$. We then write $\mathcal{L} = \langle \alpha^1, \cdots, \alpha^t \rangle$. It is easily seen that the generating set $\{\alpha^1, \cdots, \alpha^t\}$ is uniquely determined by $\mathcal{L}$. In order for a set of integer vectors $\alpha^1, \cdots, \alpha^t$ to generate a lattice cone, it is necessary and sufficient that there exist integer vectors $\alpha^{t+1}, \cdots, \alpha^s$ such that the determinant

$$(19) \qquad \begin{vmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \vdots \\ \alpha^s \end{vmatrix} = \pm 1.$$

For the absolute value of the above determinant is simply the volume of the fundamental parallelopiped spanned by the vectors $\alpha^1, \alpha^2, \cdots, \alpha^s$. (See a text on the geometry of numbers for further details.) Note that if S generates a lattice cone $\mathcal{L}$, then the boundary faces of $\mathcal{L}$ are lattice cones generated by the subsets of S.

If $\mathcal{C}$ is an NIP-cone, we define a *triangulation* of $\mathcal{C}$ to be a finite collection $\Delta = \{\mathcal{L}_i\}$ of lattice cones such that (a) if $\mathcal{L}_i \in \Delta$, then every boundary face of $\mathcal{L}_i$ is in Δ, (b) if $\mathcal{L}_i$ and $\mathcal{L}_j$ are in Δ, then their intersection $\mathcal{L}_i \cap \mathcal{L}_j$ is a common boundary face $\mathcal{L}_k$ of $\mathcal{L}_i$ and $\mathcal{L}_j$, and (c) $\cup \mathcal{L}_i = \mathcal{C}$. Hence the convex hulls $\bar{\mathcal{L}}_i$ of the $\mathcal{L}_i$'s form a triangulation of $\bar{\mathcal{C}}$ in the usual sense.

LEMMA 3.2. *Let Γ be a triangulation of a $(t - 1)$-dimensional cell $\mathcal{P}$ (so $\mathcal{P}$ is homeomorphic to a solid $(t - 1)$-sphere), $t \geq 1$. Let f_i° be the number of $(i - 1)$-dimensional simplices in Γ not contained entirely in $\partial \mathcal{P}$. Then $f_t^\circ - f_{t-1}^\circ + \cdots + (-1)^{t-1} f_1^\circ = 1$.*

Proof. Let f_i be the total number of $(i - 1)$-simplices in Γ, and let f_i' be the total number of $(i - 1)$-simplices in Γ contained in $\partial \mathcal{P}$. Thus $f_i^\circ = f_i - f_i'$. Let $\chi(\mathcal{P})$ (respectively $\chi(\partial \mathcal{P})$) denote the Euler characteristic of $\mathcal{P}$ (respectively $\partial \mathcal{P}$). Now $\mathcal{P} \neq \varnothing$ since $t \geq 1$. Thus as is well known $\chi(\mathcal{P}) = f_1 - f_2 + \cdots + (-1)^{t-1} f_t = 1$ and $\chi(\partial \mathcal{P}) = f_1' - f_2' + \cdots + (-1)^{t-1} f_t' = 1 - (-1)^{t-1}$. Hence $f_t^\circ - f_{t-1}^\circ + \cdots + (-1)^{t-1} f_1^\circ = (-1)^{t-1} [1 - (1 - (-1)^{t-1})] = 1$.

Remark. The assumption in Lemma 3.2 that $t \geq 1$ (or equivalently that $\mathcal{P} \neq \varnothing$) is not merely a matter of pedantic rigor. We shall see in the proof of Corollary 3.10 that the failure of the Euler characteristic formula for $\mathcal{P} = \varnothing$ explains the failure of Theorem 1.1 when G has no edges.

LEMMA 3.3. *Let Γ be a rectilinear triangulation of a $(t - 1)$-dimensional convex polytope $\mathcal{P} \subseteq \mathbf{R}^s$. For any $\alpha \in \mathcal{P}$ let $\Gamma(\alpha) \subseteq \Gamma$ consist of (a) every simplex S in Γ containing α and (b) every boundary face of these simplices S. Let $\mathrm{st}(\alpha) =$*

$\bigcup_{S \in \Gamma(\alpha)} S$. Then st (α) is a $(t-1)$-cell, i.e., homeomorphic to a solid $(t-1)$-sphere, and $\Gamma(\alpha)$ is a triangulation of st (α).

Sketch of proof. Since each $S \in \Gamma$ is rectilinear, it follows that st (α) is star-shaped with respect to α. Since $\mathcal{P}$ is convex, we can "expand" st (α) along the rays from α to construct a homeomorphism between st (α) and $\mathcal{P}$. Since Γ triangulates $\mathcal{P}$, $\Gamma(\alpha)$ triangulates st (α).

LEMMA 3.4. *Let Δ be a triangulation of a t-dimensional NIP-cone $\mathcal{C}$, $t \geq 1$. If $\alpha \in \mathcal{C}$, let $f_i^\circ(\alpha)$ be the number of i-dimensional lattice cones in Δ containing α and not contained entirely in $\partial \mathcal{C}$. Then*

$$(20) \qquad f_i^\circ(\alpha) - f_{i-1}^\circ(\alpha) + \cdots + (-1)^{t-1} f_1^\circ(\alpha) = 1.$$

Proof. First suppose $\alpha \neq (0, 0, \cdots, 0)$. Let $(\bar{\mathcal{C}}, \bar{\Delta})$ denote the triangulation of the convex hull $\bar{\mathcal{C}}$ of $\mathcal{C}$ obtained by taking the convex hull of each element of Δ. Let $\mathcal{P}$ be a cross section of $\bar{\mathcal{C}}$ containing α (e.g., $\mathcal{P}$ can be taken as the intersection of $\bar{\mathcal{C}}$ with the hyperplane of all β satisfying $\alpha \cdot \beta = \alpha \cdot \alpha$). Then $\mathcal{P}$ is a convex polytope, and $\bar{\Delta}$ induces a rectilinear triangulation Γ of $\mathcal{P}$. Moreover, an i-dimensional lattice cone in Δ corresponds to an $(i-1)$-simplex in Γ.

Define $\Gamma(\alpha)$ and st (α) as in Lemma 3.3. Thus by Lemma 3.3, $\Gamma(\alpha)$ triangulates the cell st (α). Now a lattice cone $\mathcal{L} \in \Delta$ corresponds to some simplex $S \in \Gamma(\alpha)$ which is not contained in ∂ st (α) if and only if $\alpha \in \mathcal{L}$ and $\mathcal{L}$ is not contained in $\partial \mathcal{C}$. Hence the number of $(i-1)$-dimensional simplices in $\Gamma(\alpha)$ which are not contained in ∂ st (α) is just $f_i^\circ(\alpha)$. Equation (20) now follows from Lemma 3.2.

If $\alpha = (0, 0, \cdots, 0)$, then every $\mathcal{L} \in \Delta$ contains α. Let $\mathcal{P}$ be any nondegenerate cross section of $\bar{\mathcal{C}}$, and let Γ be the triangulation of $\mathcal{P}$ induced by $\bar{\Delta}$. Then if f_i° is the total number of $(i-1)$-simplices in Γ which are not contained in $\partial \mathcal{P}$, we have $f_i^\circ = f_i^\circ(\alpha)$. Thus Equation (20) for $\alpha = (0, 0, \cdots, 0)$ also follows from Lemma 3.2.

LEMMA 3.5. *Let $\mathcal{L}$ be a lattice cone in $\mathbf{R}^t$ generated by the vectors $\alpha, \beta, \cdots, \delta$. Then*

$$\sum_\omega \mathbf{x}^\omega = 1/(1 - \mathbf{x}^\alpha)(1 - \mathbf{x}^\beta) \cdots (1 - \mathbf{x}^\delta)$$

where the sum is over all integer vectors ω in the convex hull of $\mathcal{L}$.

Proof. By definition of a lattice cone, the vectors ω simply range over the elements $a_\alpha \alpha + a_\beta \beta + \cdots + a_\delta \delta$ of $\mathcal{L}$, $a_\gamma \in \mathbf{N}$. But

$$\frac{1}{(1 - \mathbf{x}^\alpha)(1 - \mathbf{x}^\beta) \cdots (1 - \mathbf{x}^\delta)} = \left(\sum_{a_\alpha = 0}^\infty \mathbf{x}^{a_\alpha \alpha} \right) \left(\sum_{a_\beta = 0}^\infty \mathbf{x}^{a_\beta \beta} \right) \cdots \left(\sum_{a_\delta = 0}^\infty \mathbf{x}^{a_\delta \delta} \right)$$

$$= \sum_a \mathbf{x}^{a_\alpha \alpha + a_\beta \beta + \cdots + a_\delta \delta} = \sum_{\omega \in \mathcal{L}} \mathbf{x}^\omega.$$

LEMMA 3.6. *Let Δ be a triangulation of a t-dimensional NIP-cone $\mathcal{C}$, $t \geq 1$. Define $F(\mathbf{x}) = \sum_\omega \mathbf{x}^\omega$, where ω ranges over all elements of $\mathcal{C}$. Then*

$$(21) \qquad F(\mathbf{x}) = \sum_{i=1}^t (-1)^t$$

where for a given value of i the inner $\mathcal{L} = \langle \alpha, \cdots, \delta \rangle$ in Δ which are n

Proof. If $\omega \notin \mathcal{C}$, then clearly of (21) is 0, since the lattice cone

Now suppose that $\omega \in \mathcal{C}$. Sup $\mathcal{L}$ in Δ which are not contained in coefficient of $\mathbf{x}^\omega$ in the right-han $(-1)^{t-1} f_1^\circ(\omega)$. Hence by Lemma

We are now ready to reconsider are of course not uniquely detern in any term any two factors $1 -$ exponent of some λ_i is positive in particular, the algorithm may be rule.

(R) Choose any appropriate pa (16) simultaneously to *every* terr denominator.

Use of the rule (R) leads to a algorithm, which we now explain. $(1 - D)$ is a term appearing at a each $\lambda_i = 1$ in the term T, we get $\mathbf{x}^i$). If the vectors $\alpha, \beta, \cdots, \delta$ h $\mathcal{L}$ is the *lattice cone corresponding*

LEMMA 3.7. *If T is any term then T corresponds to some lattice*

Proof. The proof is by induct beginning of the algorithm we ha to the lattice cone $\mathcal{C}_0$ of all nonne "unit coordinate vectors"). Sup

$$T = \pm 1/(1 - A$$

has been obtained corresponding reduction (16) is applied to T, sa three new terms, viz.,

$$T_1 = \pm 1/(1 - A$$

$$T_2 = \pm 1/(1 - A$$

$$T_3 = \mp 1/(1 - A$$

homeomorphic to a solid $(t - 1)$-

ilinear, it follows that st (α) is
ex, we can "expand" st (α) along
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dimensional NIP-cone $\mathcal{C}$, $t \geq 1$.
al lattice cones in Δ *containing* α

$)^{t-1} f_1^\circ(\alpha) = 1.$

$(\bar{\mathcal{C}}, \bar{\Delta})$ denote the triangulation
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ation of $\mathcal{P}$ induced by $\bar{\Delta}$. Then
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ctors ω simply range over the
But

$\Big) \Big(\sum_{a_\beta = 0}^{\infty} \mathbf{x}^{a_\beta \beta} \Big) \cdots \Big(\sum_{a_\delta = 0}^{\infty} \mathbf{x}^{a_\delta \delta} \Big)$

$_\beta \beta + \cdots + a_\delta \delta = \sum_{\omega \in \mathcal{L}} \mathbf{x}^\omega.$

mensional NIP-cone $\mathcal{C}$, $t \geq 1$.
nents of $\mathcal{C}$. *Then*

$$(21) \qquad F(\mathbf{x}) = \sum_{i=1}^{t} (-1)^{t-i} \left[\sum \frac{1}{(1 - \mathbf{x}^\alpha) \cdots (1 - \mathbf{x}^\delta)} \right]$$

where for a given value of i *the inner sum ranges over all* i-*dimensional lattice cones* $\mathcal{L} = \langle \alpha, \cdots, \delta \rangle$ *in* Δ *which are not contained in* $\partial \mathcal{C}$.

Proof. If $\omega \notin \mathcal{C}$, then clearly the coefficient of $\mathbf{x}^\omega$ in the right-hand side of (21) is 0, since the lattice cones $\langle \alpha, \cdots, \delta \rangle$ lie in $\mathcal{C}$.

Now suppose that $\omega \in \mathcal{C}$. Suppose ω lies in $f_i^\circ(\omega)$ i-dimensional lattice cones $\mathcal{L}$ in Δ which are not contained in $\partial \mathcal{C}$. Thus using Lemma 3.5, we see that the coefficient of $\mathbf{x}^\omega$ in the right-hand side of (21) is $f_t^\circ(\omega) - f_{t-1}^\circ(\omega) + \cdots + (-1)^{t-1} f_1^\circ(\omega)$. Hence by Lemma 3.4, the coefficient of $\mathbf{x}^\omega$ is 1.

We are now ready to reconsider the EM-algorithm. The steps of the algorithm are of course not uniquely determined, since at any stage one is free to choose in any term any two factors $1 - A$ and $1 - B$ in applying (16) so long as the exponent of some λ_i is positive in either A or B and negative in the other. In particular, the algorithm may be performed in accordance with the following rule.

(R) Choose any appropriate pair $1 - A$ and $1 - B$ and apply the reduction (16) simultaneously to *every* term for which $(1 - A)(1 - B)$ appears in the denominator.

Use of the rule (R) leads to a simple geometric interpretation of the EM-algorithm, which we now explain. Suppose that $T = \pm 1/(1 - A)(1 - B) \cdots (1 - D)$ is a term appearing at any stage of the EM-algorithm. When we set each $\lambda_i = 1$ in the term T, we get an expression $\pm 1/(1 - \mathbf{x}^\alpha)(1 - \mathbf{x}^\beta) \cdots (1 - \mathbf{x}^\delta)$. If the vectors $\alpha, \beta, \cdots, \delta$ happen to generate a lattice cone $\mathcal{L}$, we say that $\mathcal{L}$ is the *lattice cone corresponding to* T.

LEMMA 3.7. *If* T *is any term appearing at any stage of the* EM-*algorithm, then* T *corresponds to some lattice cone* $\mathcal{L}$.

Proof. The proof is by induction on the steps of the EM-algorithm. At the beginning of the algorithm we have a single term (15), which clearly corresponds to the lattice cone $\mathcal{C}_0$ of all nonnegative integer vectors in $\mathbf{R}^s$ (generated by the s "unit coordinate vectors"). Suppose now that the term

$$T = \pm 1/(1 - A)(1 - B)(1 - C) \cdots (1 - D)$$

has been obtained corresponding to the lattice cone $\langle \alpha, \beta, \gamma, \cdots, \delta \rangle$. If the reduction (16) is applied to T, say to the factors $1 - A$ and $1 - B$, we obtain three new terms, viz.,

$$T_1 = \pm 1/(1 - A)(1 - AB)(1 - C) \cdots (1 - D)$$

$$T_2 = \pm 1/(1 - AB)(1 - B)(1 - C) \cdots (1 - D)$$

$$T_3 = \mp 1/(1 - AB)(1 - C) \cdots (1 - D).$$

When we set each $\lambda_i = 1$ in these three terms, we obtain

$$\pm 1/(1 - \mathbf{x}^\alpha)(1 - \mathbf{x}^{\alpha+\beta})(1 - \mathbf{x}^\gamma) \cdots (1 - \mathbf{x}^\delta)$$

$$\pm 1/(1 - \mathbf{x}^{\alpha+\beta})(1 - \mathbf{x}^\beta)(1 - \mathbf{x}^\gamma) \cdots (1 - \mathbf{x}^\delta)$$

$$\mp 1/(1 - \mathbf{x}^{\alpha+\beta})(1 - \mathbf{x}^\gamma) \cdots (1 - \mathbf{x}^\delta).$$

Thus we need to show that if $\{\alpha, \beta, \gamma, \cdots, \delta\}$ generates a lattice cone, then so do $\{\alpha, \alpha + \beta, \gamma, \cdots, \delta\}$, $\{\alpha + \beta, \beta, \gamma, \cdots, \delta\}$, and $\{\alpha + \beta, \gamma, \cdots, \delta\}$. But this is an elementary fact which can easily be proved directly or from (19) and the determinantal identity

$$\begin{vmatrix} \alpha \\ \beta \\ \cdot \\ \cdot \\ \zeta \end{vmatrix} = \begin{vmatrix} \alpha + \beta \\ \beta \\ \cdot \\ \cdot \\ \zeta \end{vmatrix}.$$

Hence the proof follows by induction.

LEMMA 3.8. *Suppose at some stage of the EM-algorithm, performed in accordance with the rule* (R), *we have*

(22) $\mathfrak{F} = \sum \pm 1/(1 - A)(1 - B) \cdots (1 - D).$

Let $\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_k$ *be the s-dimensional lattice cones corresponding to the terms in* (22) *with s factors (with s as in* (15)). *Then* $\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_k$ *together with all their boundary faces form a triangulation* Δ *of the NIP-cone* $\mathcal{C}_0$ *of all nonnegative integer vectors in* $\mathbf{R}^s$. *Moreover, any term* $\pm 1/(1 - A)(1 - B) \cdots (1 - D)$ *of* (22) *corresponds to some lattice cone* $\mathcal{L}$ *in* Δ. (*Conversely, an i-dimensional lattice cone* $\mathcal{L}$ *in* Δ *corresponds to some term* $T = \pm 1/(1 - A)(1 - B) \cdots (1 - D)$ *of* (22) *if and only if* $\mathcal{L}$ *is not contained in* $\partial \mathcal{C}_0$, *in which case the correct sign of T is* $(-1)^{s-i}$.)

Proof. As in the previous lemma, the proof is by induction on the steps of the EM-algorithm. At the beginning of the algorithm the lemma is clear, since here Δ consists of the single s-dimensional lattice cone $\mathcal{C}_0$ together with its boundary faces. Suppose now that we have reached the expression (22) and that the lemma holds in this case. Suppose that the rule (R) is now applied to factors $1 - A_1$ and $1 - A_2$, where when we set each $\lambda_i = 1$, A_1 becomes $\mathbf{x}^\alpha$ and A_2 becomes $\mathbf{x}^\beta$. Note that the following holds by Lemma 3.1.

(*) If after setting each $\lambda_i = 1$ some monomial B becomes $\mathbf{x}^\alpha$ (respectively some monomial C becomes $\mathbf{x}^\beta$), then $B = A_1$ (respectively $C = A_2$).

Let $\mathcal{L}_1', \mathcal{L}_2', \cdots, \mathcal{L}_j'$ be the s-dimensional lattice cones corresponding to the s-factor terms in the new expression for $\mathfrak{F}$. It is clear that a lattice cone $\mathcal{L}$ is one of the $\mathcal{L}_i'$ if and only if either (a) $\mathcal{L}$ is some $\mathcal{L}_i$ not containing both α and β in its set of generators or (b) for some $\mathcal{L}_i = \langle \alpha, \beta, \gamma, \cdots, \delta \rangle$, $\mathcal{L}$ is either $\langle \alpha, \alpha + \beta, \gamma, \cdots, \delta \rangle$ or $\langle \alpha + \beta, \beta, \gamma, \cdots, \delta \rangle$ (since by (*) such an $\mathcal{L}_i$ can always be reduced

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via (16)). Hence the $\mathcal{L}_i'$ are simp division Δ' of Δ, viz., the subdivisi the unique subdivision of Δ with th cones in Δ' consist of the one-dir $\langle \alpha + \beta \rangle$.) Thus the $\mathcal{L}_i'$ together wit tion Δ' of $\mathcal{C}_0$.

Similarly, it is easily seen by indu for $\mathfrak{F}$ corresponds to some lattice co

(The converse statement in par Although we have no need of this f the amount of work which needs to in specific cases.)

We are now ready for our mair

THEOREM 3.9. *Let* $P_1, P_2, \cdots$ *with integer coefficients in the variable in* $\mathbf{R}^s$ *of all* $\mathbf{N}$-*solutions to* $P_1 = P_2 = \Lambda$ *into lattice cones.*

Moreover, define $F(\mathbf{x}) = \sum \mathbf{x}^\omega$, *w*

(23) $F(\mathbf{x}) = \sum_{i=1}^{t} (-1)^{t-i} \Big[$

where for each i the inner sum range $\langle \alpha, \cdots, \delta \rangle$ *in* Λ *which are not containe*

Proof. When the crude generatir using the rule (R), we obtain $F(\mathbf{x})$ $\pm 1/(1 - A)(1 - B) \cdots (1 - D)$ factors correspond to a boundary face where Δ is the triangulation of $\mathcal{C}_0$ algorithm via Lemma 3.8.

Now since Δ is a triangulation of $\mathcal{C}$ above together with their boundary f union $\mathfrak{D}$. Moreover, if α is a generato from a factor $1 - \mathbf{x}^\alpha$ free from λ_i's in of $\mathfrak{F}$. Then by Lemma 3.1, $P_i(\alpha) = 0$ to the system (4). Since $\mathbf{N}$-combinat certainly a subset of all $\mathbf{N}$-solutions. we obtain $F(\mathbf{x}) = \sum_{\alpha \in \mathfrak{e}} \mathbf{x}^\alpha$ at the fini nonzero coefficients of the terms $\mathbf{x}^\alpha$ wi lates $\mathfrak{e}$ into lattice cones.

Equation (23) is simply an instance plete.

We at last return to magic labelings

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h an $\mathcal{L}_i$ can always be reduced

(16)). Hence the $\mathcal{L}'_i$ are simply the s-dimensional lattice cones of a sub-division Δ' of Δ, viz., the subdivision "induced" by the vector $\alpha + \beta$. (Δ' is the unique subdivision of Δ with the property that the one-dimensional lattice cones in Δ' consist of the one-dimensional lattice cones in Δ together with $\langle \alpha + \beta \rangle$.) Thus the $\mathcal{L}'_i$ together with their boundary faces form a new triangulation Δ' of $\mathcal{C}_0$.

Similarly, it is easily seen by induction that every term in the new expression for $\mathcal{F}$ corresponds to some lattice cone in Δ'.

(The converse statement in parentheses follows easily from Lemma 3.6. Although we have no need of this fact, nevertheless it is of use in cutting down the amount of work which needs to be done in carrying out the EM-algorithm in specific cases.)

We are now ready for our main result connected with the EM-algorithm.

THEOREM 3.9. *Let* $P_1, P_2, \cdots, P_p$ *be a system of homogeneous linear forms with integer coefficients in the variables* $z_1, z_2, \cdots, z_s$. *Let* $\mathcal{C}$ *be the NIP-cone in* $\mathbf{R}^s$ *of all* **N**-*solutions to* $P_1 = P_2 = \cdots = P_p = 0$. *Then* $\mathcal{C}$ *has a triangulation* Λ *into lattice cones.*

Moreover, define $F(\mathbf{x}) = \sum \mathbf{x}^\omega$, *where* ω *ranges over all elements of* $\mathcal{C}$. *Then*

$$(23) \qquad F(\mathbf{x}) = \sum_{i=1}^{t} (-1)^{t-i} \left[\sum \frac{1}{(1 - \mathbf{x}^\alpha) \cdots (1 - \mathbf{x}^\delta)} \right]$$

where for each i *the inner sum ranges over all* i-*dimensional lattice cones* $\mathcal{L} = \langle \alpha, \cdots, \delta \rangle$ *in* Λ *which are not contained in* $\partial \mathcal{C}$ *and where* t *is the dimension of* $\mathcal{C}$.

Proof. When the crude generating function $\mathcal{F}$ of (15) is reduced to (17) using the rule (R), we obtain $F(\mathbf{x})$ by picking out the factors of each term $\pm 1/(1 - A)(1 - B) \cdots (1 - D)$ free from the λ_i's. By Lemma 3.8 these factors correspond to a boundary face of some $\mathcal{L} \in \Delta$ and hence to some $\mathcal{L}' \in \Delta$, where Δ is the triangulation of $\mathcal{C}_0$ obtained at the termination of the EM-algorithm via Lemma 3.8.

Now since Δ is a triangulation of $\mathcal{C}_0$, the lattice cones $\mathcal{L}' \in \Delta$ obtained as above together with their boundary faces must form a triangulation Λ of their union $\mathfrak{D}$. Moreover, if α is a generator of one of these $\mathcal{L}'$, then it was obtained from a factor $1 - \mathbf{x}^\alpha$ free from λ_i's in the denominator of the reduced form (17) of $\mathcal{F}$. Then by Lemma 3.1, $P_i(\alpha) = 0$ for $i = 1, 2, \cdots, p$, i.e., α is an **N**-solution to the system (4). Since **N**-combinations of **N**-solutions are **N**-solutions, $\mathfrak{D}$ is certainly a subset of all **N**-solutions. But by construction of the EM-algorithm we obtain $F(\mathbf{x}) = \sum_{\alpha \in \mathcal{C}} \mathbf{x}^\alpha$ at the finish; on the other hand, we can only obtain nonzero coefficients of the terms $\mathbf{x}^\alpha$ with $\alpha \in \mathfrak{D}$. Hence $\mathcal{C} = \mathfrak{D}$; so Λ triangulates $\mathcal{C}$ into lattice cones.

Equation (23) is simply an instance of Lemma 3.6, and so the proof is complete.

We at last return to magic labelings of pseudographs.

COROLLARY 3.10. *Let G be a finite positive pseudograph with at least one edge. If $P_G(r)$ and $Q_G(r)$ are as in Corollary 2.8, then $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for all $r \in \mathbf{N}$.*

Proof. Let $e_1, \cdots, e_q$ be the edges of G, and let $F_G(x_1, \cdots, x_q, y)$ be the generating function given by (13). Theorem 3.9 now allows $F_G(x_1, \cdots, x_q, y)$ to be expressed in the form (23), provided the NIP-cone of magic labelings of G does not consist only of the zero labeling (accounting for our hypothesis that G is positive with at least one edge). (This explains the anomaly that if the only magic labeling of G is the zero labeling, then $H_G(r) = \delta_{0r}$, which does not satisfy Theorem 1.1.) But $F_G(y) = F_G(1, \cdots, 1, y)$; so from (23) we obtain an expression for $F_G(y)$ of the form

$$(24) \qquad F_G(y) = \sum_{i=1}^{t} (-1)^{t-i} \left[\sum \prod_1^i (1 - y^c)^{-1} \right].$$

Now in (24) each $c > 0$ since each vector $\alpha, \cdots, \delta$ appearing in (23) is a nonzero solution to the system $P_1 = P_2 = \cdots = P_p = 0$, and every nonzero magic labeling of G has index strictly greater than 0. Thus (24) expresses $F_G(y)$ as a linear combination of rational functions of degree strictly less than 0; so $F_G(y)$ has degree strictly less than 0. As remarked at the beginning of this section, this gives the desired result. (Note that from (18) alone we can conclude $\deg F_G(y) \leq 0$, i.e., $H_G(r) = P_G(r) + (-1)^r Q_G(r)$ for all $r > 0$. However, it does not seem *a priori* evident that the ± 1 terms in (18), corresponding to void products $(1 - X)(1 - Y) \cdots (1 - Z)$, cancel out.)

We have therefore completed the proofs of the statements in Theorems 1.1 and 1.2 about $H_G(r)$. The statements about $\bar{H}_G(r)$ then follow trivially from Theorem 1.3 and the formulas $H_G(0) = 1$, $\bar{H}_G(0) = 0$; so it remains only to prove Theorem 1.3.

Conjecture. If Λ is a triangulation into lattice cones of the NIP-cone $\mathcal{C}$ of all $\mathbf{N}$-solutions to the system (4), it is easy to see that every fundamental solution α to (4) is a generator of some $\mathcal{L} \in \Lambda$ (otherwise $\mathbf{x}^\alpha$ would not appear in the expansion of the right-hand side of (23)). We conjecture that Λ can be chosen so that *every* generator of every $\mathcal{L} \in \Lambda$ is a fundamental solution to (4). The reader can check that although this conjecture is valid for the equation $2x_1 + 3x_2 = 6x_3$, no such triangulation can be obtained via the EM-algorithm. Hence it appears that a proof of this conjecture (if true) would require new techniques.

One consequence of this conjecture is the following. Suppose the conjecture is true for the magic labelings of the complete bipartite graph K_{nn} (the Anand–Dumir–Gupta case). (One can show that the conjecture is indeed true here when $n \leq 3$.) Let Λ be a triangulation into lattice cones of the NIP-cone $\mathcal{C}$ of all magic labelings of K_{nn} such that every generator of every lattice cone in Λ has index one. Let f_i be the number of i-faces of the triangulated convex polytope $(\mathcal{P}, \Gamma)$ given by a nondegenerate cross section of $(\bar{\mathcal{C}}, \bar{\Lambda})$. Then $H_n(r+1)$

$$= \sum_{i=0}^{(n-1)^2} f_i \binom{r}{i} \text{ or, equivalently, } \Delta^i H_n(1) = f_i. \text{ Thus the numbers } \Delta^i H_n(1)$$

(assuming the validity of the co
Moreover, the equalities $H_n(-1) =$
$H_n(r) = (-1)^{n-1} H_n(-n - r)$ imp
the f_i's [8; Chapter 9]. For referen

$$H_3(r + 1) = 6 + 15($$

and

$$H_4(r + 1) = 24 + 258\binom{r}{1} + 1468($$

$$+$$

4. A reciprocity theorem.

Usir "reciprocity theorem" concerning i this reciprocity theorem, Theorem

THEOREM 4.1. *Let $P_1, \cdots, P_p$ integer coefficients in the variables z_1 $\cdots = P_p = 0$ have at least one so functions $F(\mathbf{x}) = F(x_1, \cdots, x_s)$ ar and $\bar{F}(\mathbf{x}) = \sum_{\bar{\omega}} \mathbf{x}^{\bar{\omega}}$, where ω rang in nonnegative integers while $\bar{\omega}$ rang $F(\mathbf{x})$ and $\bar{F}(\mathbf{x})$ are rational functio $(-1)^t F(1/x_1, \cdots, 1/x_s)$, where t i. to $P_1 = \cdots = P_p = 0$ in nonnega*

Proof. The assumption that the is equivalent to the statement that points (nonboundary points) of $\mathcal{C}$. cones as guaranteed by Theorem 3. point of a *unique* $\mathcal{L} \in \Lambda$ not con contained in $\partial \mathcal{C}$, then an interior 1

$$(25) \qquad\qquad \bar{F}(\mathbf{x})$$

where $\mathcal{L}$ ranges over all lattice cone all points in the interior of $\mathcal{L}$. If $\mathcal{L}$ sum in (25) is given by

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so

$$(26) \qquad\qquad \bar{F}(\mathbf{x}) = \sum_{i=1}^{t} \left[\sum \right.$$

ograph with at least one edge.

$(r) = P_G(r) + (-1)^r Q_G(r)$

$F_G(x_1, \cdots, x_q, y)$ be the

allows $F_G(x_1, \cdots, x_q, y)$

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$r) = \delta_{0r}$, which does not

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hus the numbers $\Delta^i H_n(1)$

(assuming the validity of the conjecture) have a geometric interpretation. Moreover, the equalities $H_n(-1) = H_n(-2) = \cdots = H_n(-n + 1) = 0$ and $H_n(r) = (-1)^{n-1} H_n(-n - r)$ impose a kind of Dehn–Sommerville relation on the f_i's [8; Chapter 9]. For reference we record

$$H_3(r + 1) = 6 + 15\binom{r}{1} + 19\binom{r}{2} + 12\binom{r}{3} + 3\binom{r}{4}$$

and

$$H_4(r + 1) = 24 + 258\binom{r}{1} + 1468\binom{r}{2} + 4945\binom{r}{3} + 10532\binom{r}{4} + 14620\binom{r}{5}$$
$$+ 13232\binom{r}{6} + 7544\binom{r}{7} + 2464\binom{r}{8} + 352\binom{r}{9}.$$

4. A reciprocity theorem. Using Theorem 3.9, we will prove a general "reciprocity theorem" concerning integer solutions to linear equations. From this reciprocity theorem, Theorem 1.3 will follow easily.

THEOREM 4.1. *Let $P_1, \cdots, P_p$ be a system of homogeneous linear forms with integer coefficients in the variables $z_1, \cdots, z_s$. Suppose that the equations $P_1 = \cdots = P_p = 0$ have at least one solution in positive integers. Define generating functions $F(\mathbf{x}) = F(x_1, \cdots, x_s)$ and $\bar{F}(\mathbf{x}) = \bar{F}(x_1, \cdots, x_s)$ by $F(\mathbf{x}) = \sum_\omega \mathbf{x}^\omega$ and $\bar{F}(\mathbf{x}) = \sum_{\bar{\omega}} \mathbf{x}^{\bar{\omega}}$, where ω ranges over all solutions to $P_1 = \cdots = P_p = 0$ in nonnegative integers while $\bar{\omega}$ ranges over all solutions in positive integers. Then $F(\mathbf{x})$ and $\bar{F}(\mathbf{x})$ are rational functions of the x_i's related by $\bar{F}(x_1, \cdots, x_s) = (-1)^t F(1/x_1, \cdots, 1/x_s)$, where t is the dimension of the NIP-cone $\mathcal{C}$ of solutions to $P_1 = \cdots = P_p = 0$ in nonnegative integers.*

Proof. The assumption that there is a positive solution to $P_1 = \cdots = P_p = 0$ is equivalent to the statement that the positive solutions consist of the interior points (nonboundary points) of $\mathcal{C}$. Let Λ be a triangulation of $\mathcal{C}$ into lattice cones as guaranteed by Theorem 3.9. Then an interior point of $\mathcal{C}$ is an interior point of a *unique* $\mathcal{L} \in \Lambda$ not contained in $\partial\mathcal{C}$. Conversely, if $\mathcal{L} \in \Lambda$ is not contained in $\partial\mathcal{C}$, then an interior point of $\mathcal{L}$ is an interior point of $\mathcal{C}$. Hence

$$(25) \qquad \bar{F}(\mathbf{x}) = \sum_{\mathcal{L}} \sum_{\zeta} \mathbf{x}^\zeta$$

where $\mathcal{L}$ ranges over all lattice cones in Λ not contained in $\partial\mathcal{C}$ and ζ ranges over all points in the interior of $\mathcal{L}$. If $\mathcal{L} = \langle \alpha, \cdots, \delta \rangle$, then it is clear that the inner sum in (25) is given by

$$\frac{\mathbf{x}^{\alpha + \cdots + \delta}}{(1 - \mathbf{x}^\alpha) \cdots (1 - \mathbf{x}^\delta)}$$

so

$$(26) \qquad \bar{F}(\mathbf{x}) = \sum_{i=1}^{t} \left[\sum \frac{\mathbf{x}^{\alpha + \cdots + \delta}}{(1 - \mathbf{x}^\alpha) \cdots (1 - \mathbf{x}^\delta)} \right]$$

where now for each i the inner sum ranges over all i-dimensional lattice cones $\langle \alpha, \cdots, \delta \rangle$ in Λ which are not contained in $\partial\mathbb{C}$. Comparing (26) with (23), we obtain the desired result.

The proof of Theorem 1.3 now depends on the following simple lemma, whose straightforward proof using partial fractions we omit.

LEMMA 4.2. *Let* $\{H(i)\}$, $i \in \mathbf{Z}$, *be a doubly-infinite sequence of complex numbers satisfying for all* $N \in \mathbf{Z}$ *the recurrence*

$$(27) \qquad H(N + n) + \alpha_{n-1}H(N + n - 1) + \cdots + \alpha_0 H(N) = 0$$

where n *is a fixed nonnegative integer and* α_0, α_1, $\cdots$, α_{n-1} *are fixed complex numbers. Define* $F(y) = \sum_{r=0}^{\infty} H(r)y^r$ *and* $\bar{F}(y) = \sum_{r=1}^{\infty} H(-r)y^r$. *Then* $F(y)$ *and* $\bar{F}(y)$ *are rational functions of* y *related by* $\bar{F}(y) = -F(1/y)$.

Proof of Theorem 1.3. Suppose P and Q are polynomials and $H(r) = P(r) + (-1)^r Q(r)$. Let $F(y) = \sum_0^{\infty} H(r)y^r$. Now it is a basic fact from the theory of linear difference equations with constant coefficients that a sequence $H'(r)$, $r \in \mathbf{Z}$, satisfies a recursion of the form (27) if and only if there are complex numbers β_1, $\cdots$, β_t and polynomials $P_1(r)$, $\cdots$, $P_t(r)$ such that $H'(r) = \sum_1^t P_i(r)\beta_i^r$ for all $r \in \mathbf{Z}$. In particular, $H(r)$ satisfies a recursion of the form (27), with $\beta_1 = 1$, $\beta_2 = -1$, $P_1(r) = P(r)$, and $P_2(r) = Q(r)$. Thus by Lemma 4.2,

$$(28) \qquad \sum_1^{\infty} H(-r)y^r = -F(1/y).$$

The assumption that G is a *positive* pseudograph implies the existence of a positive integer solution to (3); so Theorem 4.1 applies to the generating function $F_G(x_1, \cdots, x_q, y)$ of (13). Thus with $\bar{F}_G(x_1, \cdots, x_q, y)$ as in Theorem 4.1,

$$(29) \qquad \bar{F}_G(x_1, \cdots, x_q, y) = (-1)^t F\left(\frac{1}{x_1}, \cdots, \frac{1}{x_q}, \frac{1}{y}\right).$$

If we let $F_G(y) = \sum_0^{\infty} H_G(r)y^r$ and $\bar{F}_G(y) = \sum_1^{\infty} \bar{H}_G(r)y^r$, then $F_G(y) = F_G(1, \cdots, 1, y)$ and $\bar{F}_G(y) = \bar{F}_G(1, \cdots, 1, y)$. Hence by (29), $\bar{F}_G(y) = (-1)^t F_G(1/y)$. Comparing with (28) gives

$$(30) \qquad \bar{H}_G(r) = (-1)^{t-1}H(-r), \qquad r > 0.$$

Theorem 1.1 now implies that (30) also holds for $r < 0$. Finally, it is easily seen that $\deg P_G(r) = t - 1$ (see the proof of Proposition 5.2); so the proof is complete.

A remark on magic hypercubes. Let $H_{dn}(r)$ denote the number of $n \times n \times \cdots \times n$ (d times) d-dimensional hypercubes of nonnegative integers summing to r on all dn^{d-1} "lines". If follows from Theorem 3.9 that we can write

$$(31) \qquad H_{dn}(r) = \sum_{i=1}^{p} P_i(r)\zeta^{ir},$$

where ζ is a p-th root of unity for is a polynomial in r (depending on It now follows from Theorem 4.1 ar 1.4 follows from Theorem 1.3, tha $H_{dn}(-n - r) = (-1)^{n-1}H_{dn}(r)$. Gupta and Nath [9], who compu possible value of p in (31) (as a though of course by Theorem 1.2

5. Miscellaneous. We have ;
1.1–1.3. If, however, one wishes t graph G, it is convenient to be ab one needs only compute $H_G(r)$ f Theorems 1.2 and 1.3 and Corolla of $H_G(r)$ that need to be compu provided later. We begin with ; equations.

PROPOSITION 5.1. *Let* (4) *be integer coefficients possessing a sol of the* NIP-*cone* $\mathbb{C}$ *of all* **N**-*solution i.e.,* $t = s - \rho$, *where* s *is the nur*

Proof. By elementary linear tions to (4) has dimension κ (with **N**-solution is a rational solution; and let β be any rational solutic vector. For sufficiently large n $(1/m)(n\alpha - \gamma)$ so $t \geq \kappa$, and th

The hypothesis in Proposition For instance, the equation $x_1 +$

Remark. Let M be the $R =$ α^1, $\cdots$, α^g of nonzero **N**-solutic finitely-generated). Then it is the Krull dimension of the rin; which annihilates M, is equal solutions to (4). In particular, we have $\dim (M) = \kappa$.

If a pseudograph G is not cc $H_G(r) = H_K(r)H_L(r)$. Hence the next proposition.

PROPOSITION 5.2. *Let* G *b p vertices and* $q > 1$ *edges. The*

$$\deg P_G(r) = \begin{cases} \vdots \\ \vdots \end{cases}$$

r all i-dimensional lattice cones

c. Comparing (26) with (23),

following simple lemma, whose
omit.

ly-infinite sequence of complex

$$\cdots + \alpha_0 H(N) = 0$$

$_1, \cdots, \alpha_{n-1}$ *are fixed complex*
$= \sum_{r=1}^{\infty} H(-r)y^r$. *Then* $F(y)$
$) = -F(1/y)$.

olynomials and $H(r) = P(r) +$
a basic fact from the theory of
icients that a sequence $H'(r)$,
and only if there are complex
$\cdots, P_i(r)$ such that $H'(r) =$
atisfies a recursion of the form
$^2{}_2(r) = Q(r)$. Thus by Lemma

$/y)$.

aph implies the existence of a
plies to the generating function
$\cdots, x_q, y)$ as in Theorem 4.1,

$$, \cdots, \frac{1}{x_q}, \frac{1}{y}\right).$$

$\sum_1^{\infty} \bar{H}_G(r)y^r$, then $F_G(y) =$
$y)$. Hence by (29), $\bar{F}_G(y) =$

$-r)$, $\qquad\qquad r > 0$.

or $r < 0$. Finally, it is easily
Proposition 5.2); so the proof is

note the number of $n \times n \times$
nonnegative integers summing
m 3.9 that we can write

r,
,

where ζ is a p-th root of unity for some positive integer p and where each $P_i(r)$ is a polynomial in r (depending on d and n). Hence $H_{dn}(-r)$ can be defined. It now follows from Theorem 4.1 and Lemma 4.2, in the same way that Corollary 1.4 follows from Theorem 1.3, that $H_{dn}(-r) = 0$, $r = 1, 2, \cdots, n - 1$, and $H_{dn}(-n - r) = (-1)^{n-1}H_{dn}(r)$. When $d = 3$ this establishes a conjecture of Gupta and Nath [9], who compute $H_{33}(r)$. The problem of finding the least possible value of p in (31) (as a function of d and n) appears very difficult, though of course by Theorem 1.2 we know $p = 1$ when $d = 2$.

5. Miscellaneous. We have succeeded in our aim of proving Theorems 1.1–1.3. If, however, one wishes to compute $P_G(r)$ and $Q_G(r)$ for a given pseudograph G, it is convenient to be able to determine $\deg P_G$ and $\deg Q_G$, since then one needs only compute $H_G(r)$ for sufficiently many values of r. Note that Theorems 1.2 and 1.3 and Corollary 1.4 frequently reduce the number of values of $H_G(r)$ that need to be computed. An example of such a computation is provided later. We begin with a simple result concerning linear homogeneous equations.

PROPOSITION 5.1. *Let (4) be a system of linear homogeneous equations with integer coefficients possessing a solution in positive integers. Then the dimension t of the NIP-cone $\mathcal{C}$ of all N-solutions to (4) is equal to the corank κ of the system (4), i.e., $t = s - \rho$, where s is the number of variables and ρ is the rank of (4).*

Proof. By elementary linear algebra the vector space V of all rational solutions to (4) has dimension κ (without any hypothesis on positivity). Since every N-solution is a rational solution, we have $t \leq \kappa$. Let α be a P-solution to (4), and let β be any rational solution. For some integer $m \neq 0$, $m\beta$ is an integer vector. For sufficiently large $n \in \mathbf{P}$, $n\alpha - m\beta$ is an N-vector γ. Thus $\beta = (1/m)(n\alpha - \gamma)$ so $t \geq \kappa$, and the proof is complete.

The hypothesis in Proposition 5.1 that (4) has a P-solution cannot be removed. For instance, the equation $x_1 + x_2 = 0$ has corank $\kappa = 1$ but $t = 0$.

Remark. Let M be the $R = K[X_1, \cdots, X_q]$-module corresponding to a set $\alpha^1, \cdots, \alpha^\sigma$ of nonzero N-solutions to (4) satisfying Proposition 2.2 (ii) (so M is finitely-generated). Then it is not hard to see that the *dimension* of M, i.e., the Krull dimension of the ring $R/\text{Ann}(M)$, where $\text{Ann}(M)$ is the ideal of R which annihilates M, is equal to the dimension t of the NIP-cone of all N-solutions to (4). In particular, if (4) has a P-solution, then by Proposition 5.1 we have $\dim (M) = \kappa$.

If a pseudograph G is not connected, say $G = K + L$, then it is clear that $H_G(r) = H_K(r)H_L(r)$. Hence we consider only connected pseudographs in the next proposition.

PROPOSITION 5.2. *Let G be a finite connected positive pseudograph with p vertices and $q > 1$ edges. Then*

$$\deg P_G(r) = \begin{cases} q - p + 1 & \text{if } G \text{ is bipartite} \\ q - p & \text{if } G \text{ is not bipartite.} \end{cases}$$

Moreover, $\deg Q_G(r) \leq \deg P_G(r)$.

Proof. By, e.g., partial fractions, $\deg P_G$ is one less than the highest order to which 1 is a pole of the rational function $F_G(y) = \sum_0^\infty H_G(r)y^r$. From Equation (24) we get $\deg P_G = t - 1$, where t is the dimension of the NIP-cone $\mathcal{C}$ of all magic labelings of G. Now by Proposition 5.1, t is equal to the dimension of the vector space V_G of all rational solutions to (3). Following Stewart [20], we call an element of V_G a *semi-magic* labeling of G.

Stewart [20; Corollary 2.4] has shown that if G is not bipartite, then $\dim V_G = q - p + 1$. (Although Stewart does not allow loops and multiple edges, his proof does not depend on this fact.) Thus when G is not bipartite, $\deg P_G = \dim V_G - 1 = q - p$. Suppose G is bipartite, say with every edge connecting a vertex in V_1 to one in V_2, where $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \varnothing$. It is evident that the positivity of G implies $|V_1| = |V_2|$, say $V_1 = \{v_1, \cdots, v_n\}$ and $V_2 = \{v_1', \cdots, v_n'\}$. Let G' be the pseudograph obtained from G by adding a loop e_0 at v_1. Then G is not bipartite; so by Stewart's result $\dim V_{G'} = q - p + 2$, where q is the number of edges of G (so G' has $q + 1$ edges). Suppose S is a semi-magic labeling of G' of index r. Now $\sum_e S(e) = \sum_{e'} S(e') = nr$, where e ranges over all $e \in E$ incident to some $v \in V_1$ and e' ranges over all $e' \in E$ incident to some $v' \in V_2$. But every edge except the loop e_0 is incident to some $v \in V_1$ and some $v' \in V_2$. Thus $0 = \sum_e S(e) - \sum_{e'} S(e') = S(e_0)$. Therefore $\dim V_{G'} = \dim V_G$; so $\deg P_G = \dim V_G - 1 = q - p + 1$.

Finally, if $\deg Q_G > \deg P_G$, then for sufficiently large r of the appropriate parity $H_G(r)$ would be negative, which is absurd.

That $\deg P_G = \deg Q_G$ is possible follows, for example, from taking G to be an odd cycle. Here $P_G(r) = Q_G(r) = 1/2$. A further example is the wheel W_p, p odd, discussed after Corollary 2.10.

Problem. Find a better upper bound (in terms of the structure of G) on $\deg Q_G(r)$ than the one given in Proposition 5.2. In view of the second example following Corollary 2.10, it seems unlikely that a simple explicit expression for $\deg Q_G(r)$ exists.

Example. In the Anand–Dumir–Gupta conjecture, G is bipartite with $p = 2n$, $q = n^2$. Hence $\deg H_n(r) = q - p + 1 = (n-1)^2$. In the Carlitz conjecture, G is not bipartite and $p = n$, $q = \binom{n+1}{2}$. Hence $\deg P_G(r) = q - p = \binom{n}{2}$. (There is a misprint in [5], where Carlitz makes a conjecture equivalent to $\deg P_G(r) = \binom{n+1}{2}$.)

Example. We illustrate the application of our results to computing a specific example. Let $T_4(r)$ be the number of 4×4 matrices of nonnegative integers summing to r in every row and column, with zero trace. This corresponds to taking G to be the complete bipartite graph $K_{4,4}$ with a matching (1-factor) removed. (G is also isomorphic to the 1-skeleton of a cube.) By Theorem 1.2, $T_4(r)$ is a polynomial in r. Since $p = 8$ and $q = 12$, by Proposition 5.2 we have

$\deg T_4(r) = q - p + 1 = 5$. we have $T_4(-1) = T_4(-2) =$ $T_4(0) = 1$. Thus if we know th $r \leq 1$, and these six values de computed to be 9 (the number of from which we quickly obtain

$$T_4(r) = \binom{r+5}{5} +$$

We list some additional funct of the generating function $F_G(r)$

(a) The Petersen graph [10; F

$$F_G(y) = (1 + y + 6y$$

(b) The product $C_5 \times K_2$ (fir

$$F_G(y) = (1 + 6y + 1$$

(c) The 1-skeleton of the octa

$$F_G(y) = (1 + 2y +$$

(d) The 4×4 checkerboard connect two squares with a line i

$$F_G(y) = (1 + 26y + 131y^2 \cdot$$

Is there an *a priori* theoretical rea

(e) A triangle with two loops a matrices with line sums less than

$$F_G(y) = (1 + 8y +$$

(f) The wheel $W_5 : F_G(y) = ($

(g) The wheel $W_7 : F_G(y) = ($

(h) The wheel $W_9 : F_G(y) = ($

In general, if G is positive, the $-m$, where m is the least positive G of index m exists. The leading to lowest terms) is then equal to of index m. For example, if G is has degree $2i$ and leading coeffici

Addendum. I am grateful to work of Eugène Ehrhart, which The following papers in particula

deg $T_4(r) = q - p + 1 = 5$. Since G is regular of degree 3, by Corollary 1.4 we have $T_4(-1) = T_4(-2) = 0$ and $T_4(r) = -T_4(-3 - r)$. Of course also $T_4(0) = 1$. Thus if we know the value $T_4(1)$, we then know $T_4(r)$ for $-4 \leq r \leq 1$, and these six values determine $T_4(r)$ uniquely. Now $T_4(1)$ is easily computed to be 9 (the number of permutations without fixed points on 4 objects), from which we quickly obtain

$$T_4(r) = \binom{r + 5}{5} + 3\binom{r + 4}{5} + 3\binom{r + 3}{5} + \binom{r + 2}{5}.$$

We list some additional functions $H_G(r)$ which we have computed in terms of the generating function $F_G(r) = \sum_{r=0}^{\infty} H_G(r)y^r$.

(a) The Petersen graph [10; Figure 9.6]:

$$F_G(y) = (1 + y + 6y^2 + y^3 + y^4)/(1 - y)^6(1 + y).$$

(b) The product $C_5 \times K_2$ (first graph on the cover of [10]):

$$F_G(y) = (1 + 6y + 11y^2 + 6y^3 + y^4)/(1 - y)^6(1 + y).$$

(c) The 1-skeleton of the octahedron:

$$F_G(y) = (1 + 2y + 6y^2 + 2y^3 + y^4)/(1 - y)^7(1 + y).$$

(d) The 4×4 checkerboard (the vertices of G are the squares, and edges connect two squares with a line in common so $p = 16$, $q = 24$):

$$F_G(y) = (1 + 26y + 131y^2 + 212y^3 + 131y^4 + 26y^5 + y^6)/(1 - y)^{10}.$$

Is there an *a priori* theoretical reason why the numerator is divisible by $(1 + y)^2$?

(e) A triangle with two loops at each vertex (corresponds to 3×3 symmetric matrices with line sums less than or equal to r):

$$F_G(y) = (1 + 8y + 15y^2 + 8y^3 + y^4)/(1 - y)^7(1 + y).$$

(f) The wheel $W_5 : F_G(y) = (1 + 4y^2 + 2y^4)/(1 - y^2)^4$.

(g) The wheel $W_7 : F_G(y) = (1 + 12y^2 + 21y^4 + 4y^6)/(1 - y^2)^6$.

(h) The wheel $W_9 : F_G(y) = (1 + 24y^2 + 92y^4 + 64y^6 + 6y^8)/(1 - y^2)^8$.

In general, if G is positive, then the degree of the rational function $F_G(y)$ is $-m$, where m is the least positive integer for which a positive magic labeling of G of index m exists. The leading coefficient of the numerator of $F_G(y)$ (reduced to lowest terms) is then equal to the number $\bar{H}_G(m)$ of positive magic labelings of index m. For example, if G is the wheel W_{2i+1}, then the numerator of $F_G(y)$ has degree $2i$ and leading coefficient $2i - 2$.

Addendum. I am grateful to K. Baclawski for calling my attention to the work of Eugène Ehrhart, which is closely related to the results of this paper. The following papers in particular are significant.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720

CURRENT ADDRESS: DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139.

BERRY-ESSEEN BC ERDÖS AND TURÁN

H. NIEDERREI[

1. Introduction. Let $\langle x_n \rangle$, n contained in [0, 1). Denote by The sequence $\langle x_n \rangle$ is called uni as $N \to \infty$ for all $0 \le x \le 1$. (In of fractional parts $\{x_n\}$ is u.d.)

$$D_N^* \stackrel{\text{def}}{=} \sup_{0 \le x \le 1}$$

Another equivalent condition is

$$S_N(h) \equiv$$

for all $h \in \mathbf{Z} - \{0\}$. (For the pr ing theorem due to Erdös and version of the sufficient part of tl

THEOREM A. *For any integer*

$$D_N^* \le c_1 \frac{}{m}$$

The best constants c_1 and c_2 so reiter, unpublished). Much larg

The purpose of this paper is to and to point out their connecti prove the following theorem.

THEOREM 1. *Let $F(x)$ be nond and let $G(x)$ satisfy a Lipschitz co*

$$|G($$

for all $0 \le x, y \le 1$. Suppose that integer m

$$\sup_{0 \le x \le 1} |F(x) - G(x)| \le \frac{4M}{m+}$$