VOLUMES AND EHRHART POLYNOMIALS OF CONVEX POLYTOPES

dedicated to the memory of
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$\mathcal{P}$ = convex polytope in $\mathbb{R}^n$

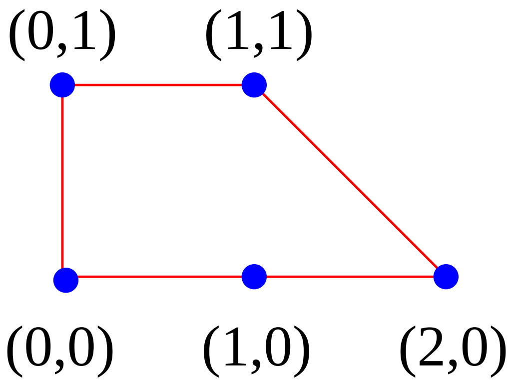
integer polytope: vertices $\in \mathbb{Z}^n$

$\mathbf{V}(\mathcal{P})$ = volume of $\mathcal{P}$

If $\mathcal{P}$ is an integer polytope, let

$$\widetilde{\mathbf{V}}(\mathcal{P}) = n!\, V(\mathcal{P}) \in \mathbb{Z},$$

the **normalized volume** of $\mathcal{P}$.



$$V(\mathcal{P}) = 3/2$$

$$\widetilde{V}(\mathcal{P}) = 3$$

A Refinement of Volume

Let $\mathcal{P}$ be an integer polytope and let $r \geq 1$. Define

$$\mathbf{r}\mathcal{P} = \{rv \; : \; v \in \mathcal{P}\}$$

$$\mathbf{i}(\mathcal{P}, \mathbf{r}) = \#\left(r\mathcal{P} \cap \mathbb{Z}^n\right),$$

the **Ehrhart polynomial** of $\mathcal{P}$.

- $i(\mathcal{P}, r)$ is a polynomial in r
- $i(\mathcal{P}, 0) = 1$
- If $r > 0$, then

  $i(\mathcal{P}, -r) = (-1)^{\dim \mathcal{P}} \#(\text{int}(r\mathcal{P}) \cap \mathbb{Z}^n)$
- $i(\mathcal{P}, r) = V(\mathcal{P})r^n + O(r^{n-1})$.

- Let $\dim \mathcal{P} = n$ and

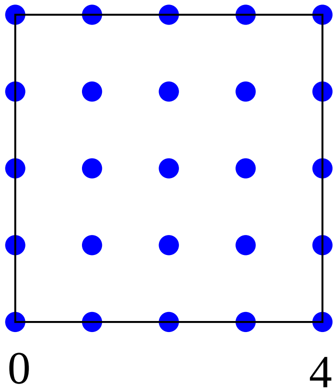
$$\sum_{r \geq 0} i(\mathcal{P}, r) x^r = \frac{h_0 + h_1 x + \cdots + h_n x^n}{(1-x)^{n+1}}.$$

Then $h_j \in \mathbb{Z}$, $h_j \geq 0$, and

$$\sum_j h_j = \widetilde{V}(\mathcal{P}).$$

Example. $\mathcal{P}$ = unit square:

$$0 \le x \le 1, \quad 0 \le y \le 1$$



$4\mathcal{P}$

$$\begin{aligned} \widetilde{V}(\mathcal{P}) &= 2! \cdot 1 \ = \ 2 \\ i(\mathcal{P}, r) &= (r+1)^2 \\ i(\mathcal{P}, -r) &= (r-1)^2 \\ \sum_{r \ge 0} i(\mathcal{P}, r) x^r &= \frac{1+x}{(1-x)^3} \end{aligned}$$

Example. $\mathcal{P}_n =$ unit n-cube

$$i(\mathcal{P}_n, r) = (r+1)^n,\ \widetilde{V}(\mathcal{P}_n) = n!$$

If $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$, then define

$$d(w) = \#\{i\ :\ w_i > w_{i+1}\},$$

the number of **descents** of w. Let

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{1+d(w)},$$

the nth **Eulerian polynomial**. E.g.,

$$A_3(x) = x + 4x^2 + x^3.$$

Then

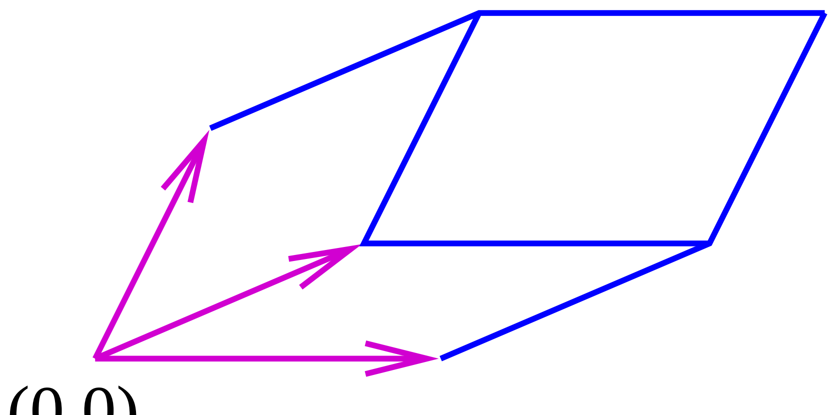
$$\sum_{r \geq 0} (r+1)^n x^n = \frac{A_n(x)/x}{(1-x)^{n+1}}.$$

Zonotopes

The **Minkowski sum** $\mathcal{P}+\mathcal{Q}$ of convex polytopes $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ is defined by

$$\mathcal{P}+\mathcal{Q} = \{\alpha+\beta \,:\, \alpha \in \mathcal{P},\ \beta \in \mathcal{Q}\}.$$

A **zonotope** is a Minkowski sum of line segments.



Let $v_1, \ldots, v_k \in \mathbb{R}^n$ and

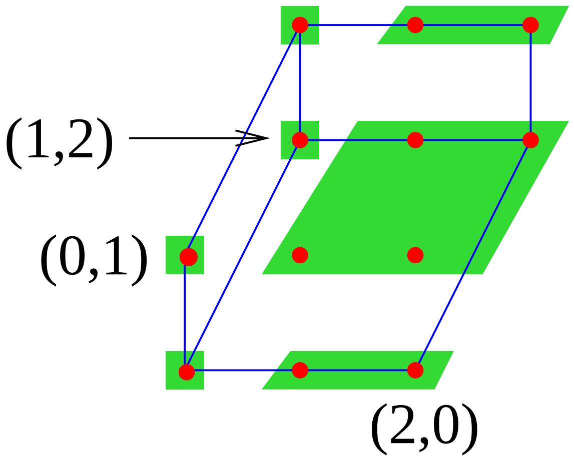
$$Z(v_1, \ldots, v_k) = [0, v_1] + \cdots + [0, v_k].$$

Theorem. *If* $v_1, \ldots, v_k \in \mathbb{Z}^n$, *then*

$$i\left(Z(v_1, \ldots, v_n), r\right) = \sum_{\substack{S \subseteq \{v_1, \ldots, v_k\} \\ (|S| = j) \\ \text{lin. indep.}}} \gcd\begin{pmatrix} j \times j \text{ minors of matrix} \\ \text{with rows } v \in S \end{pmatrix} r^j.$$

Corollary. $V(Z(v_1, \ldots, v_n)) =$

$$\sum_{\substack{S \subseteq \{v_1, \ldots, v_k\} \\ S = \text{basis for } \mathbb{R}^n}} |\det(\text{matrix with rows } v \in S)|.$$



$$7r^2 + 4r + 1$$

matrix	gcd
$\emptyset$	1
$[2\ \ 0]$	2
$[0\ \ 1]$	1
$[1\ \ 2]$	1

matrix	gcd
$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$	2
$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$	4
$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$	1

Example. Let $e_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$.

$Z_n = Z(e_i + e_j + e_{n+1} \;:\; 1 \le i < j \le n) \subset \mathbb{R}^{n+1}$

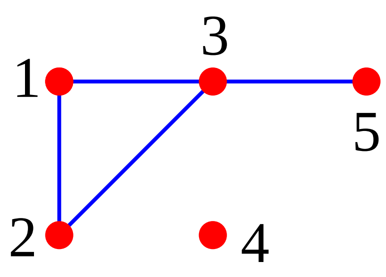
Erdős-Gallai:

$$(d_1, \ldots, d_{n+1}) \in Z_n \cap \mathbb{Z}^{n+1}$$

$\Leftrightarrow \exists$ (simple) graph on $1, 2, \ldots, n$ with

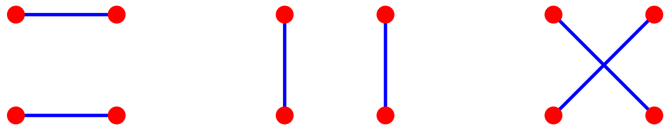
$$d_j = \deg(j),\ 1 \le j \le n$$

$$d_{n+1} = \frac{1}{2}\left(d_1 + \cdots + d_n\right).$$



(2, 2, 3, 0, 1, 4)

Let $f(n)$ = # **distinct** degree sequences $(d_1, \ldots, d_n)$, so $f(n) = i(Z_n, 1)$.

f(3)=8, f(4)=54 

Theorem. $$\sum_{n\geq} f(n)\frac{x^n}{n!} =$$

$$\frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2}\right.$$

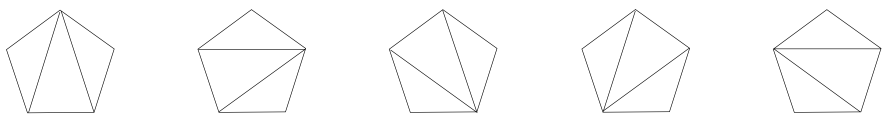
$$\left.\times\left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right]$$

$$\times \exp \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}$$

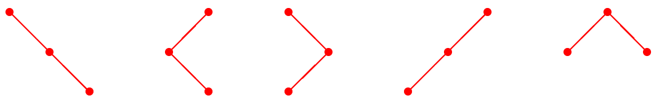
Catalan Numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

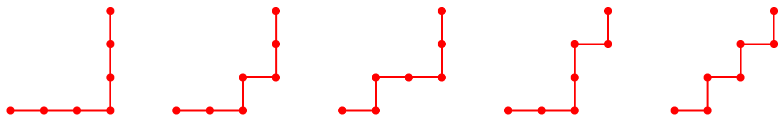
- triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



- binary trees with n vertices



- lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y = x$



- sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111---\qquad 11-1--\qquad 11--1-$$

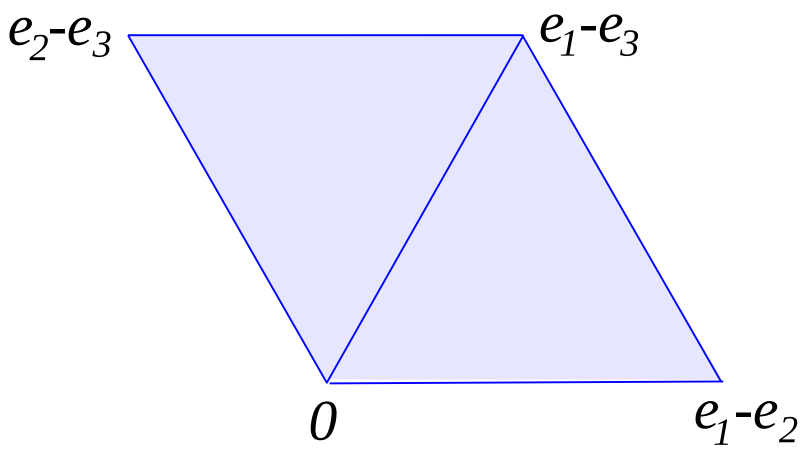
$$1-11--\qquad 1-1-1-$$

For 62 additional combinatorial interpretations of C_n, see Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, volume 2, Cambridge University Press, 1999

The Catalanotope

$$A_n^+ = \{e_i - e_j \in \mathbb{R}^{n+1} \; : \; 1 \le i < j \le n+1\}$$

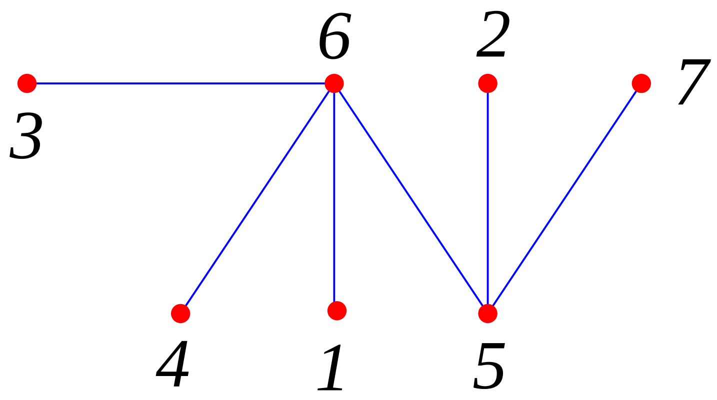
$$\mathcal{C}_{\mathbf{n}} = \mathrm{conv}(A_n^+ \cup \{0\}) \subset \mathbb{R}^{n+1}.$$



$$\dim \mathcal{C}_n = n$$

Let T be a tree with vertex set $\{1, \ldots, n+1\}$ and edge set E. Let $e_{ij} = e_i - e_j$. Define the simplex

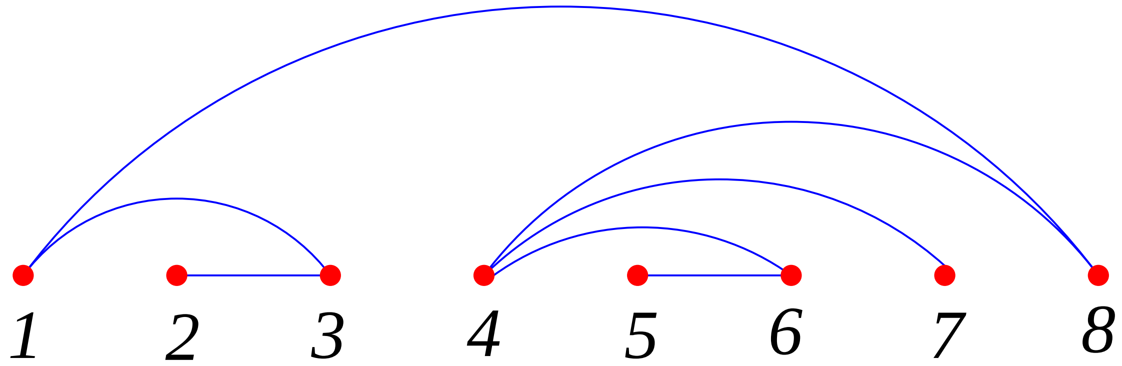
$$\sigma_T = \text{conv}\left(\{e_{ij} \,:\, ij \in E,\ i < j\} \cup \{0\}\right).$$



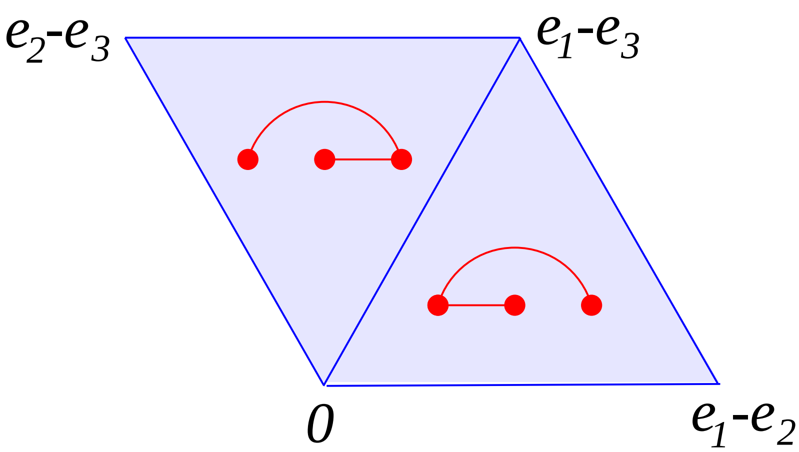
$$\sigma_T = \text{conv}\{e_{16}, e_{25}, e_{36}, e_{46}, e_{56}, e_{57}, 0\}$$

T is **alternating** if either every neighbor of vertex i is less than i or every neighbor is greater than i.

T is **noncrossing** if there are not edges ik and jl where $i < j < k < l$.

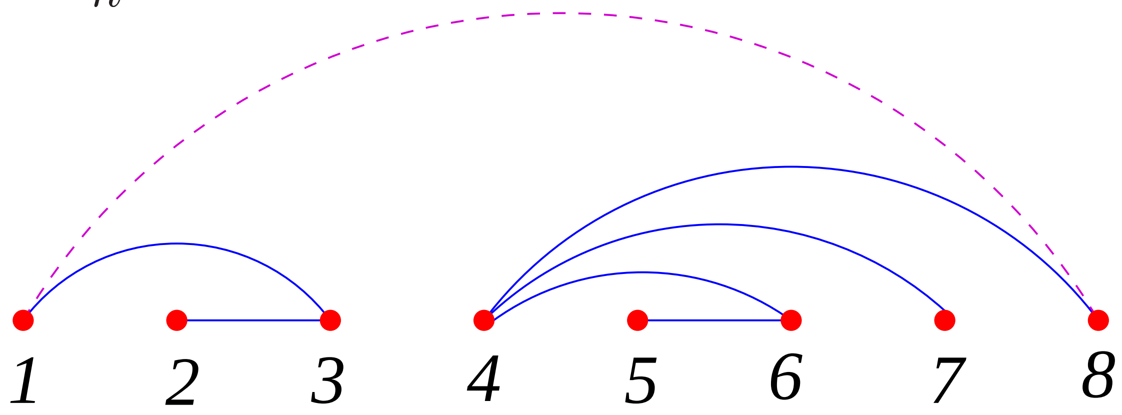


Theorem (A. Postnikov). *The simplices* σ_T*, where* T *ranges over all noncrossing alternating trees with vertex set* $\{1, \ldots, n+1\}$*, are the maximal faces of a triangulation of* $\mathcal{C}_n$*.*



Easy: $\widetilde{V}(\sigma_T) = 1.$

Lemma. *The number of noncrossing alternating trees with vertex set* $\{1, \ldots, n+1\}$ *is the Catalan number* C_n.



Corollary. $\widetilde{V}(\mathcal{C}_n) = C_n$.

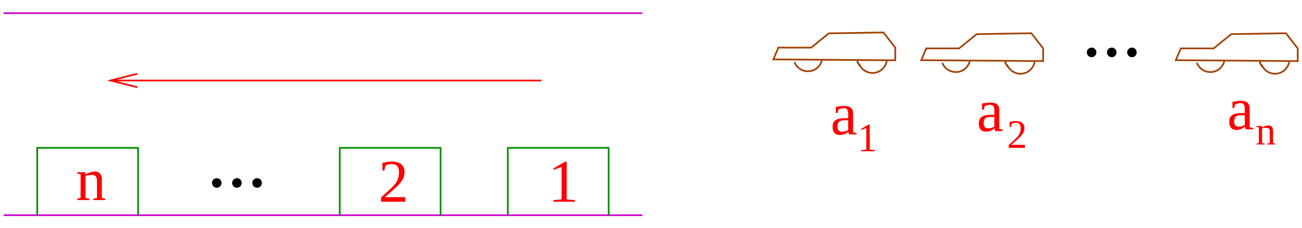
Theorem.

$$\sum_{r\geq 0} i(\mathcal{C}_n, r)x^r = \frac{\sum_{j=0}^{n-1} \frac{1}{n}\binom{n}{j}\binom{n}{j+1}x^j}{(1-x)^{n+1}}$$

Here $\frac{1}{n}\binom{n}{j}\binom{n}{j+1}$ is a **Narayana number**.

The Parking Function Polytope

(with J. Pitman)



Car C_i prefers space a_i. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

$n = 2$: 11 12 21

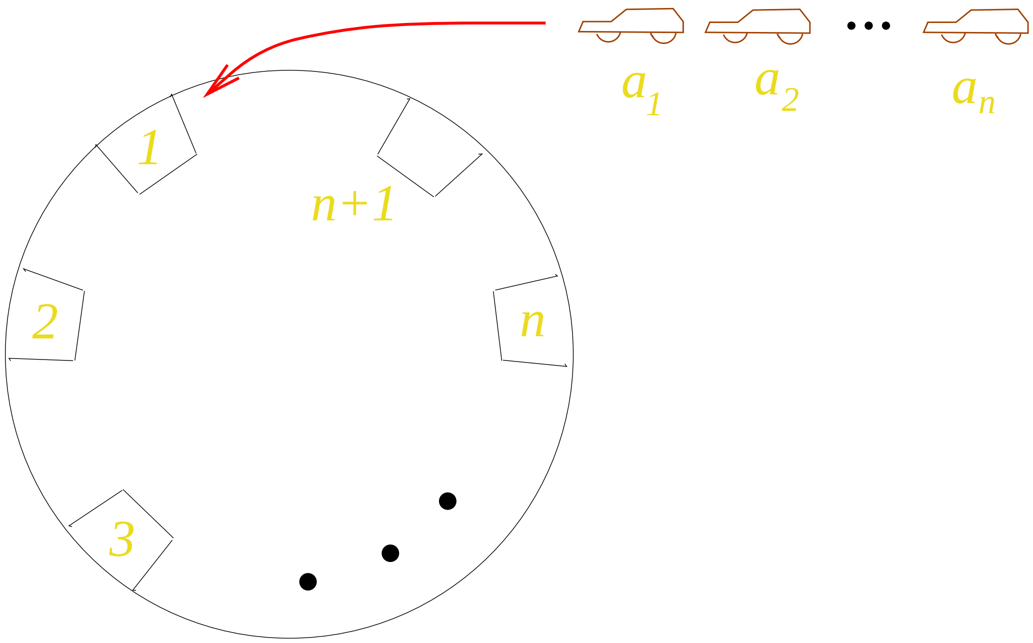
$n = 3$: 111 112 121 211 113 131 311 122
212 221 123 132 213 231 312 321

Theorem. *A sequence* $(a_1, \ldots, a_n)$ *of positive integers is a parking function if and only if its increasing rearrangement* $b_1 \leq \cdots \leq b_n$ *satisfies* $b_i \leq i$.

111
112 121 211
122 212 221
113 131 311
123 132 213 231 312 321

Theorem. (Pyke, 1959; Konheim & Weiss, 1966) The number of parking functions of length n is $(n+1)^{n-1}$.

Proof (Pollak, c. 1974). Add an additional space $n + 1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.



Now all cars can park, and there will be one empty space. A is a parking function if and only if the empty space is $n+1$. If $A = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1{+}i, a_2{+}i, \ldots, a_n{+}i) \text{ (modulo } n+1)$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

More generally, Let

$\alpha = (\alpha_1, \ldots, \alpha_n),\ \ 0 < \alpha_1 \le \cdots \le \alpha_n.$

An **α-parking function** is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ whose increasing rearrangement $b_1 \le \cdots \le b_n$ satisfies $b_i \le \alpha_i$.

E.g., $\alpha = (1,3)$: 11 12 21 13 31

Ordinary parking functions:

$$\alpha = (1, 2, \ldots, n)$$

Let $\mathbf{L}(\alpha)$ = number of α-parking functions

Theorem (Steck 1968, Gessel 1996):

$$\mathbf{L}(\alpha) = n!\ \det \left[\frac{\alpha_i^{j-i+1}}{(j-i+1)!} \right]_{i,j=1}^{n}$$

Theorem. Let $x_j = \alpha_j - \alpha_{j-1}$ (with $\alpha_0 = 0$). Then

$$L(\alpha) = \sum_{\substack{\text{parking functions} \\ (j_1,\ldots,j_n)}} x_{j_1} \cdots x_{j_n}.$$

E.g., $\alpha = (1,3)$, $(x_1, x_2) = (1,2)$:

$$x_1^2 + x_1x_2 + x_2x_1 = 5.$$

Given $x_1,\ldots,x_n \in \mathbb{R}_{\geq 0}$, define $\mathcal{P} = \mathcal{P}(\mathbf{x_1},\ldots,\mathbf{x_n}) \subset \mathbb{R}^n$ by: $(y_1,\ldots,y_n) \in \mathcal{P}_n$ if

$$y_j \geq 0, \quad y_1+\cdots+y_j \leq x_1+\cdots+x_j$$

for $1 \leq j \leq n$.

Theorem. (a) *Let $x_1,\ldots,x_n \in \mathbb{N}$. Then*

$$n!\,V(\mathcal{P}_n) = L(\alpha),$$

where $\alpha_{n-j+1} = x_1+\cdots+x_j$.

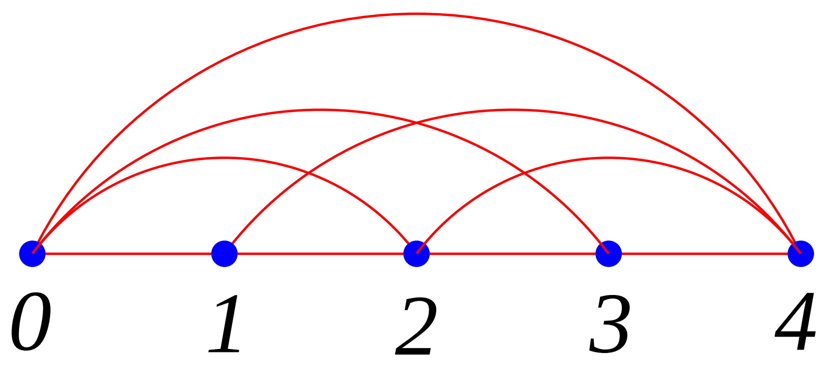
(b) Let $x_1 = a$, $x_2 = \cdots = x_n = b$. Then

$$i(\mathcal{P}_n, 1) = \frac{1}{n!}(a+1)(a+nb+2)(a+nb+3)\cdots(a+nb+n).$$

Flow Polytopes

(with A. Postnikov)

Let G a directed graph with vertices $0, 1, \ldots, m$ and edge set E such that if $i \to j$ is an edge, then $i < j$. Call G a **flow graph**. For simplicity we assume that $(0, i)$ is an edge for $1 \leq i \leq m$ and (j, m) is an edge for $0 \leq j \leq m - 1$.



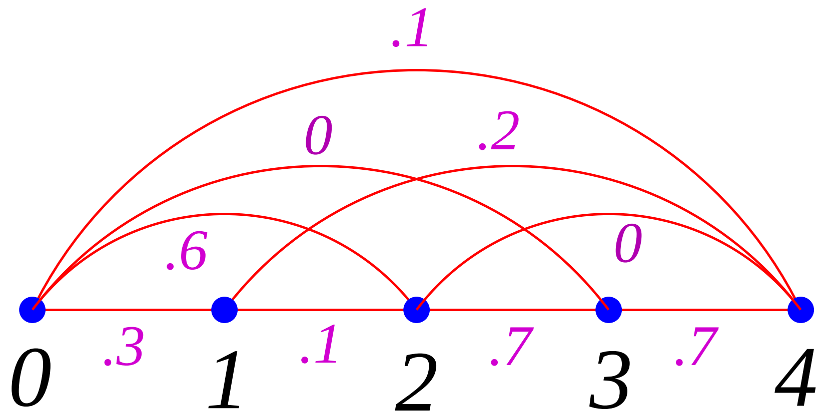
Define the **flow polytope** $\mathcal{F}_G$ to be the set of all $f \in \mathbb{R}^E_{\geq 0}$ satisfying

$$\sum_{(0,j)\in E} f(0,j) = 1$$

$$\sum_{(j,m)\in E} f(j,m) = 1$$

$$\sum_{i\,:\,(i,j)\in E} f(i,j) = \sum_{k\,:\,(j,k)\in E} f(j,k),$$

for $1 \leq j \leq m-1$.



total flow out of 0 and into m is 1

flow into an internal vertex = flow out

Triangulating $\mathcal{F}_G$. Let

$E^* = \{(i,j) \in E \,:\, 1 \le i < j \le m{-}1\}$.

Let $x^G = \displaystyle\prod_{(i,j)\in E^*} x_{ij}$.

Continually apply the relations

$$x_{ij}x_{jk} \to x_{ik}(x_{ij} + x_{jk})$$

until unable to do so.

Example.

$$x_{12}x_{13}x_{14}x_{23}x_{34}$$

$$\begin{aligned}
&\to x_{13}(x_{12}+x_{23})x_{14}(x_{13}+x_{34})x_{14} \\
&= 2\text{ terms} + (x_{12}+x_{23})x_{14}^2 x_{13}x_{34} \\
&\to 2\text{ terms} + (x_{12}+x_{23})x_{14}^3(x_{13}+x_{34}) \\
&= 5\text{ terms} + x_{14}^3 x_{23}x_{34} \\
&\to 5\text{ terms} + x_{14}^3 x_{24}(x_{23}+x_{34}) \\
&= 7\text{ terms.}
\end{aligned}$$

Each monomial u corresponds to a simplex σ_u with $\widetilde{V}(\sigma_u) = 1$ in a triangulation of $\mathcal{F}_G$. Hence $\widetilde{V}(\mathcal{F}_G) =$ number of monomials.

More generally, let

$$x_{ij}x_{jk} \to x_{ik}(x_{ij} + x_{jk} - 1)$$

until unable to do so. Then let $x_{ij} = 1/(1-x)$ and divide by $(1-x)^m$ to get $\sum_{r\geq 0} i(\mathcal{F}_G, r)x^r$.

Example. $x_{12}x_{23} \to x_{13}\,(x_{12} + x_{23} - 1)$

$$\to \frac{1}{(1-x)^4}\frac{1}{1-x}\left(\frac{2}{1-x} - 1\right)$$

$$= \frac{1+x}{(1-x)^6}.$$

Let $d_i(G)$ be the outdegree of vertex i of G.

Theorem. *$\widetilde{V}(\mathcal{F}_G)$ is the number of ways to write the vector*

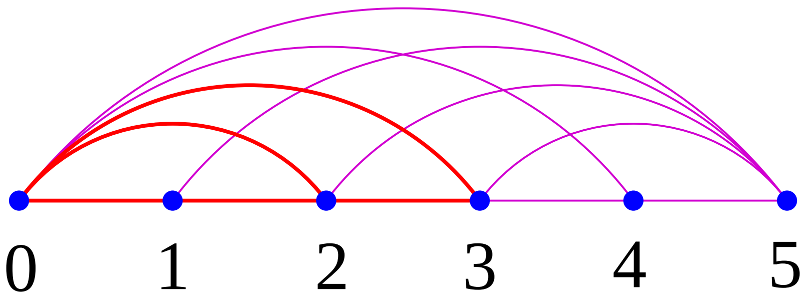
$$(d_1+\cdots+d_{m-2}-m+2, 1-d_1, \ldots, 1-d_{m-2})$$

as a sum of vectors $e_i - e_j$, where

$$(i-1, j-1) \in E(G) \text{ and } j < m$$

(without regard to order).

Example.



$$
\begin{aligned}
&(d_1 + \cdots + d_3 - 3, 1 - d_1, 1 - d_2, 1 - d_3) \\
&\quad = (3, -1, -1, -1) \\
&\quad = 3e_{12} + 2e_{23} + e_{34} \\
&\quad = e_{12} + 2e_{13} + e_{34} \\
&\quad = 2e_{12} + e_{13} + e_{23} + e_{34} \\
&\quad = 2e_{12} + e_{14} + e_{23} \\
&\quad = e_{12} + e_{13} + e_{14}.
\end{aligned}
$$

Example. G_m =**complete flow graph** on $0, 1, \ldots, m$: $E = \{(i,j) : 0 \le i < j \le m\}$.

Chan-Robbins conjecture: $\widetilde{V}(\mathcal{F}_{G_m}) = C_1 C_2 \cdots C_{m-2}$.

Equivalently: The number of ways to write the vector

$$\left(\binom{m+1}{2}, -m, -m+1, \ldots, -1 \right)$$

as a sum of vectors

$$e_i - e_j, \quad 1 \le i < j \le m+1.$$

is $C_1 C_2 \cdots C_m$ (**Kostant's partition function**).

Corollary. *Let CT denote the constant term of a Laurent series. Then*

$$\widetilde{V}(\mathcal{F}_{G_m}) =$$

$$\mathrm{CT} \prod_{i=1}^{m} (1-x_i)^{-2} \prod_{1\leq i<j\leq m} (x_j - x_i)^{-1}.$$

Theorem (Morris).

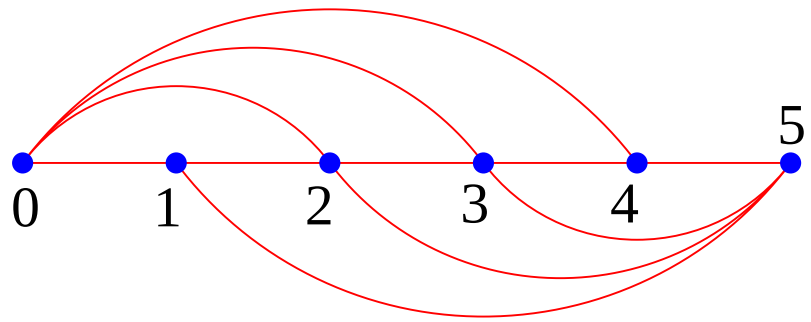
$$\mathrm{CT} \prod_{i=1}^{m} (1-x_i)^{-a} \prod_{i=1}^{m} x_i^{-b} \prod_{1\leq i<j\leq m} (x_j - x_i)^{-2c}$$

$$= \frac{1}{m!} \prod_{j=0}^{m-1} \frac{\Gamma(a+b+(m-1+j)c)\Gamma(c)}{\Gamma(a+jc)\Gamma(c+jc)\Gamma(b+jc+1)}.$$

Corollary (Zeilberger) $\widetilde{V}(\mathcal{F}_{G_m}) = C_1 \cdots C_{m-2}$.

Simpler proof?

One Further Flow Graph



$$\begin{aligned} E(G) \;=\; & \{(i, i+1) \;:\; 0 \le i \le m-1\} \\ \cup \; & \{(0, i) \;:\; 1 \le i \le m-1\} \\ \cup \; & \{(i, m) \;:\; 1 \le i \le m-1\}. \end{aligned}$$

Theorem. $\widetilde{V}(\mathcal{F}_G) = C_{m-1}$.

$$
\begin{aligned}
x_{12}x_{23}x_{34} &\to x_{13}(x_{12}+x_{23})x_{34} \\
&\to x_{14}(x_{13}+x_{34})(x_{12}+x_{23}) \\
&\to x_{14}x_{13}x_{12}+x_{14}x_{13}x_{23} \\
&\quad +x_{14}x_{34}x_{12}+x_{14}x_{24}(x_{23}+x_{34})
\end{aligned}
$$

(five terms)