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Constructions and Complexity of Secondary Polytopes

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The secondary polytope $\Sigma(\mathscr{A})$ of a configuration $\mathscr{A}$ of n points in affine $(d-1)$-space is an $(n-d)$-polytope whose vertices correspond to regular triangulations of $\mathrm{conv}(\mathscr{A})$. In this article we present three constructions of $\Sigma(\mathscr{A})$ and apply them to study various geometric, combinatorial, and computational properties of secondary polytopes. The first construction is due to Gel'fand, Kapranov, and Zelevinsky, who used it to describe the face lattice of $\Sigma(\mathscr{A})$. We introduce the universal polytope $\mathscr{U}(\mathscr{A}) \subset \bigwedge_d \mathbb{R}^n$, a combinatorial object depending only on the oriented matroid of $\mathscr{A}$. The secondary $\Sigma(\mathscr{A})$ can be obtained as the image of $\mathscr{U}(\mathscr{A})$ under a canonical linear map onto $\mathbb{R}^n$. The third construction is based upon Gale transforms or oriented matroid duality. It is used to analyze the complexity of computing $\Sigma(\mathscr{A})$ and to give bounds in terms of n and d for the number of faces of $\Sigma(\mathscr{A})$. © 1990 Academic Press, Inc.

1. INTRODUCTION AND POLYHEDRAL PRELIMINARIES

In their recent work on generalized hypergeometric functions and discriminants, Gel'fand, Kapranov, and Zelevinsky [10, 11] introduced the *secondary polytope* $\Sigma(\mathscr{A})$ of an affine point configuration $\mathscr{A}$, where the vertices of $\Sigma(\mathscr{A})$ are in one-to-one correspondence with the regular triangulations of the "primary polytope" $P = \mathrm{conv}(\mathscr{A})$. In spite of its algebraic origin as the Newton polytope of the principal $\mathscr{A}$-determinant (for $\mathscr{A} \subset \mathbb{Z}^d$), this polytope is of independent interest for combinatorial convexity. A special case which has received much attention in combinatorics [14–16] and theoretical computer science [21], as well as topology [23], is the *associahedron*, which is the secondary polytope of a convex n-gon.

It is the objective of the present paper to provide a self-contained and comprehensive study of secondary polytopes. We shall give three alternative descriptions of $\Sigma(\mathscr{A})$. Section 2 is expository, giving the original

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construction due to Gel'fand, Kapranov, and Zelevinsky, including essentially the proof that they give in [11]. This proof is direct and analytic, providing vertex coordinates for the secondary polytope and a complete description of the facial structure of $\Sigma(\mathscr{A})$.

In Section 3 we express the secondary polytope $\Sigma(\mathscr{A})$ as the projection of the *universal polytope* $\mathscr{U}(\mathscr{A})$ which is a certain polytope contained the exterior algebra $\bigwedge_* \mathbb{R}^n$. This approach is based on the techniques used in [9] and it has the important advantage that it separates the combinatorial and metrical properties of the secondary polytope in a systematic way.

In Section 4 we give a geometric description of $\Sigma(\mathscr{A})$ using *Gale transforms*. Compared to the two previous treatments, this point of view is the most constructive one because it leads to an algorithm for computing all regular triangulations of $\mathscr{A}$ and therefore all vertices of $\Sigma(\mathscr{A})$. We illustrate the effectiveness of the Gale transform approach with a complete description of the secondary of the cyclic 4-polytope with 8 vertices.

Section 5 deals with the computational complexity of secondary polytopes. We give a bound in terms of n and d for the number of faces of $\Sigma(\mathscr{A})$, and we show that our bound is sharp for the class of *Lawrence polytopes* [1]. In particular, we will see that $\Sigma(\mathscr{A})$ is a zonotope whenever $\mathscr{A}$ is the vertex set of a Lawrence polytope.

Throughout this paper $\mathscr{A} = \{a_1, a_2, ..., a_n\}$ denotes a subset of $\mathbb{R}^d$ which spans an affine hyperplane. A *triangulation* of $\mathscr{A}$ is a triangulation of the $(d-1)$-polytope $P := \text{conv}(\mathscr{A})$ with vertices in $\mathscr{A}$. We identify $\mathbb{R}^n$ with the vector space $\mathbb{R}^{\mathscr{A}}$ of real valued functions on $\mathscr{A}$. Given a fixed triangulation Δ of $\mathscr{A}$, then every $\psi \in \mathbb{R}^n$ induces a unique piecewise linear function $g_{\psi,\Delta}$ on the polytope P. More precisely, this function is defined by assigning $g_{\psi,\Delta}(a_i) := \psi_i$ for vertices a_i of Δ and by the requirement that $g_{\psi,\Delta}$ be an affine function on each simplex of Δ. Consider the set

$$\mathscr{C}(\mathscr{A}, \Delta) := \{\psi \in \mathbb{R}^n : g_{\psi,\Delta} \text{ is a convex function, and } g_{\psi,\Delta}(a_i) \leqslant \psi_i \text{ whenever } a_i \text{ is not a vertex of } \Delta\}.$$

It is easy to check that $\mathscr{C}(\mathscr{A}, \Delta)$ is a closed polyhedral cone and that the collection

$$\mathscr{F}(\mathscr{A}) := \{\mathscr{C}(\mathscr{A}, \Delta) : \Delta \text{ is a triangulation of } \mathscr{A}\}$$

covers $\mathbb{R}^n$. We call this collection the *secondary fan* of $\mathscr{A}$. This terminology will be justified in the proof of Theorem 1.3.

In the following we recall some general facts about convex polytopes and polyhedral fans. By a *complex* we mean a family of polyhedra, the intersection of any two of which is a face of each and is itself in the family. A *fan* in $\mathbb{R}^n$ is a complex of polyhedral cones that covers $\mathbb{R}^n$. At times, we will specify a fan by giving a subcomplex containing at least its maximal cells.

This is the case, for example, with the collection $\mathscr{F}(\mathscr{A})$ defined above. The *normal cone* of a polytope $Q \subset \mathbb{R}^n$ at a point $p \in Q$ is defined as

$$\mathscr{N}(Q, p) = \{v \in \mathbb{R}^n : \langle v, p\rangle \leqslant \langle v, y\rangle \text{ for all } y \in Q\}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The *normal fan* of Q, denoted $\mathscr{N}(Q)$, is the collection of cones $\mathscr{N}(Q, p)$ where $p \in Q$.

LEMMA 1.1. *The normal cone $\mathscr{N}(Q, p)$ of a polytope Q at $p \in Q$ has non-empty interior if and only if p is a vertex of Q. More generally, the codimension of $\mathscr{N}(Q, p)$ equals the dimension of the largest face of Q containing p.*

A polyhedral fan $\mathscr{F}$ in $\mathbb{R}^n$ is said to be *strongly polytopal* if there exists a polytope $Q \subset \mathbb{R}^n$ such that $\mathscr{F} = \mathscr{N}(Q)$. Suppose that Q is an n-polytope containing the origin of $\mathbb{R}^n$ in its interior. Then the collection of polyhedral cones which are obtained as positive hulls of all facets of Q is called the *interior point fan* of Q. The following proposition summarizes some known facts about strongly polytopal fans and Minkowski sums of polytopes (cf. [22, 12]). In (2) the *intersection* $\mathscr{F} \cap \mathscr{F}'$ of two polyhedral fans is understood as the fan of all intersections of cones from $\mathscr{F}$ and $\mathscr{F}'$.

PROPOSITION 1.2. (1) *A fan $\mathscr{F}$ is strongly polytopal if and only if it is the interior point fan of a polytope Q. In that case $\mathscr{F}$ is the normal fan $\mathscr{N}(Q^*)$ of the polar polytope to Q.*

(2) *The intersection of strongly polytopal fans corresponds to the Minkowski addition of polytopes, i.e., $\mathscr{N}(Q + Q') = \mathscr{N}(Q) \cap \mathscr{N}(Q')$.*

(3) *For two strongly polytopal fans $\mathscr{F} = \mathscr{N}(Q)$ and $\mathscr{F}' = \mathscr{N}(Q')$ we have $\mathscr{F} \prec \mathscr{F}'$ (i.e., $\mathscr{F}'$ refines $\mathscr{F}$) if and only if $Q \prec Q'$ (i.e., λQ is a Minkowski summand of Q' for some $\lambda > 0$).*

(4) *A strongly polytopal fan $\mathscr{F} = \mathscr{N}(Q)$ determines Q uniquely (up to homothety) if and only if Q is indecomposable (i.e., $P \prec Q$ implies $P = \lambda Q$ for some $\lambda > 0$).*

(5) *The normal fan of a zonotope is a central hyperplane arrangement.*

For examples of fans which are not strongly polytopal see [5, p. 119, Fig. 3; 19, p. 85]. Using the language of polyhedral fans, the existence of a secondary polytope can be expressed as follows.

THEOREM 1.3 (Gel'fand, Kapranov, and Zelevinsky). *The secondary fan $\mathscr{F}(\mathscr{A})$ of any affine point configuration $\mathscr{A}$ is strongly polytopal. That is, there exists a* secondary polytope *$Q = \Sigma(\mathscr{A})$ in $\mathbb{R}^n$ whose normal fan $\mathscr{N}(Q)$ equals $\mathscr{F}(\mathscr{A})$.*

Proposition 1.2 (4) tells us that we cannot expect the secondary polytope $\Sigma(\mathcal{A})$ to be unique (up to homothety) because it may be decomposable into non-trivial Minkowski summands (see Corollary 4.4). In particular, $\Sigma(\mathcal{A})$ is highly decomposable when $\mathcal{A}$ consists of n points in convex position in the affine plane (i.e., $d=3$). In [16] (see also [15]) Lee gave a geometric construction of the associahedron $\Sigma(\mathcal{A})$, which is a simple $(n-3)$-dimensional polytope with $n(n-3)/2$ facets and $(1/(n-1))\binom{2n-4}{n-2}$ vertices (the Catalan number). It follows from the results in [22] that the associahedron has $\binom{n-2}{2}$ degrees of freedom in choosing a secondary polytope for the n-gon.

We note that Lee also constructed secondary polytopes in the case $n \leqslant d+2$ (see [15] and Proposition 2.2). Around the same time, Haiman [14] gave an independent, and somewhat different, construction for the associahedron. Much earlier than this, Stasheff [23] had constructed the associahedron as a geometric cell complex, although he did not address whether it could be realized as a convex polytope.

An interesting application of the associahedron to theoretical computer science has recently been given by Sleator, Tarjan, and Thurston [21]. These authors derive a tight upper bound for the rotation distance between binary trees with n nodes by proving that the diameter of the associahedron equals $2n-10$, for large n. From Fig. 4 in [21] we can see that the secondary polytope of a hexagon is a simple 3-polytope with 14 vertices, 21 edges, and 9 facets.

2. The Analytic Construction and the Face Lattice of the Secondary

The following analytic description of the secondary polytope is the original one due to Gel'fand, Kapranov, and Zelevinsky [10, 11]. We include it here for completeness. Let

$$Q := \operatorname{conv}\{\phi_\Delta : \Delta \text{ is a triangulation of } \mathcal{A}\}, \tag{2.1}$$

where

$$\phi_\Delta := \sum_{i=1}^{n} \left(\sum \{\operatorname{vol}(\tau) : \tau \in \Delta \text{ and } i \in \tau\} \right) \cdot e_i. \tag{2.2}$$

In this formula e_i denotes the ith standard basis vector of $\mathbb{R}^n$, and $\operatorname{vol}(\tau)$ denotes the volume of the $(d-1)$-simplex $\operatorname{conv}\{a_{\tau_1}, a_{\tau_2}, \ldots, a_{\tau_d}\}$.

First Proof of Theorem 1.3 (Gel'fand, Kapranov, and Zelevinsky). Since both collections $\mathcal{F}(\mathcal{A}) = \{\mathcal{C}(\mathcal{A}, \Delta)\}$ and $\mathcal{N}(Q) = \{\mathcal{N}(Q, \phi_\Delta)\}$ cover $\mathbb{R}^n$, and since $\mathcal{N}(Q)$ is a fan, it will suffice to prove the inclusion $\mathcal{C}(\mathcal{A}, \Delta) \subseteq \mathcal{N}(Q, \phi_\Delta)$. Note that this will also show that the collection $\mathcal{F}(\mathcal{A})$ defines a polyhedral fan.

Let $\psi \in \mathcal{C}(\mathcal{A}, \Delta)$. Then $g_{\psi,\Delta}$ is a piecewise linear convex function whose graph contains or lies below the point $(a_i, \psi_i) \in \mathbb{R}^{d+1}$ for $i = 1, \ldots, n$. This implies that

$$g_{\psi,\Delta}(x) \leqslant g_{\psi,\Delta'}(x) \tag{2.3}$$

for all $x \in P = \operatorname{conv}(\mathcal{A})$ and for all other triangulations Δ' of $\mathcal{A}$. Consequently,

$$\int_{x \in P} g_{\psi,\Delta}(x)\, dx \leqslant \int_{x \in P} g_{\psi,\Delta'}(x)\, dx \tag{2.4}$$

for all triangulations Δ' of $\mathcal{A}$. We evaluate the integral on the left hand side as follows:

$$\begin{aligned} \int_{x \in P} g_{\psi,\Delta}(x)\, dx &= \sum_{\tau \in \Delta} \int_{x \in \tau} g_{\psi,\Delta}(x)\, dx \\ &= \sum_{\tau \in \Delta} \operatorname{vol}(\tau) \cdot (\text{"barycenter of the simplex } \tau\text{"}) \\ &= \sum_{\tau \in \Delta} \operatorname{vol}(\tau) \cdot \frac{1}{d} \sum_{i=1}^{d} g_{\psi,\Delta}(a_{\tau_i}) \\ &= \frac{1}{d} \sum_{i=1}^{d} \psi_i \sum_{i \in \tau \in \Delta} \operatorname{vol}(\tau) = \frac{1}{d} \langle \psi, \phi_\Delta \rangle. \end{aligned}$$

Since the same formula holds for Δ', Eq. (2.4) implies $\langle \psi, \phi_\Delta \rangle \leqslant \langle \psi, \phi_{\Delta'} \rangle$ for all triangulations Δ' of $\mathcal{A}$. But this is precisely the condition that ψ is contained in $\mathcal{N}(Q, \phi_\Delta)$, which is the normal fan at ϕ_Δ of the convex hull of the ϕ_Δ's. ∎

A triangulation Δ of $\mathcal{A}$ is said to be *regular* if there exists a function on P that is piecewise linear and strictly convex with respect to Δ. (A convex piecewise linear function over a triangulation Δ is said to be *strictly convex* if it is given by a different linear function on each maximal cell of Δ.) This condition is equivalent to $\mathcal{C}(\mathcal{A}, \Delta)$ having non-empty interior. Distinct regular triangulations must have distinct cones, since a point in the interior of one cone (coming from a strictly convex function over the corresponding triangulation) cannot belong to any other cone. Thus, we get the following corollary from Theorem 1.3.

COROLLARY 2.1. *The vertices of the secondary polytope $Q = \Sigma(\mathcal{A})$ are in one-to-one correspondence with the regular triangulations of $\mathcal{A}$.*

Suppose one knows all the vectors ϕ_Δ, but not the actual triangulations Δ. Then any regular triangulation Δ of $\mathcal{A}$ is uniquely determined by the vector ϕ_Δ. To reconstruct Δ from the set of vectors $\phi_{\Delta'}$, first note that

$$\mathcal{C}(\mathcal{A}, \Delta) = \bigcap_{\Delta' \neq \Delta} [\mathrm{pos}\{\phi_\Delta - \phi_{\Delta'}\}]^*$$

where K^* denotes the cone polar to K. A d-tuple $\tau = (\tau_1, ..., \tau_d)$ defines a facet of Δ if and only if there is a $\psi \in \mathcal{C}(\mathcal{A}, \Delta)$ with $\psi_j = 0$ for $j \in \tau$ and $\psi_j \geq 1$ for $j \notin \tau$. One may determine the existence of such a ψ by linear programming. We will see in Example 2.4 that a triangulation Δ may *not* be determined by its vector ϕ_Δ if it is not regular.

Let us first summarize a few positive results concerning the regularity of triangulations. The *lexicographic triangulations* of $\mathcal{A}$ constructed in [2] are easily seen to be regular. These triangulations have the important property that they depend only on the oriented matroid [3] of $\mathcal{A}$ and not its specific realization. It is shown in [2] that all triangulations of a convex n-gon are lexicographic, and consequently all triangulations are regular if $\mathcal{A}$ is a planar affine point configuration in convex position. If $\mathcal{A}$ is not in convex position, then there exist non-regular triangulations (cf. Fig. 1). Using Gale diagram techniques, Lee [17] has recently proved that all triangulations of point sets with small "codimension" are regular.

PROPOSITION 2.2 (Lee). *If $n \leq d+2$, then all triangulations of $\mathcal{A}$ are regular.*

A *polyhedral subdivision Π of $\mathcal{A}$* is a collection of subsets of $\mathcal{A}$, called faces of Π, such that the set of polytopes $\{\mathrm{conv}(\tau) \mid \tau \in \Pi\}$ is a polyhedral complex that covers $P = \mathrm{conv}(\mathcal{A})$. As with triangulations, we call *Π regular* if there is a function on P that is strictly convex and piecewise linear with respect to Π. Given two polyhedral subdivisions Π_1 and Π_2 of $\mathcal{A}$, we say *Π_1 refines Π_2*, written $\Pi_2 \prec \Pi_1$, if every face of Π_1 is a subset of some face of Π_2. Consider the poset $\mathcal{P}(\mathcal{A})$ of all regular polyhedral subdivisions of $\mathcal{A}$, ordered by refinement.

THEOREM 2.3. *For any configuration $\mathcal{A}$, the poset $\mathcal{P}(\mathcal{A})$ is a lattice which is anti-isomorphic to the face lattice of the secondary polytope $\Sigma(\mathcal{A})$.*

Proof. If we define

$$\mathcal{C}(\mathcal{A}, \Pi) := \{\psi \in \mathbb{R}^n\text{: there is a piecewise linear convex function } g \text{ over } \Pi \text{ with } g(a_i) = \psi_i \text{ for } a_i \in \tau \in \Pi,\ g(a_i) \leq \psi_i \text{ otherwise}\},$$

then the proof of Theorem 1.3 given above also shows that $\mathcal{C}(\mathcal{A}, \Pi)$ is the normal cone to a face of the secondary polytope $Q = \Sigma(\mathcal{A})$. This defines a map $\Pi \mapsto \mathcal{C}(\mathcal{A}, \Pi)$ from $\mathcal{P}(\mathcal{A})$ to $\mathcal{N}(Q)$ (considered as its lattice of faces).

To construct the inverse, let F be a face of Q and define $T(F)$ to be the set of all regular triangulations Δ of $\mathcal{A}$ such that $\phi_\Delta \in F$. Let $\Pi(F)$ be the finest regular subdivision of $\mathcal{A}$ refined by each $\Delta \in T(F)$. We claim that $\mathcal{C}(\mathcal{A}, \Pi(F)) = \mathcal{N}(Q, F)$. The inclusion $\mathcal{C}(\mathcal{A}, \Pi(F)) \subseteq \mathcal{N}(Q, F)$ is straightforward. To see that $\mathcal{C}(\mathcal{A}, \Pi(F)) \supseteq \mathcal{N}(Q, F)$, take $\psi \in \mathrm{relint}\, \mathcal{N}(Q, F)$. Then ψ induces a convex function g over P, piecewise linear with respect to a regular subdivision Π' of P. Now for $\Delta \in T(F)$ we have that $g_{\psi,\Delta} = g$ because $g_{\psi,\Delta}$ has the same integral as g and $g_{\psi,\Delta} \geq g$. This equality is equivalent to $\Pi' \prec \Delta$. On the other hand, if $\Delta \notin T(F)$, $g_{\psi,\Delta}$ must have a larger integral than g, implying that $\Pi' \nprec \Delta$. So $\Pi' \prec \Pi(F)$ showing $\psi \in \mathcal{C}(\mathcal{A}, \Pi(F))$.

Note that for regular Π and Π', we have $\mathcal{C}(\mathcal{A}, \Pi) \subseteq \mathcal{C}(\mathcal{A}, \Pi')$ if and only if $\Pi \preccurlyeq \Pi'$, and so the map $\Pi \mapsto \mathcal{C}(\mathcal{A}, \Pi)$ and its inverse are both order preserving. ∎

The poset of *all* polyhedral subdivisions of $\mathcal{A}$ is in general not polytopal. In fact, it may have maximal chains of unequal length. See [15] for an example. A 2-dimensional example can be made using the configuration in the following example.

EXAMPLE 2.4. Let $\mathcal{A} = \{a_1, ..., a_6\} \subset \mathbb{R}^3$ where $a_1 = (4, 0, 0)$, $a_2 = (0, 4, 0)$, $a_3 = (0, 0, 4)$, $a_4 = (2, 1, 1)$, $a_5 = (1, 2, 1)$, $a_6 = (1, 1, 2)$. We will describe two distinct triangulations Δ_1 and Δ_2 of $\mathcal{A}$ such that

(1) $\phi_{\Delta_1} = \phi_{\Delta_2}$, and

(2) both Δ_1 and Δ_2 are not regular.

First note that assertion (1) implies assertion (2). For, suppose (1) holds and Δ_1 is regular. Then $\phi_{\Delta_1} = \phi_{\Delta_2}$ is a vertex of $\Sigma(\mathcal{A})$, and Δ_2 is also regular. But then Corollary 2.1 implies $\Delta_1 = \Delta_2$. Consider the two triangulations

$$\Delta_1 := \{125, 134, 145, 236, 256, 346, 456\}$$

and

$$\Delta_2 := \{124, 136, 146, 235, 245, 356, 456\}$$

of $\mathcal{A}$ which are depicted in Fig. 1.

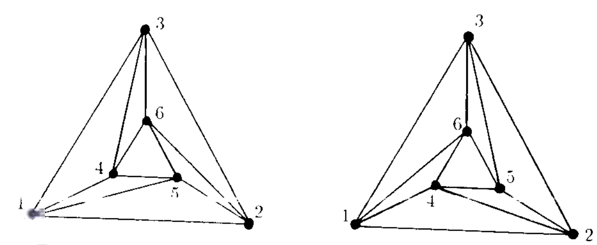

FIG. 1. Two non-regular triangulations Δ_1 and Δ_2 with $\phi_{\Delta_1} = \phi_{\Delta_2}$.

Writing $[ijk]$ for the absolute value of the determinant $\det(a_i, a_j, a_k)$, we compute

$$\begin{aligned}
\phi_{\Delta_1} &= ([125]+[134]+[145])e_1 + ([125]+[236]+[256])e_2 \\
&\quad + ([134]+[236]+[346])e_3 \\
&\quad + ([134]+[145]+[346]+[456])e_4 \\
&\quad + ([125]+[145]+[256]+[456])e_5 \\
&\quad + ([236]+[256]+[346]+[456])e_6 \\
&= 36e_1 + 36e_2 + 36e_3 + 28e_4 + 28e_5 + 28e_6 \\
= \phi_{\Delta_2} &= ([124]+[136]+[146])e_1 + ([124]+[235]+[245])e_2 \\
&\quad + ([136]+[235]+[356])e_3 \\
&\quad + ([124]+[146]+[245]+[456])e_4 \\
&\quad + ([235]+[245]+[356]+[456])e_5 \\
&\quad + ([136]+[146]+[356]+[456])e_6.
\end{aligned}$$

In this example the secondary $\Sigma(\mathscr{A})$ is a 3-dimensional polytope, and the point $\phi_{\Delta_1} = \phi_{\Delta_2}$ is contained in the relative interior of a facet of $\Sigma(\mathscr{A})$.

3. The Universal Polytope

Here we construct the secondary polytope $Q = \Sigma(\mathscr{A})$ as a projection of a certain higher-dimensional polytope. The *universal polytope* $\mathscr{U}(\mathscr{A})$ of the point configuration $\mathscr{A}$ will be defined as the convex hull in $\bigwedge_d \mathbb{R}^n$ of a set of d-vectors associated with triangulations of $P = \operatorname{conv} \mathscr{A}$. The universal polytope $\mathscr{U}(\mathscr{A})$ depends only on the oriented matroid [3] of the point configuration $\mathscr{A}$, and not on the specific embedding.

Let $\mathbf{A}$ be the $n \times d$ matrix whose ith row contains the homogeneous coordinates of a_i. Without loss of generality we may assume

$$\mathbf{A} = \begin{bmatrix} 1 & a_{11} & \cdots & a_{1,d-1} \\ 1 & a_{21} & \cdots & a_{2,d-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_{n,1} & & a_{n,d-1} \end{bmatrix}. \tag{3.1}$$

We denote by η the exterior product of the columns of $\mathbf{A}$, so η is a simple (or decomposable) d-vector in $\bigwedge_d \mathbb{R}^n$. If $\{e_1, ..., e_n\}$ is the standard basis of $\mathbb{R}^n$, then the d-vectors

$$e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d},$$
$$\lambda \in \Lambda(n, d) := \{(\lambda_1, ..., \lambda_d) \mid 1 \leqslant \lambda_1 < \cdots < \lambda_d \leqslant n\}, \tag{3.2}$$

form an orthonormal basis of $\bigwedge_d \mathbb{R}^n$. We associate to any triangulation Δ of $\mathscr{A}$ the d-vector

$$\varphi_\Delta := \sum_{\lambda \in \Delta} \operatorname{sign}\langle \eta, e_\lambda \rangle \cdot e_\lambda, \tag{3.3}$$

which is called the *projection form* of Δ. The factor $\operatorname{sign}\langle \eta, e_\lambda \rangle$ is just the orientation of the simplex $\operatorname{conv}\{a_{\lambda_1}, ..., a_{\lambda_d}\}$. Note that this orientation can also be defined intrinsically: The simplicial complex Δ is an orientable manifold with boundary, and hence each of its facets λ has a unique orientation $\operatorname{sign}_\Delta(\lambda)$ in Δ (up to a global sign change). We have $\operatorname{sign}_\Delta(\lambda) = \operatorname{sign}\langle \eta, e_\lambda \rangle$ which shows that (3.3) depends only on the triangulation Δ and not on the specific coordinates $\eta = \bigwedge_d \mathbf{A}$.

The projection forms in (3.3) have been used to solve various isoperimetric problems, including maximizing the volume of projections of the regular simplex [9]. We define the *universal polytope* $\mathscr{U}(\mathscr{A})$ of $\mathscr{A}$ as

$$\mathscr{U}(\mathscr{A}) := \operatorname{conv}\left\{ \varphi_\Delta \in \bigwedge_d \mathbb{R}^n \;\middle|\; \Delta \text{ is a triangulation of } \mathscr{A} \right\}. \tag{3.4}$$

Some basic properties of $\mathscr{U}(\mathscr{A})$ are:

(a) The oriented matroid of $\mathscr{A}$ determines the universal polytope $\mathscr{U}(\mathscr{A})$, and conversely.

(b) Every triangulation of $\mathscr{A}$ (including the non-regular ones) corresponds to a unique vertex of $\mathscr{U}(\mathscr{A})$.

(c) If the points of $\mathscr{A}$ are in general position, then the dimension of $\mathscr{U}(\mathscr{A})$ equals $\binom{n-1}{d}$.

Property (a) follows directly from the definitions. This contrasts with the secondary polytope $\Sigma(\mathscr{A})$ which may depend on the embedding of $\mathscr{A}$.

Property (b) can be proved by noting that if Δ and Δ' are distinct triangulations of $\mathscr{A}$, then

$$\langle \varphi_\Delta, \varphi_{\Delta'} \rangle = \sum_{\lambda \in \Delta \cap \Delta'} \operatorname{sign}\langle \eta, e_\lambda \rangle^2 = |\Delta \cap \Delta'| < |\Delta| = \langle \varphi_\Delta, \varphi_\Delta \rangle. \quad (3.5)$$

The proof of (c) will be postponed until we discuss bistellar operations. As an application of (c), consider the case where $\mathscr{A}$ is the vertex set of a convex pentagon. Then (b) implies $\mathscr{U}(\mathscr{A})$ has 5 vertices, and by (c) its dimension is $\binom{4}{3} = 4$. Hence the universal polytope $\mathscr{U}(\mathscr{A})$ of a convex pentagon is a 4-simplex. Note that the secondary polytope $\Sigma(\mathscr{A})$ of a convex pentagon is again a convex pentagon.

For the purpose of this paper the most important property of the universal polytope is the existence of a canonical projection onto the secondary polytope. Consider the linear map

$$\phi: \bigwedge_d \mathbb{R}^n \to \mathbb{R}^n$$
$$\varphi \mapsto \sum_{i=1}^{n} \langle (e_i \rfloor \varphi) \wedge e_i, \eta \rangle e_i, \quad (3.6)$$

where "$\rfloor$" denotes left interior multiplication, the adjoint to the linear operator given by "$\wedge$", defined by the relation $\langle a \wedge b, c \rangle = \langle a, b \rfloor c \rangle$ for $a, b, c \in \bigwedge_* \mathbb{R}^n$ of appropriate degree.

A d-vector $\eta \in \bigwedge_d \mathbb{R}^n$ is said to be *simple* if it can be written as a wedge product of vectors in $\mathbb{R}^n$, i.e., $\eta = x_1 \wedge \cdots \wedge x_d$. The set of d-vectors obtained from all possible bases of a fixed d-subspace of $\mathbb{R}^n$ form a line through the origin in $\bigwedge_d \mathbb{R}^n$. This correspondence between d-subspaces and simple d-vectors is the classical Plücker embedding of the Grassmannian. Using this, we can give a geometric interpretation of the operations $\wedge$ and $\rfloor$. If η and φ are simple, and the corresponding subspaces L and M satisfy $L \cap M = 0$, then $\eta \wedge \varphi$ represents the subspace $L \oplus M$. Also if $L^\perp + M = \mathbb{R}^n$, then $\eta \rfloor \varphi$ corresponds to $L^\perp \cap M$ [8, Chap. 0].

Now suppose $\varphi = e_\lambda$. If $i \in \lambda$, then $(e_i \rfloor e_\lambda) \wedge e_i = e_\lambda$, and $\phi_i(e_\lambda) = \langle e_\lambda, \eta \rangle$ is a Plücker coordinate of η. If $i \notin \lambda$, then $e_i \rfloor e_\lambda = 0$ and $\phi_i(e_\lambda) = 0$. For a vertex φ_Δ of the universal polytope, $(e_i \rfloor \varphi_\Delta) \wedge e_i$ thus eliminates all terms in φ_Δ except those corresponding to the link of a_i in Δ. The inner product $\langle (e_i \rfloor \varphi_\Delta) \wedge e_i, \eta \rangle$ gives the volume of this link in the realization of $\mathscr{A}$ since $\langle e_\lambda, \eta \rangle = \det \mathbf{A}_\lambda$, the maximal minor of $\mathbf{A}$ with rows in λ. Consequently, $\phi(\varphi_\Delta) = \phi_\Delta$. This discussion proves the following result.

THEOREM 3.1. *The secondary polytope $\Sigma(\mathscr{A}) \subset \mathbb{R}^n$ is the image of the universal polytope $\mathscr{U}(\mathscr{A}) \subset \bigwedge_d \mathbb{R}^n$ under the projection ϕ.*

We next prove a key property of the map in (3.6).

PROPOSITION 3.2. *The following diagram of linear maps commutes:*

$$\begin{array}{ccc} \bigwedge_{d+1} \mathbb{R}^n & \xrightarrow{e \rfloor} & \bigwedge_d \mathbb{R}^n \\ & \searrow^{(-1)^{d+1}\eta \rfloor} & \downarrow \phi \\ & & \mathbb{R}^n \end{array}$$

where $e = e_1 + \cdots + e_n$.

Proof. It is enough to check the result on a basis vector $e_\mu \in \bigwedge_{d+1} \mathbb{R}^n$. By (3.6), we obtain

$$\phi(e \rfloor e_\mu) = \sum_{i \in \mu} \langle e \rfloor e_\mu, (e_i \rfloor \eta) \wedge e_i \rangle e_i.$$

For $i \in \mu$, the coefficient of e_i is

$$\langle e \rfloor e_\mu, (e_i \rfloor \eta) \wedge e_i \rangle = \sum_{\{j \in \mu | j \neq i\}} \langle e_j \rfloor e_\mu, (e_i \rfloor \eta) \wedge e_i \rangle. \quad (3.7)$$

Since $i \in \mu \setminus j$ in (3.7), this reduces to

$$\sum_{\{j \in \mu | j \neq i\}} \langle e_j \rfloor e_\mu, \eta \rangle = \langle e \rfloor e_\mu, \eta \rangle - \langle e_i \rfloor e_\mu, \eta \rangle.$$

However, $\langle e \rfloor e_\mu, \eta \rangle = 0$, since e is a column of η, and $\langle e_i \rfloor e_\mu, \eta \rangle = (-1)^d \langle \eta \rfloor e_\mu, e_i \rangle$. Hence

$$\phi(e \rfloor e_\mu) = (-1)^{d+1} \sum_{i=1}^{n} \langle \eta \rfloor e_\mu, e_i \rangle e_i = (-1)^{d+1} \eta \rfloor e_\mu,$$

which proves the theorem. ∎

This theorem can be interpreted as showing that ϕ takes d-boundaries to the circuit space of the oriented matroid of η.

Next we will prove that the affine hull of $\Sigma(\mathscr{A})$ is orthogonal to the column space of $\mathbf{A}$. We first need a description of bistellar operations in terms of exterior algebra.

LEMMA 3.3. *If Δ and Δ' differ by a bistellar operation on $\mu = \operatorname{conv}\{a_{\mu_1}, \ldots, a_{\mu_{d+1}}\}$, then*

$$\varphi_\Delta - \varphi_{\Delta'} = \pm e \rfloor e_\mu.$$

Proof. The bistellar operation on μ consists in replacing $\alpha \cdot \partial\beta$ with $\partial\alpha \cdot \beta$, where $\{\alpha, \beta\}$ is the unique partition of (the vertex set of) μ such that

$$\text{link } \alpha = \partial\beta \quad \text{and} \quad \text{link } \beta = \partial\alpha \quad (3.8)$$

(see [20, Definition (2.2)]). In the forms φ_Δ and $\varphi_{\Delta'}$, the join operation "$\cdot$" of complexes is represented by "$\wedge$", and the boundary operation by "$e \rfloor$". Thus

$$\varphi_\Delta - \varphi_{\Delta'} = \sum_{j \in \beta} \operatorname{sign}\langle \eta, e_\alpha \wedge (e_j \rfloor e_\beta)\rangle e_\alpha \wedge (e_j \rfloor e_\beta)$$
$$- \sum_{i \in \alpha} \operatorname{sign}\langle \eta, (e_i \rfloor e_\alpha) \wedge e_\beta\rangle (e_i \rfloor e_\alpha) \wedge e_\beta. \qquad (3.9)$$

Since a permutation of the indices in μ will not change the signs of the terms in (3.9), we may assume $i < j$, for all $i \in \alpha$ and for all $j \in \beta$. In this case

$$e \rfloor e_\mu = e_\alpha \wedge (e \rfloor e_\beta) + (-1)^{|\beta|} (e \rfloor e_\alpha) \wedge e_\beta$$
$$= \sum_{j \in \beta} e_\alpha \wedge (e_j \rfloor e_\beta) + (-1)^{|\beta|} \sum_{i \in \alpha} (e_i \rfloor e_\alpha) \wedge e_\beta. \qquad (3.10)$$

Comparing (3.9) with (3.10), we see it suffices to show that

$$\operatorname{sign}\langle \eta, e_\alpha \wedge (e_j \rfloor e_\beta)\rangle = -\operatorname{sign}\langle \eta, (-1)^{|\beta|} (e_i \rfloor e_\alpha) \wedge e_\beta\rangle, \qquad (3.11)$$

for all $i \in \alpha$ and for all $j \in \beta$. But this follows from Cramer's rule, since $\{\alpha, \beta\}$ is the unique Radon partition of μ. ∎

In the following lemma, we shall determine $\operatorname{aff}(\mathscr{U}(\mathscr{A}))$ precisely when $\mathscr{A}$ is generic. This will also give a proof of (c).

PROPOSITION 3.4. *If $\mathscr{A}$ is a point configuration in general position, then $\operatorname{aff}(\mathscr{U}(\mathscr{A}))$ is a translate of $\bigwedge_d e^\perp$.*

Proof. Let $L = \operatorname{span}\{\varphi_\Delta - \varphi_{\Delta'} \mid \varphi_\Delta, \varphi_{\Delta'} \in \mathscr{U}(\mathscr{A})\}$ be the subspace parallel to $\operatorname{aff}(\mathscr{U}(\mathscr{A}))$ through the origin. We shall show that $L = \bigwedge_d e^\perp$.

$\bigwedge_d e^\perp \subset L$: The space $\bigwedge_d e^\perp$ is spanned by $\bigwedge_d e^\perp = \operatorname{span}\{e \rfloor e_\mu \mid \mu \in \Lambda(n, d+1)\}$. Since $\mathscr{A}$ is in general position, for each $\mu \in \Lambda(n, d+1)$ there exist two triangulations Δ and Δ' of conv $\mathscr{A}$ which differ by a bistellar operation on $\{a_{\mu_1}, ..., a_{\mu_{d+1}}\}$. From Lemma 3.3, $e \rfloor e_\mu = \pm(\varphi_\Delta - \varphi_{\Delta'})$ and thus $e \rfloor e_\mu \in L$.

$L \subset \bigwedge_d e^\perp$: Let Δ and Δ' be any two triangulation of $P = \operatorname{conv} \mathscr{A}$, and let $\varphi = \varphi_\Delta - \varphi_{\Delta'}$. Then

$$\langle \eta, \varphi\rangle = \langle \eta, \varphi_\Delta\rangle - \langle \eta, \varphi_{\Delta'}\rangle = \operatorname{vol}(P) - \operatorname{vol}(P) = 0. \qquad (3.12)$$

Recall that e is a column of $\mathbf{A}$ and so $\eta = \eta_0 \wedge e$ for some $\eta_0 \in \bigwedge_{d-1} \mathbb{R}^n$. Substituting in (3.12) gives

$$0 = \langle \eta_0 \wedge e, \varphi\rangle = \langle \eta_0, e \rfloor \varphi\rangle. \qquad (3.13)$$

Since $\mathscr{A}$ is in general position, both Δ and Δ' will remain triangulations for small perturbations of $\mathscr{A}$, and so (3.13) holds in an open neighborhood of η_0 on $G(d-1, n)$. It follows that $e \rfloor \varphi = 0$ (see, e.g., [9, Theorem 1]). The proof follows since $e \rfloor \varphi = 0$ if and only if $\varphi \in \bigwedge_d e^\perp$. ∎

COROLLARY 3.5. *For $\mathscr{A}$ in general position, the space $\operatorname{aff}(\Sigma(\mathscr{A}))$ is orthogonal to the column space of $\mathbf{A}$, the coordinate matrix of $\mathscr{A}$.*

Proof. Since the map ϕ which takes $\mathscr{U}(\mathscr{A})$ to $\Sigma(\mathscr{A})$ is linear, $\operatorname{aff}(\Sigma(\mathscr{A}))$ is parallel to

$$\operatorname{span}\{\phi(e \rfloor e_\mu) \mid \mu \in \Lambda(n, d+1)\} \subseteq \operatorname{span}\{\eta \rfloor e_\mu \mid \mu \in \Lambda(n, d+1)\} \qquad (3.14)$$

by Propositions 3.2 and 3.4. It was shown in [24] that the vector $\eta \rfloor e_\mu$ in (3.14) is an elementary vector of the linear subspace $\eta^\perp \subset \mathbb{R}^n$, and that all elementary vectors of $\eta^\perp$ have this form (up to scaling). In order to complete the proof, it suffices to observe that $\eta \rfloor e_\mu$ is orthogonal to η, which follows immedately from the geometric interpretation of "$\rfloor$". ∎

For arbitrary $\mathscr{A}$, the conclusion of Corollary 3.5 follows directly from the convex function point of view of Section 2 by observing that each of the cones $\mathscr{C}(\mathscr{A}, \Delta)$ contains all ψ induced by affine functions on P. These are precisely the elements of the column space of $\mathbf{A}$. That this is the largest subspace contained in these cones follows from the fact that if a function and its negative are both convex, then it must be affine. This general form of Corollary 3.5 will also be a direct conclusion of the construction in the next section.

The lexicographic triangulations considered in [2] have the property that they will be vertices of the image of ϕ for any embedding of the set $\mathscr{A}$ having the same oriented matroid. The set of all such "intrinsic" triangulations may be worth further study.

Finally, it is shown in [10] that the edges of the secondary correspond to triangulations which differ by an operation they call a *perestroika*. We note that these are precisely the "stellar exchange" operations of Pachner [20].

4. THE CONSTRUCTION USING GALE TRANSFORMS

This section gives a self-contained geometric construction of the secondary polytope. We identify $(\mathbb{R}^d)^* = \mathbb{R}^d$ with the space of affine functions on the set $\mathscr{A}$. The linear transformation $\mathbb{R}^d \to \mathbb{R}^n$ defined by the $n \times d$ matrix $\mathbf{A}$, having rows $a_1, a_2, ..., a_n$, takes affine functions to their values on $\mathscr{A}$. The image of $\mathbf{A}$ is a d-dimensional linear subspace which is clearly contained in the cone $\mathscr{C}(\mathscr{A}, \Delta)$ for each triangulation Δ of $\mathscr{A}$.

Pick an $(n-d)\times d$ matrix $\mathbf{B}$, with columns $b_1, b_2, \ldots, b_n$, such that

$$0 \longrightarrow \mathbb{R}^d \xrightarrow{\mathbf{A}} \mathbb{R}^n \xrightarrow{\mathbf{B}} \mathbb{R}^{n-d} \longrightarrow 0 \qquad (4.1)$$

is an exact sequence of $\mathbb{R}$-linear maps. The vector configuration $\mathscr{B} = \{b_1, b_2, \ldots, b_n\}$ is called a *Gale transform* of $\mathscr{A}$ (cf. [13, 18, 25]). Note that the oriented matroid of $\mathscr{B}$ is dual to the oriented matroid of $\mathscr{A}$.

LEMMA 4.1. *The convex hull of $\mathscr{B}$ contains the origin $0 \in \mathbb{R}^{n-d}$ in its interior.*

Proof. There exists a linear function on $\mathbb{R}^d$ which is strictly positive on $\mathscr{A}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ be the corresponding element of $\mathrm{Im}(\mathbf{A}) = \mathrm{Ker}(\mathbf{B})$. Then $\lambda_1 b_1 + \cdots + \lambda_n b_n$ is a positive combination of the b_i's giving the zero vector in $\mathbb{R}^{n-d}$. ∎

Fix a triangulation Δ of $\mathscr{A}$, and consider the closed convex polyhedral cone

$$\mathscr{C}'(\mathscr{A}, \Delta) := \bigcap_{\tau \in \Delta} \mathrm{pos}\{b_{\tau_1^*}, b_{\tau_2^*}, \ldots, b_{\tau_{n-d}^*}\}, \qquad (4.2)$$

where "pos" stands for the positive hull and τ^* is the complementary index set to the facet $\tau = (\tau_1, \tau_2, \ldots, \tau_d)$ of Δ, i.e., $\tau \cup \tau^* = \{1, 2, \ldots, n\}$.

LEMMA 4.2. *The map $\mathbf{B}$ induces the decomposition*

$$\mathscr{C}(\mathscr{A}, \Delta) = \mathrm{Ker}(\mathbf{B}) \oplus \mathscr{C}'(\mathscr{A}, \Delta)$$

into a d-dimensional linear subspace and an $(n-d)$-dimensional pointed cone.

Proof. It follows directly from the definition that the cone $\mathscr{C}'(\mathscr{A}, \Delta)$ is *pointed*, which means it contains no non-trivial linear subspace. We need to show that a vector $\psi \in \mathbb{R}^n$ is contained in $\mathscr{C}(\mathscr{A}, \Delta)$ if and only if its image $\mathbf{B}\psi \in \mathbb{R}^{n-d}$ is contained in $\mathscr{C}'(\mathscr{A}, \Delta)$. First observe that

$$\mathbf{B}\psi = \sum_{i=1}^{n} \psi_i b_i \in \mathrm{pos}\{b_{\tau_1^*}, b_{\tau_2^*}, \ldots, b_{\tau_{n-d}^*}\} \qquad (4.3)$$

if and only if

$$\psi'_{\tau_1} = \psi'_{\tau_2} = \cdots = \psi'_{\tau_d} = 0$$

and

$$\psi'_{\tau_1^*} \geq 0, \psi'_{\tau_2^*} \geq 0, \ldots, \psi'_{\tau_{n-d}^*} \geq 0 \qquad (4.4)$$

for some vector $\psi' \in \psi + \mathrm{Ker}(\mathbf{B})$. The piecewise linear function $g_{\psi,\Delta}$ induced by ψ is convex if and only if for each $\tau \in \Delta$ there exists a global affine function, with value vector $\lambda_\tau \in \mathrm{Im}(\mathbf{A}) = \mathrm{Ker}(\mathbf{B})$, such that $\psi' = \psi - \lambda_\tau$ satisfies (4.4). Therefore, $\psi \in \mathscr{C}(\mathscr{A}, \Delta)$ is equivalent to (4.3) holding for all $\tau \in \Delta$, and hence equivalent to $\mathbf{B}\psi \in \mathscr{C}'(\mathscr{A}, \Delta)$. ∎

We define the *pointed secondary fan* $\mathscr{F}'(\mathscr{A})$ to be the collection of cones $\mathscr{C}'(\mathscr{A}, \Delta)$ in $\mathbb{R}^{n-d}$ where Δ ranges over all triangulations of $\mathscr{A}$. By Lemma 4.2, $\mathscr{F}'(\mathscr{A})$ is strongly polytopal if and only if the secondary fan $\mathscr{F}(\mathscr{A})$ is strongly polytopal. More precisely, if $Q' \subset \mathbb{R}^{n-d}$ is a polytope with $\mathscr{N}(Q') = \mathscr{F}'(\mathscr{A})$, then $0 \oplus Q' \subset \mathbb{R}^n$ is a polytope with $\mathscr{N}(0 \oplus Q') = \mathscr{F}(\mathscr{A})$. This means that the secondary polytopes of $\mathscr{A}$ are exactly the polytopes in $\mathbb{R}^{n-d}$ with normal fan $\mathscr{F}'(\mathscr{A})$.

For each basis μ of $\mathscr{B}$ we define the cone

$$C_\mu = \mathrm{pos}\{b_{\mu_1}, b_{\mu_2}, \ldots, b_{\mu_{n-d}}\}.$$

LEMMA 4.3. *Let $x \in \mathbb{R}^{n-d}$ be such that x is not contained in the boundary of any of the C_μ. Then the set of d-tuples $\Omega_x := \{\mu^* \mid x \in C_\mu\}$ is a regular triangulation of $\mathscr{A}$.*

Proof. Pick a preimage $\psi \in \mathbb{R}^n$ of x under $\mathbf{B}$, and let Δ be any regular triangulation of $\mathscr{A}$ such that $\psi \in \mathscr{C}(\mathscr{A}, \Delta)$. It suffices to show that $\Delta = \Omega_x$.

Consider any index tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_d)$. Then τ is contained in the triangulation Δ if and only if (4.4) holds. But (4.4) is equivalent to (4.3) and therefore to $x \in C_{\tau^*}$. Hence $\tau \in \Delta$ if and only if $\tau^* \in \Omega_x$, which completes the proof. ∎

Lemma 4.3 implies that each full-dimensional polyhedral cone of the form $\bigcap_{\mu \in \Omega} C_\mu$ is a maximal cell of $\mathscr{F}'(\mathscr{A})$, and conversely. In other words, $\mathscr{F}'(\mathscr{A})$ is the multi-intersection in $\mathbb{R}^{n-d}$ of all cones C_μ, where μ ranges over all bases of $\mathscr{B}$. Note that, by matroid duality, the bases of $\mathscr{B}$ are precisely the complements of basis of $\mathscr{A}$.

Second Proof of Theorem 1.3. Let μ be any basis of $\mathscr{B}$, μ^* the complementary basis of $\mathscr{A}$, and $\varepsilon > 0$ a sufficiently small real number. Define the convex polytope

$$P_\mu := \mathrm{conv}(b_{\mu_1}, b_{\mu_2}, \ldots, b_{\mu_{n-d}}, \varepsilon \cdot b_{\mu_1^*}, \varepsilon \cdot b_{\mu_2^*}, \ldots, \varepsilon \cdot b_{\mu_d^*}).$$

We define $\mathscr{F}_\mu$ to be the interior point fan of P_μ with respect to the origin, which is contained in the interior of P_μ by Lemma 4.1. By Proposition 1.2 (1), $\mathscr{F}_\mu$ is the normal fan of the polar polytope P_μ^*.

All facets of $\mathscr{F}_\mu$ are unions of cones $C_{\mu'}$ for bases μ' of $\mathscr{B}$, which means, by Lemma 4.3, that $\mathscr{F}'(\mathscr{A})$ is a refinement of the fan $\mathscr{F}_\mu$. By the choice of ε, $\mathrm{conv}\{b_{\mu_1}, b_{\mu_2}, \ldots, b_{\mu_{n-d}}\}$ is a facet of P_μ, and hence C_μ is a maximal cone in $\mathcal{F}_\mu$. The pointed secondary fan can therefore be written as the intersection

$$\mathcal{F}'(\mathcal{A}) = \bigcap_{\mu \text{ basis of } \mathcal{A}} \mathcal{F}_\mu.$$

Proposition 1.2 (2) now implies that

$$\mathcal{F}'(\mathcal{A}) = \bigcap_{\mu \text{ basis of } \mathcal{A}} \mathcal{N}(P_\mu^*) = \mathcal{N}\left(\sum_{\mu \text{ basis of } \mathcal{A}} P_\mu^*\right). \tag{4.5}$$

We have proved that the Minkowski sum $\sum_\mu P_\mu^*$ is a secondary polytope. ∎

Actually, Proposition 1.2 implies that the P_μ^*'s in (4.5) can be replaced by arbitrary homothetic images $c_\mu P_\mu^*$. This describes the degrees of freedom in choosing a secondary polytope.

COROLLARY 4.4. *A polytope is a secondary polytope of $\mathcal{A}$ if and only if it is a translate of $\sum_\mu c_\mu P_\mu^* \in \mathbb{R}^{n-d}$ for some choice of positive numbers c_μ.*

We close this section by describing the secondary of the cyclic 4-polytope P with 8 vertices $\mathcal{A} = \{(1, i, i^2, i^3, i^4) \in \mathbb{R}^5 \colon i = 1, 2, ..., 8\}$. By Gale's evenness criterion ([13, 26]), the facets of $P = \text{conv}(\mathcal{A})$ are the following:

1234	1238	1245	1256	1267	1278	1348	1458	1568	1678
2345	2356	2367	2378	3456	3467	3478	4567	4578	5678.

Let $\mathcal{B} = \{b_1, b_2, ..., b_8\} \subset \mathbb{R}^3$ be a Gale transform of $\mathcal{A}$. We will represent $\mathcal{B}$ by an *affine Gale diagram* as in [25]. The resulting planar diagram is given in [25, Fig. 1] and in Fig. 2 below. We think of Fig. 2 as the northern hemisphere of a configuration on the 2-sphere. The points 1, 3, 5, 7 are contained in the northern hemisphere, while the points 2, 4, 6, 8 are contained in the southern hemisphere. However, these four southern points are represented on the northern hemisphere by their antipodal points $\bar{2}$, $\bar{4}$, $\bar{6}$, $\bar{8}$.

Now consider the pointed secondary fan $\mathcal{F}'(\mathcal{A})$ in $\mathbb{R}^3$, which is the multi-intersection of all cones $\text{pos}\{b_i, b_j, b_k\}$, where $1 \leq i < j < k \leq 8$. The resulting cell decomposition of the northern hemisphere is depicted in Fig. 2, while the cell decomposition of the southern hemisphere is obtained by symmetry. Altogether we get a polyhedral subdivision of the 2-sphere with 40 faces, 64 edges, and 26 vertices. Nine of the vertices (denoted 1, 3, 5, 7, a, b, c, d, e) are contained in the northern hemisphere, nine vertices (including 2, 4, 6, 8) are in the southern hemisphere, and eight vertices (denoted f, g, h, i, j, k, l, m) are on the equator, which is the line at infinity for the affine diagram in Fig. 2. Eight vertices are 7-valent, and 18 vertices are 4-valent.

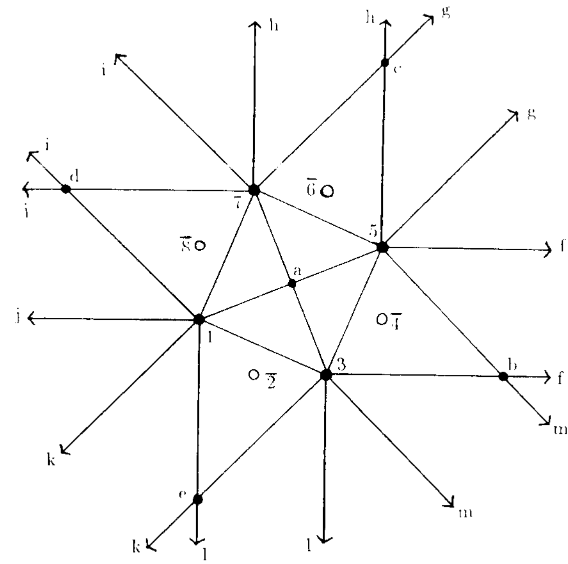

FIG. 2. Affine diagram of the secondary fan of the cyclic 4-polytope with 8 vertices.

These 40 faces are the maximal cells of the pointed secondary fan $\mathcal{F}'(\mathcal{A})$ and hence they correspond to the regular triangulations of the cyclic polytope P. Note that there are 32 triangular faces and 8 quadrilateral faces. We can use (4.2) to read off the regular triangulations Δ of P corresponding to the regions in Figure 2. Here are two examples. Consider the triangular region with vertices 3, 5, b. This region is the intersection of the positive bases

567 378 <u>358</u> <u>356</u> 237 <u>235</u> 178 158 156 123

on the sphere. The corresponding triangulation $\Delta_{3,5,b}$ of P consists of all 4-simplices with complementary index sets, i.e.,

$$\Delta_{3,5,b} = \{12348, 12456, \underline{12467}, \underline{12478}, 14568, \underline{14678}, 23456, 23467, 23478, 45678\}.$$

$\Delta_{3,5,b}$ is the *vertex triangulation* of P which is obtained by joining vertex 4 with all facets in its antistar. This can also been seen from the fact that $\bar{4}$ is contained in the region in question.

Let us now move to the adjacent region with vertices a, 3, 5. Crossing the line 35 corresponds to performing the bistellar operation supported on

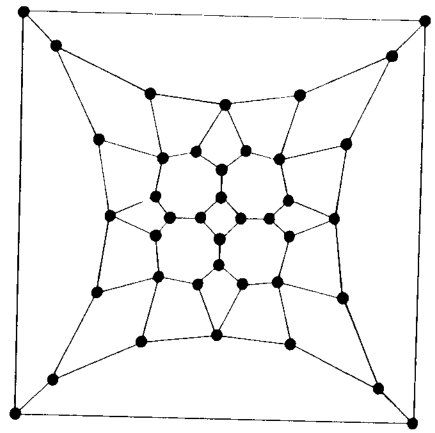

FIG. 3. Secondary polytope of the cyclic 4-polytope with 8 vertices.

the complementary index set 124678. The resulting regular triangulation of the cyclic polytope P equals

$$\Delta_{3,5,a} = \{12348, 12456, \underline{12468}, \underline{12678}, 14568, 23456, 23467, 23478, \underline{24678}, 45678\}.$$

The index sets involved in this bistellar operation are underlined in each case. In this manner we can easily construct all 40 regular triangulations of P.

The cell decomposition $\mathcal{F}'(\mathcal{A})$ is polar to the secondary polytope $\Sigma(\mathcal{A})$ of the cyclic 4-polytope with 8 vertices. This shows that $\Sigma(\mathcal{A})$ is a 3-polytope with 40 vertices, 64 edges and 26 facets. Eight of the facets are heptagons and 18 of the facets are quadrilaterals; 32 of the 40 vertices are 3-valent (corresponding to regular triangulations which admit three bistellar switches), while eight vertices are 4-valent (corresponding to regular triangulations with four possible bistellar switches). A Schlegel diagram of $\Sigma(\mathcal{A})$ is shown in Fig. 3.

5. On the Complexity of Secondary Polytopes

In this section we determine upper and lower bounds for the number of faces of the secondary polytope $\Sigma(\mathcal{A})$, and we discuss an optimal algorithm for computing its vertices and face lattice from the input data $\mathcal{A} \subset \mathbb{R}^d$. Our complexity bounds are sharp when $\mathcal{A}$ is the vertex set of a generic Lawrence polytope [1].

Here the main idea is a reduction to the well-understood case of hyperplane arrangements. As is customary, any (finite) *arrangement of hyperplanes* $\mathcal{H}$ in $\mathbb{R}^D$ is naturally identified with its polyhedral cell complex whose D-cells are the connected components of $\mathbb{R}^D \setminus (\bigcup \mathcal{H})$. An arrangement $\mathcal{H}$ is said to be *central* if all hyperplanes pass through the origin in $\mathbb{R}^D$. In this case it is convenient to identify antipodal regions and to think of $\mathcal{H}$ as an arrangement in projective $(D-1)$-space. A hyperplane arrangement in affine or projective D-space is called *simple* if every vertex is incident to precisely D hyperplanes.

For a comprehensive study of hyperplane arrangements from an enumerative point of view we refer to the monograph [27]. The following formulas due to Buck [4] follows as a special case from Zaslavsky's results (see [27, Sect. 5E]).

PROPOSITION 5.1 (Buck). (1) *The number of K-cells in a simple arrangement $\mathcal{H}$ of N hyperplanes in* projective *D-space equals* $f_K(\mathcal{H}) = \sum_{j=0}^{\lfloor D/2 \rfloor} \binom{N}{D-2j}\binom{D-2j}{D-K} = O(D^{D-K+1}N^D)$.

(2) *The number of* bounded *K-cells in a* simple *arrangement $\mathcal{H}$ of N hyperplanes in* affine *D-space equals* $f_K^{bd}(\mathcal{H}) = (D+1)/(N+K-D)\binom{D}{K}\binom{N}{D+1}$.

As in the previous section, let $\mathcal{B} = \{b_1, b_2, ..., b_n\} \subset \mathbb{R}^{n-d}$ be a Gale transform of the given affine point set $\mathcal{A} = \{a_1, a_2, ..., a_n\} \subset \mathbb{R}^d$. The k-faces of the secondary polytope $\Sigma(\mathcal{A})$ are in one-to-one correspondence with the $(n-d-k)$-cells of its normal fan, the pointed secondary fan $\mathcal{F}(\mathcal{A})$ in $\mathbb{R}^{n-d}$ (in this section we omit the "prime"). In Lemma 4.3 we saw that $\mathcal{F}(\mathcal{A})$ can be obtained as the multi-intersection of all simplicial cones $C_\mu = \text{pos}\{b_{\mu_1}, b_{\mu_2}, ..., b_{\mu_{n-d}}\}$, where μ ranges over all bases of $\mathcal{B}$.

Now let $\mathcal{H}_{\mathcal{B}}$ denote the central arrangement in $\mathbb{R}^{n-d}$ consisting of all hyperplanes which are spanned by subsets of $\mathcal{B}$ of rank $n-d-1$. If $\mathcal{A}$ and hence $\mathcal{B}$ are in general position, then the number N of hyperplanes in $\mathcal{H}_{\mathcal{B}}$ is $N = \binom{n}{n-d-1}$; otherwise we have $N < \binom{n}{n-d-1}$. Let $\mathcal{Z}_{\mathcal{B}}$ denote the zonotope which is the Minkowski sum of the N unit line segments perpendicular to the N hyperplanes in $\mathcal{H}_{\mathcal{B}}$.

LEMMA 5.2. (1) *The arrangement $\mathcal{H}_{\mathcal{B}}$ refines the pointed secondary fan $\mathcal{F}(\mathcal{A})$; i.e., $\mathcal{F}(\mathcal{A}) \prec \mathcal{H}_{\mathcal{B}}$.*

(2) *The secondary polytope $\Sigma(\mathcal{A})$ is a Minkowski summand of the zonotope $\mathcal{Z}_{\mathcal{B}}$, i.e., $\Sigma(\mathcal{A}) \prec \mathcal{Z}_{\mathcal{B}}$.*

(3) *If $\mathcal{B} = -\mathcal{B}$, then equality holds in both* (1) *and* (2).

Proof. Every linearly independent $(n-d-1)$-element subset $\{b_{v_1}, ..., b_{v_{n-d-1}}\}$ of $\mathcal{B}$ defines a linear form $l_v(x) = \det(b_{v_1}, ..., b_{v_{n-d-1}}, x)$ on $\mathbb{R}^{n-d}$. By definition, $\mathcal{H}_{\mathcal{B}}$ is the arrangement consisting of the hyperplanes $\{l_v(x) = 0\}$.

Given any basis μ of $\mathscr{B}$, then the cone C_μ is the intersection of $n-d$ supporting half-spaces of the form $\{l_\nu(x) \geq 0\}$. Each maximal cell of the pointed secondary fan $\mathscr{F}(\mathscr{A})$ is an intersection of C_μ's and can therefore be written as the intersection of half-spaces $\{l_\nu(x) \geq 0\}$. This proves claim (1).

Statement (2) follows directly from Proposition 1.2. To see statement (3), note that each cell of $\mathscr{H}_\mathscr{B}$ is of the form

$$\bigcap_\mu \{x \mid \pm l_\nu(x) \geq 0\} = \bigcap_\mu \bigcap_{\nu \subset \mu} \{x \mid \pm \det(b_{\nu_1}, ..., b_{\nu_{n-d-1}}, x) \geq 0\}$$
$$= \bigcap_\mu \operatorname{pos}\{\pm b_{\mu_1}, \pm b_{\mu_2}, ..., \pm b_{\mu_{n-d}}\}$$

for suitable choice of the signs of the b_{μ_i}. Thus if $\mathscr{B} = -\mathscr{B}$, then every maximal region of $\mathscr{H}_\mathscr{B}$ can be written as an intersection of the positive hulls C_λ of bases λ of $\mathscr{B}$, which proves (3). ∎

We remark that the converse of (3) does not hold. By adding one suitable vector to the centrally symmetric set $\mathscr{B}$ in Example 5.6, we can obtain a Gale transform $\bar{\mathscr{B}}$ of a 5-polytope $P = \operatorname{conv}(\mathscr{A})$ with 9 vertices such that P is not a Lawrence polytope (defined below) but its secondary polytope equals the zonotope $\mathscr{Z}_{\bar{\mathscr{B}}} = \mathscr{Z}_\mathscr{B} = \Sigma(\mathscr{A})$.

By combining Proposition 5.1 with Lemma 5.2 we shall obtain the desired upper bounds for the face numbers of secondary polytopes. We abbreviate $K := n-d-1-k$, $N := \binom{n}{n-d-1}$, and $D := n-d-1$. The number of k-faces of $\Sigma(\mathscr{A})$ equals the number of $(K+1)$-cells of $\mathscr{F}(\mathscr{A})$, and, by Lemma 5.2 (1), this number is bounded above by the number of $(K+1)$-cells of $\mathscr{H}_\mathscr{B}$. Since $\mathscr{H}_\mathscr{B}$ is a central arrangement of at most N hyperplanes in $\mathbb{R}^{D+1}$, the number of its $(K+1)$-cells is bounded above by twice the number of K-cells of a simple arrangement of N hyperplanes in projective D-space. This number is given in Proposition 5.1, and we conclude the following.

THEOREM 5.3. *The number of k-dimensional faces of the secondary polytope $\Sigma(\mathscr{A}) \subset \mathbb{R}^{n-d}$ of an affine point set $\mathscr{A} = \{a_1, a_2, ..., a_n\} \subset \mathbb{R}^d$ satisfies the inequality*

$$f_k(\Sigma(\mathscr{A})) \leq 2 \cdot \sum_{j=0}^{\lfloor (n-d-1)/2 \rfloor} \binom{\binom{n}{n-d-1}}{n-d-1-2j} \binom{n-d-1-2j}{k} = O(n^{(n-d-1)^2}).$$

If we regard the input dimension d as a constant, then we get a singly exponential lower bound already in the case $d=3$. If $\mathscr{A}$ is the vertex set of a convex n-gon, then by [16] the number of vertices of the *associahedron* $\Sigma(\mathscr{A})$ equals $(1/(n-1))\binom{2n-4}{n-2} = \Omega(n^{n-3})$. Here our singly exponential upper bound $O(n^{(n-2)^2})$ is only off by the square in the exponent.

We will next describe a construction which gives a tight lower bound when the dimension $r := n-d$ of the secondary polytope is considered fixed. Let $\mathscr{A} = \{a_1, a_2, ..., a_{d+r}\} \subset \mathbb{R}^d$ be an affine $(d-1)$-dimensional point set, and suppose that r is a constant. Now Theorem 5.3 can be rephrased as a polynomial upper bound in d for the size of the face lattice of $\Sigma(\mathscr{A})$.

COROLLARY 5.4. *The number of faces of the secondary polytope $\Sigma(\mathscr{A}) \subset \mathbb{R}^r$ is bounded above by $c(r) \cdot d^{(r-1)^2}$, where $c(r)$ is a constant which depends on r.*

A $(d-1)$-polytope $P = \operatorname{conv}(\mathscr{A})$ with $d+r$ vertices is called a *Lawrence polytope* if it has a centrally symmetric Gale transform $\mathscr{B} \subset \mathbb{R}^r$, i.e., if $d+r = 2s$ is even $(s \geq r)$ and $\mathscr{B} = \{b_1, b_2, ..., b_s, -b_1, -b_2, ..., -b_s\}$ for some vector configuration $\{b_1, b_2, ..., b_s\}$. (See [1] for details). We call P a *generic Lawrence polytope* if, in addition, the configuration $\{b_1, b_2, ..., b_s\}$ is in generic position in $\mathbb{R}^r$. Here we mean by "generic" that the coordinates of these s vectors are algebraically independent over the rational numbers. Note, conversely, that a generic spanning vector configuration $\{b_1, b_2, ..., b_s\} \in \mathbb{R}^r$ defines a generic Lawrence polytope of dimension $2s-r-1$ with $2s$ vertices. Hence there exist $(d-1)$-dimensional generic Lawrence polytopes with $d+r$ vertices, whenever $d+r$ is even and $r \leq d$.

LEMMA 5.5. *Let $\mathscr{A} = \{a_1, a_2, ..., a_{d+r}\} \subset \mathbb{R}^d$ be the vertex set of a generic Lawrence polytope, and let $2s = r+d$. Then the secondary polytope $\Sigma(\mathscr{A}) \subset \mathbb{R}^r$ is a zonotope with*

$$f_k(\Sigma(\mathscr{A})) = 2 \cdot \sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{\binom{s}{r-1}}{r-1-2j} \binom{r-1-2j}{k}$$
$$- 2 \cdot s \cdot \frac{r}{\binom{s-1}{r-2} - k} \binom{r-1}{r-1-k} \binom{\binom{s-1}{r-2}}{r}$$

k-dimensional faces for $k = 0, 1, ..., r-2$. The number of facets of $\Sigma(\mathscr{A})$ equals

$$f_{r-1}(\Sigma(\mathscr{A})) = 2 \cdot \sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{\binom{s}{r-1}}{r-1-2j} \binom{r-1-2j}{r-1}$$
$$- 2 \cdot s \cdot \left[\binom{\binom{s-1}{r-2}}{r-1} - 1 \right].$$

Proof. The Gale transform of $\mathscr{A}$ is a centrally symmetric vector configuration $\mathscr{B} = \{b_1, b_2, ..., b_s, -b_1, -b_2, ..., -b_s\} \subset \mathbb{R}^r$ in generic position. By Lemma 5.2 (3), the secondary fan $\mathscr{F}(\mathscr{A})$ equals the hyperplane arrangement $\mathscr{H}_\mathscr{B}$, and the secondary polytope $\Sigma(\mathscr{A})$ equals the zonotope

$\mathscr{Z}_{\#}$. We need to compute the number $f_k(\mathscr{Z}_{\#})$ which is equal to the number of $(r-k)$-cells of the central r-dimensional arrangement $\mathscr{H}_{\#}$. Let $\tilde{\mathscr{H}}_{\#}$ denote the induced arrangement in projective $(r-1)$-space. In this projective $(r-1)$-space we select a hyperplane not containing any vertex of $\mathscr{H}_{\#}$ to be "at infinity."

Let $U_i(r, s)$ be the number of i-cells in a simple arrangement of $\binom{s}{r-1}$ hyperplanes in projective $(r-1)$-dimensional space. If the arrangement $\tilde{\mathscr{H}}_{\#}$ were simple, then $f_k(\mathscr{Z}_{\#}) = 2 \cdot U_{r-k-1}(r, s)$. However, $\tilde{\mathscr{H}}_{\#}$ is not simple unless $r = s$, which is a trivial case.

Suppose for the moment that $r < s$. Then the vectors $b_1, ..., b_s$ do not correspond to simple vertices of $\tilde{\mathscr{H}}_{\#}$. However, since $\mathscr{B}$ was chosen to be generic all other vertices of $\tilde{\mathscr{H}}_{\#}$ are simple. If we perturb the arrangement $\tilde{\mathscr{H}}_{\#}$ slightly, so that it becomes simple, then we create additional *bounded* regions around each vertex b_j. The number of hyperplanes passing through each b_j equals $\binom{s-1}{r-2}$. Let $V_i(r, s)$ denote the number of *bounded* i-cells in an arrangement of $\binom{s-1}{r-2}$ hyperplanes in *affine* $(r-1)$-dimensional space. The process of perturbing $\tilde{\mathscr{H}}_{\#}$ to a simple arrangement creates $V_i(r, s)$ new i-dimensional regions around each vertex b_j for $i = 1, 2, ..., r-1$. This implies that

$$f_k(\mathscr{Z}_{\#}) = 2 \cdot U_{r-k-1}(r, s) - 2 \cdot s \cdot V_{r-k-1}(r, s) \qquad \text{for} \quad k = 0, 1, ..., r-2.$$

For $i = 0$ we have to discount the vertex b_j (which itself is a bounded 0-dimensional region), and we get

$$f_{r-1}(\mathscr{Z}_{\#}) = 2 \cdot U_0(r, s) - 2 \cdot s \cdot [V_0(r, s) - 1].$$

Since $V_i(s, s) = 0$, these two formulas are also valid in the special case $r = s$. From Proposition 5.1 we find that

$$U_i(r, s) = \sum_{j=0}^{\lfloor (r-1)/2 \rfloor} \binom{\binom{s}{r-1}}{r-1-2j} \binom{r-1-2j}{r-1-i}$$

and

$$V_i(r, s) = \frac{r}{\binom{s-1}{r-2} + i + 1 - r} \binom{r-1}{i} \binom{\binom{s-1}{r-2}}{r}.$$

This completes the proof of Lemma 5.5. ∎

To illustrate the formula in Lemma 5.5, we consider the smallest non-trivial example of a 4-dimensional Lawrence polytope.

EXAMPLE 5.6. Let $\mathscr{A} = \{a_1, a_2, ..., a_8\} \subset \mathbb{R}^5$ be the set of vertices of a prism over a tetrahedron, $\operatorname{conv}(\mathscr{A}) = \Delta_1 \times \Delta_3$. This 4-polytope is a generic Lawrence polytope because its Gale transform equals $\mathscr{B} = \{b_1, b_2, b_3, b_4, -b_1, -b_2, -b_3, -b_4\} \subset \mathbb{R}^3$, where the vectors b_1, b_2, b_3, b_4 are in general position (see [25, Fig. 4]). (In this easy example there is no difference between "general position" and "generic position".) The secondary polytope is a 3-dimensional zonotope with 6 zones. We can write

$$\Sigma(\Delta_1 \times \Delta_3) = \mathscr{Z}_{\#} = \{\lambda_{ij} \cdot b_i \times b_j \in \mathbb{R}^3 \colon 0 \leqslant \lambda_{ij} \leqslant 1,\ 1 \leqslant i < j \leqslant 4\},$$

where $b_i \times b_j$ denotes the ordinary cross product of vectors in 3-space. We compute the face numbers of $\Sigma(\Delta_1 \times \Delta_3)$ by specializing $r = 3$ and $s = 4$ in Lemma 5.5. The secondary polytope $\Sigma(\Delta_1 \times \Delta_3)$ has 24 vertices, 36 edges, and 14 facets. In particular, there are 24 regular triangulations of the prism over the tetrahedron.

With the same argument we can easily compute the f-vector of the secondary polytope of $\Delta_1 \times \Delta_d$ (the prism over the d-simplex) for any d. It is an important open problem to determine the secondary polytopes of general products of simplices [10, Sect. 7, Remark (d)].

In Example 5.6 we can see that the face numbers of $\Sigma(\Delta_1 \times \Delta_3)$ are smaller than the face numbers of the secondary polytope of the cyclic 4-polytope with 8 vertices (determined in Section 4). However, when r is fixed and $d \to \infty$ then the secondary polytopes of generic Lawrence polytopes have the maximum number of faces.

THEOREM 5.7. *Let $F_r(d)$ denote the maximum number of faces of a secondary polytope $\Sigma(\mathscr{A}) \subset \mathbb{R}^r$ as $\mathscr{A}$ ranges over all $(d+r)$-element sets in $\mathbb{R}^d$. There exist constants $c_1(r)$ and $c_2(r)$ (depending on the dimension) such that $c_1(r) \cdot d^{(r-1)^2} \leqslant F_r(d) \leqslant c_2(r) \cdot d^{(r-1)^2}$.*

Proof. The upper bound was proved in Corollary 5.4. The lower bound is clear for $r = 1$ and $r = 2$; for $r \geqslant 3$, we use Lemma 5.5. First observe that there exist generic Lawrence polytopes for fixed r and $d \to \infty$ whenever $d + r$ is even. Consider the term corresponding to $j = 0$ in the sums of Lemma 5.5. This term equals

$$\binom{\binom{r+d-2}{r-1}}{r-1} \binom{r-1}{k},$$

and hence it is bounded below by $c(r, k) \cdot d^{(r-1)^2}$. All other terms in this sum are of lower order in d. The negative correction term can easily be bounded above by $c'(r, k) \cdot d^{(r-2)r}$. Hence the number of k-faces is bounded below by $c''(r, k) \cdot d^{(r-1)^2}$. Here c, c', c'' are constants depending on r and k. ∎

From this analysis we also get an optimal algorithm for computing the face lattice of the secondary polytope $\Sigma(\mathscr{A})$ (when its dimension r is regarded as a constant). We refer to the book of Edelsbrunner [6] for a precise notion of geometric algorithms and their complexity. In particular, in [6, Chap. 7] we find the following result due to Edelsbrunner, O'Rourke, and Seidel [7].

PROPOSITION 5.8 (Edelsbrunner, O'Rourke, and Seidel). *The face lattice of an affine arrangement $\mathscr{H}$ of N hyperplanes in $\mathbb{R}^D$ can be computed in $O(N^D)$ time.*

As a result we get that the face lattice of a *central* arrangement $\mathscr{H}$ of N hyperplanes in $\mathbb{R}^r$ can be computed in $O(N^{r-1})$ time. As can be seen from [6, Chap. 7], this algorithm also generates a test point in the relative interior of each cell of $\mathscr{H}$ at the same cost.

In order to compute the face lattice of the secondary polytope $\Sigma(\mathscr{A})$, we proceed as follows. We first compute a Gale transform $\mathscr{B}$ for $\mathscr{A}$. This can be done in $O(c(r)\cdot d^3)$ time. Then we compute the arrangement $\mathscr{H}_{\mathscr{B}}$, which requires $O(\binom{r+d}{r-1}^{r-1}) = O(c'(r)\cdot d^{(r-1)^2})$ time. Finally, we need to identify k-cells of $\mathscr{H}_{\mathscr{B}}$ which correspond to the same k-cell of $\mathscr{F}(\mathscr{A})$. We now sketch a method for performing this identification in time $O(c''(r)\cdot d^{(r-1)^2})$. All details (e.g., efficient data structures, etc.) will be omitted here.

For every $(r-1)$-cell (or subfacet) F of $\mathscr{H}_{\mathscr{B}}$ we need to decide whether F should be removed. To do so, consider all linearly independent $(r-1)$-element subsets $\{b_{v_1}, ..., b_{v_{r-1}}\}$ of $\mathscr{B}$. If F is not contained in any $\mathrm{pos}\{b_{v_1}, ..., b_{v_{r-1}}\}$, then we remove F, otherwise we keep it. The time required for each of the $O(d^{(r-1)^2})$ containment tests depends only on the dimension r. We conclude this section by stating our main computational result.

COROLLARY 5.9. *The face lattice of the secondary polytope $\Sigma(\mathscr{A})\subset\mathbb{R}^r$ of an affine point set $\mathscr{A} = \{a_1, a_2, ..., a_{d+r}\}\subset\mathbb{R}^d$ can be computed in optimal $O(d^{(r-1)^2})$ time, when r is regarded as a constant.*

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