

Note

On Vector Partition Functions

BERND STURMFELS*

Department of Mathematics, University of California, Berkeley, California 94720

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We present a structure theorem for vector partition functions. The proof rests on a formula due to Peter McMullen for counting lattice points in rational convex polytopes. © 1995 Academic Press, Inc.

INTRODUCTION

Let $A = (a_1, \dots, a_n)$ be a $d \times n$ -matrix of rank d with entries in \mathbf{N} , the set of non-negative integers. The corresponding *vector partition function* $\phi_A : \mathbf{N}^d \rightarrow \mathbf{N}$ is defined as follows: $\phi_A(u)$ is the number of non-negative integer vectors $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$ such that $A \cdot \lambda = \lambda_1 a_1 + \dots + \lambda_n a_n = u$. Equivalently, the function ϕ_A is defined by the formal power series:

$$\begin{aligned} & \prod_{i=1}^n \frac{1}{1 - t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_d^{a_{id}}} \\ &= \sum_{u \in \mathbf{N}^d} \phi_A(u_1, \dots, u_d) \cdot t_1^{u_1} t_2^{u_2} \cdots t_d^{u_d}. \end{aligned} \quad (1)$$

Vector partition functions appear in many areas of mathematics and its applications, including representation theory [9], commutative algebra [14], approximation theory [4] and statistics [5].

It was shown by Brakley [2] that there exists a finite decomposition of \mathbf{N}^d such that ϕ_A is a polynomial of degree $n - d$ on each piece. Here we describe such a decomposition explicitly and we analyze how the polynomials differ from piece to piece. Our construction uses the geometric decomposition into chambers studied by Aleksevskaya, Gel'fand and Zelevinsky in [1]. Within each chamber we give a formula which refines

* E-mail address: bernd@math.berkeley.edu.

the results by Dahmen and Micchelli in [4, §3]. The objective of this note is to provide polyhedral tools for the efficient computation of vector partition functions, with a view towards applications, such as the sampling algorithms in [6].

EXAMPLE ($n = 6$, $d = 3$). Consider the vector partition function

$$\phi_A : \mathbf{N}^3 \rightarrow \mathbf{N}, (u, v, w) \mapsto \# \{ \lambda \in \mathbf{N}^6 : A \cdot \lambda = (u, v, w)^t \}$$

associated with the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

In this instance the value of ϕ_A does not depend on the permutation of (u, v, w) , so we may assume that $u \geq v \geq w$. Also, if $u + v + w \equiv 1 \pmod 2$ then $\phi_A(u, v, w) = 0$, so we shall assume that $u + v + w \equiv 0 \pmod 2$. Given these assumptions, we distinguish two cases:

Case 1. $u \geq v + w$. Then

$$\begin{aligned} \phi_A(u, v, w) &= \frac{vw}{2} + \frac{vw^2}{8} - \frac{w^3}{24} \\ &+ \begin{cases} 1 + v/2 + 2w/3 & \text{if } u \equiv 0 \pmod 2 \text{ and } v \equiv 0 \pmod 2, \\ 1/2 + v/2 + 5w/12 & \text{if } u \equiv 1 \pmod 2 \text{ and } v \equiv 1 \pmod 2, \\ 1/2 + 3v/8 + 13w/24 & \text{otherwise.} \end{cases} \end{aligned}$$

Case 2. $u < v + w$. We set

$$\begin{aligned} \psi &:= -u^2/8 + uv/4 + uw/4 - v^2/8 + vw/4 - w^2/8 \\ &+ u^3/48 - u^2v/16 - u^2w/16 + uv^2/16 + uwv/8 + uw^2/16 \\ &- v^3/48 - v^2w/16 + vw^2/16 - w^3/16. \end{aligned}$$

Then

$$\begin{aligned} \phi_A(u, v, w) &= \psi + \begin{cases} 1 + u/6 + v/3 + w/2 & \text{if } u \equiv 0 \pmod 2 \text{ and } v \equiv 0 \pmod 2, \\ 1/2 + u/6 + v/3 + w/4 & \text{if } u \equiv 1 \pmod 2 \text{ and } v \equiv 1 \pmod 2, \\ 1/2 + u/6 + 5v/24 + 3w/8 & \text{otherwise.} \end{cases} \end{aligned}$$

Already this simple example illustrates the main feature of vector partition functions, which is the interplay of a structure of convex polyhedra (as seen in the distinction of cases 1 and 2) with a structure of finite abelian groups (as seen in the "mod"-subcases).

In order to deal with the general case, we introduce some notation. Let $\text{pos}(A) = \{\sum_{i=1}^n \lambda_i a_i \in \mathbf{R}^n : \lambda_i \geq 0\}$. For $\sigma \subset [n] := \{1, \dots, n\}$ we consider the submatrix $A_\sigma := (a_i : i \in \sigma)$, the polyhedral cone $\text{pos}(A_\sigma)$, and the abelian group $\mathbf{Z}A_\sigma$ spanned by the columns of A_σ . We may assume without loss of generality that A is surjective over \mathbf{Z} , that is, $\mathbf{Z}A = \mathbf{Z}^d$. This implies that the semigroup $\mathbf{N}A := \text{pos}(A) \cap \mathbf{Z}A$ is saturated. Why? what's that?

The surjectivity assumption does not hold for the 3×6 -matrix in our example. In order to apply the results below to such a case, one must choose a rational 3×3 -matrix B which defines an isomorphism from $\mathbf{Z}A$ onto \mathbf{Z}^3 and then use the formula $\phi_A(u) = \phi_{B \cdot A}(Bu)$.

A subset σ of $[n]$ is a *basis* if $\#(\sigma) = \text{rank}(A_\sigma) = d$. The *chamber complex* is the polyhedral subdivision of the cone $\text{pos}(A)$ which is defined as the common refinement of the simplicial cones $\text{pos}(A_\sigma)$, where σ runs over all bases. Each *chamber* C (meaning: maximal cell in the chamber complex) is indexed by the set $d(C) = \{\sigma \subset [n] : C \subseteq \text{pos}(A_\sigma)\}$. For each $\sigma \in d(C)$, the group $\mathbf{Z}A_\sigma$ has finite index in \mathbf{Z}^d ; write $G_\sigma := \mathbf{Z}^d / \mathbf{Z}A_\sigma$ for the group of residue classes. We say that σ is *non-trivial* if $G_\sigma \neq \{0\}$. For $u \in \text{pos}(A) \cap \mathbf{N}^d$, let $[u]_\sigma$ denote the image of u in G_σ .

In the small example above there are 12 chambers; they are grouped into two equivalence classes with respect to the S_3 -symmetry. The following theorem is our main result.

THEOREM 1. *For each chamber C there exists a polynomial P of degree $n-d$ in $u = (u_1, \dots, u_n)$, and for each non-trivial $\sigma \in d(C)$ there exists a polynomial Q_σ of degree $\#(\sigma) - d$ in u and a function $\Omega_\sigma : G_\sigma \setminus \{0\} \rightarrow \mathbf{Q}$ such that, for all $u \in \mathbf{N}A \cap C$,*

$$\phi_A(u) = P(u) + \sum \{Q_\sigma([u]_\sigma) \cdot Q_\sigma(u) : \sigma \in d(C) \text{ and } [u]_\sigma \neq 0\}.$$

Moreover, the "corrector polynomials" Q_σ satisfy the linear partial differential equations

$$\sum_{i=1}^d a_{u_i} \frac{\partial Q_\sigma}{\partial u_i} \equiv 0 \quad \text{for all } j \in \sigma \text{ such that } \sigma \setminus \{j\} \notin d(C).$$

Remarks. (1) Theorem 1 provides a generalization of the theory of *determinants* (the $d=1$ case) which can be found in Comtet's book [3, §2.6]. A nice MAPLE package for computing determinants has been implemented by P. Lisonek [10].

That's our case!!!

(2) Another important special case occurs when the matrix A is *unimodular*, which means that $G_\sigma = \{0\}$ for each basis σ . In this case ϕ_A is a polynomial function on each chamber [4, Corollary 3.1]. This happens, for instance, in the problem of counting non-negative integer matrices with prescribed row and column sums; see [5] for a general survey and see [13] for the computational state of the art. $\leftarrow \text{Ge} + i +$

(3) The main point of our formula is the "additive decoupling" of the correction term, which generalizes Theorem C in [3, §2.6]. The results of Dahmen and Micchelli in [4, §3] generalize a (somewhat weaker) classical theorem of Bell [3, §2.6, Thm. B]. The computational advantage of the additive decoupling is explained on page 114 in [3].

THE PROOF

We shall use notation which is standard in the theory of toric varieties; see e.g. [7]. Let N be a lattice of rank m , $M = \text{Hom}(N, \mathbf{Z})$ its dual lattice, and $N_\mathbf{Q}$ and $M_\mathbf{Q}$ the corresponding rational vector spaces. Suppose we are given a complete simplicial fan Σ in N having n rays, and non-zero lattice points b_1, \dots, b_n on these rays. (The b_i need not be primitive in N !) We identify the cones of Σ with subsets of $\{b_1, \dots, b_n\}$. For each $1 \leq l \leq m$ and each cone $\tau = \{b_{\tau_1}, \dots, b_{\tau_l}\}$, we let H_τ denote the group \mathbf{Z}^τ of integer valued functions on τ modulo the subgroup of those functions on τ which are restrictions from $M = \text{Hom}(N, \mathbf{Z})$. We say that τ is *non-trivial* if $H_\tau \neq \{0\}$. Consider any convex polytope of the form

$$P_\gamma = \{x \in M_\mathbf{Q} : \langle x, b_i \rangle \leq \gamma_i \text{ for } i = 1, \dots, n\}, \quad (2)$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ ranges over the set $C(\Sigma)$ of all vectors in \mathbf{Z}^n such that the normal fan of P_γ is coarser or equal to Σ . It is well known that there exists a polynomial function $F = F(\gamma)$ on $C(\Sigma)$ of degree m such that $\#(P_\gamma \cap M) = F(\gamma)$ provided that P_γ is integral, i.e., all vertices of P_γ lie in M . For a toric proof of this fact see e.g. [8, §5]. In general, however, the polytope P_γ is not integral, since the fan Σ is not assumed to be smooth. The following proposition characterizes the difference between $\#(P_\gamma \cap M)$ and the polynomial $F(\gamma)$ in the general case. If $\gamma \in \mathbf{Z}^n$, then we write $[\gamma]_\tau$ for the image of the function $\tau \rightarrow \mathbf{Z}$, $b_i \mapsto \gamma_i$, in the group H_τ .

PROPOSITION 2. *For every non-trivial cone $\tau \in \Sigma$ there exists a polynomial R_τ of degree $m - \#(\tau)$ in the variables $\gamma = (\gamma_1, \dots, \gamma_n)$ and a function $\omega_\tau : H_\tau \setminus \{0\} \rightarrow \mathbf{Q}$ such that $\#(P_\gamma \cap M) - F(\gamma) = \sum \{\omega_\tau([\gamma]_\tau) \cdot R_\tau(\gamma) : \tau \in \Sigma \text{ and } [\gamma]_\tau \neq 0\}$ for all $\gamma \in C(\Sigma)$. Moreover, the polynomial R_τ depends only on those variables γ_i for which $\tau \cup \{b_i\} \in \Sigma$.*

Sasha has one elementary

Lothar's point counting: a case study in expected Algebraic Geometry

We shall use the following theorem of McMullen. If F is a face of a polytope $P \subset M_Q$ then $v(P, F)$ denotes the cone in N normal to F at P . Let \mathcal{L} denote the set of all pairs (τ, L) where τ is a cone in N and L is an affine subspace of M_Q which is a translate of τ^\perp .

THEOREM 3 (McMullen [11]). *There exists a function $\theta : \mathcal{L} \rightarrow \mathbb{Q}$ such that $\theta(\tau, L) = \theta(\tau, L + m)$ for all $m \in M$ and*

$$\#(P \cap M) = \sum_{F \text{ face of } P} \theta(v(P, F), \text{aff}(F)) \cdot \text{Vol}(F) \text{ for every polytope } P \text{ in } M_Q.$$

Here “Vol” denotes the standard volume form on the affine span $\text{aff}(F)$ of the face F .

Proof. This is a special case of Theorem 3 in [11], provided one passes from simple valuations to general valuations using the technique in §3 of [11]. ■

COROLLARY 4. *If P_γ is an integral polytope then the number of lattice points in P_γ equals*

$$F(\gamma) = \sum_{\tau \in \Sigma} \theta(\tau, \tau^\perp) \cdot \text{Vol}(P_\gamma), \quad (3)$$

where P_γ denotes the face of P_γ supported by τ .

Proof. If P_γ is integral then $\text{aff}(P_\gamma)$ is a lattice translate of the linear subspace τ^\perp . Therefore $\theta(\tau, \text{aff}(P_\gamma)) = \theta(\tau, \tau^\perp)$, and the claim follows directly from Theorem 3. ■

We remark that formula (3) is a valid presentation for the polynomial function $F(\gamma)$ throughout the cone $C(\Sigma)$, not just for those special values of γ for which P_γ is integral.

Proof of Proposition 2. Let τ be a cone in Σ and let F_γ be the corresponding face of P_γ . As γ runs over $C(\Sigma)$, the volume of F_γ varies as a polynomial in γ of degree $\dim(F_\gamma) = m - \#(\tau)$. We set $R_\tau(\gamma) := \text{Vol}(F_\gamma)$. This function is independent of a support parameter γ_i if the hyperplane $\langle x, b_i \rangle = \gamma_i$ does not intersect the face F_γ for general γ . The latter condition is equivalent to $\tau \cup \{b_i\}$ not being a cone of Σ . Hence R_τ has the property asserted in the second part of Proposition 2.

Consider any other vector $\gamma' \in C(\Sigma)$ and corresponding face $F_{\gamma'}$ of $P_{\gamma'}$. Note that $\text{aff}(F_\gamma) = \{y \in M : \forall b_i \in \tau : \langle y, b_i \rangle = \gamma_i\}$, and similarly for $F_{\gamma'}$. This implies

$$\begin{aligned} [\gamma]_\tau &= [\gamma']_\tau \Leftrightarrow \exists u \in M : \forall b_i \in \tau : \gamma_i = \gamma'_i + \langle u, b_i \rangle \\ &\Leftrightarrow \exists u \in M : \text{aff}(F_\gamma) = \text{aff}(F_{\gamma'}) + u. \end{aligned}$$

We can therefore define a function $\omega_\tau : H_\tau \setminus \{0\} \rightarrow \mathbb{Q}$ by setting

$$\omega_\tau([\gamma]_\tau) := \theta(\tau, \text{aff}(F_\gamma)) - \theta(\tau, \tau^\perp).$$

Proposition 2 now follows immediately from Theorem 3 and Corollary 4. ■

Proof of Theorem 1. We shall use representation techniques as in [12, §5]. Let $B = (b_1, \dots, b_n)$ be an integer $(n-d) \times n$ -matrix whose row space (over \mathbb{Z}) equals the kernel of A . In other words, we construct a short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B'} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow 0.$$

We set $m = n - d$ and $M = \mathbb{Z}^{n-d}$, and we consider the polytope P_γ in (2), for an arbitrary $\gamma \in \mathbb{Z}^n$. The map $x \mapsto \gamma - B' \cdot x$ defines a bijection between the lattice points in P_γ and the set of elements $\lambda \in \mathbb{N}^m$ such that $A \cdot \lambda = A \cdot \gamma$. Therefore we have

$$\phi_A(A \cdot \gamma) = \#(P_\gamma \cap M). \quad (4)$$

We now fix a chamber C and we consider those vectors $\gamma \in \mathbb{N}^n$ such that $A \cdot \gamma$ lies in the interior of C . This determines the normal fan Σ of P_γ as follows:

$$\Sigma = \{ \{b_{\alpha_1}, \dots, b_{\alpha_k}\} : [n] \setminus \{\tau_1, \dots, \tau_k\} \in \mathcal{D}(C) \}.$$

Let us now fix $\sigma \in \mathcal{D}(C)$ and set $\tau := [n] \setminus \sigma$. In order to derive the first part of Theorem 1 from Proposition 2, it suffices to show that there exists a group isomorphism δ between H_τ and G_σ , which takes a class $[\gamma]_\tau$ in H_τ to the class of $[u]_\sigma$ in G_σ , where $u := A \cdot \gamma$. Indeed, in view of (4), we can then simply define $P(u) := F(\gamma)$, $\Omega_\sigma([u]_\sigma) := \omega_\tau([\gamma]_\tau)$, and $Q_\sigma(u) := R_\tau(\gamma)$ to get the desired formula for $\phi_A(u)$. (Note that $\#(\sigma) - d = m - \#(\tau)$.)

To define the group isomorphism δ , we consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}^\sigma \xrightarrow{i} \mathbb{Z}^n \xrightarrow{\pi} \mathbb{Z}^\tau \longrightarrow 0,$$

where i and π are the obvious coordinate inclusion and projection respectively. We have

$$H_\tau = \text{coker}(\pi \circ B') \quad \text{and} \quad G_\sigma = \text{coker}(A \circ i).$$

Consider any element of G_σ , given by a representative $u \in \mathbb{Z}^d$. We define $\delta(u)$ to be $\pi(\gamma)$, where γ is any preimage of u under A . This defines a unique

element of H_τ because γ is well-defined up to $\text{im}(B') = \ker(A)$. We have the equivalences

$$u \in Z_{A_\sigma} \Leftrightarrow u \text{ has a preimage } \tilde{\gamma} \text{ under } A \text{ such that } \pi(\tilde{\gamma}) = 0$$

$$\Leftrightarrow \gamma - \tilde{\gamma} \in \ker(A) = \text{im}(B')$$

$$\text{for some } \tilde{\gamma} \in Z'' \text{ such that } \pi(\tilde{\gamma}) = 0$$

$$\Leftrightarrow \pi(\gamma) = \pi(B' \cdot \lambda) \text{ for some } \lambda \in Z''^{-d}.$$

This shows that u is zero in G_σ if and only if $\pi(\gamma) = \delta(u)$ is zero in H_τ . Therefore the group homomorphism δ is injective. But it is also surjective: if $v \in Z'$, then choose any $w \in Z$, consider $v + w \in Z''$ and define $u = A(v + w)$. Then $\delta(u)$ and v represent the same element of H_τ . This completes the proof of the first part of Theorem 1.

To prove the second part we note that an element $j \in \sigma$ satisfies $\sigma \setminus \{j\} \notin d(C)$ if and only if $\tau \cup \{b_j\} \notin \Sigma$. For such an index j , we apply the operator $\partial/\partial\gamma_j$ to the polynomial

$$R_\tau(\gamma) = Q_\sigma(A \cdot \gamma).$$

The result is zero, by Proposition 2, and consequently $\sum_{i=1}^d a_{ij} (\partial Q_\sigma / \partial u_i) \equiv 0$, as required. ■

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