Best Regards,
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Counterexamples to the Connectivity Conjecture of the mixed cells

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In [4] a conjecture concerning the connectivity of mixed cells of subdivisions for Minkowski sums of polytopes was formulated. This conjecture was, in fact, originally proposed by P. Pedersen [3]. It turns out that a positive confirmation of this conjecture can speed up the algorithm for the "dynamical lifting" developed in [4] substantially. In the mean time, when the polyhedral method is used for solving polynomial systems by homotopy continuation methods [2], the positiveness of this conjecture also plays an important role in the efficiency of the algorithm. Very unfortunately, we found that this conjecture is inaccurate in general. In section 2, a counterexample is presented for a general subdivision. In section 3, another counterexample shows that even restricted to "regular" subdivisions induced by liftings, this conjecture still fails to be true.

1 Counterexample for general subdivisions

For $m \leq n$, let $S = (S_1, ..., S_m)$ be a sequence of finite subsets of R^n whose union affinely spans R^n. For $Q_i = conv(S_i)$, the convex hull of $S_i, i = 1, ..., m$, their *Minkowski* sum is defined by

$$Q_1 + \cdots + Q_m = \{x_1 + \cdots + x_m | x_i \in Q_i \ for \ i = 1, ..., m\}.$$

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By a *cell* of S, we mean a tuple $C = (C_1, ..., C_m)$ of subsets $C_i \subset S_i$ for $i = 1, ..., m$. Define

$$type(C) := (dim(conv(C_1)), ..., dim(conv(C_m))) \in N^m$$
$$conv(C) := conv(C_1 + \cdots + C_m) \subset R^n.$$

Here, N is the set of natural numbers. Cells of the same type will be called the *mixed cells* of that type. A *face* of a cell C is a subcell $F = (F_1, ..., F_m)$ where $F_i \subset C_i$ and some linear functional $\alpha \in (R^n)^v$ attains its minimum over C_i at F_i for $i = 1, ..., m$. Such an α is called an *inner normal* of F. If F is a face of C then $conv(F_i)$ is a face of the polytope $conv(C_i)$ for $i = 1, ..., m$.

Definition: *A* fine mixed subdivision *of S is a collection* $A = \{C^{(1)}, ..., C^{(k)}\}$ *of cells such that*

(a) $dim(conv(C^{(j)})) = n$ *for all* $j = 1, ..., k$,

(b) $conv(C^{(j)}) \cap conv(C^{(l)})$ *is a proper common face of* $conv(C^{(j)})$ *and* $conv(C^{(l)})$ *when it is nonempty for* $j \neq l$,

(c) $\cup_{j=1}^{k} conv(C^{(j)}) = conv(S)$,

(d) For $j = 1, ..., k$, *write* $C^{(j)} = (C_1^{(j)}, ..., C_m^{(j)})$. *Then, each* $conv(C_i^{(j)})$ *is a simplex of dimension* $\#C_i^{(j)} - 1$ *and for each j,* $dim(conv(C_1^{(j)})) + \cdots + dim(conv(C_n^{(j)})) = n$.

Remark: In fact, the notation $S = (S_1, \ldots, S_m)$ is just a short hand abstraction of

$$conv(S) = conv(S_1 + \cdots + S_m). \quad (1)$$

So is a cell $C = (C_1, \ldots, C_m)$ in the subdivision of S where $C_i \subset S_i$ for $i = 1, \ldots, m$; namely, it means

$$conv(C) = conv(C_1 + \cdots + C_m). \quad (2)$$

Later, when we draw figures for $S = (S_1, \ldots, S_m)$ or subdivisions of S, they actually represent (1) and (2).

For a given fine mixed subdivision A of $S = (S_1, ..., S_m)$, let $\tilde{A} = \{B^{(1)}, ..., B^{(l)}\} \subset A$ be the set of all mixed cells in A having the same type $(k_1, ..., k_m)$ with $k_i > 0$ for all $i = 1, ..., m$. Write $B^{(j)} = (B_1^{(j)}, ..., B_m^{(j)})$ for $j = 1, ..., l$. Two cells D and E

in $\tilde{A}$ are said to be *connected* if there exists a sequence of cells $\{B^{(j_1)}, ..., B^{(j_d)}\}$ in $\tilde{A}$ such that

a. $B^{(j_1)} = D$ and $B^{(j_d)} = E$, and

b. $dim(conv(B_i^{(j_p)}) \cap conv(B_i^{(j_{p+1})})) \geq k_i - 1$ for all $i = 1, ..., m$ and $p = 1, ..., d-1$.

It is conjectured in [3, 4] that all the mixed cells in $\tilde{A}$ are connected. The following example shows that this conjecture may not hold in general.

Example 1 *Let*

$$S_1 = \left\{ a = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$$

and

$$S_2 = \left\{ e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, g = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

$Q_1 = conv(S_1)$, $Q_2 = conv(S_2)$ and $Q_1 + Q_2$ are shown in Fig. 1.

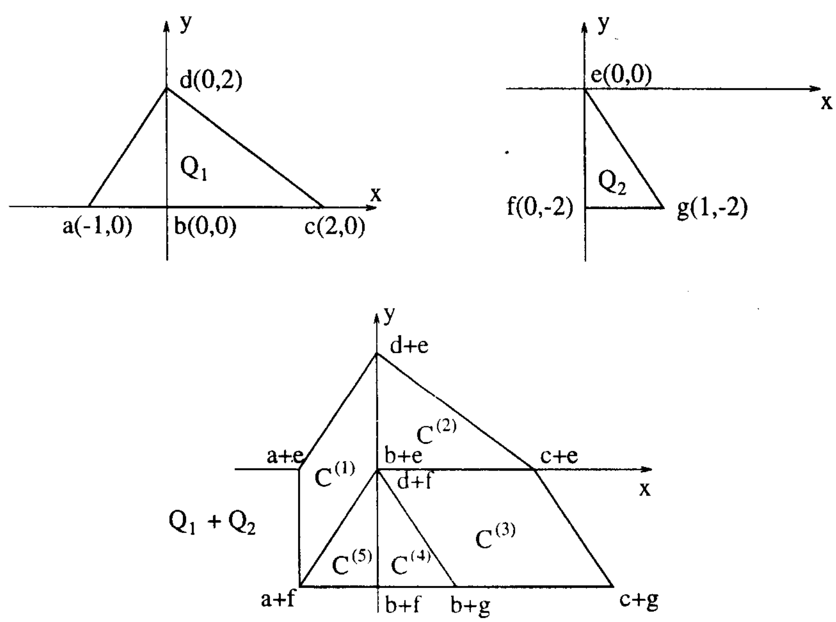


Figure 1

Consider the collection of cells $C = (C^{(1)}, C^{(2)}, C^{(3)}, C^{(4)}, C^{(5)})$, where

$$C^{(1)} = \{(a,d),(e,f)\}, \quad C^{(2)} = \{(b,d,c),(e)\}, \quad C^{(3)} = \{(b,c),(e,g)\},$$
$$C^{(4)} = \{(b),(e,f,g)\}, \quad C^{(5)} = \{(a,b,d),(f)\}.$$

Clearly, C is a fine mixed subdivision of $S = (S_1, S_2)$ and cells $C^{(1)}$ and $C^{(3)}$ are the only cells of type $(1,1)$. They are not connected because $(a,d) \cap (b,c) = \emptyset$.

2 Counterexample for regular subdivisions

For $S = (S_1, ...S_m)$ as in §1, each S_i is a finite subset of R^n, choose real-valued functions $\omega_i : S_i \to R$ for $i = 1, ..., m$. The m-tuple $\omega = (\omega_1, ..., \omega_m)$ is called a *lifting function* on S. We say that ω lifts S_i to its graph $\widehat{S_i} = \{(q, \omega_i(q)) : q \in S_i\} \subset R^{n+1}$. This notion is extended in the obvious way: $\hat{S} = (\widehat{S_1}, ..., \widehat{S_m}), \widehat{Q_i} = conv(\widehat{S_i}), \hat{Q} = \widehat{Q_1} + \cdots + \widehat{Q_m}$, etc. ...

Let S_ω be the set of cells C of S which satisfy

a. $dim(conv(\hat{C})) = n$, and

b. $\hat{C}$ is a facet, face of dimension n, of $\hat{S}$ whose inner normals $\alpha \in (R^{n+1})^v$ have positive last coordinate.

In other words, $conv(\hat{C})$ is a facet of the lower hull of $\hat{Q}$. It is known [1] that S_ω is a subdivision of S and for a generic lifting function ω, S_ω is always a fine mixed subdivision. A subdivision induced by a lifting function is called a *regular subdivision.*

When the polyhedral method is used for solving polynomial systems by homotopy continuation methods [2], the lifting function ω provides a nonlinear homotopy for the algorithm and its induced subdivision S_ω for the supports of the polynomials plays an essential role for the whole process of the algorithm. For the efficiency of the algorithm, it is very desirable that the mixed cells of a fixed type of this subdivision are connected. In fact, P. Pedersen showed [3] that when $n = 2$, mixed cells of a fixed type are indeed connected for regular subdivisions. The idea of the proof was sketched in [4]. Unfortunately, the following example shows this conjecture is incorrect when $n = 3$.

Example 2 *Let*

$$S_1 = \left\{ a_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$S_2 = \left\{ b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

$$S_3 = \left\{ c_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, c_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, c_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, c_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and its lifting

$$\widehat{S_1} = \left\{ \begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \begin{pmatrix} a_2 \\ 0 \end{pmatrix}, \begin{pmatrix} a_3 \\ 0 \end{pmatrix} \right\},$$

$$\widehat{S_2} = \left\{ \begin{pmatrix} b_1 \\ -2 \end{pmatrix}, \begin{pmatrix} b_2 \\ -2 \end{pmatrix}, \begin{pmatrix} b_3 \\ -4 \end{pmatrix} \right\},$$

$$\widehat{S_3} = \left\{ \begin{pmatrix} c_1 \\ 1 \end{pmatrix}, \begin{pmatrix} c_2 \\ 0 \end{pmatrix}, \begin{pmatrix} c_3 \\ 1 \end{pmatrix}, \begin{pmatrix} c_4 \\ 0 \end{pmatrix} \right\}.$$

The regular subdivision of $S = (S_1, S_2, S_3)$ induced by the above lifting is shown in Fig.2. In this fine mixed subdivision, there are 8 cells, they are:

$$\begin{aligned} C^{(1)} &= \{(a_1), (b_1, b_2, b_3), (c_2, c_4)\}, & C^{(2)} &= \{(a_1), (b_2, b_3), (c_2, c_3, c_4)\}, \\ C^{(3)} &= \{(a_1), (b_1, b_3), (c_1, c_2, c_4)\}, & C^{(4)} &= \{(a_1, a_3), (b_2, b_3), (c_3, c_4)\}, \\ C^{(5)} &= \{(a_1, a_2), (b_1, b_3), (c_1, c_2)\}, & C^{(6)} &= \{(a_1, a_3), (b_3), (c_2, c_3, c_4)\}, \\ C^{(7)} &= \{(a_1, a_2), (b_3), (c_1, c_2, c_4)\}, & C^{(8)} &= \{(a_1, a_2, a_3), (b_3), (c_2, c_4)\}. \end{aligned}$$

(See Fig.3) Clearly, there are two cells of type $(1,1,1)$, they are $C^{(4)}$ and $C^{(5)}$. On the other hand,

$$\alpha_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

are the inner normals of $\hat{C}^{(4)}$ and $\hat{C}^{(5)}$ respectively. Obviously, $C^{(4)}$ and $C^{(5)}$ are not connected since $(c_1, c_2) \cap (c_3, c_4) = \emptyset$.

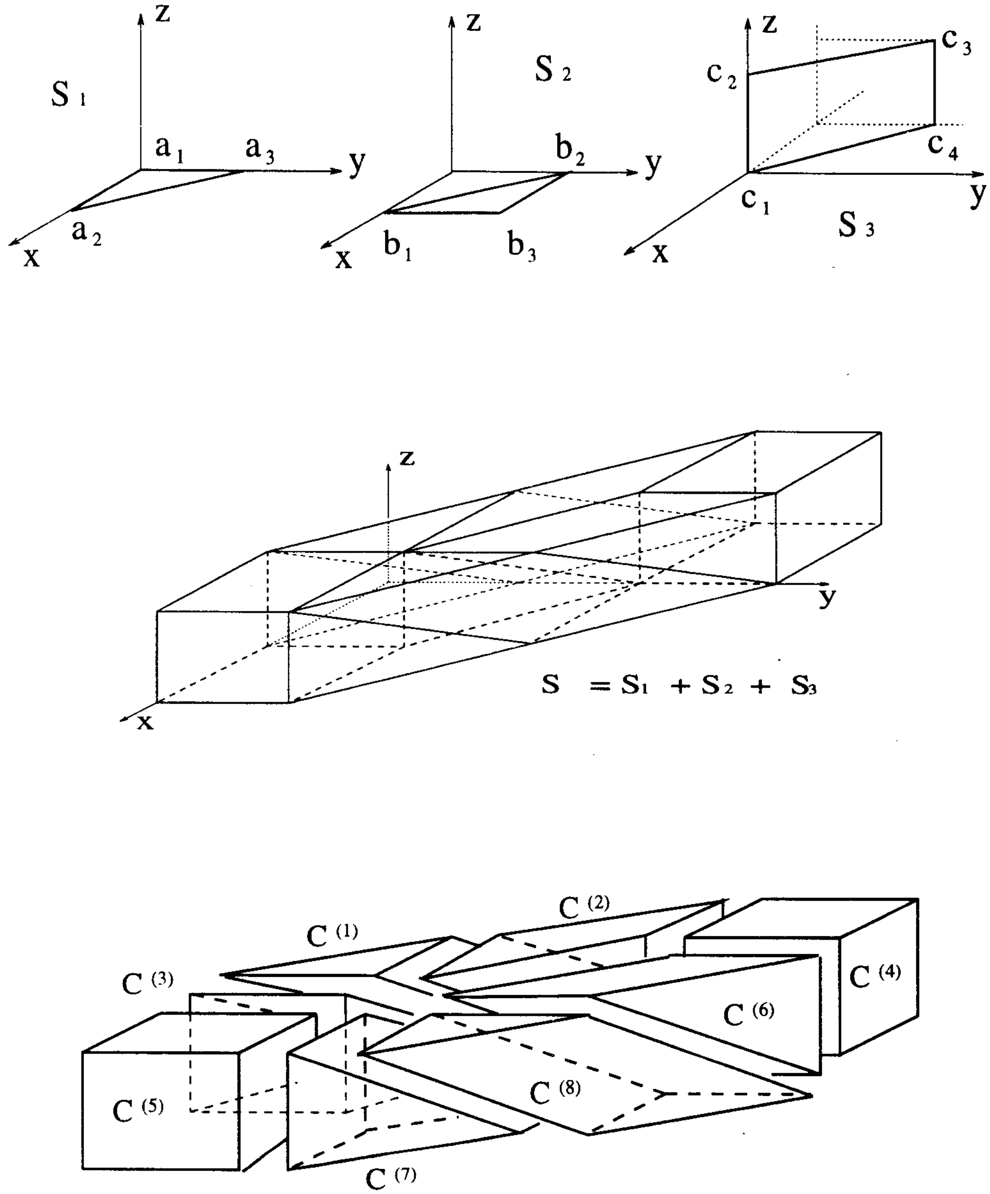
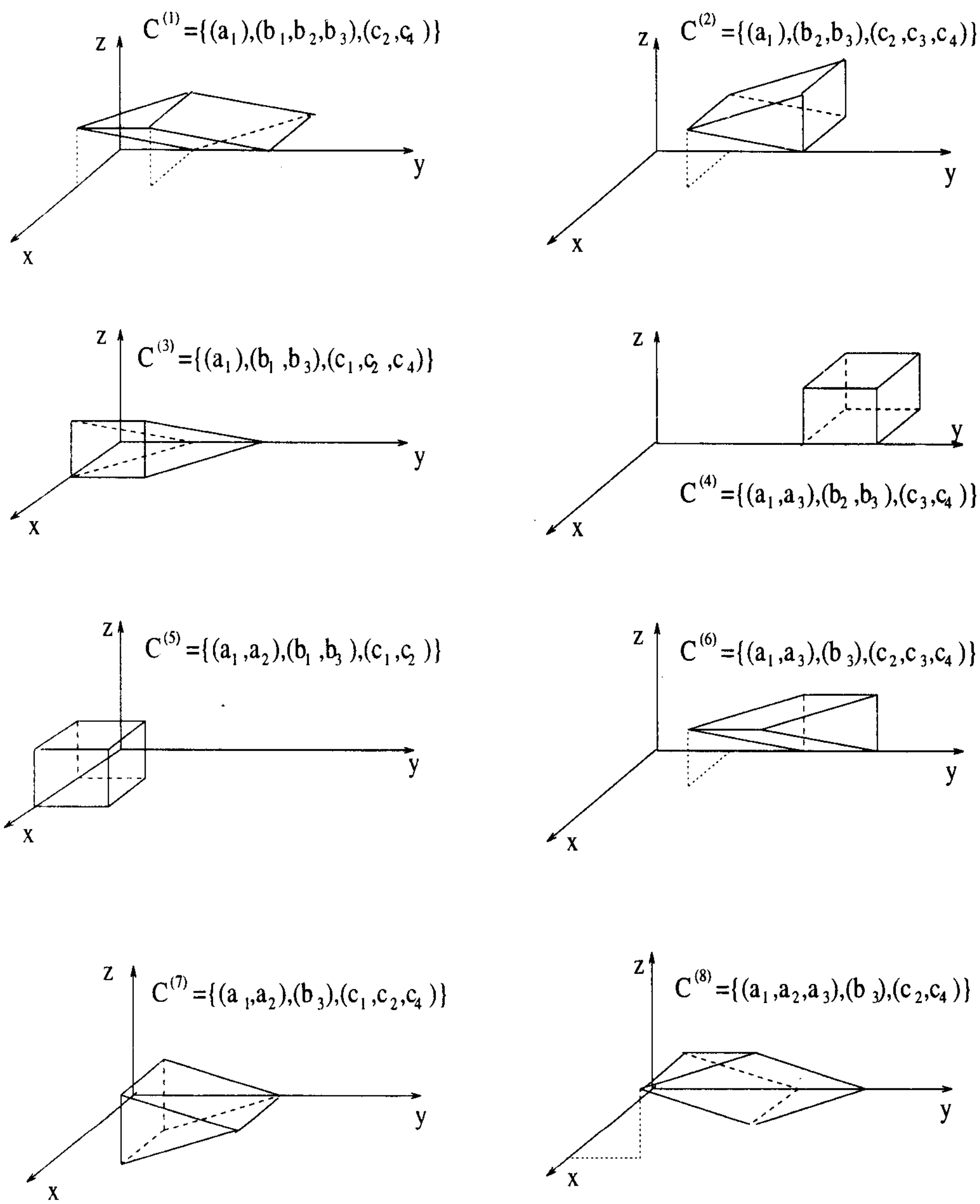

z
S 1
a 1
a 3
y
x
a 2
z
S 2
b 2
y
x
b 1
b 3
z
c 2
c 3
c 4
y
c 1
S 3
x
z
y
x
S = S1 + S2 + S3
C (1)
C (2)
C (3)
C (4)
C (5)
C (6)
C (7)
C (8)

Figure 2



$C^{(1)}=\{(a_1),(b_1,b_2,b_3),(c_2,c_4)\}$
$C^{(2)}=\{(a_1),(b_2,b_3),(c_2,c_3,c_4)\}$
$C^{(3)}=\{(a_1),(b_1,b_3),(c_1,c_2,c_4)\}$
$C^{(4)}=\{(a_1,a_3),(b_2,b_3),(c_3,c_4)\}$
$C^{(5)}=\{(a_1,a_2),(b_1,b_3),(c_1,c_2)\}$
$C^{(6)}=\{(a_1,a_3),(b_3),(c_2,c_3,c_4)\}$
$C^{(7)}=\{(a_1,a_2),(b_3),(c_1,c_2,c_4)\}$
$C^{(8)}=\{(a_1,a_2,a_3),(b_3),(c_2,c_4)\}$
x
y
z

Figure 3

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