On Multivariate Descartes' Rule — A Counterexample

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The purpose of this short note is to give a counterexample to a widely known conjecture on multivariate version of Descartes' rule [2] proposed by I. Itenberg and M.-F. Roy [3]. The famous Descartes' rule states that the number of positive real zeroes of a univariate polynomial is no greater than the number of sign changes in the list of its coefficients.

Let $P_1, ..., P_k$ be real polynomials in k variables, and let $A_1, \ldots, A_k$ and $\Delta_1, ..., \Delta_k$ be the supports and Newton polytopes of these polynomials. Each support A_i can be equipped with a distribution δ_i of signs at its integer points: a point gets the sign ("+" or " −") of the coefficient of the corresponding monomial of the polynomial P_i, Each polytope Δ_i with a distribution δ_i on A_i is called a *signed Newton diagram* and is denoted by $\tilde{\Delta}_i$.

Let ω_i be a real-valued function defined on the set A_i. By taking the lower convex hall in $\mathbf{R}^{k+1}$ of the graph of ω_i and then projecting each facet to $\mathbf{R}^k \times \{0\}$, the function ω_i defines a polyhedral subdivision τ_i of Δ_i.

Denote by Δ_M the Minkowski sum of the polytopes $\Delta_1, ..., \Delta_k$ and by A the set

$$\{a \in \Delta_M | a = a_1 + ... + a_k, \text{ where } a_i \in A_i\}.$$

Define $\omega : A \longrightarrow \mathbf{R}$ as,

$$\omega(a) = \min\{w_1(a_1) + ... + \omega_k(a_k)\}$$

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for $a_1 + ... + a_k = a$. Such a function ω defines a polyhedral subdivision τ_ω of Δ_M. The vertices of τ_ω belong to A and each polytope F of τ_ω has a unique representation

$$F = F_1 + ... + F_k,$$

where F_i is a face of τ_i. Suppose the functions $\omega_1, ..., \omega_k$ are generic; that is, for any polytope $F = F_1 + ... + F_k$ of τ_ω, we have

$$\dim(F) = \dim(F_1) + ... + \dim(F_k).$$

A polytope $V = v_1 + ... + v_k$ of τ_ω such that

$$\dim(v_1) = ... = \dim(v_k) = 1$$

is called a mixed cell. The mixed volume of A is the sum of the volumes of the mixed cells.

The *index of parity* $p(V)$ of a mixed cell $V = v_1 + ... + v_k$ is defined to be the corank (over $\mathbf{Z}/2\mathbf{Z}$) of the matrix $\widehat{V}$ whose rows are composed of the coordinates modulo 2 of the vectors $v_1, ..., v_k$. Let v_i be one of the edges of a mixed cell V. We call the edge v_i *alternating* if the distribution of signs δ_i associates different signs to the endpoints of v_i. A new $k \times (k+1)$ matrix $\bar{V}$ is defined by adding to $\widehat{V}$ a $(k+1)$-th column, its $(i, k+1)$ -th element equal to 0 if the edge v_i is alternating and equal to 1 otherwise. A mixed cell V is called *contributing* if rank($\widehat{V}$)= rank($\bar{V}$).

Let $n(\tilde{\Delta}, \omega) = \sum_V 2^{p(V)}$, where V ranges over all contributing mixed cells of τ_ω and $n(\tilde{\Delta})$ be the maximum of $n(\tilde{\Delta}, \omega)$ for all possible choices of generic functions $\omega_1, \ldots, \omega_k$.

To each open orthant m of $\mathbf{R}^k$ we associate a vector $\vec{m}$ from $\mathbf{Z}/2\mathbf{Z}$ in the following way: let $(x_1, ..., x_k)$ belong to m, then the i-th coordinate of $\vec{m}$ equal to 0 if $x_i > 0$, and equal to 1 otherwise. To each point $a_i \in A_i$ we also associate a vector $\vec{a_i}$ from $\mathbf{Z}/2\mathbf{Z}$ by replacing each coordinate of a_i by its parity (0 or 1).

The *symmetric copy* $a_i^{(m)}$ of a point a_i in an orthant m is a point having the same coordinates as a_i equipped with a sign defined by

$$\mathrm{sign}(a_i^{(m)}) = (-1)^{\vec{a_i} \cdot \vec{m}} \mathrm{sign}(a_i)$$

where $\vec{a_i} \cdot \vec{m}$ stands for the scalar product (over $\mathbf{Z}/2\mathbf{Z}$) of the vectors $\vec{a_i}$ and $\vec{m}$ and $\mathrm{sign}(a_i)$ stands for the sign associated by δ_i to a_i. For a mixed cell $V = v_1 + \ldots + v_k$ of the subdivision τ_ω, the symmetric copy $V^{(m)}$ of V in an orthant m is called *alternating* if the endpoints of the symmetric copy of each v_i in the orthant m are of different signs. Let $n(\tilde{\Delta}, \omega; m)$ be the number of alternating copies in an orthant m of all mixed cells of τ_ω and $n(\tilde{\Delta}; m)$ be the maximum of $n(\tilde{\Delta}, \omega; m)$ for all possible choices of generic functions $\omega_1, ..., \omega_k$.

For a univariate polynomial there are two orthants numbered 0 (corresponding to $X > 0$) and 1 (corresponding $X < 0$). A signed Newton diagram is simply a list of pairs of integers and associated signs, listed by increasing order of the integers. A mixed cell is given by two consecutive integers in this list. If the difference of two consecutive integers is even (resp. odd), the index of parity of the cell is 1 (resp. 0). The cell is contributing if its index of parity is 0, or if the index of parity is 1 and the two integers have different associated signs. By choosing ω with increasing slope, the number $n(\tilde{\Delta}, \omega, 0)$ counts the number of sign changes in the list of coefficients [3] and the famous Descartes' rule [2] states that the number of positive real zeroes of a polynomial is no greater than this number.

The multivariate Descartes' rule proposed by I. Itenberg and M.-F. Roy [3] is the following:

Suppose that k real polynomials in k variables with signed Newton diagrams $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_k$ have a finite number of common real zeroes. Then

(1) *the number of common zeroes of these polynomials in an open orthant m of $\mathbf{R}^k$ is not greater than $n(\tilde{\Delta}, m)$;*

(2) *the number of common zeroes of these polynomials in $(\mathbf{R}^*)^k$ is not greater than $n(\tilde{\Delta})$.*

Now, let us consider the following polynomial system in $\mathbf{C}[x_1, x_2]$:

$$\begin{aligned} p_1 &=: -1 - x_1 + x_2, \\ p_2 &=: -2 - 9x_1^3 + x_2^3 + 0.01x_1^3x_2^3 \end{aligned}$$

with supports

$$\begin{aligned} A_1 &= \{a = (0,0), b = (1,0), c = (0,1)\}, \\ A_2 &= \{d = (0,0), e = (3,0), f = (3,3), g = (0,3)\} \end{aligned}$$

(as shown in Fig. 1 with their associate signs).

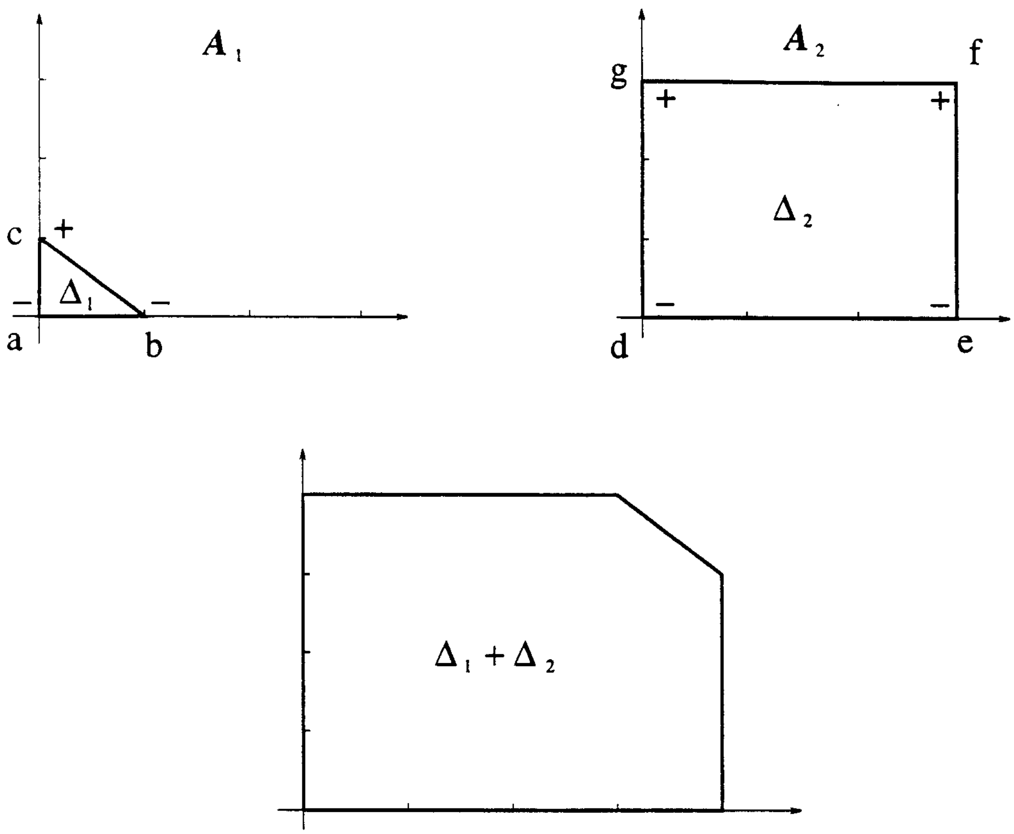

Figure 1

There are three solutions of this system in the first quadrant, they are:

$$(0.317659, 1.317659), \quad (0.659995, 1.659995), \quad \text{and} \quad (8.12058, 9.12058).$$

In the first quadrant $\vec{m} = (0, 0)$, so

$$\text{sign}(a_i^{(m)}) = (-1)^0 \text{sign}(a_i) = \text{sign}(a_i).$$

Hence, the possible alternating copies of mixed cells in the first quadrant under all possible choices of generic functions ω_1 and ω_2 are as follows:

$$V_1 = \overline{ac} + \overline{df} \quad \text{with} \quad \text{Volume}(V_1) = 3,$$

$$V_2 = \overline{ac} + \overline{ge} \quad \text{with} \quad \text{Volume}(V_2) = 3,$$

$$V_3 = \overline{bc} + \overline{ef} \quad \text{with} \quad \text{Volume}(V_3) = 3,$$

$$V_4 = \overline{bc} + \overline{df} \quad \text{with} \quad \text{Volume}(V_4) = 6,$$

$$V_5 = \overline{bc} + \overline{gd} \quad \text{with} \quad \text{Volume}(V_5) = 3.$$

However, the mixed volume of $A = A_1 + A_2$ is equal to [1]:

$$\text{Volume}(\Delta_1 + \Delta_2) - (\text{Volume}(\Delta_2) + \text{Volume}(\Delta_2)) = 15\frac{1}{2} - (9 + \frac{1}{2}) = 6.$$

Thus, no subdivisions resulting from any choices of ω_1 and ω_2 can admit more than 2 alternating copies listed above, for otherwise the sum of their volumes, being the mixed volume of A, would be more than 6. So, statement (1) in the proposed multivariate Descartes' rule is inaccurate in this case.

In addition to the possible alternating mixed cells listed above, there are eight more possible mixed cells, namely,

$$V_6 = \overline{ab} + \overline{dg} \quad \text{with} \quad \text{Volume}(V_6) = 3,$$

$$V_7 = \overline{ab} + \overline{ef} \quad \text{with} \quad \text{Volume}(V_7) = 3,$$

$$V_8 = \overline{ab} + \overline{eg} \quad \text{with} \quad \text{Volume}(V_8) = 3,$$

$$V_9 = \overline{ab} + \overline{df} \quad \text{with} \quad \text{Volume}(V_9) = 3,$$

$$V_{10} = \overline{ac} + \overline{gf} \quad \text{with} \quad \text{Volume}(V_{10}) = 3,$$

$$V_{11} = \overline{ac} + \overline{de} \quad \text{with} \quad \text{Volume}(V_{11}) = 3,$$

$$V_{12} = \overline{bc} + \overline{de} \quad \text{with} \quad \text{Volume}(V_{12}) = 3,$$

$$V_{13} = \overline{bc} + \overline{gf} \quad \text{with} \quad \text{Volume}(V_{13}) = 3.$$

It is easy to check that all those cells $V_1, \ldots, V_{13}$ are contributing and the index of parity is 0 for those cells with volumes equal to 3 and is 1 for V_4, the only one with volume 6. For any subdivision resulting from any choices of ω_1 and ω_2, it can tolerate either two mixed cells with volume 3 of each, or a single mixed cell with volume 6. In both cases,

$$n(\tilde{\Delta}, \omega) = \sum_V 2^{p(V)} \leq 2.$$

Therefore, statement (2) in the conjecture also fails in this case.

References

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[3] I. Itenberg and M.-F. Roy *Multivariate Descartes' rule*, Beiträge zur Algebra und Geometrie, **37**(1996), 337-346.