On the product of n linearly independent linear functions in n variables

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Abstract

In [10], F. Jaeger has conjectured that for every nonsingular $n \times n$ matrix over the finite field GF_q, $q \geq 4$, there exists a vector x in $(GF_q)^n$ such that neither x nor Ax have any zero components. A stronger conjecture, due to M. Tarsi (c.f. [8]), asserts that for every nonsingular $n \times n$ matrix over any field F, and for every $S_1, ..., S_n \subseteq F$ each of cardinality 3, there exists a vector $x \in S_1 \times ... \times S_n$ such that neither x nor Ax have any zero components. We apply algebraic techniques in order to prove this conjecture for $n \leq 5$.

1 Introduction

The following long standing conjecture is due to F. Jaeger (c.f. [10]):

Conjecture 1.1 ***(The Nowhere Zero Point Conjecture)***. *Let A be a nonsingular $n \times n$ matrix over the finite field GF_q, $q \geq 4$; then there exists a vector x in $(GF_q)^n$ such that neither x nor Ax have any zero components.*

The assertion of conjecture 1.1 is easily seen to be false for $q = 2, 3$, and is not so hard to prove for infinite fields.

Besides being interesting in its own right, conjecture 1.1 has some interesting combinatorial implications (c.f. [3],[8]), especially the case $q = 5$ in which F. Jaeger was originally interested (c.f. [8]). The version of his conjecture, presented here, is due to N. Alon and M. Tarsi [3], who proved it for fields whose order is a proper prime power, i.e. GF_q where $q = p^k$, $k \geq 2$ and p is a prime. Their proof makes extensive use of the multivariate polynomial $P_A(x_1, ..., x_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j$, associated with the $n \times n$ matrix $A = (a_{ij})$. They proved that if A is nonsingular over a field F of characteristic p then there exists a monomial $c \prod_{j=1}^{n} x_j^{\alpha_j}$ in the expansion of P_A such that $c \neq 0$ (in F), and $\alpha_j < p$ for all $1 \leq j \leq n$. Unfortunately $\alpha_j < p - 1$ is required in order to prove conjecture 1.1 for fields of prime order. In [14], Y. Yu proved that if $n \leq 2^{p-2}$ then the assertion of conjecture 1.1 is true for any field of characteristic p. The following, more recent, conjecture due to M. Tarsi, asserts that the existence of a non-vanishing monomial of small degree in the expansion of P_A, is independent of the order of the field F.

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Conjecture 1.2 *Let F be any field, and A an $n \times n$ nonsingular matrix over F; then there exists a monomial $c\prod_{j=1}^{n} x_j^{\alpha_j}$ in the expansion of P_A such that $c \neq 0$ and $\alpha_j \leq 2$ for all $1 \leq j \leq n$.*

Note that this conjecture is the same as the one mentioned in the abstract.
The assertion of conjecture 1.2 is true for fields of characteristic 2 and 3, as can be easily derived from [3]. We will therefore assume $char(F) \neq 2, 3$ throughout this paper, in order to avoid certain difficulties that arise in those cases.

Denote by $L_{m,n}$ the set of all ordered partitions of m into n non-negative integer summands, i.e. $L_{m,n} = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) | \sum_{i=1}^{n} \alpha_i = m,\ \alpha_i \in \mathbb{Z}^+\}$. For a positive integer k denote by $L_{m,n}^k$ the subset of $L_{m,n}$, consisting of the partitions in which all summands are at most k, i.e. $L_{m,n}^k = \{\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) | \sum_{i=1}^{n} \alpha_i = m,\ \alpha_i \in \{0, 1, ..., k\}\}$. We will abbreviate $L_{n,n}$ and $L_{n,n}^k$ to L_n and L_n^k respectively. For an $n \times n$ matrix A and $\alpha = (\alpha_1, ..., \alpha_n) \in L_{m,n}$ let A^α denote the $n \times m$ matrix whose columns are α_j copies of the j^{th} column of A ordered naturally, e.g. $A^{(1,2,1,0)}$ is a 4×4 matrix whose 1^{st} column is a copy of A's 1^{st} column, 2^{nd} and 3^{rd} columns are two copies of A's 2^{nd} column and 4^{th} column is a copy of A's 3^{rd} column. Let c_α denote the coefficient of the monomial $\prod_{j=1}^{n} x_j^{\alpha_j}$ in the expansion of P_A. It is straightforward to verify that $Per(A^\alpha) = c_\alpha \prod_{j=1}^{n} \alpha_j!$ where $Per(A^\alpha)$ is the permanent of the matrix A^α (the assertion of conjecture 1.2 can be deduced again for fields of characteristic 2, as over such fields the determinant and the permanent of a matrix coincide and thus if A is nonsingular then $Per(A) \neq 0$). We say that an $n \times n$ matrix A has the p_2 property, or that it is a p_2 matrix or simply that it is p_2 if $\forall \alpha \in L_n^2\ Per(A^\alpha) = 0$. We can now restate conjecture 1.2 in a fashion, more convenient for our purposes.

Conjecture 1.3 *If A is a p_2 matrix then it is singular.*

Our main result is a proof of conjecture 1.3 for $n \times n$ matrices where $n \leq 5$.

Theorem 1.4 *Let A be an $n \times n$ matrix over any field F where $n \leq 5$. If A is p_2 then it is singular.*

Theorem 1.4 clearly holds for $n \leq 2$. In section 2 we prove it for $n = 3, 4$ and in section 3 for $n = 5$.

2 The Hessian

Let $P \in F[x_1, ..., x_n]$ be a multivariate polynomial. The **Hessian** of P, denoted $Hes(P)$, is the determinant of the $n \times n$ matrix whose (i, j) entry is $\frac{\partial^2 P}{\partial x_i \partial x_j}$. It is known that the Hessian is a covariant of weight 2, i.e. if T is a linear transformation applied to $\bar{x} = (x_1, ..., x_n)$ then $Hes(P(T(\bar{x}))) = Det(T)^2 T(Hes(P(\bar{x})))$.
Let A be a nonsingular $n \times n$ matrix. Let $\bar{x} = (x_1, ..., x_n), \bar{y} = (y_1, ..., y_n) \in F^n$ be two vectors such that $A\bar{x} = \bar{y}$. A expresses each y_i in terms of the x_j's, thus A can be viewed as changing the old variables y_i into new ones x_j. Hence any polynomial $P(y_1, ..., y_n)$ can be transformed into a polynomial $Q(x_1, ..., x_n)$ by applying A^{-1} to $\bar{y} = (y_1, ..., y_n)$.

We can now prove theorem 1.4 for $n = 3$ (first stated and proved in [8]).

Proof:

$$Hes(P_I(y_1,y_2,y_3)) = Hes(y_1y_2y_3) = Det\begin{pmatrix} 0 & y_3 & y_2 \\ y_3 & 0 & y_1 \\ y_2 & y_1 & 0 \end{pmatrix} = 2y_1y_2y_3 = 2P_I(y_1,y_2,y_3).$$

Assume for the sake of contradiction that A is nonsingular.

$$A\bar{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \bar{y},$$

thus $P_A(\bar{x})$ can be obtained from $P_I(\bar{y})$ by applying the linear transformation T, represented by the matrix A^{-1} to $\bar{y} = (y_1,y_2,y_3)$. Hence $Hes(P_A(\bar{x})) = Hes(P_I(T(\bar{y}))) = Det(T)^2T(Hes(P_I(\bar{y}))) = Det(T)^2T(2P_I(\bar{y})) = cP_A(\bar{x})$, where $c \neq 0$, since $char(F) \neq 2$ and A is nonsingular. However, since A is a p_2 matrix, $P_A(x_1,x_2,x_3) = rx_1^3 + sx_2^3 + tx_3^3$ for some constants r,s,t not all zero since A is nonsingular (c.f. [3]). Hence

$$Hes(P_A(x_1,x_2,x_3)) = Det\begin{pmatrix} 6rx_1 & 0 & 0 \\ 0 & 6sx_2 & 0 \\ 0 & 0 & 6tx_3 \end{pmatrix} = 216rstx_1x_2x_3 \neq cP_A(x_1,x_2,x_3),$$

for every nonzero constant c. Hence A must be singular. □

A reasoning which is essentially the same, yields a similar proof of theorem 1.4 for $n = 4$. **Proof:**

$$Hes(P_I(y_1,y_2,y_3,y_4)) = Hes(y_1y_2y_3y_4) = Det\begin{pmatrix} 0 & y_3y_4 & y_2y_4 & y_2y_3 \\ y_3y_4 & 0 & y_1y_4 & y_1y_3 \\ y_2y_4 & y_1y_4 & 0 & y_1y_2 \\ y_2y_3 & y_1y_3 & y_1y_2 & 0 \end{pmatrix}$$

$$= -3(y_1y_2y_3y_4)^2 = -3(P_I(y_1,y_2,y_3,y_4))^2.$$

Assume for the sake of contradiction that A is nonsingular. Let T be the linear transformation represented by A^{-1}, then as before we have $Hes(P_A(\bar{x})) = Hes(P_I(T(\bar{y}))) = Det(T)^2T(Hes(P_I(\bar{y}))) = Det(T)^2T(-3(P_I(\bar{y}))^2) = c(P_A(\bar{x}))^2$, where $c \neq 0$, since $char(F) \neq 3$ and A is nonsingular. However, since A is a p_2 matrix,
$P_A(x_1,x_2,x_3,x_4) = a_{11}x_1^4 + a_{22}x_2^4 + a_{33}x_3^4 + a_{44}x_4^4 + a_{12}x_1^3x_2 + a_{13}x_1^3x_3 + a_{14}x_1^3x_4 + a_{21}x_2^3x_1 + a_{23}x_2^3x_3 + a_{24}x_2^3x_4 + a_{31}x_3^3x_1 + a_{32}x_3^3x_2 + a_{34}x_3^3x_4 + a_{41}x_4^3x_1 + a_{42}x_4^3x_2 + a_{43}x_4^3x_3$
for some constants a_{ij}. Hence $Hes(P_A(x_1,x_2,x_3,x_4))$ is the determinant of the 4×4 matrix M, whose (i,j) entry is $6x_i(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4) + 6a_{ii}x_i^2$ if $i = j$, and $3a_{ij}x_i^2 + 3a_{ji}x_j^2$ otherwise. We will prove that $Hes(P_A) \neq c(P_A)^2$ for every nonzero constant c, thus obtaining a contradiction to the covariant property of the Hessian and therefore proving A's singularity. If indeed A is nonsingular then P_A is not identically zero (c.f. [3]). Thus at least one of the a_{ij}'s must be nonzero. Hence, in the expansion of $(P_A)^2$ there exists a monomial $\hat{a}x_i^8$ or $\hat{a}x_i^6x_j^2$, $i \neq j$. Such a monomial cannot, however, appear in the expansion of $Hes(P_A)$ as x_i appears only in the i^{th} row, i^{th} column and diagonal of M. But without the factors on the diagonal x_i can get at most to the 4^{th} power (x_i^2 from the i^{th} row and x_i^2 from the i^{th} column), and by using the diagonal we must have at least

3 different variables (if for example we wish to obtain a monomial $x_1^k x_2^l,\ k,l \geq 0, k+l=8$ then we must use a permutation $\sigma \in S_4$ such that $\sigma(\{3,4\}) \subseteq \{1,2\}$ entailing $\sigma(\{1,2\}) \subseteq \{3,4\}$, thus σ has no fixed points). $\square$

3 The linear operator τ_α

For 5 × 5 matrices we will use a different technique. It can be successfully applied to smaller matrices as well, though not as elegantly as the Hessian.
Let $\alpha = (\alpha_1, ..., \alpha_n) \in L_{m,n}$ and $1 \leq r \leq n$. Denote by $(\alpha + r)$ the n-tuple $(\alpha_1, ..., \alpha_{r-1}, \alpha_r + 1, \alpha_{r+1}, ..., \alpha_n) \in L_{m+1,n}$. If $m = n-2$ denote by A_α the $n \times n$ matrix whose (i,j) entry is 0 if $i = j$ and, the permanent of the matrix obtained from A^α by deleting its i^{th} and j^{th} rows, otherwise. For example if $n = 4, A = (a_{ij})$ and $\alpha = (0,0,2,0)$ then

$$\mathbf{A}^\alpha = \begin{pmatrix} a_{13} & a_{13} \\ a_{23} & a_{23} \\ a_{33} & a_{33} \\ a_{43} & a_{43} \end{pmatrix}, \quad \mathbf{A}^{(\alpha+\mathbf{2})} = \mathbf{A}^{(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{0})} = \begin{pmatrix} a_{12} & a_{13} & a_{13} \\ a_{22} & a_{23} & a_{23} \\ a_{32} & a_{33} & a_{33} \\ a_{42} & a_{43} & a_{43} \end{pmatrix}$$

and

$$\mathbf{A}_\alpha = \begin{pmatrix} 0 & 2a_{33}a_{43} & 2a_{23}a_{43} & 2a_{23}a_{33} \\ 2a_{33}a_{43} & 0 & 2a_{13}a_{43} & 2a_{13}a_{33} \\ 2a_{23}a_{43} & 2a_{13}a_{43} & 0 & 2a_{13}a_{23} \\ 2a_{23}a_{33} & 2a_{13}a_{33} & 2a_{13}a_{23} & 0 \end{pmatrix}.$$

Let $A = (a_{ij})$ be an $n \times n$ matrix over the field F, with columns $c_1, ..., c_n$, and let $\alpha \in L^2_{n-2,n}$. The matrix A_α as defined above represents a linear mapping $\tau_\alpha : F^n \to F^n$. For all $1 \leq i,j \leq n,\ \tau_\alpha(c_i)$ is a vector $(q_{i1}, q_{i2}, ..., q_{in}) \in F^n$ where q_{ij} is the permanent of the matrix obtained from $A^{(\alpha+i)}$ by deleting its j^{th} row. If A is a p_2 matrix then $\tau_\alpha(c_i)$ is orthogonal to c_j for each i,j such that $\alpha_i = 0$ and $\alpha_j \leq 1$.

We begin by stating the following trivial, though very useful observation.

Proposition 3.1 *Let A,B be $n \times n$ matrices such that B can be obtained from A by permuting its rows and columns or by multiplying its rows and columns by non zero constants, then A is p_2 iff B is p_2, and $Det(A) = 0$ iff $Det(B) = 0$.*

The straightforward proof is omitted.
We will also make extensive use of the following fact, first stated and proved in [8]:

Proposition 3.2 *Let A be a nonsingular $n \times n$ matrix with entries in the field F. If A is a p_2 matrix then all its $(n-1) \times (n-1)$ minors have zero permanent.*

Proof: Let $(q_1, ..., q_n)$ be the vector whose j^{th} entry is the permanent of the matrix obtained from A by deleting its 1^{st} column and j^{th} row. Since A is a p_2 matrix, $(q_1, ..., q_n)$ is orthogonal to every column of A ($Per(A) = 0$ and $Per(A^\alpha) = 0$ for $\alpha = (0,2,1,...,1)$ and all its permutations that fix the first coordinate). Since A is nonsingular, $(q_1, ..., q_n)$ is orthogonal to a basis and is therefore the zero vector. The proposition now follows by applying the same argument to the other columns. $\square$

We are now ready to prove theorem 1.4 for $n = 5$:
Proof: Assume for the sake of contradiction that A is a nonsingular p_2 matrix. This assumption yields the following lemmas:

Lemma 3.3 *If there exists at least four zero entries in some column of A then $Det(A) = 0$.*

Proof: By proposition 3.1 we can assume that $a_{51} = a_{52} = a_{53} = a_{54} = 0$ and that $a_{55} = 1$, i.e.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{pmatrix}.$$

A is a p_2 matrix and so is

$$\mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

because $\forall\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in L_4^2 \; Per(B^\alpha) = Per(A^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,1)})$. B however, is a 4×4 matrix so by the proof of theorem 1.4 for $n = 4$ we have $Det(A) = Det(B) = 0$. $\square$

Lemma 3.4 $\forall\alpha \in L_{3,5}^2 \; rank(A_\alpha) \leq 3$.

Proof: Let $\alpha \in L_{3,5}^2$, then α is either a permutation of $(1, 1, 1, 0, 0)$ or a permutation of $(2, 1, 0, 0, 0)$. We will assume without loss of generality that α is either $(1, 1, 1, 0, 0)$ or $(2, 1, 0, 0, 0)$. Let $c_1, ..., c_5$ denote the columns of A. For all $1 \leq i, j \leq 5$ $A_\alpha c_i$ is a vector $(q_{i1}, ..., q_{i5})$, where q_{ij} is the permanent of the matrix obtained from $A^{(\alpha+i)}$ by deleting its j^{th} row. If $\alpha = (1, 1, 1, 0, 0)$ then by proposition 3.2 $A_\alpha c_4 = A_\alpha c_5 = \bar{0}$ and therefore $rank(A_\alpha) \leq 3$. If $\alpha = (2, 1, 0, 0, 0)$ then $span(\{A_\alpha c_3, A_\alpha c_4, A_\alpha c_5\}) \subseteq (span(\{c_2, c_3, c_4, c_5\}))^\perp$ which is a one dimensional vector space. Hence $rank(A_\alpha) \leq 3$. $\square$

Lemma 3.5 *If a column of A has a zero entry then it has another.*

Proof: By proposition 3.1 and for the sake of contradiction assume that $a_{11} = 0$ and $a_{21} = a_{31} = a_{41} = a_{51} = 1$. For $k = 2, 3, 4, 5$ let $\alpha_k = (2, \alpha_{k2}, ..., \alpha_{k5}) \in L_{3,5}^2$ such that $\alpha_{kk} = 1$; thus

$$\mathbf{A}_{\alpha_\mathbf{k}} = \begin{pmatrix} \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & 0 & 2a_{1k} & 2a_{1k} & 2a_{1k} \\ \varnothing & 2a_{1k} & 0 & 2a_{1k} & 2a_{1k} \\ \varnothing & 2a_{1k} & 2a_{1k} & 0 & 2a_{1k} \\ \varnothing & 2a_{1k} & 2a_{1k} & 2a_{1k} & 0 \end{pmatrix}$$

where the values of the entries denoted by a '$\varnothing$' are irrelevant to the proof. By lemma 3.4 $rank(A_{\alpha_k}) \leq 3$ so the determinant of the minor of A_{α_k} obtained by deleting its first row and first column, equals zero. Since $char(F) \neq 2, 3$ this implies $a_{1k} = 0$; thus A's first row is the zero vector and therefore $Det(A) = 0$. This is a contradiction to our assumption that A is nonsingular and thus A has at least two zero entries in its first column. □

Lemma 3.6 *If there exists at least three zero entries in some row of A then $Det(A) = 0$.*

Proof: By proposition 3.1 we can assume $a_{11} = a_{12} = a_{13} = 0$, thus $\forall \alpha = (\alpha_1, \alpha_2, \alpha_3, 0, 0) \in L^2_{3,5}$, we have

$$\mathbf{A}_\alpha = \begin{pmatrix} 0 & \beta & \gamma & \delta & \epsilon \\ \beta & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $c_1, ..., c_5$ denote the columns of A. By proposition 3.2 $A_{(1,1,1,0,0)}c_4 = A_{(1,1,1,0,0)}c_5 = \bar{0}$. If $A_{(1,1,1,0,0)}$ is not the zero matrix then $a_{14} = a_{15} = 0$ entailing $Det(A) = 0$. So assume it is the zero matrix. This yields a system of equations which we will denote by a '$\star$'. By lemma 3.5 we know that there are at least two zero entries in each of A's first three columns. We consider three cases, namely those three zero entries are in the same row, two different rows or three different rows:
case 1: $a_{21} = a_{22} = a_{23} = 0$.
For $\alpha = (2, 1, 0, 0, 0)$ we have $A_\alpha c_1 = A_\alpha c_2 = \bar{0}$. Since $dim(sp\{A_\alpha c_3, A_\alpha c_4, A_\alpha c_5\}) \leq 1$ we have $rank(A_\alpha) \leq 1$, but A_α is a symmetric matrix with zero entries along its main diagonal and is therefore the zero matrix. Similarly A_α is the zero matrix for all the permutations of $(2, 1, 0, 0, 0)$ that fix the last two coordinates. It follows that any matrix obtained from A by taking its first three columns and any three rows is p_2. Since those are 3×3 matrices, they are singular by the proof of theorem 1.4 for $n = 3$. If we expand A's determinant using its first three columns we get $Det(A) = 0$.
case 2: $a_{21} = a_{22} = a_{33} = 0$.
As in the previous case $A_{(2,1,0,0,0)}$ is the zero matrix. This yields the equation $a_{31}a_{41}a_{52} + a_{31}a_{42}a_{51} + a_{32}a_{41}a_{51} = 0$. From the system of equations $\star$ we get $a_{23}(a_{41}a_{52} + a_{42}a_{51}) = a_{23}(a_{31}a_{52} + a_{32}a_{51}) = a_{23}(a_{31}a_{42} + a_{32}a_{41}) = 0$. We can assume $a_{23} \neq 0$ as otherwise we are back to case 1. So from the above equations we get $a_{31}a_{41}a_{52} = a_{31}a_{42}a_{51} = a_{32}a_{41}a_{51} = 0$. Hence either there are four zero entries in a column of A and then $Det(A) = 0$ by lemma 3.3 or two zero entries in the same row. So without loss of generality assume $a_{51} = a_{52} = 0$. Clearly if $a_{14} = a_{15} = 0$ then we are done, so assume without loss of generality that $a_{14} \neq 0$. Hence $0 = Per(A^{(2,0,2,1,0)}) = 4a_{14}a_{23}a_{31}a_{41}a_{53}$, and therefore either $a_{23}a_{53} = 0$ and we are back to case 1 or $a_{31}a_{41} = 0$ and then there are four zero entries in A's first column, entailing $Det(A) = 0$ by lemma 3.3.
case 3: $a_{21} = a_{32} = a_{43} = 0$.
$0 = Per(A^{(2,2,0,1,0)}) = a_{14}a_{22}a_{31}(a_{41}a_{52} + a_{42}a_{51})$. Once again we can assume without loss of generality that $a_{14} \neq 0$ thus either $a_{22}a_{31} = 0$ and we are back to case 2, or $a_{41}a_{52} + a_{42}a_{51} = 0$. From the system of equations $\star$ we get $a_{31}a_{42}a_{53} + a_{33}(a_{41}a_{52} + a_{42}a_{51}) = 0$ and so either $a_{31}a_{42} = 0$ and we are back to case 2, or $a_{53} = 0$. Assuming the latter we get $0 = Per(A^{(2,0,2,1,0)}) = a_{14}a_{23}a_{33}a_{41}a_{51}$ and we are back to case 2. □

Lemma 3.7 *Let* $(1,1,1,1,1)^t$ *and* $(0,0,0,a,b)^t$ *be columns of* A*, then* $a+b=0$.

Proof: By proposition 3.1 we can assume that these are the first and second columns of A respectively; thus for $\alpha=(2,1,0,0,0)$ we have

$$\mathbf{A}_\alpha = 2\begin{pmatrix} 0 & a+b & a+b & b & a \\ a+b & 0 & a+b & b & a \\ a+b & a+b & 0 & b & a \\ b & b & b & 0 & 0 \\ a & a & a & 0 & 0 \end{pmatrix}.$$

If $a=b=0$ then we are done. Assume then, without loss of generality that $b\neq 0$. By Lemma 3.4 $rank(A_\alpha)\leq 3$ so the determinant of the minor of A_α obtained by deleting its last row and last column is zero. Since $char(F)\neq 3$ this implies $a+b=0$. □

Lemma 3.8 *Let* $(1,1,1,1,1)^t$ *and* $(1,1,1,a,b)^t$ *where* $a+b=-1$ *be the first and second columns of* A *respectively; then* $Det(A)=0$.

Proof:

$$\mathbf{A}_{(\mathbf{1,2,0,0,0})} = \begin{pmatrix} 0 & 2(a+b+ab) & 2(a+b+ab) & 2+4b & 2+4a \\ 2(a+b+ab) & 0 & 2(a+b+ab) & 2+4b & 2+4a \\ 2(a+b+ab) & 2(a+b+ab) & 0 & 2+4b & 2+4a \\ 2+4b & 2+4b & 2+4b & 0 & 6 \\ 2+4a & 2+4a & 2+4a & 6 & 0 \end{pmatrix}.$$

Dividing each row by 2 and substituting -1 for $a+b$ yields

$$\begin{pmatrix} 0 & ab-1 & ab-1 & 1+2b & 1+2a \\ ab-1 & 0 & ab-1 & 1+2b & 1+2a \\ ab-1 & ab-1 & 0 & 1+2b & 1+2a \\ 1+2b & 1+2b & 1+2b & 0 & 3 \\ 1+2a & 1+2a & 1+2a & 3 & 0 \end{pmatrix}.$$

Adding row 5 to row 4 and substituting -1 for $a+b$ yields

$$\begin{pmatrix} 0 & ab-1 & ab-1 & 1+2b & 1+2a \\ ab-1 & 0 & ab-1 & 1+2b & 1+2a \\ ab-1 & ab-1 & 0 & 1+2b & 1+2a \\ 0 & 0 & 0 & 3 & 3 \\ 1+2a & 1+2a & 1+2a & 3 & 0 \end{pmatrix}.$$

Adding column 5 to column 4 and substituting -1 for $a+b$ yields

$$\mathbf{M} = \begin{pmatrix} 0 & ab-1 & ab-1 & 0 & 1+2a \\ ab-1 & 0 & ab-1 & 0 & 1+2a \\ ab-1 & ab-1 & 0 & 0 & 1+2a \\ 0 & 0 & 0 & 6 & 3 \\ 1+2a & 1+2a & 1+2a & 3 & 0 \end{pmatrix}.$$

Elementary operations preserves rank and so $rank(M) = rank(A_{(1,2,0,0,0)})$ which is at most 3 by lemma 3.4. Since $char(F) \neq 2, 3$ this implies $ab = 1$.
For $k = 3, 4, 5$ let us denote the k^{th} column of A by $c_k = (x_k, y_k, z_k, u_k, v_k)^t$. Since A is p_2 we have

$$0 = Per \begin{pmatrix} 1 & 1 & 1 & 1 & x_k \\ 1 & 1 & 1 & 1 & y_k \\ 1 & 1 & 1 & 1 & z_k \\ 1 & 1 & a & a & u_k \\ 1 & 1 & b & b & v_k \end{pmatrix}.$$

Expanding the latter, using the first two columns yields

$$0 = 2Per\begin{pmatrix} 1 & 1 & x_k \\ 1 & 1 & y_k \\ 1 & 1 & z_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & x_k \\ 1 & 1 & y_k \\ a & a & u_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & x_k \\ 1 & 1 & z_k \\ a & a & u_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & y_k \\ 1 & 1 & z_k \\ a & a & u_k \end{pmatrix}$$

$$+ 2Per\begin{pmatrix} 1 & 1 & x_k \\ 1 & 1 & y_k \\ b & b & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & x_k \\ 1 & 1 & z_k \\ b & b & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & y_k \\ 1 & 1 & z_k \\ b & b & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & x_k \\ a & a & u_k \\ b & b & v_k \end{pmatrix}$$

$$+ 2Per\begin{pmatrix} 1 & 1 & y_k \\ a & a & u_k \\ b & b & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & 1 & z_k \\ a & a & u_k \\ b & b & v_k \end{pmatrix}$$

$$\begin{aligned} &= 4[x_k + y_k + z_k + 3u_k + 3v_k + 2a(x_k + y_k + z_k) + 2b(x_k + y_k + z_k) + 3av_k + 3bu_k + ab(x_k + y_k + z_k)] \\ &= 12(u_k + v_k + bu_k + av_k) \Rightarrow (u_k + v_k + bu_k + av_k) = 0, \end{aligned}$$

where the last equality is obtained by substituting 1 for ab and -1 for $a + b$.
Similarly we have

$$0 = Per\begin{pmatrix} 1 & 1 & 1 & x_k & x_k \\ 1 & 1 & 1 & y_k & y_k \\ 1 & 1 & 1 & z_k & z_k \\ 1 & 1 & a & u_k & u_k \\ 1 & 1 & b & v_k & v_k \end{pmatrix} + Per\begin{pmatrix} 1 & 1 & 1 & x_k & x_k \\ 1 & 1 & 1 & y_k & y_k \\ 1 & 1 & 1 & z_k & z_k \\ 1 & a & a & u_k & u_k \\ 1 & b & b & v_k & v_k \end{pmatrix}$$

$$= 2Per\begin{pmatrix} 1 & x_k & x_k \\ 1 & y_k & y_k \\ 1 & z_k & z_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & x_k & x_k \\ 1 & y_k & y_k \\ a & u_k & u_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & x_k & x_k \\ 1 & z_k & z_k \\ a & u_k & u_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & y_k & y_k \\ 1 & z_k & z_k \\ a & u_k & u_k \end{pmatrix}$$

$$+ 2Per\begin{pmatrix} 1 & x_k & x_k \\ 1 & y_k & y_k \\ b & v_k & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & x_k & x_k \\ 1 & z_k & z_k \\ b & v_k & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & y_k & y_k \\ 1 & z_k & z_k \\ b & v_k & v_k \end{pmatrix} + 2Per\begin{pmatrix} 1 & x_k & x_k \\ a & u_k & u_k \\ b & v_k & v_k \end{pmatrix}$$

$$+2Per\begin{pmatrix} 1 & y_k & y_k \\ a & u_k & u_k \\ b & v_k & v_k \end{pmatrix} +2Per\begin{pmatrix} 1 & z_k & z_k \\ a & u_k & u_k \\ b & v_k & v_k \end{pmatrix} +Per\begin{pmatrix} 1 & 1 & y_k & y_k \\ 1 & 1 & z_k & z_k \\ a & a & u_k & u_k \\ b & b & v_k & v_k \end{pmatrix} +Per\begin{pmatrix} 1 & 1 & x_k & x_k \\ 1 & 1 & z_k & z_k \\ a & a & u_k & u_k \\ b & b & v_k & v_k \end{pmatrix}$$

$$+\,Per\begin{pmatrix}1&1&x_k&x_k\\1&1&y_k&y_k\\a&a&u_k&u_k\\b&b&v_k&v_k\end{pmatrix}+Per\begin{pmatrix}1&1&x_k&x_k\\1&1&y_k&y_k\\1&1&z_k&z_k\\b&b&v_k&v_k\end{pmatrix}+Per\begin{pmatrix}1&1&x_k&x_k\\1&1&y_k&y_k\\1&1&z_k&z_k\\a&a&u_k&u_k\end{pmatrix}$$

$$\begin{aligned}
&= 4[x_ky_k+x_kz_k+y_kz_k+2u_k(x_k+y_k+z_k)+2v_k(x_k+y_k+z_k)+a(x_ky_k+x_kz_k+y_kz_k)\\
&+ b(x_ky_k+x_kz_k+y_kz_k)+3u_kv_k+av_k(x_k+y_k+z_k)+bu_k(x_k+y_k+z_k)]+4[3u_kv_k\\
&+ ab(x_ky_k+x_kz_k+y_kz_k)+(a+b)(x_ky_k+x_kz_k+y_kz_k)+u_k(x_k+y_k+z_k)+v_k(x_k+y_k+z_k)\\
&+ 2av_k(x_k+y_k+z_k)+2bu_k(x_k+y_k+z_k)]\\
&= 4[(x_k+y_k+z_k)(2u_k+2v_k+bu_k+av_k)+3u_kv_k]+4[(x_k+y_k+z_k)(u_k+v_k+2bu_k+2av_k)+3u_kv_k]\\
&= 4[6u_kv_k+3(x_k+y_k+z_k)(u_k+v_k+bu_k+av_k)]\\
&= 24u_kv_k \Rightarrow u_kv_k=0,
\end{aligned}$$

where the second equality is obtained by substituting -1 for $a+b$, 1 for ab; and the fourth equality by $(u_k+v_k+bu_k+av_k)=0$.
$a,b\neq -1$ because otherwise we will have $ab=1\Rightarrow a=b=-1\Rightarrow -2=a+b=-1$ which is absurd. Hence we have $u_k=v_k=0$ for $k=3,4,5$. This yields 3 zero entries in A's last row and so by lemma 3.6 we have $Det(A)=0$. □

Lemma 3.9 *Let K be the set of all 5×5 matrices with exactly two zero entries in every row and every column. Define an equivalence relation R on K as follows: $\forall A,B\in K,\ ARB \Leftrightarrow$ there exists a permutation of the rows and columns of A that yields a matrix with the same zero pattern as B. In this case*

$$\mathbf{A_1}=\begin{pmatrix}0&0&\bullet&\bullet&\bullet\\0&0&\bullet&\bullet&\bullet\\\bullet&\bullet&0&\bullet&0\\\bullet&\bullet&0&0&\bullet\\\bullet&\bullet&\bullet&0&0\end{pmatrix},\mathbf{A_2}=\begin{pmatrix}0&\bullet&\bullet&\bullet&0\\0&0&\bullet&\bullet&\bullet\\\bullet&0&0&\bullet&\bullet\\\bullet&\bullet&0&0&\bullet\\\bullet&\bullet&\bullet&0&0\end{pmatrix}$$

where the entries denoted by a '$\bullet$' have some arbitrary non-zero value, is a complete system of representatives for $^{K}\!/_{R}$.

Proof: We can view each element of K as a disjoint union of two perfect matchings forming a bipartite graph with 5 vertices on each part (each zero entry denotes an edge). Since the two matchings are disjoint there are no 2-cycles. Thus up to isomorphism there are only two possibilities, namely the disjoint union of a 4-cycle and a 6-cycle, and the 10-cycle. These are represented by A_1 and A_2 respectively. □

Lemma 3.10 *There exists a column of A with no zero entries.*

Proof: Assume for the sake of contradiction that there is a zero entry in every column of A. By lemma 3.5 A has at least two zero entries in every column. By lemma 3.6 A has at most two zero entries in every row as otherwise we will have $Det(A)=0$ contrary to our assumption that A is nonsingular. This implies that A has exactly two zero entries in every row and in every column. Hence we need only consider the two matrices of lemma 3.9.

$$\mathbf{A_1} = \begin{pmatrix} 0 & 0 & g & j & m \\ 0 & 0 & h & k & n \\ a & d & 0 & l & 0 \\ b & e & 0 & 0 & p \\ c & f & i & 0 & 0 \end{pmatrix}.$$

Since A_1 is a p_2 matrix we have the following equations:

$$0 = Per\begin{pmatrix} 0 & j & j & g & g \\ 0 & k & k & h & h \\ a & l & l & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & i & i \end{pmatrix} = bPer\begin{pmatrix} j & j & g & g \\ k & k & h & h \\ l & l & 0 & 0 \\ 0 & 0 & i & i \end{pmatrix} = 4bli(gk+hj) \Rightarrow gk+hj=0,$$

where the last implication follows since $char(F) \neq 2$ and we assumed $b, l, i \neq 0$.

$$0 = Per\begin{pmatrix} 0 & 0 & j & g & g \\ 0 & 0 & k & h & h \\ a & a & l & 0 & 0 \\ b & b & 0 & 0 & 0 \\ c & c & 0 & i & i \end{pmatrix} = 2bPer\begin{pmatrix} 0 & j & g & g \\ 0 & k & h & h \\ a & l & 0 & 0 \\ c & 0 & i & i \end{pmatrix} \Rightarrow 0 = cghl + ai(gk+hj) = cghl.$$

This yields another zero entry in A_1, which is a contradiction to our assumption on the structure of A.

$$\mathbf{A_2} = \begin{pmatrix} 0 & d & g & j & 0 \\ 0 & 0 & h & k & m \\ a & 0 & 0 & l & n \\ b & e & 0 & 0 & p \\ c & f & i & 0 & 0 \end{pmatrix}.$$

Since A_2 is a p_2 matrix we have the following equations:

$$0 = Per\begin{pmatrix} 0 & 0 & d & d & g \\ 0 & 0 & 0 & 0 & h \\ a & a & 0 & 0 & 0 \\ b & b & e & e & 0 \\ c & c & f & f & i \end{pmatrix} = hPer\begin{pmatrix} 0 & 0 & d & d \\ a & a & 0 & 0 \\ b & b & e & e \\ c & c & f & f \end{pmatrix} = 4adh(bf+ce) \Rightarrow bf+ce=0,$$

where the last implication follows since $char(F) \neq 2$ and we assumed $a, d, h \neq 0$. By proposition 3.2 all of A_2's 4×4 minors have zero permanent and so we have

$$0 = Per\begin{pmatrix} 0 & 0 & h & k \\ a & 0 & 0 & l \\ b & e & 0 & 0 \\ c & f & i & 0 \end{pmatrix} = aeik + hl(bf+ce) = aeik.$$

This yields another zero entry in A_2, which is a contradiction to our assumption on the structure of A. □

Lemma 3.11 *Let K be the set of all 5×5 matrices with their first column being $(1,1,1,1,1)^t$, having at least two zero entries in any other column and at most two zero entries in any row. Let $A \in K$, each entry of A will be denoted by a ’1’, or a ’0’ - the only one referred to as a zero entry, or a ’$\star$’ which means that we don’t know the value of this entry. Define an equivalence relation R on K as follows: $\forall A, B \in K,\ ARB \Leftrightarrow$ there exists a permutation of the rows and columns of A that yields a matrix with the same zero pattern as B (note that the ’$\star$’ entries do not affect the zero pattern). In this case*

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 & \star & \star \\ 1 & 0 & 0 & \star & \star \\ 1 & \star & \star & \star & \star \\ 1 & \star & \star & 0 & 0 \\ 1 & \star & \star & 0 & 0 \end{pmatrix}, \mathbf{A_2} = \begin{pmatrix} 1 & 0 & 0 & \star & \star \\ 1 & 0 & 0 & \star & \star \\ 1 & \star & \star & 0 & \star \\ 1 & \star & \star & 0 & 0 \\ 1 & \star & \star & \star & 0 \end{pmatrix}, \mathbf{A_3} = \begin{pmatrix} 1 & 0 & \star & \star & \star \\ 1 & 0 & 0 & \star & \star \\ 1 & \star & 0 & 0 & \star \\ 1 & \star & \star & 0 & 0 \\ 1 & \star & \star & \star & 0 \end{pmatrix},$$

$$\mathbf{A_4} = \begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 1 & 0 & 0 & \star & \star \\ 1 & \star & 0 & 0 & \star \\ 1 & \star & \star & \star & 0 \\ 1 & \star & \star & \star & 0 \end{pmatrix}, \mathbf{A_5} = \begin{pmatrix} 1 & 0 & \star & \star & 0 \\ 1 & 0 & 0 & \star & \star \\ 1 & \star & 0 & 0 & \star \\ 1 & \star & \star & 0 & 0 \\ 1 & \star & \star & \star & \star \end{pmatrix}$$

is a complete system of representatives for ${}^{K}\!/_{R}$.

Proof: We can view each element of K as a simple bipartite graph $G = (V_1 \cup V_2, E)$, where V_1 has 5 vertices each of which has degree ≤ 2, $\sum_{v \in V_1} deg(v) = |E| = 8$ and V_2 has one isolated vertex and 4 vertices of degree 2. Thus up to isomorphism there are only 5 possibilities, namely: the disjoint union of two 4-cycles represented by A_1, the disjoint union of a 4-cycle and a simple path of length 4 represented by A_2, a simple path of length 8 represented by A_3, the disjoint union of a 6-cycle and a simple path of length 2 represented by A_4 and an 8-cycle represented by A_5. $\square$

Lemma 3.12 *There are at least two columns of A with no zero entries.*

Proof: By lemma 3.10 we know that there exists at least one such column; thus we assume for the sake of contradiction that there is exactly one column with no zero entries. We will show that this assumption contradicts A’s non-singularity. By proposition 3.1 we may assume that A’s first column is $(1,1,1,1,1)^t$. By lemma 3.5 A has at least two zero entries in any column other than the first. By lemma 3.6 we may assume that A has at most two zero entries in every row. Hence we need only consider the five matrices of lemma 3.11.

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 & g & j \\ 1 & 0 & 0 & h & k \\ 1 & a & d & i & l \\ 1 & b & e & 0 & 0 \\ 1 & c & f & 0 & 0 \end{pmatrix}.$$

We may assume $g, j, h, k, b, e, c, f \neq 0$ because otherwise A_1 will have 3 zero entries in one row and therefore by lemma 3.6 we will have $Det(A_1) = 0$. Since A_1 is a p_2 matrix we have the following

equations:
By proposition 3.2 all of A_1's 4×4 minors have zero permanent, hence

$$0 = Per \begin{pmatrix} 0 & 0 & g & j \\ 0 & 0 & h & k \\ b & e & 0 & 0 \\ c & f & 0 & 0 \end{pmatrix} = (gk + hj)(bf + ce).$$

Assume without loss of generality that $bf + ce = 0$ (the case $gk + hj = 0$ can be proved similarly).

$$0 = Per \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & a & d & d \\ 1 & 1 & b & e & e \\ 1 & 1 & c & f & f \end{pmatrix} = 4(aef + cde + bdf) \Rightarrow aef + d(bf + ce) = 0 \Rightarrow a = 0,$$

where the last implication is by our assumptions $bf + ce = 0$ and $e, f \neq 0$.

$$0 = Per \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & a & a & d \\ 1 & 1 & b & b & e \\ 1 & 1 & c & c & f \end{pmatrix} = Per \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & d \\ 1 & 1 & b & b & e \\ 1 & 1 & c & c & f \end{pmatrix} = 4bcd \Rightarrow d = 0,$$

where the last implication is by our assumption $b, c \neq 0$. Since $a = d = 0$ we have by lemma 3.7 that the second and third columns of A_1 are linearly dependent, hence $Det(A_1) = 0$.

$$\mathbf{A_2} = \begin{pmatrix} 1 & 0 & 0 & g & j \\ 1 & 0 & 0 & h & k \\ 1 & a & d & 0 & l \\ 1 & b & e & 0 & 0 \\ 1 & c & f & i & 0 \end{pmatrix}.$$

We may assume $g, j, h, k, b, e \neq 0$ because otherwise A_2 will have 3 zero entries in one row and therefore by lemma 3.6 we will have $Det(A_2) = 0$. We may further assume $i, l \neq 0$ because otherwise we are back with the equivalence class of A_1. Since A_2 is a p_2 matrix we have the following equations:

$$0 = Per \begin{pmatrix} 1 & g & g & j & j \\ 1 & h & h & k & k \\ 1 & 0 & 0 & l & l \\ 1 & 0 & 0 & 0 & 0 \\ 1 & i & i & 0 & 0 \end{pmatrix} = Per \begin{pmatrix} g & g & j & j \\ h & h & k & k \\ 0 & 0 & l & l \\ i & i & 0 & 0 \end{pmatrix} = 4il(gk + hj) \Rightarrow gk + hj = 0,$$

where the last implication is by our assumption $i, l \neq 0$.

$$0 = Per \begin{pmatrix} 0 & 0 & g & g & j \\ 0 & 0 & h & h & k \\ a & a & 0 & 0 & l \\ b & b & 0 & 0 & 0 \\ c & c & i & i & 0 \end{pmatrix} = 2bPer \begin{pmatrix} 0 & g & g & j \\ 0 & h & h & k \\ a & 0 & 0 & l \\ c & i & i & 0 \end{pmatrix} = 4b[cghl + ai(gk + hj)] = 4bcghl \Rightarrow c = 0,$$

where the last implication is by our assumption $b, g, h, l \neq 0$.

$$0 = Per\begin{pmatrix} 0 & 0 & 0 & j & j \\ 0 & 0 & 0 & k & k \\ a & a & d & l & l \\ b & b & e & 0 & 0 \\ c & c & f & 0 & 0 \end{pmatrix} = Per\begin{pmatrix} 0 & 0 & 0 & j & j \\ 0 & 0 & 0 & k & k \\ a & a & d & l & l \\ b & b & e & 0 & 0 \\ 0 & 0 & f & 0 & 0 \end{pmatrix} = fPer\begin{pmatrix} 0 & 0 & j & j \\ 0 & 0 & k & k \\ a & a & l & l \\ b & b & 0 & 0 \end{pmatrix} = 4abfjk.$$

By our assumption $b, j, k \neq 0$, so either $a = 0$ or $f = 0$. If $a = 0$ then A_2 has 4 zero entries in its first column, hence $Det(A_2) = 0$ by lemma 3.3. Otherwise $f = 0$ and so A_2 has 3 zero entries in its last row, hence $Det(A_2) = 0$ by lemma 3.6.

$$\mathbf{A_3} = \begin{pmatrix} 1 & 0 & d & g & j \\ 1 & 0 & 0 & h & k \\ 1 & a & 0 & 0 & l \\ 1 & b & e & 0 & 0 \\ 1 & c & f & i & 0 \end{pmatrix}.$$

We may assume $a, b, e, h, k, l \neq 0$ because otherwise A_3 will have 3 zero entries in one row and therefore by lemma 3.6 we will have $Det(A_3) = 0$. We may further assume $d, i \neq 0$ because otherwise we are back with the equivalence class of A_2. Since A_3 is a p_2 matrix we have the following equations:

$$0 = Per\begin{pmatrix} 1 & 0 & 0 & d & d \\ 1 & 0 & 0 & 0 & 0 \\ 1 & a & a & 0 & 0 \\ 1 & b & b & e & e \\ 1 & c & c & f & f \end{pmatrix} = Per\begin{pmatrix} 0 & 0 & d & d \\ a & a & 0 & 0 \\ b & b & e & e \\ c & c & f & f \end{pmatrix} = 4ad(bf + ce) \Rightarrow bf + ce = 0,$$

where the last implication is by our assumption $a, d \neq 0$. By proposition 3.2 all of A_3's 4×4 minors have zero permanent, hence

$$0 = Per\begin{pmatrix} 0 & 0 & h & k \\ a & 0 & 0 & l \\ b & e & 0 & 0 \\ c & f & i & 0 \end{pmatrix} = aeik + hl(bf + ce) \Rightarrow aeik = 0,$$

which contradicts our assumption $a, e, i, k \neq 0$.

$$\mathbf{A_4} = \begin{pmatrix} 1 & 0 & d & 0 & j \\ 1 & 0 & 0 & g & k \\ 1 & a & 0 & 0 & l \\ 1 & b & e & h & 0 \\ 1 & c & f & i & 0 \end{pmatrix}.$$

We may assume $a, d, g, j, k, l \neq 0$ because otherwise A_4 will have 3 zero entries in one row and therefore by lemma 3.6 we will have $Det(A_4) = 0$. Since A_4 is a p_2 matrix we have the following equations:

$$0 = Per\begin{pmatrix} 0 & 0 & d & d & j \\ 0 & 0 & 0 & 0 & k \\ a & a & 0 & 0 & l \\ b & b & e & e & 0 \\ c & c & f & f & 0 \end{pmatrix} = kPer\begin{pmatrix} 0 & 0 & d & d \\ a & a & 0 & 0 \\ b & b & e & e \\ c & c & f & f \end{pmatrix} = 4adk(bf + ce) \Rightarrow bf + ce = 0,$$

where the last implication is by our assumption $a, d, k \neq 0$.

$$0 = Per\begin{pmatrix} 0 & 0 & d & j & j \\ 0 & 0 & 0 & k & k \\ a & a & 0 & l & l \\ b & b & e & 0 & 0 \\ c & c & f & 0 & 0 \end{pmatrix} = 2kPer\begin{pmatrix} 0 & 0 & d & j \\ a & a & 0 & l \\ b & b & e & 0 \\ c & c & f & 0 \end{pmatrix} = 4ajk(bf+ce)+4bcdkl = 4bcdkl \Rightarrow bc = 0,$$

where the last implication is by our assumption $d, k, l \neq 0$. Assume without loss of generality that $c = 0$ (the case $b = 0$ can be proved similarly). $c = 0, bf + ce = 0 \Rightarrow bf = 0$. If $b = 0$ then A_4 has 4 zero entries in its first column, thus $Det(A_4) = 0$ by lemma 3.3. Otherwise $f = 0$, hence A_4 has 3 zero entries in its last row and thus $Det(A_4) = 0$ by lemma 3.6.

$$\mathbf{A_5} = \begin{pmatrix} 1 & 0 & d & g & 0 \\ 1 & 0 & 0 & h & j \\ 1 & a & 0 & 0 & k \\ 1 & b & e & 0 & 0 \\ 1 & c & f & i & l \end{pmatrix}.$$

We may assume $a, b, d, e, g, h, j, k \neq 0$ because otherwise A_5 will have 3 zero entries in one row and therefore by lemma 3.6 we will have $Det(A_5) = 0$. Since A_5 is a p_2 matrix we have the following equations:

$$0 = Per\begin{pmatrix} 0 & 0 & d & d & g \\ 0 & 0 & 0 & 0 & h \\ a & a & 0 & 0 & 0 \\ b & b & e & e & 0 \\ c & c & f & f & i \end{pmatrix} = hPer\begin{pmatrix} 0 & 0 & d & d \\ a & a & 0 & 0 \\ b & b & e & e \\ c & c & f & f \end{pmatrix} = 4adh(bf + ce) \Rightarrow bf + ce = 0,$$

where the last implication is by our assumption $a, d, h \neq 0$.

$$0 = Per\begin{pmatrix} 0 & 0 & d & g & g \\ 0 & 0 & 0 & h & h \\ a & a & 0 & 0 & 0 \\ b & b & e & 0 & 0 \\ c & c & f & i & i \end{pmatrix} = 2hPer\begin{pmatrix} 0 & 0 & d & g \\ a & a & 0 & 0 \\ b & b & e & 0 \\ c & c & f & i \end{pmatrix} = 4agh(bf + ce) + 4abdhi \Rightarrow i = 0,$$

where the last implication is by our assumption $a, b, d, h \neq 0$.

$$0 = Per\begin{pmatrix} 1 & g & g & 0 & 0 \\ 1 & h & h & j & j \\ 1 & 0 & 0 & k & k \\ 1 & 0 & 0 & 0 & 0 \\ 1 & i & i & l & l \end{pmatrix} = Per\begin{pmatrix} g & g & 0 & 0 \\ h & h & j & j \\ 0 & 0 & k & k \\ 0 & 0 & l & l \end{pmatrix} = 4ghkl \Rightarrow l = 0,$$

where the last implication is by our assumption $g, h, k \neq 0$. $l = 0$ brings us back to the equivalence class of A_3, thus $Det(A_5) = 0$ and we are done. □

Lemma 3.13 *Let* $(1,1,1,1,1)^t$ *and* $(x,y,z,u,v)^t$ *be the first and second columns of* A *respectively then* $A_{(2,1,0,0,0)}$ *has the same rank as*

$$\mathbf{M} = \begin{pmatrix} y+z+u+v & x & x & x & x \\ y & x+z+u+v & y & y & y \\ z & z & x+y+u+v & z & z \\ u & u & u & x+y+z+v & u \\ v & v & v & v & x+y+z+u \end{pmatrix}.$$

Proof: Let

$$\mathbf{B} = \frac{1}{6}\begin{pmatrix} -2 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 & -2 \end{pmatrix}.$$

Since $char(F) \neq 2,3$, B is well defined and nonsingular. The theorem now follows as $M = BA_{(2,1,0,0,0)}$. □

Lemma 3.14 *There is at most one column of* A *with no zero entries.*

Proof: Assume for the sake of contradiction that A has at least two such columns. By proposition 3.1 we can assume that $(1,1,1,1,1)^t$ is A's first column and $(x_1,x_2,x_3,x_4,x_5)^t$ where $x_1,x_2,x_3,x_4,x_5 \neq 0$ its second. We have

$$\mathbf{A_{(2,1,0,0,0)}} = \begin{pmatrix} 0 & 2(x_3+x_4+x_5) & 2(x_2+x_4+x_5) & 2(x_2+x_3+x_5) & 2(x_2+x_3+x_4) \\ 2(x_3+x_4+x_5) & 0 & 2(x_1+x_4+x_5) & 2(x_1+x_3+x_5) & 2(x_1+x_3+x_4) \\ 2(x_2+x_4+x_5) & 2(x_1+x_4+x_5) & 0 & 2(x_1+x_2+x_5) & 2(x_1+x_2+x_4) \\ 2(x_2+x_3+x_5) & 2(x_1+x_3+x_5) & 2(x_1+x_2+x_5) & 0 & 2(x_1+x_2+x_3) \\ 2(x_2+x_3+x_4) & 2(x_1+x_3+x_4) & 2(x_1+x_2+x_4) & 2(x_1+x_2+x_3) & 0 \end{pmatrix}.$$

By lemma 3.4 $rank(A_{(2,1,0,0,0)}) \leq 3$ and so by lemma 3.13 all of M's 4×4 minors are singular, where

$$\mathbf{M} = \begin{pmatrix} x_2+x_3+x_4+x_5 & x_1 & x_1 & x_1 & x_1 \\ x_2 & x_1+x_3+x_4+x_5 & x_2 & x_2 & x_2 \\ x_3 & x_3 & x_1+x_2+x_4+x_5 & x_3 & x_3 \\ x_4 & x_4 & x_4 & x_1+x_2+x_3+x_5 & x_4 \\ x_5 & x_5 & x_5 & x_5 & x_1+x_2+x_3+x_4 \end{pmatrix}.$$

Let us look at the minor obtained from M by deleting its last row and first column:

$$\begin{pmatrix} x_1 & x_1 & x_1 & x_1 \\ x_1+x_3+x_4+x_5 & x_2 & x_2 & x_2 \\ x_3 & x_1+x_2+x_4+x_5 & x_3 & x_3 \\ x_4 & x_4 & x_1+x_2+x_3+x_5 & x_4 \end{pmatrix}.$$

By its singularity and our assumption $x_1 \neq 0$ we get

$$0 = Det\begin{pmatrix} x_1 & x_1 & x_1 & x_1 \\ x_1+x_3+x_4+x_5 & x_2 & x_2 & x_2 \\ x_3 & x_1+x_2+x_4+x_5 & x_3 & x_3 \\ x_4 & x_4 & x_1+x_2+x_3+x_5 & x_4 \end{pmatrix}$$

$$= x_1 Det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 + x_3 + x_4 + x_5 & x_2 & x_2 & x_2 \\ x_3 & x_1 + x_2 + x_4 + x_5 & x_3 & x_3 \\ x_4 & x_4 & x_1 + x_2 + x_3 + x_5 & x_4 \end{pmatrix}$$

$$= x_1 Det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 - x_2 + x_3 + x_4 + x_5 & 0 & 0 & 0 \\ 0 & x_1 + x_2 - x_3 + x_4 + x_5 & 0 & 0 \\ 0 & 0 & x_1 + x_2 + x_3 - x_4 + x_5 & 0 \end{pmatrix}$$

$$\begin{aligned} &= \quad -x_1(x_1 - x_2 + x_3 + x_4 + x_5)(x_1 + x_2 - x_3 + x_4 + x_5)(x_1 + x_2 + x_3 - x_4 + x_5) \\ &\Rightarrow \quad (x_1 - x_2 + x_3 + x_4 + x_5)(x_1 + x_2 - x_3 + x_4 + x_5)(x_1 + x_2 + x_3 - x_4 + x_5) = 0. \end{aligned}$$

For $1 \leq k \leq 5$ let $S_k = x_1 + x_2 + x_3 + x_4 + x_5 - 2x_k$ then the last equality can be abbreviated to $S_2 S_3 S_4 = 0$. Similarly, by deleting the i^{th} column and j^{th} row of M for some $1 \leq i < j \leq 5$ we get the equation $0 = \prod_{k \neq i,j} S_k$. The product of any 3 of S_1, S_2, S_3, S_4, S_5 equals 0 and therefore at least 3 of them must vanish. Assume without loss of generality that these are S_1, S_2 and S_3. Solving the equations $S_1 = S_2 = S_3 = 0$ for the x_i's yields $x_1 = x_2 = x_3$ and $x_4 + x_5 = -x_1$. By proposition 3.1 we can multiply A's second column by x_1^{-1} which yields $(1, 1, 1, a, b)^t$ where $a + b = -1$. By lemma 3.8 we now have $Det(A) = 0$, a contradiction. $\Box$

Lemma 3.14 contradicts lemma 3.12 and therefore the theorem is proved. $\Box$

4 Concluding Remarks

- Conjecture 1.3 asserts that if $Per(A^\alpha) = 0$ for all $\alpha \in L_n^2$ then $Det(A) = 0$. If the assertion of the conjecture is true, then by Hilbert's NullStellenSatz Theorem, there exists some $r \in \mathbb{N}$ such that $Det(A)^r$ can be expressed as a linear combination, with polynomial coefficients, of the elements of $\{Per(A^\alpha) | \alpha \in L_n^2\}$. We found two such formulas for 3×3 matrices - one for $r = 4$ and the other for $r = 3$. The former formula was found by the computer algebra system MATHEMATICA, using a simple coefficient equating argument. The latter formula uses the larger set of polynomials $\{Per(A^\alpha) | \alpha \in L_n^2\} \bigcup \{Per(A_{ij}) | 1 \leq i, j \leq 3\}$ where A_{ij} is the minor of A obtained by deleting its i^{th} row and j^{th} column. The formula is $Det(A)^3 = \sum_{\alpha \in L_3^2} \frac{1}{h(\alpha)} Per((adjA^t)^\alpha) Per(A^\alpha) + 3! \sum_{i,j=1}^{3} (-1)^{i+j} Det(A_{ij}) Per(A_{ij}) \prod a_{ij}$, where the a_{ij}'s appearing in the last product are those components of A that do not appear in A_{ij} and $adjA$ is the adjoint matrix of A. If we assume for the sake of contradiction that A is a nonsingular p_2 matrix then the right hand side of the above formula vanishes (by proposition 3.2 $Per(A_{ij}) = 0$ for all $1 \leq i, j \leq 3$) contrary to $Det(A) \neq 0$. For proofs of the formulas see [9].
- Conjecture 1.1 is not hard to prove if we assume that F is an infinite field (as no infinite linear space can be covered by a finite collection of its proper subspaces). No proof is known for conjecture 1.3 in that case. We do however believe that the special case in which F is a field of characteristic zero (and is therefore infinite) is indeed easier. We believe that in that

case the matrix at hand is not just singular, but all the summands in the expansion of its determinant vanish, i.e. if A is p_2 then $\forall \sigma \in S_n$ $\prod_{i=1}^{n} a_{i\sigma(i)} = 0$. This is easily seen to be false if $char(F) \neq 0$. Indeed if $char(F) = p$ and $n \geq p$, then J_n is clearly a counter example (even if F is infinite); in fact in this example $Per(A^{\alpha}) = 0$ for all $\alpha \in L_n$ and not just for $\alpha \in L_n^2$. We will verify its correctness for $n \leq 4$. First we need some terminology:
Let $S_n = \{\sigma_1, \sigma_2, ..., \sigma_{n!}\}$. For all $1 \leq j \leq n!$ set $x_j = \prod_{i=1}^{n} a_{i\sigma_j(i)}$. $\forall \alpha \in L_n^2$ define its complement by $\alpha^c = \bar{2} - \alpha = (2-\alpha_1, ..., 2-\alpha_n)$. Clearly $\alpha \in L_n^2 \Leftrightarrow \alpha^c \in L_n^2$.
Let x be any one of the $n!$ summands in the expansion of $Per(A^{\alpha})$, and let y be such a summand in the expansion of $Per(A^{\alpha^c})$. The product xy consists of $2n$ entries of A, exactly two of every row and of every column. As such, it can be viewed as a product $x_i x_j$ of two summands of the expansion of $Per(A)$ (i and j may not be uniquely determined). Accordingly, if A is p_2 and $\alpha \neq (1, 1, ..., 1)$, we get a set of quadratic equations of the form $yPer(A^{\alpha}) = \sum x_i x_j = 0$, each obtained from an arbitrary selection of $\alpha \in L_n^2$ and a summand y from the expansion of $Per(A^{\alpha^c})$. To that set of equations one should add the equation $\sum_{i=1}^{n!} x_i = 0$ resulting from $Per(A) = 0$, and the relations $x_i x_j = x_k x_l$ corresponding to the various factorizations of each product $x_i x_j$. Note that all of those equations, with the exception of $Per(A) = 0$, are linear homogenous in the $\binom{n!}{2}$ variables $y_{ij} = x_i x_j$. We will denote this system of equations by a $(\star)$. If the system of equations $(\star)$ has only the trivial solution then we are done. The standard procedure for solving a system of multivariate polynomial equations is that of Groebner bases (solving the system of linear equations in the variables $y_{ij} = x_i x_j$, yields many non-trivial solutions). Using the computer algebra system MATHEMATICA this was seen to be true for the case $n = 3$. The case $n = 4$ was too space consuming but the program was able to compute the reduced Groebner basis G for the ideal $\hat{\mathcal{I}}_2$, generated by the polynomials corresponding to the equations $(\star)$. By ordering the variables $x_1 > x_2 > ... > x_{n!}$ the program yields $x_{n!}^n \in G$ entailing $x_{n!}^n \in \hat{\mathcal{I}}_2$. Since the ordering of the variables was arbitrary, it follows that $x_i^n \in \hat{\mathcal{I}}_2$ for all $1 \leq i \leq n!$, and therefore the only solution of the system $(\star)$ is the trivial one.

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