ALGEBRAIC METHODS IN INTEGER PROGRAMMING

Introduction. This article highlights some of the recent results in theoretical integer programming that have been obtained by studying *integer programs* using tools from *commutative algebra* and *algebraic geometry*. The main computational tool involved in the discussion here is the *Gröbner basis* of a special polynomial ideal called a *toric ideal* [30],[14]. For connections between Gröbner bases of toric ideals and polytopes see [30] and for Gröbner basis theory for general polynomial ideals see [1] and [11]. Toric ideals and more generally, *lattice ideals* [33], have been the subject of much research in the past few years. The discussion in this article follows a specific route through the work done in this area. All effort will be made to include references needed for details and further reading.

Toric ideals and integer programming. We will be concerned with integer programs of the form $IP_{A,c}(b) := \min\{c \cdot x \, : \, Ax = b, \, x \in \mathbf{N}^n\}$ where A is a fixed $d \times n$ integer matrix of rank d. Here $\mathbf{N}$ denotes the non-negative integers. The right hand side vector b will be assumed to lie in the monoid $pos_{\mathbf{Z}}(A) := \{Ax \, : \, x \in \mathbf{N}^n\}$ which guarantees that $IP_{A,c}(b)$ is always feasible. Let $ker_{\mathbf{Z}}(A)$ denote the $(n-d)$-dimensional saturated *lattice* $\{u \in \mathbf{Z}^n : Au = 0\}$. For simplicity we assume that $ker_{\mathbf{Z}}(A) \cap \mathbf{N}^n = \{0\}$ which implies that $P_b := conv\{x \in \mathbf{N}^n : Ax = b\}$ is a polytope for all $b \in pos_{\mathbf{Z}}(A)$. For $b \in pos_{\mathbf{Z}}(A)$ and a $v \in P_b \cap \mathbf{N}^n$ the set of lattice points in P_b is precisely the congruence class in $\mathbf{N}^n$ of v modulo $ker_{\mathbf{Z}}(A)$.

The toric ideal of A is the d-dimensional binomial prime ideal $I_A := \langle x^{u^+} - x^{u^-} \, : \, u := u^+ - u^- \in ker_{\mathbf{Z}}(A), \, u^+, u^- \in \mathbf{N}^n \rangle$ in $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ where k is a field. The cost vector c can be any vector in $\mathbf{R}^n$ and for each polynomial $f = \sum_{i=1}^{l} k_i x^{\alpha_i} \in I_A$ the *initial term* of f with respect to c, denoted as $in_c(f)$, is the sum of those terms in f for which $c \cdot \alpha_i$ is maximal. The *initial ideal* of I_A with respect to c is then the ideal $in_c(I_A) := \langle in_c(f) : f \in I_A \rangle \subset k[\mathbf{x}]$. We will assume unless stated otherwise that the cost vector c is such that $in_c(I_A)$ is a *monomial ideal*, i.e, $in_c(I_A)$ can be generated by monomials. Such a c is said to be *generic* with respect to IP_A. Equivalently, c is generic with respect to IP_A if and only if each integer program in the family $IP_{A,c}$ has a unique optimal solution. Note that each lattice point $\alpha \in \mathbf{N}^n$ is a solution to a unique integer program in $IP_{A,c}$ since α lies in $P_{A\alpha}$ and in no other polytope of the form P_b. The following theorem relates $in_c(I_A)$ to $IP_{A,c}$.

Lemma 1 *The lattice point* $\alpha \in \mathbf{N}^n$ *is a non-optimal solution to* $IP_{A,c}(A\alpha)$ *if and only if the monomial* x^α *lies in the initial ideal* $in_c(I_A)$.

Proof: The lattice point $\alpha \in \mathbf{N}^n$ is a non-optimal solution to $IP_{A,c}(A\alpha)$ if and only if there exists β in $P_{A\alpha} \cap \mathbf{N}^n$ such that $c \cdot \alpha > c \cdot \beta$. This is equivalent to the statement that $x^\alpha - x^\beta$ is a non-zero element of I_A with $in_c(x^\alpha - x^\beta) = x^\alpha$. $\square$

The *standard monomials* of $in_c(I_A)$ are precisely all the monomials in $k[\mathbf{x}]$ that do not lie in $in_c(I_A)$.

Corollary 2 *A monomial* $x^\gamma \in k[\mathbf{x}]$ *is a standard monomial of* $in_c(I_A)$ *if and only if* γ *is the unique optimal solution to the integer program* $IP_{A,c}(A\gamma)$.

By Corollary 2, there is a bijection between the standard monomials of $in_c(I_A)$ and the elements of the monoid $pos_{\mathbf{Z}}(A)$.

The Conti-Traverso algorithm. In [9], Conti and Traverso gave an algorithm to solve integer programs using Gröbner bases of toric ideals. A Gröbner basis with respect to c, of the toric ideal I_A, is any finite subset $\mathcal{H}$ of I_A such that $in_c(I_A) = \langle in_c(f) \, : \, f \in \mathcal{H} \rangle$. A Gröbner basis $\mathcal{H}$ is *reduced* if it has the additional property that for each $f \in \mathcal{H}$, the coefficient of $in_c(f)$ is the identity in k and $in_c(f)$ does not divide any term in another element g of $\mathcal{H}$. Reduced Gröbner bases are unique.

Let $\mathcal{G}_c$ denote the reduced Gröbner basis of I_A with respect to c. Then $\mathcal{G}_c$ has the form $\{x^{\alpha_i} - x^{\beta_i} \, : \, i = 1, \ldots, t\}$ where $\alpha_i - \beta_i \in ker_{\mathbf{Z}}(A)$, $\alpha_i, \beta_i \in \mathbf{N}^n$ and $supp(\alpha_i) \cap supp(\beta_i) = \emptyset$ for all $i = 1, \ldots, t$. For $p \in \mathbf{Z}^n$, $supp(p) := \{i \in [n] :=$

$\{1, \ldots, n\} : p_i \neq 0\}$ denotes the support of p. If $x^{\alpha_i} - x^{\beta_i} \in \mathcal{G}_c$ then we always assume that $c \cdot \alpha_i > c \cdot \beta_i$.

Lemma 3 *If $\mathcal{G}_c = \{x^{\alpha_i} - x^{\beta_i} : i = 1, \ldots, t\}$ is the reduced Gröbner basis of I_A with respect to c then (i) $\{x^{\alpha_i} : i = 1, \ldots, t\}$ is the minimal generating set of the initial ideal $in_c(I_A)$ and (ii) for each binomial $x^{\alpha_i} - x^{\beta_i} \in \mathcal{G}_c$, β_i is the unique optimal solution to the integer program $IP_{A,c}(A\alpha_i)$.*

Proof: Part (i) follows from the definition of reduced Gröbner bases. For each binomial $x^{\alpha_i} - x^{\beta_i} \in \mathcal{G}_c$ we have $A\alpha_i = A\beta_i$, $\alpha_i, \beta_i \in \mathbf{N}^n$ and $c \cdot \alpha_i > c \cdot \beta_i$. If β_i is a non-optimal solution to $IP_{A,c}(A\alpha_i)$ then x^{β_i} lies in $in_c(I_A)$ by Lemma 1 and hence some x^{α_j} for $j = 1, \ldots, t$ will divide x^{β_i} contradicting the definition of a reduced Gröbner basis. □

The conditions in Lemma 3 are in fact also sufficient for a finite subset of binomials in I_A to be the reduced Gröbner basis of I_A with respect to c. Given $f \in I_A$, the *normal form* of f with respect to $\mathcal{G}_c$ is the unique remainder obtained upon dividing f by $\mathcal{G}_c$. See [11] for details on the *division algorithm* in $k[\mathbf{x}]$. The structure of $\mathcal{G}_c$ implies that the normal form of a monomial x^v with respect to $\mathcal{G}_c$ is a monomial $x^{v'}$ such that both v and v' are solutions to $IP_{A,c}(Av)$. The Conti-Traverso algorithm for $IP_{A,c}$ can be summarized as follows.

Algorithm 4 *How to solve programs in $IP_{A,c}$.*
Input: The matrix A and cost vector c.

Pre-processing:
1. Find a generating set for the toric ideal I_A.
2. Compute the reduced Gröbner basis, $\mathcal{G}_c$, of I_A with respect to the cost vector c.

To solve $IP_{A,c}(b)$:
3. Find a solution v to $IP_{A,c}(b)$.
4. Compute the normal form x^{v^*} of the monomial x^v with respect to the reduced Gröbner basis $\mathcal{G}_c$. Then v^* is the optimal solution to $IP_{A,c}(b)$.

Proof: In order to prove the correctness of this algorithm, it suffices to show that for each solution v of $IP_{A,c}(b)$, the normal form of x^v modulo $\mathcal{G}_c$ is the monomial x^{v^*} where v^* is the unique optimal solution to $IP_{A,c}(b)$. Suppose x^w is the normal form of the monomial x^v. Then w is also a solution to $IP_{A,c}(b)$ since the exponent vectors of all monomials $x^{w'}$ obtained during division of x^v by $\mathcal{G}_c$ satisfy $b = Av = Aw'$, $w' \in \mathbf{N}^n$. If $w \neq v^*$, then $x^w - x^{v^*} \in I_A$ and $in_c(x^w - x^{v^*}) = x^w$ since $c \cdot w > c \cdot v^*$. This implies that $x^w \in in_c(I_A)$ and hence can be further reduced by $\mathcal{G}_c$ contradicting the definition of the normal form. □

Computational Issues. Algorithm 4 raises several computational issues. In Step 1, we require a generating set of the toric ideal I_A which can be a computationally challenging task as the size of A increases. The original Conti-Traverso algorithm starts with the ideal $J_A := \langle x_j t^{a_j^-} - t^{a_j^+} : j = 1, \ldots n, t_0 t_1 \cdots t_d - 1 \rangle$ in the larger polynomial ring $k[t_0, t_1, \ldots, t_d, x_1, \ldots, x_n]$ where $a_j = a_j^+ - a_j^-$ is the jth column of the matrix A. The toric ideal $I_A = J_A \cap k[\mathbf{x}]$ and hence the reduced Gröbner basis of I_A with respect to c can be obtained by *elimination* (see Chapter 3 in [11]). Although conceptually simple, this method has its limitations as the size of A increases since it requires $d+1$ extra variables over those present in I_A and the *Buchberger algorithm* for computing Gröbner bases [8] is sensitive to the number of variables involved. Two different algorithms for computing a generating set for I_A without introducing extra variables can be found in [5] and [19] respectively.

Once the generating set of I_A has been found, one needs to compute the reduced Gröbner basis $\mathcal{G}_c$ of I_A. This can be done by any *computer algebra* package that does Gröbner basis computations like Macaulay2, Maple, Reduce, Singular or Cocoa to name a few. Cocoa has a dedicated implementation for toric ideals [6]. As the size of the problem increases, a straightforward computation of reduced Gröbner bases of I_A can become expensive and even impossible. Several tricks can be applied to help the computation, many of which are problem specific.

In Step 3 of Algorithm 4 one requires an initial solution to $IP_{A,c}(b)$. The original Conti-Traverso algorithm achieves this indirectly during the elimination procedure. Theoretically this task can be as hard as solving $IP_{A,c}(b)$, although in practice this depends on the specific problem at hand. The last step — to compute the normal form of a monomial with respect to the current reduced Gröbner basis — is (relatively speaking) a computationally easy task. Toric Gröbner bases for integer programming can be found in "GRIN" [19] by Hosten and "BASTAT" by Pottier (available by anonymous ftp from *zenon.inria.fr*). Both these packages exploit the special structural properties of the underlying ideals.

In practice, one is often only interested in solving $IP_{A,c}(b)$ for a fixed b. In this situation, the Buchberger algorithm can be *truncated* to produce a sufficient set of binomials that will solve this integer program [36]. This idea was originally introduced in [37] in the context of 0/1 *integer programs* in which all the data is non-negative. See also [10]. A "non-toric" algorithm for solving integer programs with fixed right hand sides was recently proposed in [4].

Test sets in integer programming. A geometric interpretation of Algorithm 4 and more generally of the Buchberger algorithm for toric ideals can be found in [35]. A *test set* for $IP_{A,c}$ is a finite subset of vectors in $ker_{\mathbf{Z}}(A)$ such that for an integer program $IP_{A,c}(b)$ and a non-optimal solution v to this program, there is some u in the test set such that $c \cdot v > c \cdot (v - u)$. By interpreting a binomial $x^{\alpha_i} - x^{\beta_i} \in \mathcal{G}_c$ as the vector $\alpha_i - \beta_i \in ker_{\mathbf{Z}}(A)$, it can be seen that $\mathcal{G}_c$ is the unique minimal test set for the family $IP_{A,c}$. A closely related test set for integer programming is the set of *neighbors of the origin* introduced by Scarf [27].

The binomial $x^{\alpha_i} - x^{\beta_i} \in \mathcal{G}_c$ can also be viewed as the directed line segment $[\alpha_i, \beta_i]$ directed from α_i to β_i. For each $b \in pos_{\mathbf{Z}}(A)$ we now construct a directed graph $\mathcal{F}_{b,c}$ as follows: the vertices of this graph are the solutions to $IP_{A,c}(b)$ and the edges of this graph are all possible directed line segments from $\mathcal{G}_c$ that connect two vertices of this graph. Then $\mathcal{G}_c$ is a necessary and sufficient set of directed line segments such that $\mathcal{F}_{b,c}$ is a connected graph with a unique sink (at the optimal solution) for each $b \in pos_{\mathbf{Z}}(A)$. This geometric interpretation of $\mathcal{G}_c$ can be used to solve several problems. By reversing the directions on all edges in $\mathcal{F}_{b,c}$, one obtains a directed graph with a unique root. One can enumerate all lattice points in P_b by searching this graph starting at its root. This idea was used in [34] to solve a class of manufacturing problems. The graphs $\mathcal{F}_{b,c}$ provide a way to connect all the feasible solutions to an integer program by lattice moves. This idea was applied to statistical sampling in [13].

Universal Gröbner bases. A subset $\mathcal{U}_A$ of I_A is a *universal Gröbner basis* for I_A if $\mathcal{U}_A$ contains a Gröbner basis of I_A with respect to all (generic) cost vectors $c \in \mathbf{R}^n$. The *Graver basis* of A [17] is a finite universal Gröbner basis of I_A that can be described as follows. For each $\sigma \in \{+,-\}^n$, let $\mathcal{H}_\sigma$ be the unique minimal generating set (over $\mathbf{N}$) of the semigroup $ker_{\mathbf{Z}}(A) \cap \mathbf{R}^n_\sigma$. Then the Graver basis, $Gr_A := \cup_\sigma \mathcal{H}_\sigma \backslash \{0\}$. An algorithm to compute Gr_A can be found in [31]. It was shown in [35] that all reduced Gröbner bases of I_A are contained in Gr_A which implies that there are only finitely many distinct reduced Gröbner bases for I_A as c varies over generic cost vectors. Let UGB_A denote the union of all the distinct reduced Gröbner bases of I_A. Then UGB_A is a universal Gröbner basis of I_A that is contained in the Graver basis Gr_A. The following theorem from [31] characterizes the elements of UGB_A and thus allows one to test whether a binomial $x^{\alpha_i} - x^{\beta_i} \in Gr_A$ belongs UGB_A. A second test can also be found in [31]. A vector $u \in \mathbf{Z}^n$ is said to be *primitive* if the g.c.d. of its components is one.

Theorem 5 *For a primitive vector $u \in ker_{\mathbf{Z}}(A)$, the binomial $x^{u^+} - x^{u^-}$ belongs to UGB_A if and only if the line segment $[u^+, u^-]$ is a primitive edge in the polytope P_{Au^+}.*

The *degree* of a binomial $x^{\alpha_i} - x^{\beta_i} \in I_A$, is defined to be $\sum \alpha_{ij} + \sum \beta_{ij}$. The degree of the universal Gröbner basis UGB_A is then simply the maximum degree of any binomial in UGB_A. This number is an important complexity measure for the family of integer programs that have A as coefficient matrix. The current best bound for the degree of UGB_A is as follows. See Chapter 4 in [30] for a full discussion.

Theorem 6 *The degree of a binomial $x^{\alpha_i} - x^{\beta_i} \in UGB_A$, is at most $(n-d)(d+1)D(A)$ where $D(A)$ is the maximum absolute value of the determinant of a $d \times d$ submatrix of A.*

It has been conjectured that this bound can be improved to $(d+1)D(A)$ and some partial results in this direction can be found in [18].

The universal Gröbner bases of several special instances of A have been investigated in the literature, a few of which we mention here. For the family of $1 \times n$ matrices $A(n) := [1\,2 \cdots n]$ it was shown in [12] that the Graver basis of $A(n)$ is in bijection with the *primitive partition identities* with largest part n. A matrix $A \in \mathbf{Z}^{d \times n}$ is *unimodular* if the absolute values of the determinants of all its non-singular maximal minors are the same positive constant. For $u \in ker_{\mathbf{Z}}(A)$, the binomial $x^{u^+} - x^{u^-} \in I_A$ is a *circuit* of A if u is primitive and has minimal support with respect to inclusion. Let $\mathcal{C}_A$ denote the set of circuits of A. Then in general, $\mathcal{C}_A \subseteq UGB_A \subseteq Gr_A$. If A is unimodular, then all of the above containments hold at equality although the converse is false: there are non-unimodular matrices for which $\mathcal{C}_A = Gr_A$. If A_n is the node-edge incidence matrix of the complete graph K_n then the elements in UGB_{A_n} can be identified with certain subgraphs of K_n. Gröbner bases of these matrices were investigated in [24]. The integer programs associated with A_n are the *b-matching* problems in the literature [25]. See Chapter 14 in [30] for some other specific examples of Gröbner bases.

Variation of cost functions in integer programming. We now consider all cost vectors in $\mathbf{R}^n$ (not just the generic ones) and study the effect of varying them. As seen earlier I_A has only finitely many distinct reduced Gröbner bases as c is varied over the generic cost vectors. We say that two cost vectors c_1 and c_2 are *equivalent* with respect to IP_A if for each $b \in pos_{\mathbf{Z}}(A)$, the integer programs $IP_{A,c_1}(b)$ and $IP_{A,c_2}(b)$ have the same set of optimal solutions. The Gröbner basis approach to integer programming allows a complete characterization of the structure of these equivalence classes of cost vectors.

Theorem 7 *[31] (i) There exists only finitely many equivalence classes of cost vectors with respect to IP_A.*
(ii) Each equivalence class is the relative interior of a convex polyhedral cone in $\mathbf{R}^n$.
(iii) The collection of all these cones defines a complete polyhedral fan in $\mathbf{R}^n$ called the Gröbner fan *of A.*
(iv) Let db denote any probability measure with support $pos_{\mathbf{Z}}(A)$ such that $\int_b b\,db < \infty$. Then the Minkowski integral $St(A) = \int_b P_b\,db$ is an $(n-d)$-dimensional convex polytope, called the state polytope *of A. The normal fan of $St(A)$ equals the Gröbner fan of A.*

Gröbner fans and state polytopes of graded polynomial ideals were introduced in [26] and [2] respectively. For a toric ideal both these entities have self contained construction methods that are rooted in the combinatorics of these ideals [31]. For a software system for computing Gröbner fans of toric ideals see [22].

We call P_b for $b \in pos_{\mathbf{Z}}(A)$ a *Gröbner fiber* of A if there is some $x^{u^+} - x^{u^-} \in UGB_A$ such that $b = Au^+ = Au^-$. Since there are only finitely many elements in UGB_A the matrix A has only finitely many Gröbner fibers. Then the *Minkowski sum* of all Gröbner fibers of A is a state polytope of A. For a survey of algorithms to construct state polytopes and Gröbner fans of graded polynomial ideals see Chapters 2 and 3 in [30]. The Gröbner fan of A provides a model for global *sensitivity analysis* for the family of integer programs $IP_{A,c}$.

We now briefly discuss a theory analogous to the above for linear programming based on results in [7] and [15]. For a comparison of integer and linear programming from this point of view see [31]. Let $LP_{A,c}(b) := min\{c \cdot x : Ax = b, x \geq 0\}$ where A and c are as before and b is any vector in the rational polyhedral cone $pos(A) := \{Ax : x \geq 0\}$. We define two cost vectors c_1 and c_2 to be *equivalent* with respect to LP_A if the linear programs $LP_{A,c_1}(b)$ and $LP_{A,c_2}(b)$ have the same set of optimal solutions for all $b \in pos(A)$. Let $\mathcal{A} := \{a_1, \dots, a_n\}$ be the vector configuration in $\mathbf{Z}^d$ consisting of the columns of A. For a subset $\sigma \subseteq \mathcal{A}$, we let $pos(\sigma)$ denote the cone generated by σ. A *polyhedral subdivision* Δ of $\mathcal{A}$ is a collection of subsets of $\mathcal{A}$ such that $\{pos(\sigma) : \sigma \in \Delta\}$ is a set of cones in a polyhedral fan whose support is $pos(\mathcal{A})$. The elements of Δ are called the *faces* or *cells* of Δ. For convenience we identify $\mathcal{A}$ with the set of indices $[n]$ and any subset of $\mathcal{A}$ by the corresponding subset $\sigma \subseteq [n]$. A cost vector $c \in \mathbf{R}^n$ induces the *regular subdivision* Δ_c of $\mathcal{A}$ [7],[15] as follows: σ is a face of Δ_c if there exists a vector $y \in \mathbf{R}^d$ such that $a_j \cdot y = c_j$ whenever $j \in \sigma$ and $a_j \cdot y < c_j$ otherwise. A cost vector $c \in \mathbf{R}^n$ is said to be *generic* with respect to LP_A if every linear program in the family $LP_{A,c}$ has a unique optimal solution. When c is generic for LP_A, the regular subdivision Δ_c is in fact a *triangulation* called the *regular triangulation* of $\mathcal{A}$ with respect to c.

Two cost vectors c_1 and c_2 are equivalent with respect to LP_A if and only if $\Delta_{c_1} = \Delta_{c_2}$. The equivalence class of c with respect to LP_A is hence $\{c' \in \mathbf{R}^n : \Delta_{c'} = \Delta_c\}$ which is the relative interior of a polyhedral cone in $\mathbf{R}^n$ called the *secondary cone* of c, denoted as $\mathcal{S}_c$. The cone $\mathcal{S}_c$ is n-dimensional if and only if c is generic with respect to LP_A. The set of all equivalence classes of cost vectors fit together to form a complete polyhedral fan in $\mathbf{R}^n$ called the *secondary fan* of A. This fan is the normal fan of a polytope called the *secondary polytope* of A. See [7] for construction methods for both the secondary fan and polytope of A.

We conclude this section by showing that the Gröbner and secondary fans of A are related. The *Stanley-Reisner* ideal of Δ_c is the square-free monomial ideal $\langle x_{i_1} \cdots x_{i_r} : \{i_1, \dots, i_r\}$ is a *non-face* of $\Delta_c\rangle \subset k[\mathbf{x}]$.

Theorem 8 *[29] The radical of the initial ideal $in_c(I_A)$ is the Stanley-Reisner ideal of the regular triangulation Δ_c.*

Corollary 9 *[29] (i) The Gröbner fan of A is a refinement of the secondary fan of A.*
(ii) A secondary polytope of A is a summand of a state polytope of A.

Corollary 9 reaffirms the view that integer programming is an arithmetic refinement of linear programming.

Group relaxations in integer programming. We now investigate *group relaxations* of integer programs in the family $IP_{A,c}$ from an algebraic point of view. The results in this section are taken from [21],[20] and [33], sometimes after an appropriate translation into polyhedral language. We refer the reader to these papers for the algebraic motivations that led to these results.

The group relaxation of $IP_{A,c}(b)$ [16] is the program $Group^\sigma(b) := min\{\tilde{c}_{\bar{\sigma}} \cdot x_{\bar{\sigma}} : A_\sigma x_\sigma + A_{\bar{\sigma}} x_{\bar{\sigma}} = b,\ x_{\bar{\sigma}} \geq 0,\ x = (x_\sigma, x_{\bar{\sigma}}) \in \mathbf{Z}^n\}$, where A_σ, the submatrix of A whose columns are indexed by $\sigma \subseteq [n]$, is the optimal basis of the linear program $LP_{A,c}(b)$ and $\tilde{c}_{\bar{\sigma}} = c_{\bar{\sigma}} - c_\sigma A_\sigma^{-1} A_{\bar{\sigma}}$. Here the cost vector c has also been partitioned as $c = (c_\sigma, c_{\bar{\sigma}})$ using the set $\sigma \subseteq [n]$.

Definition 10 *Suppose $\mathcal{L}$ is any sublattice of $\mathbf{Z}^n$, $w \in \mathbf{R}^n$ and $v \in \mathbf{N}^n$. The* lattice program *$Lat_{\mathcal{L},w}(v)$ defined by this data is*

$$minimize\ \ w \cdot u :\ u \equiv v\ mod\ \mathcal{L},\ \ u \in \mathbf{N}^n.$$

Lattice programs are a generalization of integer programs: $IP_{A,c}(b) = \mathrm{Lat}_{\mathcal{L},c}(v)$ where $\mathcal{L} = ker_{\mathbf{Z}}(A)$ and v is any feasible solution to $IP_{A,c}(b)$. Gröbner basis methods for integer programs can be extended to solve lattice programs (see [20], [33]). Given the lattice $\mathcal{L}$ and a cost vector w, we first construct the *lattice ideal* $I_{\mathcal{L}} = \langle \mathbf{x}^\alpha - \mathbf{x}^\beta :\ \alpha - \beta \in \mathcal{L}, \alpha, \beta \in \mathbf{N}^n\rangle \subset k[\mathbf{x}]$. We then compute the reduced Gröbner basis of $I_{\mathcal{L}}$ with respect to w denoted as $\mathcal{G}_w(I_{\mathcal{L}})$. (If w does not induce a total order on $\mathbf{N}^n$ via the inner product $w \cdot x$, $x \in \mathbf{N}^n$, then we use a tie breaking term order to refine the order induced by w.) For a particular lattice program $\mathrm{Lat}_{\mathcal{L},w}(v)$, the optimal solution is the exponent vector of the normal form of x^v with respect to $\mathcal{G}_w(I_{\mathcal{L}})$.

Let $\tau \subseteq [n]$ and $\pi_\tau : \mathbf{Z}^n \rightarrow \mathbf{Z}^{|\bar{\tau}|}$ be the coordinate projection map where the coordinates indexed by τ are eliminated. Consider the lattice $\mathcal{L}_\tau := \pi_\tau(\mathcal{L})$ where $\mathcal{L} = ker_{\mathbf{Z}}(A)$. Given a basis $\{b_1, \ldots, b_{n-d}\}$ of $\mathcal{L}$, the set $\{\pi_\tau(b_1), \ldots, \pi_\tau(b_{n-d})\}$ forms a basis for $\mathcal{L}_\tau$. Further, $\pi_\tau : \mathcal{L} \rightarrow \mathcal{L}_\tau$ is an isomorphism whenever $rank(A_\tau) = |\tau|$.

Proposition 11 *[33] Let v be a feasible solution to $IP_{A,c}(b)$ and A_σ be the optimal basis of $LP_{A,c}(b)$. Then the* group relaxation $Group^\sigma(b)$ *of $IP_{A,c}(b)$ is the lattice program $Lat_{\mathcal{L}_\sigma, \tilde{c}_\sigma}(\pi_\sigma(v))$ where $\tilde{c}_\sigma = \pi_\sigma(c - c_\sigma(A_\sigma)^{-1}A) = c_{\bar{\sigma}} - c_\sigma A_\sigma^{-1} A_{\bar{\sigma}}$.*

The program $Group^\sigma(b)$ can be solved by Gröbner basis methods as explained earlier or by dynamic programming [16]. The optimal solution $x^*_{\bar{\sigma}}$ to $Group^\sigma(b)$ is then lifted to the unique vector $x^* = (x^*_\sigma, x^*_{\bar{\sigma}}) \in \mathbf{Z}^n$ by solving the equation $A_\sigma x_\sigma + A_{\bar{\sigma}} x^*_{\bar{\sigma}} = b$. If all components of x^*_σ are non-negative then x^* is the optimal solution to $IP_{A,c}(b)$. Otherwise $c \cdot x^*$ is a lower bound to the optimal value of $IP_{A,c}(b)$.

When $Group^\sigma(b)$ fails to solve $IP_{A,c}(b)$, Wolsey [38] suggested using *extended group relaxations* of $IP_{A,c}(b)$. We introduce a more general set of extended group relaxations of $IP_{A,c}(b)$ inspired by the following close relationship between the linear programs in $LP_{A,c}$ and the regular triangulation Δ_c.

Proposition 12 *[31] The optimal solutions x to $LP_{A,c}(b)$ are the solutions to the problem : Find $x \in \mathbf{R}^n$ such that $Ax = b$, $x \geq 0$, and $supp(x)$ is a subset of a face of Δ_c.*

Proposition 12 says that the set σ in $Group^\sigma(b)$ is a maximal face of Δ_c.

Definition 13 *Consider the integer program $IP_{A,c}(b)$ and a feasible solution v to this program. Let τ be a face of Δ_c and σ be any maximal face of Δ_c containing τ. Then the* group relaxation *of $IP_{A,c}(b)$ with respect to τ denoted as $Group^\tau(b)$ is the lattice program $Lat_{\mathcal{L}_\tau, \tilde{c}_\tau}(\pi_\tau(v))$ where $\tilde{c}_\tau := \pi_\tau(c - c_\sigma A_\sigma^{-1} A)$.*

The extended group relaxations in [38] are precisely those $Group^\tau(b)$s where τ is a subset of the maximal face σ of Δ_c that gives the optimal basis of $LP_{A,c}(b)$. Clearly, one such relaxation will solve $IP_{A,c}(b)$. However, we consider all relaxations of $IP_{A,c}(b)$ of the form $Group^\tau(b)$ as τ varies over all faces of Δ_c in order to avoid keeping track of which b is being considered and what the optimal basis of $LP_{A,c}(b)$ is.

It was shown in [33] that the lattice program $Group^\tau(b)$ is related to the *localization* of the initial ideal $in_c(I_A)$ at the prime ideal $p_\tau := \langle x_j : j \notin \tau \rangle$ in $k[\mathbf{x}]$. Since group relaxations are always defined with respect to a face τ of Δ_c, we are guaranteed that $rank(A_\tau) = |\tau|$ which allows a unique lifting of the optimal solution of $Group^\tau(b)$ to a vector in the same congruence class modulo $\mathcal{L}$ as the solutions to $IP_{A,c}(b)$.

Theorem 14 *[20] Suppose $u' \in \mathbf{N}^{|\bar{\tau}|}$ is the optimal solution to the group relaxation $Group^\tau(b)$. Then there exists a unique $u \in \mathbf{Z}^n$ such that $A(u - v) = 0$ for any feasible solution v to $IP_{A,c}(b)$ and $\pi_\tau(u) = u'$. If $u \geq 0$ then it is the optimal solution to $IP_{A,c}(b)$.*

A group relaxation $Group^\tau(b)$ is easiest to solve when τ is a maximal face of Δ_c. In this situation, the lattice ideal $I_{\mathcal{L}_\tau}$ is *zero dimensional* and hence their Gröbner bases are easier to compute than otherwise. We call such group relaxations the *Gomory* relaxations of $IP_{A,c}(b)$. In general one is most interested in those group relaxations $Group^\tau(b)$ that solve $IP_{A,c}(b)$ with $|\tau|$ as large as possible. In the rest of this section we study several structural properties of these "least tight" extended group relaxations that solve programs in $IP_{A,c}$. We first need a diversion into combinatorics.

For $m \in \mathbf{N}^n$, we define *support* of $x^m \in k[\mathbf{x}]$ to be $supp(m)$.

Definition 15 *For a monomial $x^m \in k[\mathbf{x}]$ and $\sigma \subseteq [n]$, we say that (x^m, σ) is an* admissible pair *of a monomial ideal M if (i) $supp(m) \cap \sigma = \emptyset$ and (ii) every monomial in $x^m \cdot k[x_j : j \in \sigma]$ is a standard monomial of M.*

There is a natural partial order on the set of all admissible pairs of M given by $(x^m, \sigma) \leq (x^{m'}, \sigma')$ if and only if x^m divides $x^{m'}$ and $supp(x^{m'}/x^m) \cup \sigma' \subseteq \sigma$.

Definition 16 *An admissible pair (x^m, σ) of M is called a* standard pair *of M if it is a minimal* *element in the poset of all admissible pairs with respect to the above partial order.*

The standard pairs of M induce a unique covering of the set of standard monomials of M which we refer to as the *standard pair decomposition* of M. This decomposition was introduced in [32] to study the *associated primes* of M and their *multiplicities* and thus the *arithmetic degree* [3] of M. When M is the initial ideal of a toric ideal stronger conclusions can be drawn. In our exposition below we bypass much of the algebraic results associated with the standard pair decomposition of M, but instead use these results to motivate appropriate definitions to continue our discussion of group relaxations.

Definition 17 *(i) For $\tau \subseteq [n]$, we define the multiplicity of τ, denoted as $mult(\tau)$, to be the number of standard pairs of the form (x^m, τ) in the standard pair decomposition of M.*
(ii) The sum of the multiplicities of τ as τ varies over the subsets of $[n]$ is called the arithmetic degree of M, denoted as $arithdeg(M)$.

In the rest of this section we let $M = in_c(I_A)$.

Proposition 18 *(see Section 12.D in [30])*
(i) If (x^m, τ) is a standard pair of $in_c(I_A)$ then τ is a face of Δ_c.
(ii) The standard pair $(1, \sigma)$ occurs in the standard pair decomposition of $in_c(I_A)$ if and only if σ is a maximal face of Δ_c. In this case, $mult(\sigma)$ is the normalized volume of σ in Δ_c.

The *normalized volume* of a maximal face $\sigma \in \Delta_c$ is the quotient $|det(A_\sigma)|/T$ where T is the g.c.d. of all $|det(A_{\sigma'})|$ as σ' varies over the maximal faces of Δ_c. We note that the converse to Proposition 18 (i) is false. If τ is a non-maximal face of Δ_c then there may not be a standard pair of the form (x^m, τ) in the standard pair decomposition of $in_c(I_A)$.

The standard pair decomposition of $in_c(I_A)$ reduces the problem of solving integer programs in $IP_{A,c}$ to solving systems of linear equations: if β is the optimal solution to the program $IP_{A,c}(b)$, then the monomial x^β is covered by some standard pair (x^u, τ). Thinking of u as a vector in $\mathbf{N}^{|\bar{\tau}|}$ (by adding zero components if necessary), we get $\beta_{\bar{\tau}} = u$ and β_τ is the unique solution to the linear system $A_\tau x_\tau = b - A_{\bar{\tau}} u$. Therefore, if the standard pairs of $in_c(I_A)$ are known apriori, then one can set up $arithdeg(in_c(I_A))$-many systems of linear equations - one for each standard pair. For each $b \in pos_{\mathbf{Z}}(A)$, one then solves for β_τ as above. Whenever the β_τ obtained this way is both integral and non-negative, we have found the optimal solution to $IP_{A,c}(b)$. Hence $arithdeg(in_c(I_A))$ can be seen as a complexity measure of $IP_{A,c}$. See [23] for another pre-processing of $IP_{A,c}$ that reduces solving $IP_{A,c}(b)$ to solving a sequence of subproblems involving super additive functions.

Theorem 19 *[20] (i) The integer program $IP_{A,c}(b)$ is solved by the group relaxation $Group^\tau(b)$ if and only if the monomial x^β, where β is the optimal solution to $IP_{A,c}(b)$, is covered by a standard pair (x^α, τ') of $in_c(I_A)$ for some $\tau' \supseteq \tau$.*

In order to state the main results, we need yet another interpretation of group relaxations of programs in $IP_{A,c}$.

Let $\phi_A : \mathbf{N}^n \to \mathbf{Z}^d$ be the linear map $x \mapsto Ax$. Then P_b is the convex hull of $\phi_A^{-1}(b)$ for each $b \in pos_{\mathbf{Z}}(A)$. Consider a matrix $B \in \mathbf{Z}^{n\times(n-d)}$ such that the columns of B form a basis for $ker_{\mathbf{Z}}(A)$ (as an abelian group). For $v \in \phi_A^{-1}(b)$ we can identify $\phi_A^{-1}(b)$ with the lattice points in the polytope

$$Q_v := \{u \in \mathbf{R}^{n-d} : Bu \le v\}, \qquad (1)$$

via the bijection $Q_v \cap \mathbf{Z}^{n-d} \to \phi_A^{-1}(b)$ such that $u \to v - Bu$. Under this bijection, $v \in \phi_A^{-1}(b)$ corresponds to $0 \in Q_v$. We refer to Q_v as a *Scarf formulation* of $P_b = conv(\phi_A^{-1}(b))$. If $v, v' \in \phi_A^{-1}(b)$, then Q_v and $Q_{v'}$ are simply lattice translates of each other.

Proposition 20 *If v is a feasible solution to $IP_{A,c}(b)$, then $IP_{A,c}(b)$ is equivalent to*

$$minimize\{-(cB)\cdot u : u \in Q_v \cap \mathbf{Z}^{n-d}\} \qquad (2)$$

Proof: A lattice point v^* is the optimal solution to $IP_{A,c}(b)$
$\iff$ there exists $u^* \in \mathbf{Z}^{n-d}$ such that $v^* = v - Bu^* \ge 0$ and $c(v - Bu^*) < c(v - Bu)$ for all $u \ne u^* \in \mathbf{Z}^{n-d}$ with $v - Bu \ge 0$
$\iff$ there exist $u^* \in Q_v \cap \mathbf{Z}^{n-d}$ such that $-(cB)\cdot u^* < -(cB)\cdot u$ for all $u \in Q_v \cap \mathbf{Z}^{n-d}$
$\iff$ u^* is the optimal solution of the integer

program (2). $\square$

We will refer to the integer program (2) as a *Scarf formulation* of $IP_{A,c}(b)$. Using the optimal solution u^* of the Scarf formulation (2), we define the following subpolytope of Q_v:

$$Q_v(u^*) := \begin{array}{l} \{u \in \mathbf{R}^{n-d} : Bu \leq v, \\ \quad -(cB) \cdot u \leq -(cB) \cdot u^*\}. \end{array} \tag{3}$$

Theorem 21 *Let $v \in \mathbf{N}^n$ be a feasible solution to $IP_{A,c}(b)$. Then u^* is the optimal solution to (2) if and only if u^* is the unique lattice point in $Q_v(u^*)$. In particular, v is the optimal solution to $IP_{A,c}(b)$ if and only if 0 is the unique lattice point in $Q_v(0) = \{u \in \mathbf{R}^{n-d} : Bu \leq v, -(cB) \cdot u \leq 0\}$.*

Proof: A vector $u^* \in \mathbf{Z}^{n-d}$ is the optimal solution to (2) if and only if u^* is in Q_v and there is no $u \in Q_v \cap \mathbf{Z}^{n-d}$ such that $-(cB) \cdot u < -(cB) \cdot u^*$. Since c is a generic cost vector, this is equivalent to u^* being the unique lattice point in $Q_v(u^*)$. The second statement follows immediately. $\square$

Corollary 22 *A monomial x^v is a standard monomial of $in_c(I_A)$ if and only if 0 is the unique lattice point in $Q_v(0)$.*

Let B^τ denote the submatrix of B whose *rows* are indexed by the set $\tau \subseteq [n]$.

Lemma 23 *Suppose σ is a maximal face of Δ_c and τ a subface of σ. Then $\tilde{c}_\tau B^{\bar{\tau}} = cB$ where $\tilde{c}_\tau = \pi_\tau(c - c_\sigma(A_\sigma)^{-1}A)$.*

Proof: Since the support of $c - c_\sigma(A_\sigma)^{-1}A$ is contained in $\bar{\tau}$, $\tilde{c}_\tau B^{\bar{\tau}} = (c - c_\sigma(A_\sigma)^{-1}A)B = cB$. $\square$

Theorem 24 *Let v be a feasible solution to $IP_{A,c}(b)$, and suppose that σ is a maximal face of Δ_c and τ a subface of σ. Then the group relaxation $Group^\tau(b)$ is the integer program minimize $\{-(cB) \cdot u : B^{\bar{\tau}} u \leq \pi_\tau(v),\ u \in \mathbf{Z}^{n-d}\}$.*

Proof: Since $\mathcal{L}_\tau = \{B^{\bar{\tau}} u : u \in \mathbf{Z}^{n-d}\}$, we have:

$$\begin{aligned}
&Lat_{\mathcal{L}_\tau, \tilde{c}_\tau}(\pi_\tau(v)) \\
&\quad := \min\{\tilde{c}_\tau \cdot w : w \equiv \pi_\tau(v) \ (mod\ \mathcal{L}_\tau), \\
&\qquad\qquad w \in \mathbf{N}^{|\bar{\tau}|}\} \\
&\quad = \min\{\tilde{c}_\tau \cdot w : w \in \mathbf{N}^{|\bar{\tau}|}, \\
&\qquad\qquad w = \pi_\tau(v) - B^{\bar{\tau}} u, u \in \mathbf{Z}^{n-d}\} \\
&\quad = \min\{\tilde{c}_\tau \cdot w : \pi_\tau(v) - B^{\bar{\tau}} u \geq 0, \\
&\qquad\qquad u \in \mathbf{Z}^{n-d}\} \\
&\quad = \min\{\tilde{c}_\tau \cdot (\pi_\tau(v) - B^{\bar{\tau}} u) : B^{\bar{\tau}} u \leq \pi_\tau(v), \\
&\qquad\qquad u \in \mathbf{Z}^{n-d}\} \\
&\quad = \min\{(-\tilde{c}_\tau B^{\bar{\tau}}) \cdot u : B^{\bar{\tau}} u \leq \pi_\tau(v), \\
&\qquad\qquad u \in \mathbf{Z}^{n-d}\} \\
&\quad = \min\{-(cB) \cdot u : B^{\bar{\tau}} u \leq \pi_\tau(v), \\
&\qquad\qquad u \in \mathbf{Z}^{n-d}\} \ (by\, Lemma\ \ 23)
\end{aligned}$$
$\square$

We will denote the polyhedron obtained from Q_v by removing the inequalities corresponding to τ by $Q_v^{\bar{\tau}}$. By the above theorem, solving the group relaxation of $IP_{A,c}(b)$ with respect to $\tau \in \Delta_c$ is equivalent to minimizing the linear functional $-(cB) \cdot u$ over the lattice points in $Q_v^{\bar{\tau}}$. Now we can characterize which group relaxations will solve $IP_{A,c}(b)$.

Corollary 25 *(i) Let v be a feasible solution to $IP_{A,c}(b)$. Then $Group^\tau(b)$ solves $IP_{A,c}(b)$ if and only if the programs $min\{-(cB) \cdot u : u \in Q_v \cap \mathbf{Z}^{n-d}\}$ and $min\{-(cB) \cdot u : u \in Q_v^{\bar{\tau}} \cap \mathbf{Z}^{n-d}\}$ have the same optimal solutions.*
(ii) If v is optimal for $IP_{A,c}(b)$, then $Group^\tau(b)$ solves $IP_{A,c}(b)$ if and only if 0 is the unique lattice point in $Q_v^{\bar{\tau}}(0) := \{u \in \mathbf{R}^{n-d} : B^{\bar{\tau}} u \leq \pi_\tau(v), -(cB) \cdot u \leq 0\}$.

For a polyhedron $P = \{x \in \mathbf{R}^p : Tx \leq t\}$ we say that an inequality $T_i x \leq t_i$ is *essential* if the relaxation of the polyhedron obtained by removing $T_i x \leq t_i$ contains a new lattice point.

Theorem 26 *[21] An admissible pair (x^v, τ) is a standard pair of $in_c(I_A)$ if and only if 0 is the unique lattice point in $Q_v^{\bar{\tau}}(0)$ and all of the inequalities in the system $B^{\bar{\tau}} u \leq \pi_\tau(v)$ are essential.*

Using the above characterization of the standard pairs of $in_c(I_A)$ we obtain a combinatorial interpretation for $mult(\tau)$ and $arithdeg(in_c(I_A))$.

Corollary 27 *(i) The multiplicity of* τ *is the number of polytopes of the form* $Q_v^{\bar{\tau}}(0) := \{u \in \mathbf{R}^{n-d} : B^{\bar{\tau}}u \leq v, -(cB)\cdot u \leq 0\}$ *where* $v \in \mathbf{N}^{|\bar{\tau}|}$, 0 *is the unique lattice point in* $Q_v^{\bar{\tau}}(0)$ *and all inequalities in* $B^{\bar{\tau}}u \leq v$ *are essential.*
(ii) The arithmetic degree of $in_c(I_A)$ *is the total number of such polytopes* $Q_v^{\bar{\tau}}(0)$ *as* τ *ranges over the subsets of* $[n]$.

The result that $mult(\sigma)$ is the normalized volume of σ when σ is a maximal face of Δ_c is a special case of the above more general interpretation of multiplicity. See [21].

Corollary 28 *For the initial ideal* $in_c(I_A)$*, the following are equivalent:*
(i) The initial ideal $in_c(I_A)$ *has no standard pairs of the form* (x^m, τ) *where* τ *is a non-maximal face of* Δ_c.
(ii) For a face $\tau \in \Delta_c$*, if there exists a* $v \in \mathbf{N}^{|\bar{\tau}|}$ *such that* $Q_v^{\bar{\tau}}(0)$ *contains the origin as its unique lattice point and all inequalities are essential then* τ *is a maximal face of* Δ_c *and* $Q_v^{\bar{\tau}}(0)$ *is a simplex.*
(iii) All programs in $IP_{A,c}$ *can be solved by group relaxations with respect to maximal faces of* Δ_c.
(iv) The arithmetic degree of $in_c(I_A)$ *is* $vol(conv(\mathcal{A}))$.

Proposition 18 shows that the set of all τ in Δ_c that index standard pairs of $in_c(I_A)$ is a sub poset (with respect to inclusion) of the face lattice of Δ_c. We denote this subposet by $Std(\Delta_c)$. Note that both (face lattice of) Δ_c and $Std(\Delta_c)$ have the same maximal elements. We now show that the elements of $Std(\Delta_c)$ come in chains.

Theorem 29 *[21] Let* τ, $|\tau| < d$ *be a non-maximal face of* Δ_c *such that* $\tau \in Std(\Delta_c)$*. Then there exists some* $\tau' \in \Delta_c$ *such that* $\tau' \in Std(\Delta_c)$ *with the property that (i)* $\tau' \supset \tau$ *and (ii)* $|\tau'| = |\tau| + 1$.

We refer the reader to [21] for a proof of this theorem. The tools needed in the proof are polyhedral and depend heavily on the polyhedral interpretation of a standard pair as given in Theorem 26. In terms of group relaxations, Theorem 29 is saying that whenever there is a $b \in pos_{\mathbf{Z}}(A)$ that is solved by a "least tight" $Group^{\tau}(b)$, then there exists a $b' \in pos_{\mathbf{Z}}(A)$ that is solved by a "least tight" $Group^{\tau'}(b)$ where (i) $\tau' \supset \tau$ and (ii) $|\tau'| = |\tau| + 1$. Hence the "least tight" extended group relaxations that solve the programs in $IP_{A,c}$ form saturated chains in the poset $Std(\Delta_c)$.

Since a maximal face of Δ_c has dimension d, the length of a maximal chain in $Std(\Delta_c)$, which we denote as $length(Std(\Delta_c))$, is at most d. However, when $n-d$ which is the the corank of A is small compared to d, $length(Std(\Delta_c))$ has a stronger upper bound as shown below. We need the following result (Corollary 16.5a in [28]).

Theorem 30 *Let* $Ax \leq b$ *be a system of linear inequalities in* n *variables, and let* $c \in \mathbf{R}^n$*. If max* $\{c \cdot x : Ax \leq b, x \in \mathbf{Z}^n\}$ *is finite, then max* $\{c \cdot x : Ax \leq b, x \in \mathbf{Z}^n\}$ *= max* $\{c \cdot x : A'x \leq b', x \in \mathbf{Z}^n\}$ *for some subsystem* $A'x \leq b'$ *of* $Ax \leq b$ *with at most* $2^n - 1$ *inequalities.*

Theorem 31 *The length of a maximal chain in* $Std(\Delta_c)$ *is at most* $min(d, 2^{n-d} - (n-d+1))$.

Proof: Suppose v is the optimal solution to $IP_{A,c}(b)$ which is equivalent to $min\{-(cB)\cdot u : Bu \leq v, u \in \mathbf{Z}^{n-d}\}$. By Theorem 30, we need at most $2^{n-d} - 1$ inequalities to describe the same integer program. This means we can remove at least $n - (2^{n-d} - 1)$ inequalities from $Bu \leq v$ without changing the optimal solution. Therefore by Theorem 24, $IP_{A,c}(b)$ can be solved by a group relaxation with respect to a $\tau \in \Delta_c$ of size at least $n - (2^{n-d} - 1)$. This implies that the maximal length of a chain in $Std(\Delta_c)$ is at most $d - (n - (2^{n-d} - 1)) = 2^{n-d} - (n-d+1)$. □

Corollary 32 *If* $A \in \mathbf{Z}^{d\times n}$ *has corank two, then* $length(Std(\Delta_c)) \leq 1$.

Proof: In this situation, $2^{n-d} - (n-d+1) = 4 - (4-2+1) = 4 - 3 = 1$. □

Corollary 33 *All programs in the family* $IP_{A,c}$ *can be solved by group relaxations with respect to a* $\tau \in \Delta_c$ *of size at least* $max(0, n - (2^{n-d} - 1))$.

We conclude by remarking that the bound in Theorem 31 is sharp. See [21] for details.

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