SIAM J. COMPUT.
Vol. 26, No. 3, pp. 675–677, June 1997
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ON THE COMPLEXITY OF FINDING A MINIMUM CYCLE COVER OF A GRAPH*

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Abstract. We prove that the problem of finding a cycle cover of smallest total length is NP-hard. This confirms a conjecture of Itai, Lipton, Papadimitriou, and Rodeh from 1981.

Key words. complexity, minimum cycle cover

AMS subject classifications. 05C38, 68Q25

PII. S0097539794267255

1. Introduction. A *path* $x_1x_2\dots x_n$ is a graph with distinct vertices x_1, $x_2,\dots,$ x_n and edges $x_1x_2, x_2x_3,\dots,x_{n-1}x_n$. If we add the edge x_nx_1, we obtain a *cycle*. The *length* of a path or a cycle is the number of edges in it. A *cycle cover* of a graph G is a collection of cycles in G such that every edge is in at least one cycle. The *length* of a cycle cover is the sum of lengths of the cycles. The smallest length of a cycle cover of G is denoted $\mathrm{cc}(G)$.

There are several good upper bounds on $\mathrm{cc}(G)$ that are computable by polynomial-time algorithms; for references, see [4, 5]. However, we show that the question "Is $\mathrm{cc}(G) \le k$?" is *NP*-complete as conjectured by Itai et al. [3]. The problem is also mentioned in [4, 5] and discussed in the survey of Bondy [1].

2. Covers by paths and cycles. The subgraph induced by the vertices p_1, $p_2,\dots,p_9$ in Figure 1 together with a new vertex p_{10} joined to p_1,p_2,p_3 is the Petersen graph P.

LEMMA 2.1. $\mathrm{cc}(P) > 20$.

Proof. Consider a cycle cover of length $\mathrm{cc}(P)$. Since all vertices of P have odd degree, each vertex of P must be incident with at least one edge which is covered an even number of times. Hence $\mathrm{cc}(P) \ge 20$. Suppose equality holds. Then each vertex of P is incident with precisely one edge which is covered twice. Thus P minus the edges that are covered twice is a graph—say H—in which each vertex has degree 2. Hence H is the union of disjoint cycles. Since P has no cycle of length 10 and no cycle of length smaller than 5, H is the union of two disjoint cycles C_1 and C_2, each of length 5. If C: $z_1z_2z_3\dots$ is a cycle in the cycle cover and z_1z_2 is in C_1, say, then z_2z_3 goes from C_1 to C_2, z_3z_4 is in C_2, etc. Hence C has the same number of edges in C_1 as in C_2. Moreover, the edges of $C \cap C_1$ must be nonadjacent and hence C has at most two edges in C_1. It follows that C has length 8. Then, however, 8 divides $\mathrm{cc}(P)$, a contradiction. □

It is easy to show that $\mathrm{cc}(P) = 21$, but we shall not use this fact.

We denote by $H(k,q)(k \ge 1, q \ge 1)$ the graph obtained from the graph of Figure 1 by replacing each edge y_iz_i by a path of length k $(i = 1,2,3)$ and each edge of the form $p_ip_j(1 \le i < j \le 9)$ by a path of length q. By a *path-cycle cover* of $H(k,q)$, we mean a collection of cycles and paths covering the edges of $H(k,q)$ such that each path in the collection connects two vertices in $\{x_1,x_2,x_3\}$. If s_i is the number of

*Received by the editors May 9, 1994; accepted for publication (in revised form) June 28, 1995.
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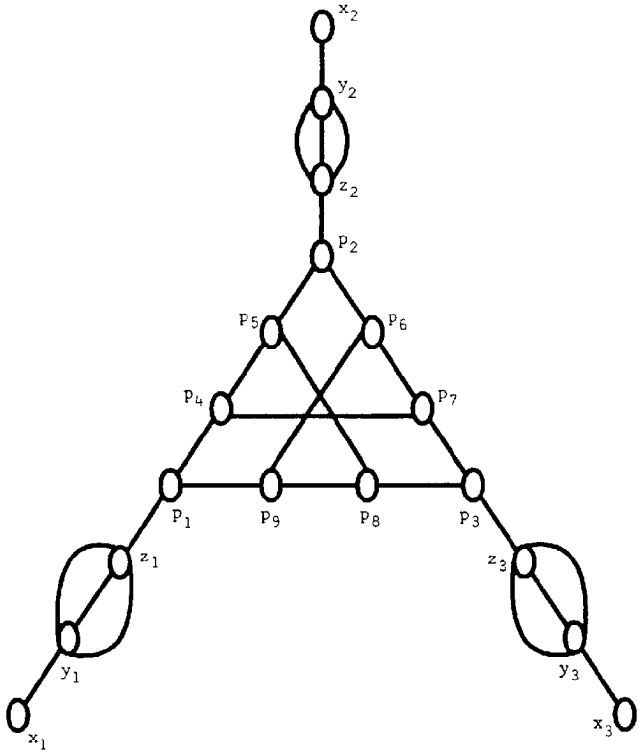

FIG. 2.1.

paths with an end vertex in x_i $(i = 1, 2, 3)$, we speak of an (s_1, s_2, s_3)-*cover*. Because of symmetry, we can assume that $1 \leq s_1 \leq s_2 \leq s_3$.

LEMMA 2.2. *If $k \geq 16(q+1) > 80$, then the sum of lengths of the paths and cycles in an (s_1, s_2, s_3)-cover of $H(k, q)$ is at least $16q + 10k + 12$. Equality can be achieved only for $(s_1, s_2, s_3) = (1, 2, 3)$.*

Proof. The paths $x_1y_1z_1p_1p_9p_6p_7p_4p_5p_8p_3z_3y_3x_3$, $x_2y_2z_2p_2p_5p_4p_1p_9p_8p_3z_3y_3x_3$, $x_2y_2z_2p_2p_6p_7p_3z_3y_3x_3$ and two cycles $y_1z_1y_1$ and $y_2z_2y_2$ form a $(1, 2, 3)$-cover of $H(k, q)$ with total length $16q + 10k + 12$.

Now suppose there exists an (s_1, s_2, s_3)-cover of total length $\leq 16q + 10k + 12$ such that $(s_1, s_2, s_3) \neq (1, 2, 3)$. Since $s_1 + s_2 + s_3$ is the total number of ends of the paths, $s_1 + s_2 + s_3$ is even and hence some s_i $(1 \leq i \leq 3)$ is even. Therefore one of the three paths between y_i and z_i is covered twice. If there exists a $j \in \{1, 2, 3\} \backslash \{i\}$ such that s_j is also even, then one of the paths between y_j and z_j is also covered twice and hence the total length of the path-cycle cover is at least $11k$, which is greater than $16q + 10k + 12$, a contradiction. Thus precisely one s_i is even.

A similar argument shows that the total length of the path-cycle cover is at least $11k$ if $s_3 \geq 5$. Hence $1 \leq s_1 \leq s_2 \leq s_3 \leq 4$.

Suppose now that $s_3 = 4$. Then either both of p_3p_8 and p_3p_7 are covered twice or one is covered three times. Since s_1 and s_2 are odd and every vertex in $\{p_1, p_2, \ldots, p_9\}$ is incident with an edge covered an even number of times, it follows that the total length of the path-cycle cover is at least $17q + 10k + 12$, a contradiction. Thus $1 \leq s_1 \leq s_2 \leq s_3 \leq 3$. This leaves the possibilities $(s_1, s_2, s_3) = (2, 3, 3)$ or $(1, 1, 2)$.

In the former case, the total length of the path-cycle cover is at least $16q + 10k + 16$, a contradiction.

Assume finally that $(s_1, s_2, s_3) = (1, 1, 2)$. It follows from Lemma 1 that the path-cycle cover restricted to the graph induced by $\{p_1, p_2, \ldots, p_9\}$ has total length at least $17q$. Hence the path-cycle cover has total length at least $17q + 10k + 8$, a contradiction. □

3. Cycle covers and the 3-edge-coloring problem. The 3-*edge-coloring* problem is the following: Given a cubic graph G (i.e., every vertex of G has degree 3), is it possible to assign a color in $\{1, 2, 3\}$ to each edge such that no two edges of the same color are incident with the same vertex? Holyer [2] proved that the 3-edge-color problem is *NP*-complete.

THEOREM 3.1. *The* 3*-edge-color problem can be reduced by a polynomial-time transformation to the question "Is* $\mathrm{cc}(G) \leq k$*?". Since the former is NP-complete, so is the latter.*

Proof. Let G be a cubic graph (which we would like to 3-edge-color). Let n be the number of vertices of G. We insert a vertex of degree 2 on every edge of G. Call the resulting graph G_1. If v is a vertex of degree 3 in G_1, then we delete v and replace it by a copy of $H(100,5)$ in such a way that x_1, x_2, and x_3 get degree 2 in the resulting graph, which we call M. We claim that G is 3-edge-colorable if and only if

$$\mathrm{cc}(M) \leq 1092n.$$

Every path-cycle cover of $H(100,5)$ has total length at least 1092 by Lemma 2. Hence $\mathrm{cc}(M) \geq 1092n$. Moreover, if equality holds, then each $H(100,5)$ is covered by a $(1, 2, 3)$-cover, and this results in a 3-edge-coloring of G. (An edge e in G is colored by i, where $i \in \{1, 2, 3\}$, and the new vertex of degree 2 on e is contained in precisely i cycles in the cycle cover of M.) Conversely, if G has a 3-edge-coloring, then we let $C_1, C_2, \ldots, C_r$ denote the cycles of color 1, 3 and $C'_1, C'_2, \ldots, C'_s$ denote the cycles of color 2, 3. Now the system $C_1, C_2, \ldots, C_r, C'_1, C'_1, C'_2, C'_2, \ldots, C'_s, C'_s$ covers each edge of color i precisely i times $(i = 1, 2, 3)$. By modifying this collection of cycles using the paths in the beginning of the proof of Lemma 2, we obtain a cycle cover of M of length $1092n$. □

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