Some Structural and Algorithmic Properties of the Maximum Feasible Subsystem Problem

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Abstract. We consider the problem MAX FS: For a given infeasible linear system, determine a *largest feasible subsystem*. This problem has interesting applications in linear programming as well as in fields such as machine learning and statistical discriminant analysis. MAX FS is NP-hard and also difficult to approximate. In this paper we examine structural and algorithmic properties of MAX FS and of *irreducible infeasible subsystems* (IISs), which are intrinsically related, since one must delete at least one constraint from each IIS to attain feasibility. In particular, we establish: (i) that finding a smallest cardinality IIS is NP-hard as well as very difficult to approximate; (ii) a new *simplex decomposition* characterization of IISs; (iii) that for a given clutter, realizability as the IIS family for an infeasible linear system subsumes the Steinitz problem for polytopes; (iv) some results on the *feasible subsystem polytope* whose vertices are incidence vectors of feasible subsystems of a given infeasible system.

1 Introduction

We consider the following combinatorial problem related to infeasible linear inequality systems.

Max FS: *Given an infeasible system* $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ *with* $A \in \mathbb{R}^{m\times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$*, find a feasible subsystem containing as many inequalities as possible.*

This problem has several interesting applications in various fields such as statistical discriminant analysis, machine learning and linear programming (see [2, 26, 22] and the references therein). In the latter case, it arises when the LP formulation phase yields infeasible models and one wishes to diagnose and resolve infeasibility by deleting as few constraints as possible, which is the complementary version of MAX FS [19, 27, 12]. In most situations this cannot be done by inspection and the need for effective algorithmic tools has become more acute with the considerable increase in model size. In fact, MAX FS turns out to be NP-hard [10] and it does not admit a polynomial time approximation scheme unless $P = NP$ [3]. The above complementary version, in which the goal is to delete as few inequalities as possible in order to achieve feasibility, is equivalent to solve to optimality, but is much harder to approximate than MAX FS [5, 4].

G. Cornuéjols, R.E. Burkard, G.J. Woeginger (Eds.): IPCO'99, LNCS 1610, pp. 45–59, 1999.

Not surprisingly, minimal infeasible subsystems, first discussed in Motzkin's thesis [25], play a key role in the study of MAX FS. A subsystem Σ' of Σ is an *irreducible infeasible subsystem* (IIS) when Σ' is infeasible, but every proper subsystem of Σ' is feasible. In order to help the modeler resolve infeasibility of large linear inequality systems, attention was first devoted to the problem of identifying IISs with a small and possibly minimum number of inequalities [19]; see [14, 13] for several heuristics, now available in commercial solvers such as CPLEX and MINOS [11]. Clearly, when there are many overlapping IISs, this does not provide enough information to repair the original system. To achieve feasibility, one must delete an inequality from each IIS. If all IISs were known, the complementary version of MAX FS could be formulated as the following covering problem [17].

Min IIS Cover: *Given an infeasible system* $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ *with* $A \in \mathbb{R}^{m\times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$ *and the set* $\mathcal{C}$ *of all its IISs, minimize* $\sum_{i=1}^m y_i$ *subject to* $\sum_{i\in C} y_i \geq 1$ $\forall\, C \in \mathcal{C}$, $y_i \in \{0,1\}$, $1 \leq i \leq m$.

Note that $|\mathcal{C}|$ can grow exponentially with m and n [10].

An exact algorithm based on a partial cover formulation is proposed in [26, 27] and heuristics are described in [22, 12]; a collection of infeasible LPs is maintained in the Netlib library. In [29, 30] the class of hypergraphs representing the IISs of infeasible systems is studied and it is shown that in some special cases MAX FS and MIN IIS COVER can be solved in polynomial time in the number of IISs.

In this paper we investigate some structural and algorithmic properties of IISs and of the polytope defined by the convex hull of incidence vectors of feasible subsystems of a given infeasible system. It is worth noting that, although MAX FS with $0-1$ variables can be easily shown to admit as a special case the graphical problem of finding a maximum independent node set, it has a different structure when the variables are real-valued. Recent work on problems related to MAX FS and IISs includes, for instance, Håstad's breakthrough [20] which bridges the approximability gap for MAX FS on $GF(p)$, as well as the investigation of the problems of determining minimum or minimal witnesses of infeasibility in network flows [1].

Below we denote the ith row of the matrix $A \in \mathbb{R}^{m\times n}$ by $\boldsymbol{a}^i \in \mathbb{R}^n$, $1 \leq i \leq m$; for $S \subseteq [m] := \{1, \ldots, m\}$, A_S denotes the $|S|\times n$ matrix consisting of the rows of A indexed by S. By identifying the ith inequality of the system Σ (i.e., $\boldsymbol{a}^i\boldsymbol{x} \leq b_i$) with index i itself, $[m]$ may also refer to Σ.

2 Irreducible Infeasible Subsystems

First we briefly recall the main known structural results regarding IISs. For notational simplicity, we use the same A and $\boldsymbol{b}$, with $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, to denote either the original system Σ or one of its IISs.

The known characterizations of IISs are based on the following version of the Farkas Lemma. *For any system* $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$, *either* $A\boldsymbol{x} \leq \boldsymbol{b}$ *is feasible or* $\exists\, \boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{y} \geq \boldsymbol{0}$, *such that* $\boldsymbol{y}A = \boldsymbol{0}$ *and* $\boldsymbol{y}\boldsymbol{b} < 0$, *but not both.*

Theorem 1 (Motzkin [25], Fan [16]). *The system $\Sigma : \{A\boldsymbol{x} \le \boldsymbol{b}\}$ with $A, \boldsymbol{b}$ as above is an IIS if and only if rank(A) $= m - 1$ and $\exists\, \boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{y} > 0$, such that $\boldsymbol{y}A = \boldsymbol{0}$ and $\boldsymbol{y}\boldsymbol{b} < 0$.*

The rank condition obviously implies that $m \le n + 1$.

Now let $\Sigma : \{A\boldsymbol{x} \le \boldsymbol{b}\}$ be an infeasible system which is not necessarily an IIS. The following result relates the IISs of Σ to the vertices of a given *alternative polyhedron*. Recall that the *support* of a vector is the set of indices of its nonzero components.

Theorem 2 (Gleeson and Ryan [17]). *Let $\Sigma : \{A\boldsymbol{x} \le \boldsymbol{b}\}$ be an infeasible system with $A, \boldsymbol{b}$ as above. Then the IISs of Σ are in one-to-one correspondence with the supports of the vertices of the polyhedron*

$$P := \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}A = \boldsymbol{0},\ \boldsymbol{y}\boldsymbol{b} \le -1,\ \boldsymbol{y} \ge \boldsymbol{0}\}\,.$$

The inequality in the alternative system can obviously be replaced by the equation $\boldsymbol{y}\boldsymbol{b} = -1$. Note that, by using the transformation into Karmarkar's standard form, any polytope can be expressed as $\{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}A = \boldsymbol{0},\ \boldsymbol{y}\mathbb{1} = 1,\ \boldsymbol{y} \ge \boldsymbol{0}\}$ for an appropriate matrix A. Theorem 2 can also be stated in terms of rays [27] and elementary vectors [18].

Definition 1. *An* elementary vector *of a subspace $L \subseteq \mathbb{R}^m$ is a nonzero vector $\boldsymbol{y}$ that has a minimal number of nonzero components (when expressed with respect to the standard basis of $\mathbb{R}^m$). In other words, if $\boldsymbol{x} \in L$ and $\mathrm{supp}(\boldsymbol{x}) \subset \mathrm{supp}(\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{0}$, where $\mathrm{supp}(\boldsymbol{y})$ denotes the support of $\boldsymbol{y}$.*

Corollary 1 (Greenberg [18]). *Let $\Sigma : A\boldsymbol{x} \le \boldsymbol{b}$, $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ be an infeasible system. Then $S \subseteq [m]$ corresponds to an IIS of Σ if and only if there exists an elementary vector $\boldsymbol{y}$ in the subspace $L := \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}A = \boldsymbol{0}\}$ with $\boldsymbol{y}\boldsymbol{b} < 0,\ \boldsymbol{y} \ge \boldsymbol{0}$ such that $S = \mathrm{supp}(\boldsymbol{y})$.*

The following result establishes an interesting geometric property of the polyhedra obtained by deleting any inequality from an IIS.

Theorem 3 (Motzkin [25]). *Let $\Sigma : \{A\boldsymbol{x} \le \boldsymbol{b}\}$ be an IIS and let $\sigma \in \Sigma$ be an arbitrary inequality of Σ. Then the polyhedron corresponding to $\Sigma \setminus \sigma$, i.e., the subsystem obtained by removal of σ, is an affine convex cone.*

2.1 Minimum Cardinality IISs

We now determine the complexity status of the following problem for which heuristics have been proposed in [14, 13, 26, 27].

Min IIS: *Given an infeasible system $\Sigma : \{A\boldsymbol{x} \le \boldsymbol{b}\}$ as above, find a minimum cardinality IIS.*

To settle the issue left open in [19, 14, 27], we prove that Min IIS is not only NP-hard to solve optimally but also hard to approximate. Note that, where $DTIME(T(m))$ denotes the class of problems solvable in time $T(m)$, the assumption $NP \not\subseteq DTIME(m^{\text{polylog } m})$ is stronger than $NP \not\subseteq P$, but it is also believed to be extremely unlikely. Results that hold under such an assumption are often referred to as *almost NP-hard*.

Theorem 4. *Assuming $P \neq NP$, no polynomial time algorithm is guaranteed to yield an IIS whose cardinality is at most c times larger than the minimum one, for any constant $c \geq 1$. Assuming $NP \not\subseteq DTIME(m^{polylog\, m})$,* Min IIS *cannot be approximated within a factor $2^{\log^{1-\varepsilon} m}$, for any $\varepsilon > 0$, where m is the number of inequalities.*

Proof. We proceed by reduction from the following problem: Given a feasible linear system $D\boldsymbol{z} = \boldsymbol{d}$, with $D \in \mathbb{R}^{m' \times n'}$ and $\boldsymbol{d} \in \mathbb{R}^{m'}$, find a solution $\boldsymbol{z}$ satisfying all equations with as few nonzero components as possible. In [4] we establish that it is (almost) NP-hard to approximate this problem within the same type of factors, but with m replaced by n, the number of variables. Note that the above nonconstant factor grows faster than any polylogarithmic function, but slower than any polynomial one.

For each instance of the latter problem with an optimal solution containing s nonzero components, we construct a particular instance of Min IIS with a minimum cardinality IIS containing $s+1$ inequalities. Given any instance $(D,\boldsymbol{d})$, consider the system

$$\begin{pmatrix} D & -D & -\boldsymbol{d} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}^+ \\ \boldsymbol{z}^- \\ z_0 \end{pmatrix} = \mathbf{0}, \qquad \begin{pmatrix} \mathbf{0}^t & \mathbf{0}^t & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{z}^+ \\ \boldsymbol{z}^- \\ z_0 \end{pmatrix} < 0, \qquad \boldsymbol{z}^+, \boldsymbol{z}^- \geq \mathbf{0},\ z_0 \geq 0. \tag{1}$$

Since the strict inequality implies $z_0 > 0$, the system $D\boldsymbol{z} = \boldsymbol{d}$ has a solution with s nonzero components if and only if (1) has one with $s+1$ nonzero components. Now, applying Corollary 1, (1) has such a solution if and only if the system

$$\begin{pmatrix} D^t \\ -D^t \\ -\boldsymbol{d}^t \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ -1 \end{pmatrix} \tag{2}$$

has an IIS of cardinality $s+1$. Since (2) is the alternative system of (1), the Farkas Lemma implies that exactly one of these is feasible; as (1) is feasible, (2) must be infeasible. Thus (2) is a particular instance of Min IIS with $m = 2n'+1$ inequalities in $n = m'$ variables.

Given that the polynomial time reduction preserves the objective function modulo an additive unit constant, we obtain the same type of non-approximability factors for Min IIS. □

Note that for the similar (but not directly related) problem of determining minimum witnesses of infeasibility in network flows, NP-hardness is established in [1].

2.2 IIS Simplex Decomposition

Here we provide a new geometric characterization of IISs. For $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, let $A^i := A_{[m]\setminus\{i\}}$ and $\boldsymbol{b}^i := \boldsymbol{b}_{[m]\setminus\{i\}}$ denote the $(m-1)\times n$ submatrix and, respectively, the $(m-1)$–dimensional vector obtained by removing the ith row of A and ith component of $\boldsymbol{b}$. The following result strengthens the initial part of Theorem 1.

Lemma 1. *For any IIS $\{A\boldsymbol{x} \le \boldsymbol{b}\}$, A^i has linearly independent rows, $\forall\ i$; i.e.,* $\mathrm{rank}(A^i) = m-1$.

Proof. According to Theorem 1, there exists a $\boldsymbol{y} > \boldsymbol{0}$ such that $\boldsymbol{y}A = \boldsymbol{0}$ and $\boldsymbol{y}\boldsymbol{b} = -1$ (by scaling $\boldsymbol{y}\boldsymbol{b} < 0$). Suppose some proper subset of rows is linearly dependent; i.e. $\exists \boldsymbol{z}$, such that $\boldsymbol{z}A = \boldsymbol{0}$, $\boldsymbol{z}\boldsymbol{b} \ge 0$ (without loss of generality) and some $z_k = 0$.

If some $z_i > 0$, consider $(\boldsymbol{y}-\epsilon\boldsymbol{z})A = 0$, $(\boldsymbol{y}-\epsilon\boldsymbol{z})\boldsymbol{b} \le -1$, where $\epsilon = \min\{y_i/z_i > 0 \,|\, 1 \le i \le m,\ z_i > 0\}$ (and $\boldsymbol{y}$ is as above). Then $\boldsymbol{y}-\epsilon\boldsymbol{z} \ge 0$, the ith component of $\boldsymbol{y}-\epsilon\boldsymbol{z}$ is 0 and the Farkas Lemma contradicts minimality of the system ($\boldsymbol{y}-\epsilon\boldsymbol{z}$ fulfills the requirements).

If all $z_i \le 0$, then $-\boldsymbol{z} \ge \boldsymbol{0}$, $-\boldsymbol{z}A = \boldsymbol{0}$ and $-\boldsymbol{z}\boldsymbol{b} \le 0$; so setting $\boldsymbol{y} = -\boldsymbol{z}$ in the Farkas Lemma leads to a contradiction of minimality, provided $-\boldsymbol{z}\boldsymbol{b} < 0$. If $-\boldsymbol{z}\boldsymbol{b} = \boldsymbol{0}$, then $(\boldsymbol{y}+\epsilon\boldsymbol{z})A = \boldsymbol{0}$, $(\boldsymbol{y}+\epsilon\boldsymbol{z})b = -1$, with $\epsilon = \min\{y_i/(-z_i) \,|\, 1 \le i \le m,\ -z_i > 0\}$ leads to a contradiction as above. $\square$

It is interesting to note that this lemma together with Theorem 1 imply that an infeasible system $\{A\boldsymbol{x} \le \boldsymbol{b}\}$ is an IIS if and only if $\mathrm{rank}(A^i) = m-1$ for all i, $1 \le i \le m$.

We thus have the following *simplex decomposition* result for IISs.

Theorem 5. *The system $\{A\boldsymbol{x} \le \boldsymbol{b}\}$ is an IIS if and only if $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} \ge \boldsymbol{b}\} = L + Q$, where L is the lineality subspace $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} = \boldsymbol{0}\}$ and Q is an $(m-1)$–simplex with vertices determined by maximal proper subsystems of $\{A\boldsymbol{x} = \boldsymbol{b}\}$; namely, each vertex of Q is a solution for a subsystem $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$, $1 \le i \le m$.*

Proof. ($\Rightarrow$) To see feasibility of $\{A\boldsymbol{x} \ge \boldsymbol{b}\}$, delete constraint $\boldsymbol{a}^i\boldsymbol{x} \ge b_i$ to get the equality system $\{A^i\boldsymbol{x} = \boldsymbol{b}^i\}$. By Lemma 1, this system has a solution, say $\boldsymbol{x}^i$, and we must have $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$, else $\boldsymbol{x}^i$ satisfies $\{A\boldsymbol{x} \le \boldsymbol{b}\}$. Applying the polyhedral resolution theorem, $P := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} \ge \boldsymbol{b}\} \ne \emptyset$ can be written as $P = K + Q$, where $K = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} \ge \boldsymbol{0}\}$ is its recession cone and $Q \subseteq P$ is a polytope generated by representatives of its minimal nonempty faces.

If $\boldsymbol{x}$ satisfies $A\boldsymbol{x} \ge \boldsymbol{0}$ and $\boldsymbol{a}^i\boldsymbol{x} > 0$ for row $\boldsymbol{a}^i$ then $\boldsymbol{x}^i - \epsilon\boldsymbol{x}$ satisfies $A(\boldsymbol{x}^i - \epsilon\boldsymbol{x}) \le \boldsymbol{b}$ for sufficiently large $\epsilon > 0$ and the original system $\{A\boldsymbol{x} \le \boldsymbol{b}\}$ would be feasible. Therefore we must have that each $\boldsymbol{a}^i\boldsymbol{x} = 0$ for $1 \le i \le m$, $\boldsymbol{x} \in K$ and we get that in fact $K = L := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, A\boldsymbol{x} = \boldsymbol{0}\}$.

For Q, minimal nonempty faces of P are given by changing a maximal set of inequalities into equalities (all but one relation). Thus the vectors $\boldsymbol{x}^i$ obtained by solving $\{A^i \boldsymbol{x} = \boldsymbol{b}^i\}$ determine Q; i.e., $Q = \operatorname{conv}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\})$. For $A \in \mathbb{R}^{m \times n}$, Q is the $(m-1)$–simplex generated by the m points $\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\}$. To see that the $\boldsymbol{x}^i$ generate an $(m-1)$–simplex, we must only show that they are affinely independent. But if $\boldsymbol{x}^i$ is affinely dependent on the other $\boldsymbol{x}^j$, then $\boldsymbol{x}^i = \sum_{j \neq i} \lambda_j \boldsymbol{x}^j$ with $\sum_{j \neq i} \lambda_j = 1$. Thus we have $\boldsymbol{a}^i \boldsymbol{x}^i > b_i$, but also $\boldsymbol{a}^i \boldsymbol{x}^i = \boldsymbol{a}^i (\sum_{j \neq i} \lambda_j \boldsymbol{x}^j) = \sum_{j \neq i} \lambda_j (\boldsymbol{a}^i \boldsymbol{x}^j) = \sum_{j \neq i} \lambda_j b_i = b_i$, which is a contradiction.

($\Leftarrow$) If the system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is infeasible, then the minimality is obvious, because the simplex conditions on Q imply that every proper subsystem has an equality solution.

To show that $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ is infeasible, assume for the sake of contradiction that $\hat{\boldsymbol{x}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\} \neq \emptyset$ and $\hat{\boldsymbol{x}}$ satisfies a maximal number of these relations at equality. Let $\boldsymbol{a}^i \hat{\boldsymbol{x}} < b_i$ and note that for $\boldsymbol{x}^i$ defined as above, we have $\boldsymbol{a}^i \boldsymbol{x}^i > b_i$. Thus we can set $\lambda = (\boldsymbol{a}^i \boldsymbol{x}^i - b_i)/(\boldsymbol{a}^i \boldsymbol{x}^i - \boldsymbol{a}^i \hat{\boldsymbol{x}})$ and have $0 < \lambda < 1$, so that $\boldsymbol{a}^i(\lambda \hat{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}^i) = b_i$. But then at $\lambda \hat{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}^i$ more relations of $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ hold at equality than at $\hat{\boldsymbol{x}}$, contradicting the choice of $\hat{\boldsymbol{x}}$. $\square$

According to the above proof, we can take the $\boldsymbol{x}^i$'s as the representatives of the minimal nonempty faces of $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$ that lie in $L^\perp$; i.e., $Q \subset L^\perp$. By Lemma 1, we know that $\{\boldsymbol{x} \in \mathbb{R}^n \mid A^i \boldsymbol{x} = \boldsymbol{b}^i\} = \boldsymbol{x}^i + L$, where L is the lineality space of the original linear system $\{A\boldsymbol{x} \geq \boldsymbol{b}\}$.

It is worth observing that Theorem 5 handles the following special cases. If $m = 1$, then A has only one row, say $\{A\boldsymbol{x} \leq \boldsymbol{b}\} = \{\boldsymbol{0}\boldsymbol{x} \leq -1\}$. Thus $L = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{0}\boldsymbol{x} = 0\} = \mathbb{R}^n$ and $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{0}\boldsymbol{x} \geq -1\} = \mathbb{R}^n + \{0\} = L + Q = L$.
If $m = n+1$, then A has $n+1$ rows. Assuming A to be of full column rank, $L = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = 0\} = \{0\}$ and $Q = \operatorname{conv}(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1}\})$ is an n–simplex and $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b}\} = \{0\} + Q$.

3 IIS-Hypergraphs

Consider for any infeasible system the following hypergraph.

Definition 2. *Given an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with $A \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, $H = (V, \mathcal{E})$ is the* IIS-hypergraph *of Σ if*

i. the nodes in V are in one-to-one correspondence with the inequalities of Σ,
ii. the hyperedges in $\mathcal{E}$ are in one-to-one correspondence with the IISs of Σ and each hyperedge contains precisely the nodes associated to the inequalities contained in the corresponding IIS.

Investigations on the structure of *IIS-hypergraphs* began with [29, 30]. In particular, it was shown that IIS-hypergraphs do not share many properties with other known classes of hypergraphs generalizing bipartite graphs. Indeed, IIS-hypergraphs (with no trivial IISs of cardinality 1) just turn out to be *bi-colourable*; i.e., their nodes can be partitioned into two subsets so that neither subset contains a hyperedge. Note, however, that there is more structure for

IIS-hypergraphs than simply bicolourability, as there will generally exist many different bipartitions into two feasible subsystems [29, 18].

In IIS-hypergraph terminology, Min IIS Cover amounts to finding a minimum cardinality *transversal*, i.e., a subset of nodes having nonempty intersection with every hyperedge. The special structure of the IIS-hypergraphs accounts for the fact that the greedy algorithm is guaranteed to find a minimum transveral for those with nondegenerate alternative polyhedra [30] (a subclass of uniform hypergraphs) while the problem is NP-hard even for simple graphs, i.e., for 2-uniform hypergraphs.

Here we address the fundamental problem of recognizing IIS-hypergraphs. For the definitions of a poset and (face) lattice see, e.g., [31].

Let E be a finite set and $\mathcal{F}$ a clutter on E. The poset $\mathcal{L}(\mathcal{F}) = (S, \leq)$ can be constructed as follows. $S \subseteq 2^E$ and the relation "$\leq$" on S is the set inclusion. A subset U of E is in S if U is the intersection of elements of $\mathcal{F}$. The element $\hat{1} := \bigcup\{F \in \mathcal{F}\}$ is also in S. Notice that the zero $\hat{0} := \bigcap\{F \in \mathcal{F}\}$ is always in S and is possibly the empty set. Then $\mathcal{L}(\mathcal{F})$ is a lattice with the meet defined by intersection. Note that the size of $\mathcal{L}(\mathcal{F})$ can be exponential in the size of $\mathcal{F}$.

The *face lattice* of a polytope P is its set of faces, ordered by inclusion, with the meet defined by intersection. It is well known (see, e.g., [31]) that the face lattice of P has a rank function $r(\cdot)$ satisfying $r(F) = \dim(F) - 1$ for any face F, and is both atomic and coatomic. Two polytopes with isomorphic face lattices are *combinatorially equivalent*.

Let R denote either $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{A}$ (the real algebraic numbers over $\mathbb{Q}$) or $\mathbb{R}$.

IIS Realizability problem *for R: Given a clutter $\mathcal{C}$ over a finite ground set of cardinality m, does there exist an infeasible linear system $\{A\boldsymbol{x} \leq \boldsymbol{b}\}$, with $A \in R^{m\times n}$ and $\boldsymbol{b} \in R^m$, such that the sets in $\mathcal{C}$ index the IISs of this system?*

In the above definition, infeasibility is meant with respect to $\mathbb{R}$ and n is free. If such a system exists, the clutter $\mathcal{C}$ is *IIS-realizable*. The IIS Realizability problem is obviously equivalent to that of recognizing IIS-hypergraphs. In the sequel we also consider the restricted version of the IIS Realizability problem in which the right-hand side of the linear system is fixed, namely, in which $\boldsymbol{b} = -\mathbb{1}$.

Steinitz problem *for R: Given a lattice $\mathcal{L}$, does there exist a polytope $P \subset \mathbb{R}^d$ with vertices in R^d such that the face lattice of P is isomorphic to $\mathcal{L}$?*

If the answer is affirmative, $\mathcal{L}$ is *realizable* as a polytope. In this case d can be assumed to be the dimension of $\mathcal{L}$. P can be given either as a (complete) list of vertices or facets. See [9] for related material.

Theorem 6. *The IIS Realizability problem is at least as hard as the Steinitz problem.*

Proof. We show that for any instance of the Steinitz problem we can construct in polynomial time a special instance of the above-mentioned restricted IIS Realizability problem such that the answer to the first instance is affirmative if and only if the answer to the second instance is affirmative. Since face lattices of polytopes need to be ranked as well as atomic and these properties can be checked in polynomial time, we focus attention on this type of lattices.

Given an arbitrary instance of the Steinitz problem defined by a ranked atomic lattice $\mathcal{L}$, we construct the following special instance of the restricted IIS Realizability problem with $\boldsymbol{b} = -\mathbb{1}$. Suppose $\mathcal{L}$ contains k atoms and m coatoms. Label arbitrarily the coatoms with the sets $\{1\}, \ldots, \{m\}$ and the atoms with the sets $C_1, \ldots, C_k$, where C_i includes all the elements in the labels of the coatoms that contain the corresponding atom. Define $\mathcal{C} = \{C_1, \ldots, C_k\}$ and $\overline{\mathcal{C}} := \{\overline{C}_1, \ldots, \overline{C}_k\}$, where $\overline{C}_i = \{1, \ldots, m\} \setminus C_i$. Thus the arbitrary choices of the labeling just correspond to a permutation of coordinates and hence do not change the structure.

If the original instance of the Steinitz problem has a positive answer, there exists a polytope P such that $\mathcal{L}$ is the face lattice of P. According to the remark following Theorem 2, this polytope can be expressed in the special form $P = \{\boldsymbol{y} \in \mathbb{R}^{m'} \mid \boldsymbol{y}A = \boldsymbol{0},\ \boldsymbol{y}\mathbb{1} = 1,\ \boldsymbol{y} \geq \boldsymbol{0}\}$, with $A \in \mathbb{R}^{m' \times n}$ and suitable m', n. Hence $m' > m$, the number of facets, and $\{A\boldsymbol{x} \leq -\mathbb{1}\}$ is the infeasible system associated to P.
Since the face lattice of a polytope is coatomic, each face of P can be identified with the set of facets it is contained in. If these sets corresponding to all faces are ordered by set inclusion, one obtains a lattice $\mathcal{L}'$ which is anti-isomorphic to the face lattice of P. The meet is defined by intersection. It is easy to see that the lattice $\mathcal{L}(\mathcal{C})$ is isomorphic to $\mathcal{L}'$. The atoms correspond to the facets of P and the coatoms to its vertices.
By construction, each set C_i (atom of $\mathcal{L}$) corresponds to a vertex $\boldsymbol{v}^i$ of P. All facets of P are defined by inequalities of the form $y_i \geq 0$. Up to relabeling of the coatoms in the definition of $\mathcal{C}$, the facet defined by $y_i \geq 0$ can be identified with $\{i\}$. Thus $C_i = \{j \in \{1, \ldots, m\} \mid v^i_j = 0\}$ and $\overline{C}_i$ is the support of the vertex $\boldsymbol{v}^i$. By Theorem 2, each $\overline{C}_i$ corresponds to an IIS of the associated infeasible system $\{A\boldsymbol{x} \leq -\mathbb{1}\}$ and hence $\overline{\mathcal{C}}$ is IIS-realizable with the restricted type of right-hand side and with a polytope as alternative polyhedron.

Conversely, suppose that the corresponding instance of the restricted IIS Realizability problem with $\boldsymbol{b} = -\mathbb{1}$ defined by $\overline{\mathcal{C}}$ has a positive answer and consider the alternative polyhedron $P = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{y}A = \boldsymbol{0},\ \boldsymbol{y}\mathbb{1} = 1,\ \boldsymbol{y} \geq \boldsymbol{0}\}$ with $A \in \mathbb{R}^{m \times n}$. As seen above, each $\overline{C}_i$ corresponds to the support of a vertex of P and each C_i corresponds to the set of facets that this vertex lies on, i.e., $\mathcal{L}(\mathcal{C})$ is anti-isomorphic to the face lattice of P. Now the vertex-facet incidence information encoded in $\mathcal{C}$ and the fact that $\mathcal{L}$ is atomic, imply the whole structure of the lattice $\mathcal{L}$. Therefore $\mathcal{L}(\mathcal{C})$ is anti-isomorphic to $\mathcal{L}$ and hence P is a realization of $\mathcal{L}$. $\square$

Given polynomials $f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t \in \mathbb{Z}[x_1, \ldots, x_l]$, the problem to decide whether the polynomial system $f_1 = \cdots = f_r = 0, g_1 \geq 0, \ldots, g_s \geq 0, h_1 > 0, \ldots, h_t > 0$ has a solution in $R^l = \mathbb{A}^l$ is called the *Existential theory of the reals* (ETR). ETR is polynomial time equivalent to the Steinitz problem for 4-Polytopes over $\mathbb{A}$ [28]. (All polytopes realizable over $\mathbb{R}$ are realizable over

$\mathbb{A}$.) Moreover, ETR is polynomial time equivalent to the Steinitz problem for d-Polytopes with $d + 4$ vertices over $\mathbb{A}$ [24]. Since ETR is easily verified to be NP-hard, the same is valid for the general Steinitz problem (over $\mathbb{A}$) and for the IIS Realizability problem.

According to Theorem 2.7 of [9], for $R = \mathbb{Q}$ or $\mathbb{A}$, to decide whether an arbitrary polynomial $f \in \mathbb{Z}[x_1, \ldots, x_l]$ has zeros in R^l, where l is a positive integer, is equivalent to solve the Steinitz problem for R. For $R = \mathbb{Q}$, it is not even clear whether the Steinitz problem (and therefore the IIS Realizability problem) is decidable since finding roots in $R = \mathbb{Q}$ of a single polynomial $f \in \mathbb{Z}[x_1, \ldots, x_l]$ is the unsolved rational version of Hilbert's 10th problem. By the well known theorem of Matiyasevic, there does not exist an algorithm for deciding whether f has roots in $\mathbb{Z}$. By the quantifier elimination result of Tarski, the problem is decidable for $R = \mathbb{A}$. Note that, unlike $\mathbb{R}$, $\mathbb{A}$ admits a finite representation. For $R = \mathbb{A}$, it is unkown whether the Steinitz problem is in NP. See [23, 8] and references therein for this and related issues.

4 Feasible Subsystem (FS) Polytope

Consider an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ and let $[m] = \{1, \ldots, m\}$ be the set of indices of all inequalities in Σ. If $\mathcal{I}$ denotes the set of all feasible subsystems of Σ, $([m], \mathcal{I})$ is clearly an independence system and its set of circuits $\mathcal{C}(\mathcal{I})$ corresponds to the set of all IISs. We denote by P_{FS} the polytope generated by the convex hull of all the incidence vectors of feasible subsystems.

Let us first briefly recall some definitions and facts about independence system polytopes. To any independence system $(E, \mathcal{I})$ with the family of circuits denoted by $\mathcal{C}(\mathcal{I})$ we can associate the polytope $P(\mathcal{I}) = P(\mathcal{C}(\mathcal{I})) = \text{conv}(\{\boldsymbol{y} \in \{0,1\}^{|E|} \,|\, \boldsymbol{y}$ is the incidence vector of an $I \in \mathcal{I}\})$. The *rank function* is defined by $r(S) = \max\{|I| \;\, | \, I \subseteq S,\ I \in \mathcal{I}\}$ for all $S \subseteq E$. For any $S \subseteq E$, the *rank inequality* for S is $\sum_{e \in S} y_e \leq r(S)$, which is clearly valid for $P(\mathcal{I})$. A subset $S \subseteq E$ is *closed* if $r(S \cup \{t\}) \geq r(S) + 1$ for all $t \in E - S$ and *nonseparable* if $r(S) < r(T) + r(S - T)$ for all $T \subset S,\ T \neq \emptyset$. For any set $S \subseteq E$, S must be closed and nonseparable for the corresponding rank inequality to define a facet of $P(\mathcal{I})$. These conditions generally are only necessary, but sufficient conditions can be stated using the following concept [21]. For $S \subseteq E$, the *critical graph* $G_S(\mathcal{I}) = (S, F)$ is defined as follows: $(e, e') \in F$, for $e,\ e' \in S$, if and only if there exists an independent set I such that $I \subseteq S$, $|I| = r(S)$ and $e \in I$, $e' \notin I$, $I - e + e' \in \mathcal{I}$. It is shown in [21] that if S is a closed subset of E and the critical graph $G_S(\mathcal{I})$ of $\mathcal{I}$ on S is connected, then the corresponding rank inequality induces a facet of the polytope $P(\mathcal{I})$. See references in [15].

4.1 Rank-Facets of the FS Polytope

As P_{FS} is an independence system polytope, it is full-dimensional if and only if there are no trivially infeasible inequalities in Σ. The inequalities $y_i \geq 0$ are facet defining for all $1 \leq i \leq m$. Moreover, it is easy to verify that for each i the inequality $y_i \leq 1$ defines a facet of P_{FS} if and only if there is no IIS of cardinality 2 that includes i and P_{FS} is full-dimensional.

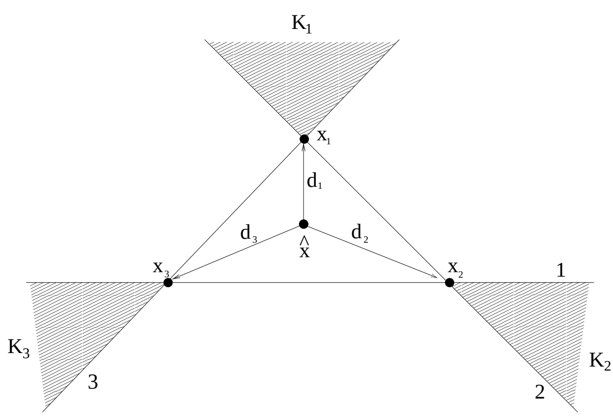
In fact, Parker [26] began an investigation of the polytope associated to the MIN IIS COVER problem, considering it as a special case of the general set covering polytope (see references in [15]). Since there is a simple correspondence between set covering polytopes and the complementary independence system polytopes [21], the results in [26] can be translated so that they apply to P_{FS}.

Let S be an arbitrary IIS of Σ, $A_S\boldsymbol{x} \leq \boldsymbol{b}_S$ be its corresponding subsystem, and $\sum_{i \in S} y_i \leq r(S) = |S| - 1$ the corresponding (rank) *IIS-inequality*. Since the complementary covering inequality $\sum_{i \in C} y_i \geq 1$ induced by every IIS C is proved to be facet defining in [26], we have:

Theorem 7. *The IIS-inequality arising from any IIS defines a facet of P_{FS}.*

We give here a geometric proof (based on the above-mentioned sufficient conditions [21]), which is simpler than that of [26] and which provides additional insight into the IIS structure.

Proof. It is easy to verify that IIS-inequalities are valid for P_{FS}. Since the critical graph corresponding to any IIS is clearly connected (in fact, a complete graph), we just need to show that every IIS is closed.
a) First consider the case of maximal IISs, i.e. with $|S| = n + 1$.



For each $i \in S$, consider the unique $\boldsymbol{x}^i = A^{-1}_{S\setminus\{i\}}\boldsymbol{b}_{S\setminus\{i\}}$. By the proof of Theorem 5, we know that $\boldsymbol{x}^1, \dots, \boldsymbol{x}^{n+1}$ are affinely independent. If $\boldsymbol{d_i} := (\boldsymbol{x}^i - \hat{\boldsymbol{x}})$ for all i, $1 \leq i \leq n+1$, where $\hat{\boldsymbol{x}} := \frac{1}{n+1}\sum_{i=1}^{n+1} \boldsymbol{x}^i$, $\boldsymbol{d}_1, \dots, \boldsymbol{d}_{n+1}$ are also affinely independent. Clearly $\sum_{i=1}^{n+1} \boldsymbol{d}_i = \boldsymbol{0}$ and the $\boldsymbol{d}_i$'s generate $\mathbb{R}^n$. Since each $\boldsymbol{x}^i$ satisfies exactly n of the $n+1$ inequalities in S with equality and $\boldsymbol{a}^i\boldsymbol{x}^i > b_i$ (otherwise S would be feasible), we have $\hat{\boldsymbol{x}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid A_S\boldsymbol{x} \geq \boldsymbol{b}_S\}$, i.e., $\hat{\boldsymbol{x}}$ satisfies the reversed inequalities of the IIS. In fact, $\hat{\boldsymbol{x}}$ is an interior point of the above "reversed" polyhedron.

According to Theorem 3, deleting any inequality from an IIS yields a feasible subsystem that defines an affine cone. For maximal IISs, we have $n+1$ affine cones $K_i := \boldsymbol{x}^i + K_i'$, where $K_i' = \{\boldsymbol{x} \in \mathbb{R}^n \mid A_{S\setminus\{i\}}\boldsymbol{x} \leq \boldsymbol{0}\}$ for $1 \leq i \leq n+1$. Note that the ray generated by $\boldsymbol{d}_i$ and passing through $\boldsymbol{x}^i$, i.e., $R_i := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{x}^i + \alpha \boldsymbol{d}_i,\ \alpha \geq 0\}$, is contained in K_i because we have

$$A_{S\setminus\{i\}}(\alpha \boldsymbol{d}_i) = \alpha A_{S\setminus\{i\}}(\boldsymbol{x}^i - \hat{\boldsymbol{x}}) = \alpha(\boldsymbol{b}_{S\setminus\{i\}} - A_{S\setminus\{i\}}\hat{\boldsymbol{x}}) \leq \boldsymbol{0},$$

where we used the fact that $A_{S\setminus\{i\}}\hat{\boldsymbol{x}} \geq \boldsymbol{b}_{S\setminus\{i\}}$. Now consider an arbitrary inequality $\tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}$ with $\tilde{\boldsymbol{a}} \neq \boldsymbol{0}$. We will verify that $H := \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}\}$ has a nonempty intersection with at least one of the K_i's, $1 \leq i \leq n+1$. Thus, for any $t \in E - S$ we have $\operatorname{rank}(S \cup \{t\}) = \operatorname{rank}(S) + 1 = n+1$, which means that the IIS defined by S is closed.

Since $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{n+1}$ generate $\mathbb{R}^n$ and $\sum_{i=1}^{n+1} \boldsymbol{d}_i = \boldsymbol{0}$, we have $\sum_{i=1}^{n+1} \tilde{\boldsymbol{a}}\boldsymbol{d}_i = \tilde{\boldsymbol{a}}(\sum_{i=1}^{n+1} \boldsymbol{d}_i) = 0$ and therefore $\tilde{\boldsymbol{a}} \neq \boldsymbol{0}$ implies that we cannot have $\tilde{\boldsymbol{a}}\boldsymbol{d}_i = 0\ \forall i,\ 1 \leq i \leq n+1$. Thus there exists at least one i, such that $\tilde{\boldsymbol{a}}\boldsymbol{d}_i < 0$. But this implies that $R_i \cap H \neq \emptyset$. In other words, $K_i \cap H \neq \emptyset$ and this proves the theorem for maximal IISs.

b) The result can be easily extended to non-maximal IISs, i.e., with $|S| < n+1$. From Theorem 5 we know that $P := \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b}\} = L + Q$ with $Q \subseteq L^\perp$. Since P is full-dimensional ($\hat{\boldsymbol{x}}$ is an interior point), $n = \dim(P) = \dim(L) + \dim(Q)$ and $\dim(Q) = \operatorname{rank}(A_S) = |S| - 1 < n$ imply that $\dim(L) \geq 1$.

Two cases can arise:
i) If the above-mentioned $\tilde{\boldsymbol{a}}$ is in $\operatorname{lin}(\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^m\}) = L^\perp$, the linear hull of the rows of A, then since $\dim(L^\perp) = \dim(Q)$, we can apply the above result to $L^\perp$.
ii) If $\tilde{\boldsymbol{a}} \notin \operatorname{lin}(\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^m\}) = L^\perp$, then the projection of $H^= := \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} = \tilde{b}\}$ onto L yields the whole L and therefore $H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \tilde{\boldsymbol{a}}\boldsymbol{x} \leq \tilde{b}\}$ must have a nonempty intersection with all the cones corresponding to the maximal consistent subsystems of $\{A_S\boldsymbol{x} \leq \boldsymbol{b}_S\}$. □

It is worth noting that closedness of every IIS makes P_{FS} quite special among all independence system polyhedra, since the circuits of a general independence system need not be closed.

The **separation problem for IIS-inequalities** is defined as follows: *Given an infeasible system Σ and an arbitrary vector $\boldsymbol{y} \in \mathbb{R}^m$, show that $\boldsymbol{y}$ satisfies all IIS-inequalities or find at least one violated by $\boldsymbol{y}$.*

In view of the trivial valid inequalities, we can assume that $\boldsymbol{y} \in [0,1]^m$. Moreover, we may assume with no loss of generality, that the nonzero components of $\boldsymbol{y}$ correspond to an infeasible subsystem of Σ.

Proposition 1. *The separation problem for IIS-inequalities is NP-hard.*

Proof. We proceed by polynomial time reduction from the decision version of the Min IIS problem, which is NP-hard according to Theorem 4. Given an infeasible system $\Sigma : \{A\boldsymbol{x} \leq \boldsymbol{b}\}$ with m inequalities, n variables and a positive integer K with $1 \leq K \leq n+1$, does it have an IIS of cardinality at most K?

Let $(A, \boldsymbol{b})$ and K define an arbitrary instance of the above decision problem. Consider the particular instance of the separation problem given by the same infeasible system together with the vector $\boldsymbol{y}$ such that $y_i = 1 - 1/(K+1)$ for all i, $1 \leq i \leq m$.

Suppose that Σ has an IIS of cardinality at most K which is indexed by the set S. Then the corresponding IIS-inequality $\sum_{i \in S} y_i \leq |S| - 1$ is violated by the vector $\boldsymbol{y}$ because

$$\sum_{i \in S} y_i = \sum_{i \in S} (1 - \frac{1}{K+1}) = |S| - \frac{|S|}{K+1} > |S| - 1,$$

where the strict inequality is implied by $|S| \leq K$. Thus the vector $\boldsymbol{y}$ can be separated from P_{FS}.

Conversely, if there exists an IIS-inequality violated by $\boldsymbol{y}$, then

$$\sum_{i \in S} y_i = |S| - \frac{|S|}{(K+1)} > |S| - 1$$

implies that the cardinality of the IIS defined by S is at most K.

Therefore, the original infeasible system Σ has an IIS of cardinality at most K if and only if some IIS-inequality is violated by the given vector $\boldsymbol{y}$. $\square$

In [21] the concept of generalized antiwebs, which includes as special cases generalized cliques, generalized odd holes and generalized antiholes, is introduced. Necessary and sufficient conditions are also established for the corresponding rank inequalities to define facets of the associated independence system polytope.

Let m, t, q be integers such that $2 \leq q \leq t \leq m$, let $E = \{e_1, \ldots, e_m\}$ be a finite set, and define for each $i \in M = \{1, \ldots, m\}$ the subset $E^i = \{e_i, \ldots, e_{i+t-1}\}$ (where the indices are taken modulo m) formed by t consecutive elements of E. An *(m,t,q)-generalized antiweb* on E is the independence system having the following family of subsets of E as circuits:

$$\mathcal{AW}(m,t,q) = \{C \subseteq E \,|\, C \subseteq E^i \;\text{ for some }\; i \in M,\; |C| = q\}.$$

As mentioned in [21], $\mathcal{AW}(m,t,q)$ corresponds to *generalized cliques* when $m = t$, to *generalized odd holes* when $q = t$ and t does not divide m, and to *generalized antiholes* when $m = qt+1$. The rank inequality induced by a generalized antiweb $\sum_{i \in E} y_i \leq \lfloor m(q-1)/t \rfloor$ defines a canonical facet of the independence system polytope $P(\mathcal{AW}(m,t,q))$ if and only if $m = t$ or t does not divide $m(q-1)$ [21].

In the case of P_{FS}, the ground set is the set of indices of inequalities in the infeasible system Σ under consideration.

Proposition 2. *No facets of P_{FS} are induced by generalized cliques other than simple IISs (i.e., $m = t = q$).*

Proof. We invoke the following result (Proposition 3.15 of [21]). For any $S \subseteq E$, let $\mathcal{C}_S = \{C \in \mathcal{C} \,|\, C \subseteq S\}$ denote the family of circuits of the independence system induced by $(E, \mathcal{I})$ on S. Then the rank inequality $\sum_{e \in S} y_e \leq r(S)$ induces a facet of $P(\mathcal{C})$ if and only if S is closed and the rank inequality induces a facet of $P(\mathcal{C}_S)$. Hence it suffices to consider the case $S = E$ and $\mathcal{C}_S = \mathcal{AW}(m, t, q))$.

It is easy to verify that the only (m, t, q)-generalized antiwebs that can arise in IIS-hypergraphs are those with $q = t$. Suppose that $q < t$ and consider E^1, an arbitrary circuit $C \in \mathcal{AW}(m, t, q)$ with $C \subseteq E^1$ and an arbitrary element $e \in E^1 \setminus C$. By definition of $\mathcal{AW}(m, t, q)$, any q subset of E^1 is a circuit. This must be true in particular for all subsets containing e and $q - 1$ elements of C. But then C cannot be closed because $r(C \cup \{e\}) = r(C)$ and thus we have a contradiction to the fact that all IISs are closed (Theorem 7). Hence the only generalized cliques that can arise are those with $m = t = q$, that is, in which the whole ground set E is an IIS. □

The generalized antiwebs which are not ruled out by the above proof, i.e, $\mathcal{AW}(m, t, q)$ with $q = t$, clearly correspond to simple circular sequences of IISs of cardinality t given by the subsets E^i, $i \in M$, of the definition. For $t = q = 2$, it is easy to see that the only possible cases that can arise as induced hypergraphs of IIS-hypergraphs are those with $m = 4$ and $m = 2$. In fact, we conjecture that no other (m, t, q)-generalized antiwebs can occur besides the cases $m = t = q$ with $q \geq 2$, $m = 4$ and $t = q = 2$ as well as the trivial cases in which $q = 1$. In this respect it is interesting to note that the remark following Theorem 5 implies that the lineality spaces L associated to all the IISs E^i, $i \in M$, in any given generalized antiweb are identical. Therefore we can assume that they are all maximal IISs contained in $L^\perp$ and exploit the special geometric structure of such IISs revealed by the proof of Theorem 7. An intermediate step would then be to show that no sequence of more than 3 such successive IISs E^i can occur without other additional IISs involving t nonsequential elements. In the case $m = 5$ and $t = 2$, this observation is clearly valid.

Besides settling the above-mentioned issue, we are investigating other rank and non-rank facets of P_{FS}. For rank facets, it is also of interest to consider the extent to which the sufficient condition involving connectedness of the critical graph could also be necessary. By enumerating all independence systems on at most 6 elements, we have verified that all cases with rank facets different from IIS-inequalities and with a nonconnected critical graph occur in independence systems which cannot be realized as P_{FS}.

For non-rank facets, we can specialize some known facet classes for general independence system polytopes and set covering polytopes, e.g., the class of all facets $(0, 1, 2)$-valued coefficients characterized in [7]. A simple example of P_{FS} polytope with such a non-rank facet is as follows. The original system contains six inequalities in three variables. In addition to the rank inequalities defined by the five maximal IISs ($\{3456\}$, $\{2345\}$, $\{1346\}$, $\{1246\}$, $\{1245\}$) and to the trivial $(0, 1)$–bounding inequalities, the single additional constraint $x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 \leq 5$ is required to provide the full description.

We have also constructed numerous examples of facets of P_{FS} having coefficients larger than 2. These examples come from full descriptions of small-to-medium size problems which we have analyzed using the software PORTA.

Acknowledgement The authors would like to thank G. M. Ziegler for helpful discussions regarding the material of Section 3.

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