

PARTITION FUNCTIONS FOR CLASSICAL ROOT SYSTEMS

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ABSTRACT. This paper presents an algorithm to compute the value of the inverse Laplace transforms of rational functions with poles on arrangements of hyperplanes. As an application, we present an efficient computation of the partition function for classical root systems.

1. INTRODUCTION

The aim of this work is to present an algorithm for a fast computation of the partition function of classical root systems.

EXPLAIN WHAT WE DO.

Let U be a finite dimensional real vector space of dimension r . Denote its dual vector space U^* by V .

Consider a set of elements

$$\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$$

of non zero vectors of V . We assume that the convex cone $\mathcal{C}(\mathcal{A})$ generated by non-negative linear combinations of the elements α_i is an acute convex cone in V with non empty interior. The elements ℓ in V produce linear functions $u \mapsto \ell(u)$ on the complexified vector space $U_{\mathbb{C}}$. In particular to the set \mathcal{A} is associated an arrangement of hyperplanes

$$\mathcal{H}_{\mathbb{C}}(\mathcal{A}) := \bigcup_{i=1}^r \{u \in U_{\mathbb{C}} \mid \alpha_i(u) = 0\}$$

in $U_{\mathbb{C}}$ and its complement

$$U_{\mathbb{C}}(\mathcal{A}) := \left\{ u \in U_{\mathbb{C}} \mid \prod_{i=1}^r \alpha_i(u) \neq 0 \right\}.$$

We denote by $\mathcal{R}_{\mathcal{A}}$ the ring of rational functions on $U_{\mathbb{C}}$ with poles along $\mathcal{H}_{\mathbb{C}}(\mathcal{A})$. Then each element $\phi \in \mathcal{R}_{\mathcal{A}}$ can be written as P/Q where P is a polynomial function on r complex variables and Q is a product of elements, not necessarily distinct, of \mathcal{A} .

In this work, we first present an algorithm to compute the value of the inverse Laplace transform of functions in $\mathcal{R}_{\mathcal{A}}$ at a point $h \in V$. In other words we study the value at a point $h \in V$ of convolutions of a number of

Date: January, 06, 2005.

Heaviside distributions $\phi \mapsto \int_0^\infty \phi(t\alpha_i)dt$. The first theoretical ingredient is the notion of Jeffrey-Kirwan residue residues [8]. Then DeConcini-Procesi [6] proved that one can compute Jeffrey-Kirwan residue using maximal nested sets (in short MNS), a combinatorial tool related to no broken circuit bases of the set of vectors \mathcal{A} .

The applications in view are the study of volumes of polytopes, the number of integral points in polytopes and, more generally, discrete or continuous integration of polynomial functions over polytopes. Indeed, Szenes-Vergne [12], refining a formula of Brion-Vergne [4], stated a formula for volume and number of integral points in polytopes involving Jeffrey-Kirwan residues [8].

In this paper, we describe an efficient algorithm for MNS computation, for classical root systems. This algorithm for MNS gives rise to programs for Kostant partition function for classical root systems (A_n , B_n , C_n and D_n). The important point is that our programs are still extremely efficient for volume and integral points of polytopes

$$\Pi_{\mathcal{A}}(h) = \left\{ x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i \alpha_i = h, x_i \geq 0 \right\}$$

with huge coordinates in the initial data h .

Another remarkable fact is that our algorithm can work with formal parameters, thus giving the quasipolynomial counting the number of integral points in $\Pi_{\mathcal{A}}(h)$ when h runs over the chamber it defines.

This paper is organized as follows. Section 2 introduces Laplace transforms and polytopes. In Section 3, we recall Jeffrey-Kirwan residue and its link with counting formulae. Maximal nested sets are described in Section 4, as well as how they are related to Jeffrey-Kirwan residue. Section 5 describes our general algorithm for MNS computations. Details of particular cases of the algorithm for root systems A_n , B_n , C_n and D_n are examined in Sections 7–10.

LATTE [10] and [11] ALSO NEW IMPLEMENTATION OF BARVINOK'S ALGORITHM [3] The `LatTE` team implemented Barvinok's algorithm

2. LAPLACE TRANSFORM AND POLYTOPES

We start by briefly recalling the notations of the introduction, aiming to relate the inverse of the Laplace transform with various *counting formulae* for a polytope. A good introduction on this theme is the survey article [13].

2.1. Laplace transform. Let U be a finite dimensional real vector space of dimension r with dual space V . We fix the choice of a Lebesgue measure dh on V .

Consider a set

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$$\mathcal{A} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

of non zero vectors of V .

For any subset S of V , we denote by $\mathcal{C}(S)$ the convex cone generated by non negative linear combinations of elements of S .

We assume that the convex cone $\mathcal{C}(\mathcal{A})$ is acute in V with non empty interior.

Let $\mathcal{V}_{sing}(\mathcal{A})$ be the union of the boundaries of the cones $\mathcal{C}(S)$, where S ranges over all the subsets of \mathcal{A} . The complement of $\mathcal{V}_{sing}(\mathcal{A})$ in $\mathcal{C}(\mathcal{A})$ is by definition the open set $\mathcal{C}_{reg}(\mathcal{A})$ of *regular* elements. A connected component \mathfrak{c} of $\mathcal{C}_{reg}(\mathcal{A})$ is called a *chamber* of $\mathcal{C}(\mathcal{A})$. Figures 1 and 2 represent slices of the cones $\mathcal{C}(A_3)$ and $\mathcal{C}(B_3)$, where the dots represents the intersection of a slice with a ray $\mathbb{R}^+\alpha$ hence showing the chambers. Remark that the chambers for B_3 and C_3 are the same (as roots in B_3 and C_3 are proportional). In dimension 3, the root system A_3 is isomorphic to D_3 .

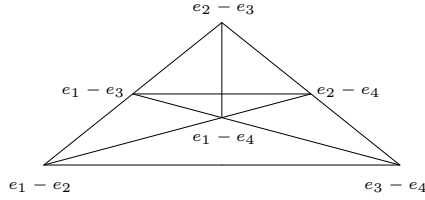


FIGURE 1. The 7 chambers for A_3

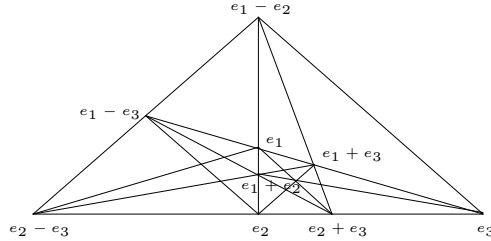


FIGURE 2. The 23 chambers for B_3

Consider now a cone $\mathcal{C}(S)$ spanned by a subset S of \mathcal{A} and let p be a function on $\mathcal{C}(S)$. We assume that p is the restriction to $\mathcal{C}(S)$ of a polynomial function on V . By superposing such functions p_i defined on cones $\mathcal{C}(S_i)$, we obtain a space $\mathcal{LP}(V, \mathcal{A})$ of locally polynomial functions on $\mathcal{C}(\mathcal{A})$. For $f \in \mathcal{LP}(V, \mathcal{A})$, the restriction of f to any chamber \mathfrak{c} of $\mathcal{C}(\mathcal{A})$ is given by a polynomial function.

The Laplace transform $L(f)$ of such a function f is defined as usual. Consider the dual cone $\mathcal{C}(\mathcal{A})^* \subset U$ of $\mathcal{C}(\mathcal{A})$ defined by:

$$\mathcal{C}(\mathcal{A})^* = \{u \in U \mid \langle h, u \rangle \geq 0 \text{ for all } h \in \mathcal{C}(\mathcal{A})\}.$$

Then for u in the interior of the cone $\mathcal{C}(\mathcal{A})^*$, the integral

$$L(f)(u) = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h, u \rangle} f(h) dh$$

is convergent. It is easy to see that the function $L(f)$ is the restriction to $\mathcal{C}(\mathcal{A})^*$ of a function in $\mathcal{R}_{\mathcal{A}}$. (Recall that $\mathcal{R}_{\mathcal{A}}$ is the ring of rational functions P/Q on U where P is a polynomial function on U and Q is a product of elements of \mathcal{A} .) It is easy [4] to characterize the functions $L(f)$ on U arising this way.

Before stating the result we need to introduce some other notations.

Let ν be a subset of $\{1, 2, \dots, n\}$. We will say that ν is *generating* (respectively *basic*) if the set $\{\alpha_i \mid i \in \nu\}$ generates (respectively is a basis of) the vector space V .

Every basic subset is of cardinality r and we write $\text{Bases}(\mathcal{A})$ for the set of basic subsets. Given $\sigma \in \text{Bases}(\mathcal{A})$ the associated *basic* fraction is the fraction defined by

$$(1) \quad f_{\sigma} = \frac{1}{\prod_{i \in \sigma} \alpha_i}.$$

In a system of coordinates (depending on σ) on U where $\alpha_i(u) = u_i$ (for $i \in \sigma$), such a basic fraction is simply of the form

$$\frac{1}{u_1 u_2 \cdots u_r}.$$

Define $\mathcal{G}(U, \mathcal{A}) \subset \mathcal{R}_{\mathcal{A}}$ as the linear span of functions $\frac{1}{\prod_{i \in \nu} \alpha_i^{n_i}}$, where ν is generating and n_i are positive integers. The following Proposition gives the characterization we were speaking of and is easy to prove:

Proposition 2.1. [4] *If f is a locally polynomial function on $\mathcal{C}(\mathcal{A})$, the Laplace transform $L(f)$ of f is the restriction to $\mathcal{C}(\mathcal{A})^*$ of a function in $\mathcal{G}(U, \mathcal{A})$. Reciprocally, for any generating set ν and every set of positive integers $n_i > 0$, there exists a locally polynomial function f on V such that*

$$\frac{1}{\prod_{i \in \nu} \alpha_i(u)^{n_i}} = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h, u \rangle} f(h) dh$$

for any u in the interior of $\mathcal{C}(\mathcal{A})^*$.

We define the inverse Laplace transform $L^{-1} : \mathcal{G}(U, \mathcal{A}) \rightarrow \mathcal{LP}(V, \mathcal{A})$ as follows. For $\phi \in \mathcal{G}(U, \mathcal{A})$, the function $L^{-1}\phi$ is the unique locally polynomial function that satisfies

$$\phi(u) = \int_{\mathcal{C}(\mathcal{A})} e^{-\langle h, u \rangle} (L^{-1}\phi)(h) dh$$

for any $u \in \mathcal{C}(\mathcal{A})^*$.

In the next Sections, we will explain the relation between Laplace transform and enumeration of integral points of families of polytopes. We will see in Section 4 that one can write efficient formulae for the inversion of Laplace transforms in terms of residues, whose algorithmic implementation is working in a quite impressive way, at least for low dimension.

2.2. Volume and number of integral points of a polytope. In this Subsection we consider a sequence

$$\mathcal{A}^+ = [\alpha_1, \alpha_2, \dots, \alpha_N]$$

of non zero elements of \mathcal{A} . We assume that each element $\alpha \in \mathcal{A}$ occurs in the sequence. In particular $N \geq n$. We introduce now the notion of a partition polytope.

We consider the space \mathbb{R}^N with its standard basis ω_i and Lebesgue measure dx .

If $x = \sum_{i=1}^N x_i \omega_i \in \mathbb{R}^N$ with $x_i \geq 0$ ($1 \leq i \leq N$) then we will simply write $x \geq 0$.

Consider the surjective map $A : \mathbb{R}^N \rightarrow V$ defined by $A(\omega_i) = \alpha_i$ and denote by K its kernel. Then K is a vector space of dimension $d = N - r$ equipped with a quotient Lebesgue measure dx/dh .

If $h \in V$, we define

$$\Pi_{\mathcal{A}^+}(h) = \{x \in \mathbb{R}^N \mid Ax = h; x \geq 0\}.$$

The set $\Pi_{\mathcal{A}^+}(h)$ is a convex polytope. It is the intersection of the positive quadrant in \mathbb{R}^N with an affine translate of the vector space K . This polytope consists of all positive solutions of the system of r linear equations

$$\sum_{i=1}^N x_i \alpha_i = h.$$

Remark 2.2. *It might be appropriate to recall that any full dimensional polytope P in a vector space E of dimension d , defined by a system of N linear inequations*

$$P = \{y \in E \mid \langle u_i, y \rangle + \lambda_i \geq 0\}$$

(where $u_i \in E^$ and λ_i are real numbers), can be canonically realized as a partition polytope $\Pi_{\mathcal{A}^+}(h)$. Here \mathcal{A}^+ is a set of N elements in a vector space of dimension $r = N - d$. Indeed, consider the diagram*

$$E \xrightarrow{i} \mathbb{R}^N \xrightarrow{A} V = \mathbb{R}^N / i(E)$$

where $i : y \mapsto \sum_{i=1}^N \langle u_i, y \rangle \omega_i$ and A is the projection map $\mathbb{R}^N \rightarrow V$. Let α_i be the images of the canonical basis ω_i of \mathbb{R}^N . Define $\mathcal{A}^+ = [\alpha_1, \dots, \alpha_N]$ and consider the point $h := A(\sum_{i=1}^N \lambda_i \omega_i)$. Then the polytope $\Pi_{\mathcal{A}^+}(h)$ is isomorphic to P . Indeed, the points in $\Pi_{\mathcal{A}^+}(h)$ are exactly the points x_i such that $\sum_{i=1}^N (x_i - \lambda_i) A(\omega_i) = 0$ with $x_i \geq 0$. By definition of the space

$V = \mathbb{R}^N / i(E)$, there exists $y \in i(E)$ such that $x_i - \lambda_i = \langle u_i, y \rangle$. As $x_i \geq 0$, this means exactly that $\langle u_i, y \rangle + \lambda_i \geq 0$, so that the point y is in P .

More concretely, to determine the partition polytope $Ax = b$ starting from a polytope P given by $Qy^T \geq \lambda$ (where Q is a $N \times d$ matrix whose i^{th} -row is given by a vector $u_i \in E^*$ and $\lambda \in \mathbb{R}^N$) we choose among the elements u_i a basis of E^* . Thus after relabeling the indices and doing an appropriate translation, we may assume the inequations of the polytope P are given in the form

$$\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \\ \dots \\ y_d \geq 0 \\ Cy^T + \lambda^T \geq 0 \end{cases} \quad \text{where} \quad \begin{aligned} & C \text{ is a } r \times d \text{ matrix,} \\ & \lambda \in \mathbb{R}^r, \\ & \text{and } y = (y_1, \dots, y_d) \in \mathbb{R}^d. \end{aligned}$$

Then the polytope P is isomorphic to the polytope defined by

$$\{x \geq 0 \mid Ax^T = \lambda^T\}$$

where A is the $r \times N$ matrix given by

$$A = \left(\underbrace{-C}_{r \times d} \quad \underbrace{I_r}_{r \times r} \right), \quad I_r \text{ being the identity matrix.}$$

Let us examine an example.

Example 2.3. Let $P \subset \mathbb{R}^2$ be the polytope defined by the system of inequalities:

$$\begin{cases} -x_1 + 1 \geq 0, \\ -x_2 + 2 \geq 0, \\ -x_1 - x_2 + 2 \geq 0, \\ 2x_1 + x_2 - 1 \geq 0. \end{cases}$$

Choosing the basis $u_1 = (1, 0)$, $u_2 = (0, 1)$ and using the translation $y_1 = -x_1 + 1$ and $y_2 = -x_2 + 2$ we can rewrite the system as:

$$\begin{cases} y_1 \geq 0 \\ y_2 \geq 0 \\ C \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \geq 0 \end{cases} \quad \text{where } C = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Therefore P is isomorphic to

$$\Pi_{\mathcal{A}^+}(h) = \left\{ y = (y_1, \dots, y_4) \in \mathbb{R}^4, y \geq 0 \mid \begin{array}{l} -y_1 - y_2 + y_3 = -1 \\ 2y_1 + y_2 + y_4 = 3 \end{array} \right\}$$

with $h = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

We continue with our review. If h is in the interior of the cone $\mathcal{C}(\mathcal{A})$, then the polytope $\Pi_{\mathcal{A}^+}(h)$ is of dimension d . It lies in a translate of the vector space K , and this translated space is provided with the quotient measure dx/dh .

Definition 2.4. We write $\text{vol}_{\mathcal{A}^+}(h)$ for the volume of $\Pi_{\mathcal{A}^+}(h)$ computed with respect to this measure.

Going on, suppose further that V is provided with a lattice $V_{\mathbb{Z}}$ and that

$$\mathcal{A}^+ := [\alpha_1, \alpha_2, \dots, \alpha_N]$$

is a sequence of non zero elements of $V_{\mathbb{Z}}$ spanning $V_{\mathbb{Z}}$.

In this case, the lattice $V_{\mathbb{Z}}$ determines a measure $d_{\mathbb{Z}}h$ on V so that the fundamental domain of the lattice $V_{\mathbb{Z}}$ is of measure 1 for $d_{\mathbb{Z}}h$. However, for reasons which will be clear later on, we keep our initial measure dh . We introduce the normalized volume.

Definition 2.5. The normalized volume $\text{vol}_{\mathbb{Z}, \mathcal{A}^+}(h)$ is the volume of $\Pi_{\mathcal{A}^+}(h)$ computed with respect to the measure $dx/d_{\mathbb{Z}}h$.

Remark 2.6. The reason for keeping our initial dh is that the root systems B_n, C_n, D_n live on the same standard vector space $V = \mathbb{R}^n$, where the most natural measure is the standard one. This measure is twice the measure given by the root lattice in the case of C_n and D_n .

If $\text{vol}(V/V_{\mathbb{Z}}, dh)$ is the volume of a fundamental domain of $V_{\mathbb{Z}}$ for dh , clearly $\text{vol}_{\mathbb{Z}, \mathcal{A}^+}(h) = \text{vol}(V/V_{\mathbb{Z}}, dh) \text{vol}_{\mathcal{A}^+}(h)$.

Let now $h \in V_{\mathbb{Z}}$. A discrete analogue of the normalized volume of $\Pi_{\mathcal{A}^+}(h)$ is the number of integral points inside this polytope.

Definition 2.7. Let $N_{\mathcal{A}^+}(h)$ be the number of integral points in $\Pi_{\mathcal{A}^+}(h)$, that is the number of solutions $x = (x_1, \dots, x_N)$ of the equation $\sum_{i=1}^N x_i \alpha_i = h$ where x_i are non negative integral numbers.

We will see after stating Theorem 3.3 that functions $h \mapsto \text{vol}_{\mathcal{A}^+}(h)$ and $h \mapsto N_{\mathcal{A}^+}(h)$ are respectively polynomial and quasi-polynomial on each chamber of $\mathcal{C}(\mathcal{A})$.

The following formulae (see for example [13]) compute the Laplace transform of the locally polynomial function $\text{vol}_{\mathcal{A}^+}(h)$ and the discrete Laplace transform of the quasi-polynomial function $N_{\mathcal{A}^+}(h)$.

Proposition 2.8. Let $u \in \mathcal{C}(\mathcal{A})^*$. Then:

- (1) $\int_{\mathcal{C}(\mathcal{A})} e^{-\langle h, u \rangle} \text{vol}_{\mathcal{A}^+}(h) dh = \frac{1}{\prod_{i=1}^N \alpha_i(u)}.$
- (2) $\sum_{h \in V_{\mathbb{Z}} \cap \mathcal{C}(\mathcal{A})} e^{-\langle h, u \rangle} N_{\mathcal{A}^+}(h) = \frac{1}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})}.$

3. JEFFREY-KIRWAN RESIDUE

The aim of this Section is to explain some theoretical results due to Jeffrey and Kirwan which are fundamental for our work. They described an efficient scheme for computing the inverse Laplace transforms in the context of hyperplane arrangements.

Let's go back to the space of rational functions $\mathcal{R}_{\mathcal{A}}$. It is \mathbb{Z} -graded by degree. Of great importance for our exposition will be certain functions in

$\mathcal{R}_{\mathcal{A}}$ of degree $-r$. Every function in $\mathcal{R}_{\mathcal{A}}$ of degree $-r$ may be decomposed into a sum of basic fractions f_{σ} (see Equation (1)) and degenerate fractions; degenerate fractions are those for which the linear forms in the denominator do not span V . Now having fixed a chamber \mathfrak{c} , we define a functional $JK_{\mathfrak{c}}(\phi)$ on $\mathcal{R}_{\mathcal{A}}$ called the Jeffrey-Kirwan residue (or *JK-residue*) as follows. Let

$$(2) \quad JK_{\mathfrak{c}}(f_{\sigma}) = \begin{cases} \text{vol}(\sigma)^{-1}, & \text{if } \mathfrak{c} \subset \mathcal{C}(\sigma), \\ 0, & \text{if } \mathfrak{c} \cap \mathcal{C}(\sigma) = \emptyset. \end{cases}$$

By setting the value of the JK-residue of a degenerate fraction or that of a rational function of pure degree different from $-r$ equal to zero, we have defined the JK-residue on $\mathcal{R}_{\mathcal{A}}$.

We may go further and extend the definition to the space $\widehat{\mathcal{R}}_{\mathcal{A}}$ which is the space consisting of functions P/Q where Q is a product of powers of the linear forms α_i and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series. Indeed suppose that $P/Q \in \widehat{\mathcal{R}}_{\mathcal{A}}$ where we may assume that Q is of degree q , and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with P_k of degree k . Then we just define

$$JK_{\mathfrak{c}}(P/Q) = JK_{\mathfrak{c}}(P_{q-r}/Q)$$

as the JK-residue of the component of degree $-r$ of P/Q . In particular if $\phi \in \mathcal{R}_{\mathcal{A}}$ and $h \in V$, the function

$$e^{\langle h, u \rangle} \phi(u) = \sum_{k=0}^{\infty} \frac{\langle h, u \rangle^k}{k!} \phi(u)$$

is in $\widehat{\mathcal{R}}_{\mathcal{A}}$ and we may compute its JK residue.

Let's now make a short digression that should clarify why JK-residues compute inverse Laplace transforms. Given $\sigma \in \text{Bases}(\mathcal{A})$, we write $\mathcal{C}(\sigma)$ for the cone generated by α_i ($i \in \sigma$) and by $\text{vol}(\sigma) > 0$ for the volume of the parallelotope $\sum_{i=1}^r [0, 1] \alpha_i$ computed for the measure dh . Observe that $\text{vol}(\sigma) = |\det(\sigma)|$, where σ is the matrix which columns are the α_i 's.

Then for $u \in \mathcal{C}(\mathcal{A})^*$ we have:

$$\frac{1}{\text{vol}(\sigma)} \int_{\mathcal{C}(\sigma)} e^{-\langle h, u \rangle} dh = f_{\sigma}(u).$$

In other words the inverse Laplace transform of f_{σ} computed at the point $h \in \mathcal{C}(\mathcal{A})$ is $\frac{1}{\text{vol}(\sigma)} \chi_{\sigma}(h)$, where χ_{σ} is the characteristic function of the cone $\mathcal{C}(\sigma)$. We state this as a formula:

$$\frac{1}{\text{vol}(\sigma)} \chi_{\sigma}(h) = L^{-1}(f_{\sigma})(h).$$

Since the JK-residue can be written in terms of basic fractions, the following theorem [8] is not surprising:

Theorem 3.1 (Jeffrey-Kirwan). *If $\phi \in \mathcal{R}_{\mathcal{A}}$, then for any $h \in \mathfrak{c}$ we have:*

$$(L^{-1}\phi)(h) = JK_{\mathfrak{c}}\left(e^{\langle h, \cdot \rangle} \phi\right).$$

Assume that $\Psi : U \rightarrow U$ is an holomorphic transformation defined on a neighborhood of 0 in U and invertible. We also assume that $\alpha_j(F(u)) = \alpha_j(u)f_j(u)$, where $f_j(u)$ is holomorphic in a neighborhood of 0 and $f_j(0) \neq 0$.

If ϕ is a function in $\widehat{\mathcal{R}}_{\mathcal{A}}$, the function $\Psi^*\phi(u) = \phi(\Psi(u))$ is again in $\widehat{\mathcal{R}}_{\mathcal{A}}$. Let $\text{Jac}(\Psi)$ be the Jacobian of the map Ψ . The function $\text{Jac}(\Psi)$ is calculated as follows: write $\Psi(u) = (\Psi_1(u_1, u_2, \dots, u_r), \dots, \Psi_r(u_1, u_2, \dots, u_r))$. Then $\text{Jac}(\Psi)(u) = \det((\frac{\partial}{\partial u_i} \Psi_j)_{i,j})$. We assume that $\text{Jac}(\Psi)(u)$ does not vanish at $u = 0$. For any ϕ in $\widehat{\mathcal{R}}_{\mathcal{A}}$ the following change of variable formula, which will be useful in our calculations later on, holds:

Proposition 3.2. *The Jeffrey-Kirwan residue obeys the rule of change of variables:*

$$JK_{\mathfrak{c}}(\phi) = JK_{\mathfrak{c}}(\text{Jac}(\Psi)(\Psi^*\phi)).$$

We conclude this Section by recalling the formula for $N_{\mathcal{A}^+}(h)$. We need more notations.

Consider the dual lattice $U_{\mathbb{Z}} = \{u \in U \mid \langle u, V_{\mathbb{Z}} \rangle \subset \mathbb{Z}\}$ and the torus $T = U/U_{\mathbb{Z}}$. Choosing a basis $\{u_1, \dots, u_r\}$ of $U_{\mathbb{Z}}$ we may identify T with the subset of U defined by the fundamental domain for translation by $U_{\mathbb{Z}}$:

$$\left\{ \sum_{j=1}^r t_j u_j \right\}$$

with $0 \leq t_j < 1$.

Every element g in $T = U/U_{\mathbb{Z}}$ produces a function on $V_{\mathbb{Z}}$ by $h \mapsto e^{\langle h, 2\pi\sqrt{-1}G \rangle}$, where we denote by G a representative of $g \in U/U_{\mathbb{Z}}$. For $\sigma \in \text{Bases}(\mathcal{A})$ we denote by $T(\sigma)$ the subset of T defined by

$$T(\sigma) = \left\{ g \in T \mid e^{\langle \alpha, 2\pi\sqrt{-1}G \rangle} = 1 \text{ for all } \alpha \in \sigma \right\}.$$

This is a finite subset of T . In particular if σ is a \mathbb{Z} basis of $V_{\mathbb{Z}}$, then this set is reduced to the identity. More generally, consider the lattice $\mathbb{Z}\sigma$ generated by the elements α in σ . If p is such that $\mathbb{Z}\sigma \subset pV_{\mathbb{Z}}$, then all elements of $T(\sigma)$ are of order p .

For $g \in T$ and $h \in V_{\mathbb{Z}}$, consider the *Kostant function* $F(g, h)$ on U defined by

$$(3) \quad F(g, h)(u) = \frac{e^{\langle h, 2\pi\sqrt{-1}G+u \rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G+u \rangle})}.$$

For example when $g = 1$,

$$F(1, h)(u) = \frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})}.$$

The function $F(g, h)(u)$ is an element of $\widehat{\mathcal{R}}_{\mathcal{A}}$. Indeed if we write

$$I(g) = \left\{ i \mid 1 \leq i \leq N, e^{-\langle \alpha_i, 2\pi\sqrt{-1}G \rangle} = 1 \right\},$$

then

$$(4) \quad F(g, h)(u) = e^{\langle h, 2\pi\sqrt{-1}G \rangle} \frac{e^{\langle h, u \rangle}}{\prod_{i \in I(g)} \langle \alpha_i, u \rangle} \psi^g(u)$$

where $\psi^g(u)$ is the holomorphic function of u (in a neighborhood of zero) defined by

$$\psi^g(u) = \prod_{i \in I(g)} \frac{\langle \alpha_i, u \rangle}{(1 - e^{-\langle \alpha_i, u \rangle})} \times \prod_{i \notin I(g)} \frac{1}{(1 - e^{-\langle \alpha_i, 2\pi\sqrt{-1}G + u \rangle})}.$$

If \mathfrak{c} is a chamber of $\mathcal{C}(\mathcal{A})$, the Jeffrey-Kirwan residue $JK_{\mathfrak{c}}(F(g, h))$ is well defined.

The following Theorem is due to Szenes-Vergne [12]. If the set \mathcal{A} is unimodular (that is all $r \times r$ minors of the matrix which columns are the α_i 's have determinant 0 or ± 1), it is a reformulation of Khovanskii-Pukhlikhov Riemann-Roch calculus on simple polytopes [9]. For a general set \mathcal{A} , this refines the formula of Brion-Vergne [4].

Theorem 3.3. *Let \mathfrak{c} be a chamber of the cone $\mathcal{C}(\mathcal{A})$ and $\overline{\mathfrak{c}}$ its closure. Then:*

(1) *For $h \in \overline{\mathfrak{c}}$ we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{A}^+}(h) = \text{vol}(V/V_{\mathbb{Z}}, dh) JK_{\mathfrak{c}} \left(\frac{e^{\langle h, \cdot \rangle}}{\prod_{i=1}^N \alpha_i} \right).$$

(2) *Assume that F is a finite subset of T such that for any $\sigma \in \text{Bases}(\mathcal{A})$, we have $T(\sigma) \subset F$. Then for $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$, we have*

$$N_{\mathcal{A}^+}(h) = \text{vol}(V/V_{\mathbb{Z}}, dh) \sum_{g \in F} JK_{\mathfrak{c}}(F(g, h)).$$

Let us explain the behavior of these functions on a chamber \mathfrak{c} . By definition, a quasi polynomial function on a lattice L is a linear combination of products of polynomial functions and of periodic functions (functions constants on cosets $h + pL$ where p is an integer). We now show that the normalized volume $\text{vol}_{\mathbb{Z}, \mathcal{A}^+}(h)$ is given by a polynomial formula, when h varies in a chamber $\overline{\mathfrak{c}}$, while $N_{\mathcal{A}^+}(h)$ is given by a quasi-polynomial formula when h varies in $V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$.

The residue vanishes except on degree $-r$, so that

$$JK_{\mathfrak{c}} \left(\frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N \langle \alpha_i, u \rangle} \right) = \frac{1}{(N-r)!} JK_{\mathfrak{c}} \left(\frac{\langle h, u \rangle^{N-r}}{\prod_{i=1}^N \langle \alpha_i, u \rangle} \right)$$

and as expected the normalized volume is a polynomial homogeneous function of h of degree $N-r$ on each chamber.

Now if \mathcal{A} is unimodular then the above defined set F is $\{1\}$. Hence the number of integral points in the polytope $\Pi_{\mathcal{A}^+}(h)$ satisfies

$$N_{\mathcal{A}^+}(h) = JK_{\mathfrak{c}} \left(\frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})} \right)$$

and is a polynomial of degree $N-r$ which homogeneous component of degree $N-r$ is the normalized volume. More precisely, write

$$\frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})} = \frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N \langle \alpha_i, u \rangle} \times \frac{\prod_{i=1}^N \langle \alpha_i, u \rangle}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})}$$

where

$$\frac{\prod_{i=1}^N \langle \alpha_i, u \rangle}{\prod_{i=1}^N (1 - e^{-\langle \alpha_i, u \rangle})} = \sum_{k=0}^{+\infty} \psi_k(u)$$

is a holomorphic function of u with $\psi_0(u) = 1$. Consequently

$$\begin{aligned} N_{\mathcal{A}^+}(h) &= JK_{\mathfrak{c}} \left(\frac{e^{\langle h, u \rangle}}{\prod_{i=1}^N \langle \alpha_i, u \rangle} \times \sum_{k=0}^{+\infty} \psi_k(u) \right) \\ &= \sum_{k=0}^{N-r} \frac{1}{(N-r-k)!} JK_{\mathfrak{c}} \left(\frac{\langle h, u \rangle^{N-r-k} \psi_k(u)}{\prod_{i=1}^N \langle \alpha_i, u \rangle} \right). \end{aligned}$$

This will be the case of the root system A_n .

Finally in the non-unimodular case (for example for the root systems B_n , C_n , D_n) the set F is no longer equal to $\{1\}$. Let us denote by $\psi^g(u) = \sum_{k=0}^{+\infty} \psi_k^g(u)$ the series development of the holomorphic function ψ^g appearing in formula (4). Then we see that $JK_{\mathfrak{c}}(F(g, h))$ equals

$$\begin{aligned} & JK_{\mathfrak{c}} \left(e^{\langle h, 2\pi\sqrt{-1}G \rangle} \frac{e^{\langle h, u \rangle}}{\prod_{i \in I(g)} \langle \alpha_i, u \rangle} \psi^g(u) \right) \\ &= e^{\langle h, 2\pi\sqrt{-1}G \rangle} \sum_{k=0}^{|I(g)|-r} \frac{1}{(|I(g)|-r-k)!} JK_{\mathfrak{c}} \left(\frac{\langle h, u \rangle^{|I(g)|-r-k}}{\prod_{i \in I(g)} \langle \alpha_i, u \rangle} \psi_k^g(u) \right). \end{aligned}$$

If g is of order p , the function $h \mapsto e^{\langle h, 2\pi\sqrt{-1}G \rangle}$ is constant on each coset $h + pV_{\mathbb{Z}}$ of the lattice $pV_{\mathbb{Z}}$, while the function $h \mapsto JK_{\mathfrak{c}} \left(\frac{\langle h, u \rangle^{|I(g)|-r-k}}{\prod_{i \in I(g)} \langle \alpha_i, u \rangle} \psi_k^g(u) \right)$

is a polynomial function of h of degree $|I(g)| - r - k$. Thus the function

$$N_{\mathcal{A}^+}(h) = \sum_{g \in F} JK_{\mathfrak{c}}(F(g, h))$$

is given by a quasi polynomial formula when h varies in the closure of a chamber. Note that its highest degree component is polynomial and is the normalized volume as expected.

Example 3.4. *Let us compute the normalized volume and number of integral points for the root system of type B_2 , that is for $\mathcal{A}^+ = \mathcal{B}_2 = \{e_1, e_2, e_1 + e_2, e_1 - e_2\}$. Fix a chamber \mathfrak{c} and an integral vector $h = (h_1, h_2)$ in the cone $C(\mathcal{B}_2)$. Then the normalized volume expresses as*

$$\frac{1}{2!} JK_{\mathfrak{c}} \left(\frac{(h_1 u_1 + h_2 u_2)^2}{u_1 u_2 (u_1 + u_2)(u_1 - u_2)} \right).$$

Note that

$$\frac{u_1^2}{u_1 u_2 (u_1 + u_2)(u_1 - u_2)} = \frac{1}{u_2 (u_1 + u_2)} + \frac{1}{(u_1 + u_2)(u_1 - u_2)}$$

(and similarly for quotients of u_2^2 and $u_1 u_2$), then we get that the normalized volume is

$$\frac{1}{2} JK_{\mathfrak{c}} \left(\frac{h_1^2}{u_2 (u_1 + u_2)} + \frac{h_1^2 + 2h_1 h_2 + h_2^2}{(u_1 + u_2)(u_1 - u_2)} - \frac{h_2^2}{u_1 (u_1 + u_2)} \right).$$

There are three chambers, namely $\mathfrak{c}_1 = C(\{e_2, e_1 + e_2\})$, $\mathfrak{c}_2 = C(\{e_1, e_1 + e_2\})$, $\mathfrak{c}_3 = C(\{e_1 - e_2, e_1\})$ (see Figure 3). Now let us compute Jeffrey-Kirwan residues on chambers. As

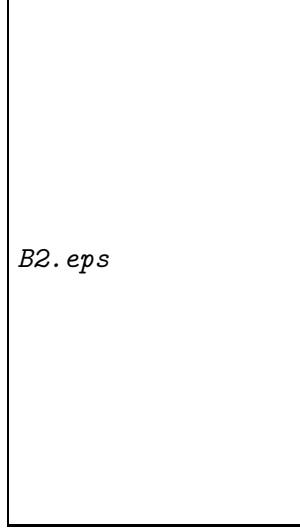
$$\begin{aligned} JK_{\mathfrak{c}_1} \left(\frac{1}{u_2 (u_1 + u_2)} \right) &= 1, & JK_{\mathfrak{c}_2} \left(\frac{1}{(u_1 + u_2)(u_1 - u_2)} \right) &= \frac{1}{2}, \\ JK_{\mathfrak{c}_2} \left(\frac{1}{u_1 (u_1 - u_2)} \right) &= 1, & JK_{\mathfrak{c}_3} \left(\frac{1}{(u_1 + u_2)(u_1 - u_2)} \right) &= \frac{1}{2}, \end{aligned}$$

we obtain

$$\begin{aligned} \text{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{2} h_1^2 & \text{if } h \in \mathfrak{c}_1, \\ \text{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{4} (h_1 + h_2)^2 - \frac{1}{2} h_2^2 & \text{if } h \in \mathfrak{c}_2, \\ \text{vol}(\Pi_{\mathcal{B}_2}(h)) &= \frac{1}{4} (h_1 + h_2)^2 & \text{if } h \in \mathfrak{c}_3. \end{aligned}$$

Remark that formulae agree on walls $\mathfrak{c}_1 \cap \mathfrak{c}_2$ and $\mathfrak{c}_2 \cap \mathfrak{c}_3$.

For the number of integral points, first remark that we have $F = \{(1, 1), (-1, -1)\}$. Consequently $N_{\mathcal{B}_2}(h)$ is equal to the Jeffrey-Kirwan residue of $f_1 = F((1, 1), h)$ plus $f_2 = F((-1, -1), h)$. Series f_j ($j = 1, 2$) rewrite as $f_j = f'_j \times$

FIGURE 3. The 3 chambers for B_2

$e^{u_1 h_1 + u_2 h_2} / u_1 u_2 (u_1 + u_2)(u_1 - u_2)$ where

$$\begin{aligned} f'_1 &= \frac{u_1}{1 - e^{-u_1}} \times \frac{u_2}{1 - e^{-u_2}} \times \frac{u_1 + u_2}{1 - e^{-(u_1 + u_2)}} \times \frac{u_1 - u_2}{1 - e^{-(u_1 - u_2)}}, \\ f'_2 &= \frac{u_1}{1 + e^{-u_1}} \times \frac{u_2}{1 + e^{-u_2}} \times \frac{u_1 + u_2}{1 - e^{-(u_1 + u_2)}} \times \frac{u_1 - u_2}{1 - e^{-(u_1 - u_2)}} \times (-1)^{h_1 + h_2}. \end{aligned}$$

Using series developments $\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + o(x^3)$ and $\frac{x}{1 + e^{-x}} = \frac{1}{2}x + o(x^2)$, we obtain that the number of integral points is the JK residue of

$$\begin{aligned} & \frac{u_1(1 + \frac{3}{2}h_1 + \frac{1}{2}h_1^2)}{u_2(u_1 - u_2)(u_1 + u_2)} + \frac{\frac{3}{4} + h_1 h_2 + \frac{3}{2}h_2 + \frac{1}{2}h_1}{(u_1 - u_2)(u_1 + u_2)} + \frac{u_2(\frac{1}{2}h_2^2 + \frac{1}{2}h_2)}{u_1(u_1 - u_2)(u_1 + u_2)} \\ & + (-1)^{h_1 + h_2} \frac{\frac{1}{4}}{(u_1 + u_2)(u_1 - u_2)} \\ & = \frac{(1 + \frac{3}{2}h_1 + \frac{1}{2}h_1^2)}{u_2(u_1 + u_2)} + \frac{\frac{7}{4} + 2(h_1 + h_2) + \frac{1}{2}h_2^2 + h_1 h_2 + \frac{1}{2}h_1^2}{(u_1 - u_2)(u_1 + u_2)} - \frac{(\frac{1}{2}h_2^2 + \frac{1}{2}h_2)}{u_1(u_1 + u_2)} \\ & + (-1)^{h_1 + h_2} \frac{\frac{1}{4}}{(u_1 + u_2)(u_1 - u_2)}. \end{aligned}$$

We then obtain:

$$\begin{aligned}
N_{\mathcal{B}_2}(h) &= 1 + \frac{3}{2}h_1 + \frac{1}{2}h_1^2 \\
&\quad \text{if } h \in \mathfrak{c}_1, \\
N_{\mathcal{B}_2}(h) &= \frac{1}{4}h_1^2 + \frac{1}{2}h_1h_2 - \frac{1}{4}h_2^2 + h_1 + \frac{1}{2}h_2 + \frac{7}{8} + (-1)^{h_1+h_2}\frac{1}{8} \\
&\quad \text{if } h \in \mathfrak{c}_2, \\
N_{\mathcal{B}_2}(h) &= \frac{1}{4}h_1^2 + \frac{1}{2}h_1h_2 + \frac{1}{4}h_2^2 + h_1 + h_2 + \frac{7}{8} + (-1)^{h_1+h_2}\frac{1}{8} \\
&\quad \text{if } h \in \mathfrak{c}_3.
\end{aligned}$$

Remark that the functions $N_{\mathcal{B}_2}$ agree on walls, and the formulae above are valid on closure of the chambers.

Of course, our general method to implement Theorem 3.3 for root systems is more systematic and will be explained in the course of this article.

Remark 3.5. *In the case where \mathcal{A}^+ is an arbitrary set of vectors in $V_{\mathbb{Z}}$, the straightforward implementation of Theorem 3.3 above is of exponential complexity. Indeed we make a summation on the set F , which can become arbitrarily large. Barvinok uses a signed cone decomposition to obtain an algorithm of polynomial complexity, when the number of elements of \mathcal{A}^+ is fixed, to compute the number $N_{\mathcal{A}^+}(h)$; the LattE team implemented Barvinok's algorithm ([11] and [10]) in the language C. Our work will be dealing either with volumes of polytopes, where the set F does not enter, either with partition function of classical root systems, where the set F is reasonably small. Then we obtain a fast algorithm, implemented for the moment in the formal calculation software MAPLE. This algorithm for these particular cases can reach examples not obtainable by the LattE program.*

In next Section 4 we will give the basic formula for $JK_{\mathfrak{c}}$ involving maximal proper nested sets, as developed in [6], and iterated residues. These formulae are implemented in our algorithms.

4. A FORMULA FOR THE JEFFREY-KIRWAN RESIDUE

If f is a meromorphic function of one variable z with a pole of order less or equal to h at $z = 0$ then we can write $f(z) = Q(z)/z^h$, where $Q(z)$ is a holomorphic function near $z = 0$. If the Taylor series of Q is given by $Q(z) = \sum_{k=0}^{\infty} q_k z^k$, then as usual the residue at $z = 0$ of the function $f(z) = \sum_{k=0}^{\infty} q_k z^{k-h}$ is the coefficient of $1/z$, that is q_{h-1} . We will denote it by $\text{res}_{z=0}f(z)$. To compute this residue we can either expand Q in power series and search for the coefficient of z^{-1} , or employ the formula

$$(5) \quad \text{res}_{z=0}f(z) = \frac{1}{(h-1)!} (\partial_z)^{h-1} \left(z^h f(z) \right) \Big|_{z=0}.$$

We now introduce the notion of iterated residue on the space $\mathcal{R}_{\mathcal{A}}$.

Let $\vec{\nu} = [\alpha_1, \alpha_2, \dots, \alpha_r]$ be an ordered basis of V consisting of elements of \mathcal{A} (here we have implicitly renumbered the elements of \mathcal{A} in order that the elements of our basis are listed first). We choose a system of coordinates on U such that $\alpha_i(u) = u_i$. A function $\phi \in \mathcal{R}_{\mathcal{A}}$ is thus written as a rational fraction $\phi(u_1, u_2, \dots, u_r) = \frac{P(u_1, u_2, \dots, u_r)}{Q(u_1, u_2, \dots, u_r)}$ where the denominator Q is a product of linear forms.

Definition 4.1. *If $\phi \in \mathcal{R}_{\mathcal{A}}$, the iterated residue $\text{Ires}_{\vec{\nu}}(\phi)$ of ϕ for $\vec{\nu}$ is the scalar*

$$\text{Ires}_{\vec{\nu}}(\phi) = \text{res}_{u_r=0} \text{res}_{u_{r-1}=0} \cdots \text{res}_{u_1=0} \phi(u_1, u_2, \dots, u_r)$$

where each residue is taken assuming that the variables with higher indices are considered as constants.

Keep in mind that at each step the residue operation augments the homogeneous degree of a rational function by $+1$ (as for example $\text{res}_{x=0}(1/xy) = 1/y$) so that the iterated residue vanishes on homogeneous elements $\phi \in \mathcal{R}_{\mathcal{A}}$, if the homogeneous degree of ϕ is different from $-r$.

Observe that the value of $\text{Ires}_{\vec{\nu}}(\phi)$ depends on the order of $\vec{\nu}$. For example, for $f = 1/(x(y-x))$ we have $\text{res}_{x=0} \text{res}_{y=0}(f) = 0$ and $\text{res}_{y=0} \text{res}_{x=0}(f) = 1$.

Remark 4.2. *Choose any basis $\gamma_1, \gamma_2, \dots, \gamma_r$ of V such that $\bigoplus_{k=1}^j \alpha_j = \bigoplus_{k=1}^j \gamma_j$ and such that $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_r = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r$. Then, by induction, it is easy to see that for $\phi \in \mathcal{R}_{\mathcal{A}}$*

$$\text{res}_{\alpha_r=0} \cdots \text{res}_{\alpha_1=0} \phi = \text{res}_{\gamma_r=0} \cdots \text{res}_{\gamma_1=0} \phi.$$

Thus given an ordered basis, we may modify α_2 by $\alpha_2 + c\alpha_1, \dots$, with the purpose of getting easier computations.

The following lemma will be useful later on.

Lemma 4.3. *Let $\vec{\nu} = [\alpha_1, \alpha_2, \dots, \alpha_r]$ and $f_{\beta} = \frac{1}{\prod_{i=1}^r \beta_i}$ be a basic fraction. Then the iterated residue $\text{Ires}_{\vec{\nu}}(f_{\beta})$ is non zero if and only if there exists a permutation w of $\{1, 2, \dots, r\}$ such that:*

$$\begin{aligned} \beta_{w(1)} &\in \mathbb{R}\alpha_1, \\ \beta_{w(2)} &\in \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2, \\ &\dots \\ \beta_{w(r)} &\in \mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_r. \end{aligned}$$

Definition 4.4. *Let $\vec{\nu} = [\alpha_1, \alpha_2, \dots, \alpha_r]$ and let $u_j = \alpha_j(u)$. Choose a sequence of real numbers: $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_r$. Then define the torus*

$$(6) \quad T(\vec{\nu}) = \{u \in U_{\mathbb{C}} \mid |u_j| = \epsilon_j, j = 1, \dots, r\}.$$

The torus $T(\vec{\nu})$ is identified via the basis α_j with the product of r circles oriented as usual. The sequence ϵ_j is chosen in order that elements α_q not in $\bigoplus_{k=1}^j \mathbb{R}\alpha_k$ do not vanish on the domain $\{u \in U_{\mathbb{C}} \mid |u_k| \leq \epsilon_k, 1 \leq k \leq j; |u_i| =$

$\epsilon_i, i = j + 1, \dots, r\}$. This is achieved by choosing the ratios $\epsilon_j/\epsilon_{j+1}$ very small. The torus $T(\vec{\nu})$ is contained in $U(\mathcal{A})$ and the homology class $[T(\vec{\nu})]$ of this torus is independent of the choice of the sequence of the ordered ϵ_j (see [12]).

Choose an ordered basis e_1, e_2, \dots, e_r of V of volume 1 with respect to the measure dh . For $z \in U_{\mathbb{C}}$, define $z_j = \langle z, e_j \rangle$ and $dz = dz_1 \wedge dz_2 \wedge \dots \wedge dz_r$. Denote by $\det(\vec{\nu})$ the determinant of the basis $\alpha_1, \alpha_2, \dots, \alpha_r$ with respect to the basis e_1, e_2, \dots, e_r .

Lemma 4.5. *For $\phi \in \mathcal{R}_{\mathcal{A}}$, we have*

$$\frac{1}{\det(\vec{\nu})} \text{res}_{\alpha_r=0} \dots \text{res}_{\alpha_1=0} \phi = \frac{1}{(2\pi\sqrt{-1})^r} \int_{T(\vec{\nu})} \phi(z) dz.$$

Thus, as for the usual residue, the iterated residue can be expressed as an integral.

We now introduce the notion of maximal proper nested set, MPNS in short.

De Concini-Procesi [6] prove that the set of MPNS is in bijection with the so-called no broken circuits bases of \mathcal{A} (with respect to a order to be specified). This is helpful as the JK-residue can be computed in terms of iterated residues with respect to these bases.

So we proceed in our overview.

If S is a subset of \mathcal{A} , we denote by $\langle S \rangle$ the vector space spanned by S . We say that S is *complete* if $S = \langle S \rangle \cap \mathcal{A}$ or in other words if any linear combination of elements of S belongs to S . A complete subset S is called *reducible* if we can find a decomposition $V = V_1 \oplus V_2$ such that $S = S_1 \cup S_2$ with $S_1 \subset V_1$ and $S_2 \subset V_2$. Otherwise S is said to be *irreducible*.

Definition 4.6. *Let \mathcal{I} be the set of irreducible subsets of \mathcal{A} . A set $M = \{I_1, I_2, \dots, I_k\}$ of irreducible subsets of \mathcal{A} is called *nested*, if given any subfamily $\{I_1, \dots, I_m\}$ of M such that there exists no i, j with $I_i \subset I_j$, then the set $I_1 \cup \dots \cup I_m$ is **complete** and the elements I_j are the irreducible components of $I_1 \cup I_2 \cup \dots \cup I_m$.*

Example 4.7. *Let E be a n -dimensional vector space with basis e_i ($i = 1, \dots, n$). We consider the set*

$$\mathcal{K}_n = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

These are the positive roots for a system of type A_{n-1} . The irreducible subsets of \mathcal{K}_n are indexed by subsets S of $\{1, 2, \dots, n\}$, the corresponding irreducible subset being $\{e_i - e_j \mid i, j \in S, i < j\}$. For instance for $n = 5$ the set $S = \{1, 2, 4\}$ parametrizes the set of roots given by $\{e_1 - e_2, e_2 - e_4, e_1 - e_4\}$.

A nested set is represented by a collection $M := \{S_1, S_2, \dots, S_k\}$ of subsets of $\{1, 2, \dots, n\}$ such that if $S_i, S_j \in M$ then either $S_i \cap S_j$ is empty, either one of them is contained in another.

Definition 4.8. A maximal nested set (in short MNS) M is a nested set such that for every irreducible set \mathcal{I} of \mathcal{A} the set $M \cup \mathcal{I}$ is no longer nested.

Assume now that \mathcal{A} is irreducible, otherwise just take the irreducible components. Then every maximal nested set M contains \mathcal{A} . Let I_1, I_2, \dots, I_k be the maximal elements of the set $M \setminus \mathcal{A}$. We see that the vector space spanned by $\langle I_1 \rangle \oplus \langle I_2 \rangle \oplus \dots \oplus \langle I_k \rangle$ is of codimension 1 (Proposition 1.3 of [6]).

Definition 4.9. A hyperplane H in V is \mathcal{A} -admissible if it is spanned by a set of vectors of \mathcal{A} .

Therefore to classify maximal nested sets (MNS) of an irreducible set \mathcal{A} we proceed by running over the set of \mathcal{A} -admissible hyperplanes, as described in Figure 4.

- Take a hyperplane H spanned by a set of vectors of \mathcal{A} .
- Break $\mathcal{A} \cap H$ into irreducible subsets $I_1 \cup I_2 \cup \dots \cup I_k$.
- For each irreducible I_i construct the set $M_i = \{M_1^i, \dots, M_{k_i}^i\}$ of maximal nested sets for I_i .
- Set $C_i = \{1, \dots, k_i\}$.
- A maximal nested set is then given by the union $M_{c_1}^1 \cup M_{c_2}^2 \cup \dots \cup M_{c_k}^k \cup \{\mathcal{A}\}$ where $c_1 \in C_1, \dots, c_k \in C_k$, and all of them are obtained by letting c_i vary.

FIGURE 4. Building of all MNS's for a \mathcal{A} -admissible hyperplane H

The whole algorithm will be described in details in Figure 6, Section 5. It turns out that each maximal nested subset of \mathcal{A} has r elements.

We describe now the notion of maximal proper nested subset of \mathcal{A} .

Fix a total order on the set \mathcal{A} . For example, we can choose a linear functional ht on V so that the values $\text{ht}(\alpha_i)$ are all distinct and positive. Thus the value $\text{ht}(\alpha)$ is larger if α is deeper in the interior of the cone.

Let $M = \{S_1, S_2, \dots, S_k\}$ be a set of subsets of \mathcal{A} . In each S_j we choose the element α_j maximal for the order given by ht . This defines a map θ from M to \mathcal{A} .

Definition 4.10. A maximal nested set M is called proper if $\theta(M)$ is a basis of V . We denote by $\mathcal{P}(\mathcal{A})$ the set of maximal proper nested sets, in short MPNS.

If $M = \{I_1, I_2, \dots, I_r\}$ is a maximal nested set, we transform M to a list $[\theta(I_1), \dots, \theta(I_r)]$ using the total order on the elements $\theta(M)$; that is we have $\text{ht}(\theta(I_1)) < \text{ht}(\theta(I_2)) < \dots < \text{ht}(\theta(I_r))$. Observe that, if \mathcal{A} is irreducible, for every maximal nested set, I_r is always equal to \mathcal{A} and $\theta(I_r)$ is the highest element of \mathcal{A} .

So we have associated to every maximal proper nested set M an ordered basis $\overrightarrow{\theta(M)} = [\alpha_1, \alpha_2, \dots, \alpha_r]$ of elements of \mathcal{A} . In all implementations, we calculate $\overrightarrow{\theta(M)}$ from a MNS M with the procedure **ThetaMNS**(M). We denote by $\text{vol}(M) > 0$ the volume of the parallelepiped $\oplus_{i=1}^r [0, 1]\alpha_i$ with respect to our measure, and by $\mathcal{C}(M) = \sum_{i=1}^r \mathbb{R}_{\geq 0}\alpha_i \subset \mathcal{C}(\mathcal{A})$ the cone generated by $\overrightarrow{\theta(M)}$.

If v is a regular element of V , let

$$(7) \quad \mathcal{P}(v, \mathcal{A}) = \{M \in \mathcal{P}(\mathcal{A}) \mid v \in \mathcal{C}(M)\}.$$

The set $\mathcal{P}(v, \mathcal{A})$ depends only of the chamber \mathfrak{c} where v belongs.

We are now ready to state the basic formula for our calculations.

Theorem 4.11 ([6]). *Let \mathfrak{c} be a chamber and let $v \in \mathfrak{c}$. Then, for $\phi \in \mathcal{R}_{\mathcal{A}}$, we have*

$$JK_{\mathfrak{c}}(\phi) = \sum_{M \in \mathcal{P}(v, \mathcal{A})} \frac{1}{\text{vol}(M)} \text{Ires}_{\overrightarrow{\theta(M)}} \phi.$$

We will use also the corresponding integration formula.

Each maximal proper nested set $M \in \mathcal{P}(v, \mathcal{A})$ determines an oriented cycle $[T(\overrightarrow{\theta(M)})]$ contained in the open set $U(\mathcal{A})$, as described in Definition 4.4.

Definition 4.12. *Let \mathfrak{c} be a chamber. Define the oriented cycle:*

$$H(\mathfrak{c}) = \sum_{M \in \mathcal{P}(v, \mathcal{A})} \text{sign} \left(\det \left(\overrightarrow{\theta(M)} \right) \right) [T(\overrightarrow{\theta(M)})].$$

The following integral version of the theorem above will be useful.

Theorem 4.13. *Let \mathfrak{c} be a chamber. Then, for $\phi \in \mathcal{R}_{\mathcal{A}}$, we have*

$$JK_{\mathfrak{c}}(\phi) = \frac{1}{(2\pi\sqrt{-1})^r} \int_{H(\mathfrak{c})} \phi(z) dz.$$

The following example should help clarifying the notions introduced.

Example 4.14. *We consider the set \mathcal{K}_4 of positive roots for A_3 (see Figure 5) defined by*

$$\mathcal{K}_4 = \{e_i - e_j \mid 1 \leq i < j \leq 4\}.$$

We let V be the vector space generated by the elements in \mathcal{K}_4 . Then V has dimension 3 and we write an element of V as

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 - (a_1 + a_2 + a_3) e_4.$$

We consider the height function defined by

$$\text{ht}(e_1 - e_2) = 10, \quad \text{ht}(e_2 - e_3) = 11, \quad \text{ht}(e_3 - e_4) = 12.$$

This choice gives the following order on the roots

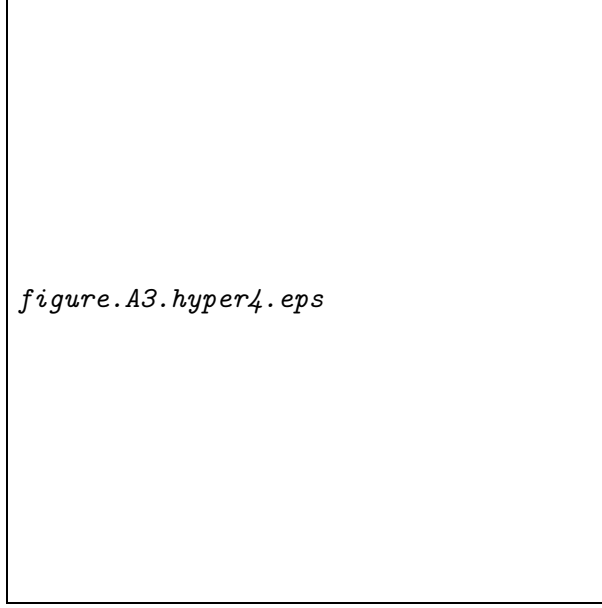


FIGURE 5. Hyperplanes for A_3 with $a = a_1e_1 + a_2e_2 + a_3e_3 - (a_1 + a_2 + a_3)e_4$

$$e_1 - e_2 < e_2 - e_3 < e_3 - e_4 < e_1 - e_3 < e_2 - e_4 < e_1 - e_4.$$

Take an hyperplane H in V spanned by two linearly independent elements of \mathcal{K}_4 . Therefore it is the kernel of a linear form $\sum_{i \in I_H} a_i$, where I_H is a proper subset of $\{1, 2, 3, 4\}$. The set of complementary indices gives the same hyperplane. Thus each admissible hyperplane partitions the set of indices $\{1, 2, 3, 4\}$ in two sets Z_1 and Z_2 , where $Z_1 := \{i \in I_H\}$ and Z_2 is the set of complementary indices. In our example we have 7 choices of admissible hyperplane corresponding to the following partitions:

$$\begin{aligned} H_1 &= \{[1, 2, 3], [4]\}, & H_2 &= \{[1, 2, 4], [3]\}, & H_3 &= \{[1, 3, 4], [2]\}, \\ H_4 &= \{[2, 3, 4], [1]\}, & H_5 &= \{[1, 2], [3, 4]\}, & H_6 &= \{[1, 3], [2, 4]\}, \\ H_7 &= \{[1, 4], [2, 3]\}. \end{aligned}$$

Now observe that if the hyperplane H_i already contains the highest root $e_1 - e_4$ then it cannot lead to a maximal proper nested set. Indeed we must get a basis if we add the highest root to a set of vectors contained in H_i . Thus H_2, H_3, H_7 can be excluded. It remains to consider the hyperplanes H_1, H_4, H_5, H_6 .

Hyperplanes H_1 and H_4 give rise to two MPNSs each one, while H_5 and H_6 give rise to only one. So we obtain a list of 6 maximal nested sets (as described in Example 4.7, we identify an irreducible subset I with a subset S of $[1, 2, 3, 4]$):

$$\begin{aligned}
M_1 &= \{[1, 2], [1, 2, 3], [1, 2, 3, 4]\}, & M_2 &= \{[2, 3], [1, 2, 3], [1, 2, 3, 4]\}, \\
M_3 &= \{[2, 3], [2, 3, 4], [1, 2, 3, 4]\}, & M_4 &= \{[3, 4], [2, 3, 4], [1, 2, 3, 4]\}, \\
M_5 &= \{[1, 3], [2, 4], [1, 2, 3, 4]\}, & M_6 &= \{[1, 2], [3, 4], [1, 2, 3, 4]\}.
\end{aligned}$$

5. SEARCH FOR MAXIMAL PROPER NESTED SETS ADAPTED TO A VECTOR : THE GENERAL CASE

Given a vector v in the cone $\mathcal{C}(\mathcal{A})$, we describe how to search for all maximal proper nested sets belonging to $\mathcal{P}(v, \mathcal{A})$, without enumerating all MPNS.

We use as height function a linear form that is positive and that takes different values on all elements α_i . Let H be a \mathcal{A} -admissible hyperplane in V , that is a hyperplane spanned by a set of vectors of \mathcal{A} . Then the cone $\mathcal{C}(\mathcal{A} \cap H)$ generated by the elements of \mathcal{A} belonging to H is a cone with non empty interior in H .

We have already seen that to list all the MPNS, we have to first list all admissible hyperplanes H and then find the irreducible components J_1, J_2, \dots, J_s of $\mathcal{A} \cap H$. Then we choose a MPNS $M_i := \{I_i^a\}$ for J_i , and define $M = M_1 \cup M_2 \cup M_3 \cup \dots \cup M_s \cup \{\mathcal{A}\}$.

Definition 5.1. *All MNPS's coming from a given \mathcal{A} -admissible hyperplane H are said attached to H .*

As we have seen in Example 4.14 we can discard some of the hyperplanes *a priori*, because they cannot lead to a maximal proper nested set. The next Lemma examines the general situation. Let θ be the highest element in \mathcal{A} and H a hyperplane of \mathcal{A} .

Lemma 5.2. *There exists a maximal proper nested set $M \in \mathcal{P}(v, \mathcal{A})$ attached to H , if and only if θ does not belong to H and if v belongs to the cone generated by θ and $\mathcal{A} \cap H$.*

Proof. The condition is necessary. Indeed v must belong to the cone generated by the elements $\theta(I_i^a)$ and θ , and all the elements $\theta(I_i^a)$ are in $\mathcal{A} \cap H$. Reciprocally consider the projection $v - \frac{\langle u, v \rangle}{\langle u, \theta \rangle} \theta$, where u is the equation of the hyperplane H . This can be written as $v_1 \oplus v_2 \oplus \dots \oplus v_s$, where each v_i is in the cone $\mathcal{C}(J_i)$. Let now $M_i \in \mathcal{P}(v_i, J_i)$ be a MPNS in J_i . The element v_i belongs to $\mathcal{C}(\theta(M_i))$. We can write

$$v = t\theta + \sum_{i=1}^s \sum_{I_i^a \in M_i} t_i^a \theta(I_i^a)$$

with $t_i^a > 0$. Thus we see that the collection $M_1 \cup \dots \cup M_s \cup \mathcal{A}$ is a maximal proper nested set in $\mathcal{P}(v, \mathcal{A})$. Moreover in this way we list all elements of $\mathcal{P}(v, \mathcal{A})$. \square

Our search for maximal proper nested sets in $\mathcal{P}(v, \mathcal{A})$ will then be pursued by constructing all possible admissible hyperplanes H for which v is in the convex hull of $\mathcal{C}(\mathcal{A} \cap H)$ and θ . We denote by $\text{Hyp}(v, \mathcal{A})$ the set of such \mathcal{A} -admissible hyperplanes.

The following easy Lemma lists some obvious conditions for the set $\text{Hyp}(v, \mathcal{A})$.
 Let $u_H \in U$ be the normal vector to a \mathcal{A} -admissible hyperplane, meaning that $H := \{h \in V \mid \langle u_H, v \rangle = 0\}$.

Lemma 5.3. *If $H \in \text{Hyp}(v, \mathcal{A})$, then it satisfies the following conditions:*

- (1) $\langle u_H, \theta \rangle \neq 0$.
- (2) $\langle u_H, v \rangle \times \langle u_H, \theta \rangle \geq 0$.

Thus if a hyperplane H satisfies the above conditions we define

$$\text{proj}_H(v) = v - \frac{\langle u_H, v \rangle}{\langle u_H, \theta \rangle} \theta.$$

Hence to decide if $H \in \text{Hyp}(v, \mathcal{A})$ we simply have to test if $\text{proj}_H(v)$ is in the cone generated by $\mathcal{A} \cap H$, which is done by standard methods. Our search for the hyperplanes $H \in \text{Hyp}(v, \mathcal{A})$ will also be considerably sped up by the following Remark.

Proposition 5.4. *Let H be a \mathcal{A} -admissible hyperplane. Let $u \in U$ be a linear form on V which is non negative on $\mathcal{A} \cap H$ and on θ . If $\langle u, v \rangle < 0$, then H is not in $\text{Hyp}(v, \mathcal{A})$.*

Proof. Indeed if v was in the cone generated by $\mathcal{A} \cap H$ and θ , the value of u would be non negative on v . \square

The point of this Remark is that in classical examples of root systems, an *a priori* description of the \mathcal{A} -admissible hyperplanes is available, together with the defining equations of the cone $\mathcal{C}(\mathcal{A} \cap H)$. This condition will allow us to disregard right away many \mathcal{A} -admissible hyperplanes.

Let us summarize the scheme of the algorithm in Figure 6. Recall that we have as input a vector v , and as output the list of all MPNS's belonging to $\mathcal{P}(v, \mathcal{A})$.

We will explain our algorithm in more details for each classical root system (see Sections 7–10).

6. TREES AND ORDER OF POLES

Let M be a maximally nested proper set of the system $\mathcal{A} := [\alpha_1, \alpha_2, \dots, \alpha_N]$.
 In our algorithms, we will need to take an iterated residue with respect to a basis $\overrightarrow{\theta(M)}$ of a function of the form $\phi = \frac{P}{\prod_{i=1}^N \alpha_i}$, where P is a polynomial function on U . It is thus important to understand the order of the poles of the function obtained after performing a certain number of residues.

We associate to a maximally nested set M a tree T as follows. Let $M = \{I_1, \dots, I_r\}$ be a maximally nested set. The vertices of T are the elements of M and the edges are determined from the order relation by inclusion:

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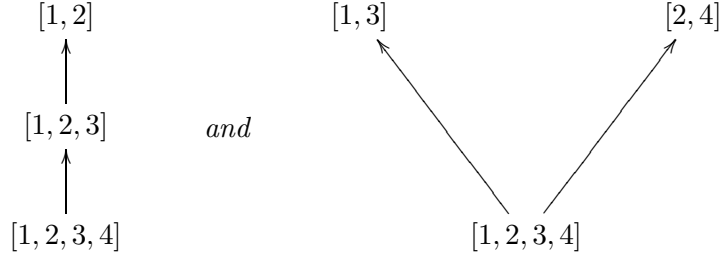
check if  $v \in \mathcal{C}(\mathcal{A})$ 
for each hyperplane  $H$  do
  check if  $v$  and  $\theta$  are on the same side of  $H$ 
  if not, then skip this hyperplane
  define the projection  $proj_H(v)$  of  $v$  on  $H$  along  $\theta$ 
  write  $\mathcal{A} \cap H$  as the union of its irreducible components  $I_1 \cup \dots \cup I_k$ 
  write  $v$  as  $v_1 \oplus \dots \oplus v_k$  according to the previous decomposition
  for each  $I_j$  do
    compute all MPNS's for  $v_j$  and  $I_j$ 
    collect all these MPNS's for  $v_j$  and  $I_j$ 
  end of loop running across  $I_j$ 's
  collect all MPNS's for the hyperplane  $H$ 
end of loop running across  $H$ 's
return the set of all MPNS's for all hyperplanes

```

FIGURE 6. Algorithm for MPNS's computation (general case)

the ends of the tree are irreducible sets with just one element and if \mathcal{A} is irreducible, the base is the set \mathcal{A} . A subset S of M will be called saturated if it contains all elements above elements of S in the tree order. Thus if S contains an element s , it contains all the elements s' of M which are contained in s .

Example 6.1. *The two MNSs named M_1 and M_5 described in Example 4.14 rewrite respectively as*



The algorithm implemented in our programs leads to the following result:

Proposition 6.2. *Let T be the graph associated to an irreducible classical root system. Then T is a connected tree for which every vertex is connected to at most two other vertices.*

We assume our total order on \mathcal{A} to have the following property: if two irreducible subsets I_1 and I_2 are contained in each other: $I_1 \subset I_2$ and distincts, then $\theta(I_1) < \theta(I_2)$. This is the case for the order defining the highest root as highest element of irreducible root systems, in all our examples.

From our order assumption, we obtain the following lemma.

Lemma 6.3. *Let $M = \{I_1, I_2, \dots, I_r\}$ be a MPNS. Here we have numbered our irreducible sets such that $\theta(I_1) < \theta(I_2) < \dots < \theta(I_r)$. Let k be an integer smaller or equal to r . Then the set $\{I_1, I_2, \dots, I_k\}$ is saturated.*

Proposition 6.4. *Let $M = [I_1, I_2, \dots, I_r]$ be a maximally nested proper family. Let $[I'_1, I'_2, \dots, I'_r]$ be a reordering of the sequence $[I_1, I_2, \dots, I_r]$. We assume that this reordering is compatible with the partial order given by inclusion: if $I'_j \subset I'_k$ then $j < k$. Let*

$$\vec{\nu} = [\theta(I_1), \theta(I_2), \dots, \theta(I_r)]$$

and

$$\vec{\nu'} = [\theta(I'_1), \theta(I'_2), \dots, \theta(I'_r)].$$

Then we have $\text{Ires}_{\vec{\nu}} = \text{Ires}_{\vec{\nu'}}$.

Proof. We prove this proposition by induction on r .

If \mathcal{A} is irreducible, then necessarily $I_r = I'_r = \mathcal{A}$ and $[I'_1, I'_2, \dots, I'_{r-1}]$ is a reordering of the sequence $[I_1, I_2, \dots, I_{r-1}]$. Furthermore the families $\{I_1, I_2, \dots, I_{r-1}\}$ and $\{I'_1, I'_2, \dots, I'_{r-1}\}$ are maximal proper nested sets for $\mathcal{A}_0 = \cup_{j=1}^{r-1} I_j$. The set \mathcal{A}_0 spans a codimension 1 vector space in V .

To prove that $\text{Ires}_{\vec{\nu}} = \text{Ires}_{\vec{\nu'}}$, it suffices to test it on basic fractions f_σ . Let $\sigma = \{\beta_1, \beta_2, \dots, \beta_r\}$ be a basic subset of \mathcal{A} . By Lemma 4.3, if $\text{Ires}_{\vec{\nu}} f_\sigma \neq 0$, then the set $\sigma \cap \langle \mathcal{A}_0 \rangle$ is of cardinal $r - 1$, and there exists an element of σ , say β_r , of the form $c\theta + \xi$ where ξ belongs to $\langle \mathcal{A}_0 \rangle$, c is a non zero constant and θ is the highest element of \mathcal{A} .

Let

$$\vec{\nu}_0 = [\theta(I_1), \theta(I_2), \dots, \theta(I_{r-1})]$$

and

$$\vec{\nu'_0} = [\theta(I'_1), \theta(I'_2), \dots, \theta(I'_{r-1})].$$

Then we have

$$\text{Ires}_{\vec{\nu}} f_\sigma = \frac{1}{c} \text{Ires}_{\vec{\nu}_0} f_{\sigma \cap \langle \mathcal{A}_0 \rangle}$$

and

$$\text{Ires}_{\vec{\nu'}} f_\sigma = \frac{1}{c} \text{Ires}_{\vec{\nu'_0}} f_{\sigma \cap \langle \mathcal{A}_0 \rangle}.$$

We conclude by induction.

When \mathcal{A} is not irreducible, we write $\mathcal{A} = \cup_{a=1}^s J_a$ where J_a are irreducibles. We have $V = \oplus_{a=1}^s \langle J_a \rangle$. Every basic subset σ of \mathcal{A} is the union of basic subsets for the irreducible sets J_a . Define

$$\vec{\nu}_a = [\theta(I_a^{i_1}), \theta(I_a^{i_2}), \dots, \theta(J_a)]$$

where $[I_a^{i_1}, I_a^{i_2}, \dots, J_a]$ is the subsequence of irreducible sets contained in J_a extracted (with conserving order) from the sequence $[I_1, I_2, \dots, I_r]$. Similarly let

$$\vec{\nu}'_a = [\theta(I_a'^{i_1}), \theta(I_a'^{i_2}), \dots, \theta(J_a)]$$

where $[I_a'^{i_1}, I_a'^{i_2}, \dots, I_a']$ is the subsequence of irreducible sets contained in J_a extracted from the sequence $[I_1', I_2', \dots, I_r']$. Then, as the calculation takes place with respect to independent variables, we have

$$\begin{aligned} \text{Ires}_{\vec{\nu}'}(f_\sigma) &= \prod_{a=1}^s (\text{Ires}_{\vec{\nu}'_a} f_{\sigma \cap \langle J_a \rangle}), \\ \text{Ires}_{\vec{\nu}'}(f_\sigma) &= \prod_{a=1}^s (\text{Ires}_{\vec{\nu}'_a} f_{\sigma \cap \langle J_a \rangle}). \end{aligned}$$

Each of the vector space $\langle J_a \rangle$ is of dimension less than r , so that by induction hypothesis $\text{Ires}_{\vec{\nu}'_a} = \text{Ires}_{\vec{\nu}'_a}$. This concludes the proof. \square

Let us now consider partial iterated residues. To a set ν of elements of \mathcal{A} , we associate the vector space

$$H_\nu := \{u \in U \mid \langle \alpha, u \rangle = 0 \text{ for all } \alpha \in \nu\}.$$

A linear function $\alpha \in \mathcal{A}$ produces a linear function on H_ν by restriction.

If $\vec{\nu} := [\alpha_1, \alpha_2, \dots, \alpha_k]$ is a sequence of elements of \mathcal{A} , the partial iterated residue

$$\text{Ires}_{\vec{\nu}} \phi := \text{res}_{\alpha_k=0} \cdots \text{res}_{\alpha_1=0} \phi$$

associates to a rational function ϕ in $R_{\mathcal{A}}$ a rational function on H_ν of the form $G / \prod_{i, \bar{\alpha}_i \neq 0} \bar{\alpha}_i^{n_i}$ where G is a polynomial function on H_ν and $\bar{\alpha}$ is the restriction of α to H_ν .

Let M be a MPNS and consider the tree associated to M . Given a saturated subset S of M , we can define the iterated residue with respect to this saturated set: we choose any order $S := [I_1, I_2, \dots, I_k]$ on S compatible with the inclusion relation and define $\text{Ires}_S := \text{Ires}_{\vec{\nu}}$ with $\vec{\nu} = [\theta(I_1), \theta(I_2), \dots, \theta(I_k)]$. With same proof as Proposition 6.4, this partial residue depends only of the set S . We denote by H_S the intersection of the kernels of the elements α for α in S . It is also the intersection of the kernels of the elements $\theta(I_k)$, as the set ν is a basic sequence in S .

Let ϕ be a function in $R_{\mathcal{A}}$ of the form

$$\phi = \frac{P}{\prod_{i=1}^N \alpha_i}.$$

Let M be a MPNS and J_1, J_2, \dots, J_h be elements of M . We consider the saturated subset S of M consisting of the elements of the tree **strictly above** J_1, J_2, \dots, J_h . The iterated residue $\text{Ires}_S \phi$ is a function on H_S . Denote by u_a the restriction of the function $\theta(J_a)$ to H_S .

Proposition 6.5. *The pole of the linear function u_a in the iterated residue $\text{Ires}_S \phi$ is of order less or equal to $|J_a| - \dim \langle J_a \rangle + 1$.*

See Figures 7 and 8 for an application of the Proposition.

Proof. Choose a vector space E such that

$$V = \langle J_1 \rangle \oplus \cdots \oplus \langle J_s \rangle \oplus E.$$

Let $B = \bigcup_{a=1}^s J_a$ and $C = \mathcal{A} \setminus B$.

Write $C := \{\beta_1, \beta_2, \dots, \beta_q\}$ and

$$\phi = P\phi_1\phi_2 \cdots \phi_s Q$$

with $\phi_a = \frac{1}{\prod_{\alpha \in J_a} \alpha}$ and $Q = \frac{1}{\prod_{j=1}^q \beta_j}$.

For $\beta_j \in C$, we write $\beta_j = \sum_{i=1}^s \beta_j^i + \gamma_j$ with $\beta_j^i \in \langle J_i \rangle$ and $\gamma_j \in E$. The element γ_j is necessarily non zero, as the set B is complete. Thus we write

$$\frac{1}{\beta_j} = \frac{1}{\gamma_j \left(1 + \frac{\sum_{i=1}^s \beta_j^i}{\gamma_j}\right)}$$

and the iterated residue is by definition

$$\text{Ires}_S(\phi) = \text{Ires}_S \left(P(\phi_1 \cdots \phi_s) \prod_{j=1}^q \frac{1}{\gamma_j} \sum_{k=0}^{\infty} \left((-1)^k \frac{\sum_{i=1}^s \beta_j^i}{\gamma_j} \right)^k \right).$$

Here, when taking the residue, elements γ_j are considered as constants and this sum is finite.

Consider the subset M_a of elements of M contained in J_a . This is a MPNS for the set J_a . Let J_a^+ be the saturated subset of M_a consisting of all elements of M_a different from J_a . Then J_a^+ has $\langle J_a \rangle - 1$ elements. If $g = \frac{P_a}{\prod_{\alpha \in J_a} \alpha^{n_\alpha}}$, the iterated residue $\text{Ires}_{J_a^+} g$ is a Laurent polynomial in u_a .

Now $\text{Ires}_S \phi$ is a sum of products of residues of the form $\text{Ires}_{J_a^+} g_a$ where $g_a = \frac{P_a}{\prod_{\alpha \in J_a} \alpha}$ and P_a is a polynomial. Thus we obtain a Laurent polynomial in $u_a, a = 1, \dots, s$ (with coefficients rational functions on the vector space E^*). Now the homogeneous degree of g_a is greater or equal to $-|J_a|$. The number of residues we are taken is equal to the dimension of $\langle J_a \rangle$ minus 1. So we obtain a function of u_a of homogeneous degree greater or equal to $-|J_a| + \dim \langle J_a \rangle - 1$. This means the pole in u_a is less or equal to $|J_a| - \dim \langle J_a \rangle + 1$. \square

Let us consider the MPNS which tree representation is given by Figure 7. Then orders of poles of its nodes are given by Figure 8.

Remark 6.6. *In our program for calculating iterated residues for root systems of type A, we will reorder roots according to the tree order: we take the residue first with respect to the elements $\theta(I_k)$ appearing at the end of*

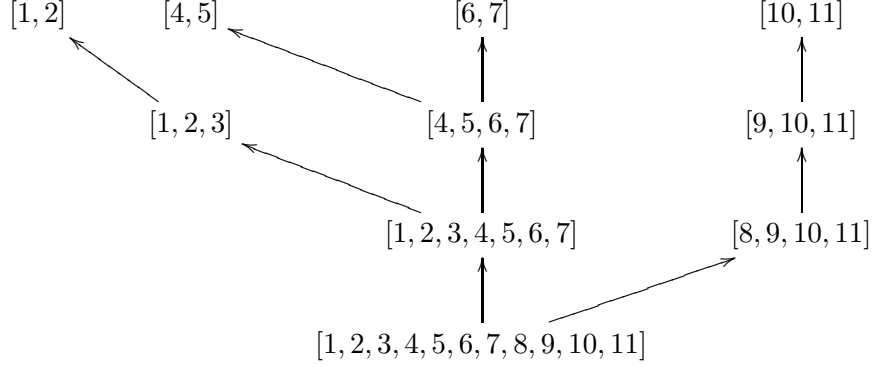
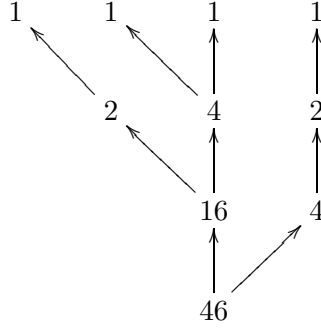
FIGURE 7. Irreducible components of a MPNS in A_{10} 

FIGURE 8. Order of nodes in the tree represented in Figure 7, according to Proposition 6.5

the tree in arbitrary order, and we remove these variables. Then we take the variables appearing at the end of the tree when we have removed these irreducible sets. Here an irreducible set I is indexed by a subset S of $[1, 2, \dots, n]$. A subset S of cardinal 2, for example $[1, 3]$, corresponds to the irreducible set with one element (here $e_1 - e_3$). Thus given a MNS M represented as $M = \{S_1, S_2, \dots, S_r\}$ we will first take the residues with respect to the roots $\theta(I_k)$, for sets S_k of cardinal 2, in arbitrary order, then with respect to irreducible sets associated to sets S_k of cardinal 3, etc. The procedure of ordering roots $\theta(I_k)$ coming from a MNS $M = \{S_j\}$ according to the cardinal of the set S_k is called **OrderThetas**. Furthermore we will at the same time keep track of the order of pole for calculating an iterated residue of a function $\phi = P / \prod_{i=1}^N \alpha_i$ in the procedure **FormalPathAwithorders**.

7. VOLUME AND PARTITION FUNCTION FOR THE SYSTEM A_{n-1}

7.1. The formulae to be implemented. Let E be a n -dimensional vector space with basis e_i ($i = 1, \dots, n$) and consider the set

$$\mathcal{K}_n = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

These are positive roots for a system of type A_{n-1} . The number of elements in \mathcal{K}_n is $N = n(n-1)/2$. Remark that \mathcal{K}_n is also the set of vectors considered in the complete graph with n nodes.

We let V be the vector space generated by the elements in \mathcal{K}_n . Then V has dimension $n-1$ and it is defined by:

$$V = \left\{ v = \sum_{i=1}^n v_i e_i \in E \mid \sum_{i=1}^n v_i = 0 \right\}.$$

In our procedures, a vector v of length n such that $\sum_{i=1}^n v_i = 0$ will be called a *A-vector* and written as $v = [v_1, v_2, \dots, v_n]$. The lattice spanned by \mathcal{K}_n is simply

$$V_{\mathbb{Z}} = \left\{ h = \sum_{i=1}^n h_i e_i \in \mathbb{Z}^n \mid \sum_{i=1}^n h_i = 0 \right\}.$$

It is well known and easy to prove that \mathcal{K}_n is unimodular. The cone $\mathcal{C}(\mathcal{K}_n)$ generated by \mathcal{K}_n is simplicial with generators the $n-1$ simple roots $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$. This cone is described as:

$$\mathcal{C}(\mathcal{K}_n) = \{ \text{A-vector } v = [v_1, v_2, \dots, v_n] \mid v_1 + v_2 + \dots + v_i \geq 0 \text{ for all } i \}.$$

Keep in mind that our vector v satisfies the condition

$$v_1 + v_2 + \dots + v_{n-1} + v_n = 0.$$

We choose on V the measure dh determined by $V_{\mathbb{Z}}$.

Let v be in the cone $\mathcal{C}(\mathcal{K}_n)$. We are interested to compute the volume $\text{vol}_{\mathbb{Z}, \mathcal{K}_n}(v)$ of the polytope

$$\Pi_{\mathcal{K}_n}(v) = \left\{ (x_{\alpha})_{\alpha} \in \mathbb{R}^N \mid x \geq 0, \sum_{\alpha \in \mathcal{K}_n} x_{\alpha} \alpha = v \right\}.$$

If h is a point in V with integral coordinates then we are also interested in computing the number $N_{\mathcal{K}_n}(h)$ of integral points in $\Pi_{\mathcal{K}_n}(h)$.

We apply formulae from Theorem 3.3. Since \mathcal{K}_n is unimodular, the set F can be taken as the identity $F := \{1\}$ (Remark 3.5).

Since V is contained in E , then we have a canonical map $E^* \longrightarrow V^*$ given by restriction. Define $U = V^*$ as in the general setting. We identify U with \mathbb{R}^{n-1} by sending $u \in \mathbb{R}^{n-1}$ to $u = \sum_{i=1}^{n-1} u_i e^i \in E^*$, where e^i is the dual basis to e_i . Thus the root $e_i - e_j$ ($1 \leq i < j < n$) produces the linear function $u_i - u_j$ on U , while the root $e_i - e_n$ produces the linear function u_i .

We now give another definition.

Definition 7.1. Let $v = \sum_{i=1}^n v_i e_i \in V$ be a vector with real coordinates. Let $h = \sum_{i=1}^n h_i e_i \in V$ be a vector with integral coordinates. Then for $u \in U$ define:

•

$$J_A(v)(u) = \frac{e^{\sum_{i=1}^{n-1} u_i v_i}}{\prod_{i=1}^{n-1} u_i \prod_{1 \leq i < j \leq n-1} (u_i - u_j)}.$$

•

$$K_A(h)(u) = \frac{\prod_{i=1}^{n-1} (1 + u_i)^{h_i + n - i}}{\prod_{i=1}^{n-1} u_i \prod_{1 \leq i < j \leq n-1} (u_i - u_j)}.$$

Theorem 7.2. Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{K}_n)$.

- For $v \in \bar{\mathfrak{c}}$, we have

$$\text{vol}_{\mathbb{Z}, \mathcal{K}_n}(v) = JK_{\mathfrak{c}}(J_A(v)).$$

- For $h \in \mathbb{Z}^n \cap \bar{\mathfrak{c}}$, we have

$$N_{\mathcal{K}_n}(h) = JK_{\mathfrak{c}}(K_A(h)).$$

Proof. The first assertion is the general formula.

The function $F(1, h)(u) = e^{\langle h, u \rangle} / \prod_{\alpha \in \mathcal{A}} (1 - e^{-\langle \alpha, u \rangle})$ for the system \mathcal{K}_n is

$$F(1, h)(u) = \frac{e^{\sum_{i=1}^{n-1} u_i v_i}}{\prod_{i=1}^{n-1} (1 - e^{-u_i}) \prod_{1 \leq i < j \leq n-1} (1 - e^{-(u_i - u_j)})}.$$

Remark that the change of variable $1 + z_i = e^{u_i}$ preserves the hyperplanes $u_i = 0$ and $u_i = u_j$. After the change of variable, we get

$$(8) \quad F(1, h)(u) = \frac{\prod_{i=1}^{n-1} (1 + z_i)^{h_i + n - i}}{\prod_{1 \leq i < j \leq n-1} (z_i - z_j) \times \prod_{i=1}^{n-1} z_i}.$$

But $z_i = e^{u_i} - 1$ leads to $dz_i = e^{u_i} du_i = (1 + z_i) du_i$ and hence we obtain the desired exponent $h_i + n - i - 1$ thanks to the formula involving Jacobians in Proposition 3.2. \square

In order to implement these formulae, we first have to describe the set $\mathcal{P}(v, \mathcal{K}_n)$ (Section 7.2), then calculate the iterated residue formulae associated to these paths (Section 7.3). After, we explain how these computations fit together to get a global procedure for Kostant partition function for A_{n-1} (Section 7.4). As a short digression, we will explain how we adapted our program to deal with formal parameters (Section 7.5).

7.2. The search for maximal proper nested sets adapted to a vector. We now look for maximal proper nested sets adapted to a vector following the general method as outlined in Figure 6: we will begin by listing all possible \mathcal{K}_n -admissible hyperplanes.

The usual height function is

$$\text{ht}(v) = \sum_{i=1}^{n-1} (n-i)v_i$$

which takes the value 1 on all the simple roots, and hence the value $j-i$ on $e_i - e_j$. We deform ht slightly in order to have a function taking different values on all roots: If two elements $e_i - e_j$ and $e_k - e_\ell$ are such that $j-i = \ell-k$, we decide that $\text{ht}(e_i - e_j) < \text{ht}(e_k - e_\ell)$ if $i < k$.

If P is a proper subset of $\{1, 2, \dots, n\}$ and v is a A -vector, we denote by $\langle u_P, v \rangle$ the linear form $\sum_{i \in P} v_i$, and by H_P the hyperplane

$$H_P := \{v \in V \mid \langle u_P, v \rangle = 0\}.$$

We will shortly see that all \mathcal{K}_n -admissible hyperplanes are obtained in this way, that is giving a proper subset P of $\{1, \dots, n\}$. Observe that the hyperplane H_P is equal to the hyperplane H_Q determined by the complement Q of P .

We denote by

$$\mathcal{K}(P) := \{e_i - e_j \mid 1 \leq i < j \leq n; i, j \in P\} \subset \mathcal{K}_n.$$

Remark that $\mathcal{K}(P)$ is the positive system of type $A_{|P|-1}$, where the positivity is induced by the lexicographic order.

Lemma 7.3. • *The hyperplane H_P is a \mathcal{K}_n -admissible hyperplane.*
 • *The set $\mathcal{K}_n \cap H_P$ is the union of $\mathcal{K}(P)$ and $\mathcal{K}(Q)$, where Q is the complement of P in $\{1, 2, \dots, n\}$.*
 • *Every \mathcal{K}_n -admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. We prove the third by induction on n , the case $n = 2$ being trivial. Let H be a \mathcal{K}_n -admissible hyperplane. Let α be a root in H . Renumbering the roots, we may assume that $\alpha = e_{n-1} - e_n$. The map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} sends the set $\mathcal{K}_n \setminus \{\alpha\}$ to \mathcal{K}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{K}_{n-1} -admissible hyperplane. It is thus determined by a subset P' of $\{1, 2, \dots, n-1\}$. If P' does not contain $n-1$, the hyperplane H is equal to the hyperplane determined by the subset P' of $\{1, 2, \dots, n-1\}$. If P' does contain $n-1$, then the hyperplane H is equal to the hyperplane determined by $P = P' \cup \{n\}$. \square

We now proceed to the detailed description of our algorithm.

Recall our description of an A -vector as an array $v = [v_1, v_2, \dots, v_n]$ with $\sum_{i=1}^n v_i = 0$. Referring to the Figure 6 we need to check if the vector is in the cone $\mathcal{C}(\mathcal{K}_n)$, that is $\sum_{j=1}^i v_j \geq 0$ for $1 \leq i \leq n-1$. This is done by using the procedure **CheckVector**(v), which gives an answer true or false.

For the system \mathcal{K}_n the highest root θ is equal to

$$\theta = [1, 0, \dots, 0, -1] \in \mathbb{R}^n$$

and computed with the procedure **theta**.

At this point, we need to list all hyperplanes that are in $Hyp(v, \mathcal{K}_n)$. This is done in the procedure **TwoSets**(**v**), that we are about to describe. As explained in Lemma 7.3, each hyperplane is determined by an equation $\sum_{i \in I} a_i = 0$. It therefore produces a set of two lists P, Q , where $P = [i \in I]$ and $Q = [i \notin I \mid 1 \leq i \leq n]$. Note that P and Q are sorted. To verify that such a hyperplane is in $Hyp(v, \mathcal{K}_n)$, we need to test if $\langle u_P, \theta \rangle$ is not zero (θ is not in the hyperplane) and if $\langle u_P, v \rangle \times \langle u_P, \theta \rangle$ is non-negative (θ and v are on the same side of the hyperplane).

Furthermore, the procedure **ProjH**(**v, H**) constructs the vector

$$proj_H(v) = v - \frac{\langle u_P, v \rangle}{\langle u_P, \theta \rangle} \theta$$

that we represent as $\{[v_1, P], [v_2, Q]\}$. Each of the vectors v_1, v_2 is a A -vector (sum of coordinates equal to zero). So the last condition for H being in $Hyp(v, \mathcal{K}_n)$ is that $v_1 \in \mathcal{C}(\mathcal{K}(P))$ and $v_2 \in \mathcal{C}(\mathcal{K}(Q))$.

Hence a hyperplane H is in $Hyp(v, \mathcal{K}_n)$ if it satisfies the series of conditions:

$$\begin{aligned} \langle u_P, \theta \rangle &\neq 0 && \text{with } Hvalue(\theta, P) \neq 0, \\ \langle u_P, v \rangle \times \langle u_P, \theta \rangle &\geq 0 && \text{with } CheckSide(v, P) = \text{true}, \\ v_1 &\in \mathcal{C}(\mathcal{K}_{|P|-1}) && \text{with } CheckVector(v_1) = \text{true}, \\ v_2 &\in \mathcal{C}(\mathcal{K}_{|Q|-1}) && \text{with } CheckVector(v_2) = \text{true}. \end{aligned}$$

The procedure **CheckList**(**v, H**) implements all these sub-routines. It is used in the procedure **TwoSets**(**v**), computing all elements of $Hyp(v, \mathcal{K}_n)$.

We combine **TwoSets** with a procedure named **TwoVector**, to finally get the procedure **TwoVectors**(**v**) determining all hyperplanes in $Hyp(v, \mathcal{K}_n)$ and projections of v on these hyperplanes.

We now have to perform the next step of our algorithm. Let $\{[v_1, K_1], [v_2, K_2]\}$ be the output of **TwoVectors**(**v**). Then we construct the MNSs for $[v_1, K_1]$ and $[v_2, K_2]$, and go on recursively until the procedure stops. These iterated steps are done by the procedure **Splits**.

Finally the procedure **MNSs**(**v**), computing all MNSs for a given vector v , works as follows. We begin by building the first seed of MNSs with the procedure **MNS1**, containing the regularization of the result of **TwoVectors**. We then call repeatedly the procedure **AllNewMNSs**, which performs the regularization of the output of **Splits**.

7.3. Residues associated to maximal proper nested sets. An element M in $\mathcal{P}(v, \mathcal{K}_n)$ is represented as a collection $M = \{K_1, K_2, \dots, K_{n-1}\}$ of $(n-1)$ subsets of $[1, 2, \dots, n]$.

As we have said in Remark 6.6 given a maximal proper nested set $M := \{K_1, K_2, \dots, K_{n-1}\}$, we associate to it an ordered basis $\overrightarrow{\theta(M)}$ of V (procedure **OrderThetas**). If $p = [\alpha_1, \alpha_2, \dots, \alpha_{n-1}]$ is the list of roots singled out by our procedure, then α_1 is an element associated to a set K_i of cardinal 2 and $\alpha_{n-1} = \theta$. We identify the root $e_i - e_n$ to the linear function z_i on \mathbb{C}^{n-1} and the root $e_i - e_j$ to $z_i - z_j$.

Let h be a A -vector with integral coordinates. Let us consider the Kostant function

$$K_A(h)(z_1, z_2, \dots, z_{n-1}) = \frac{\prod_{i=1}^{n-1} (1 + z_i)^{h_i + n - i}}{\prod_{1 \leq i < j \leq n-1} (z_i - z_j) \times \prod_{i=1}^{n-1} z_i}$$

defined in 7.1. To compute $N_{\mathcal{K}_n}(h)$, we will have to compute

$$\text{res}_M \phi := \text{res}_{\alpha_{n-1}=0} \text{res}_{\alpha_2=0} \cdots \text{res}_{\alpha_1=0} \phi$$

with $\phi = K_A(h)$.

Using Proposition 6.5, we know in advance the order of the pole in $\alpha_k = 0$ of the function obtained after taking the first $(k-1)$ residues. These orders are recorded in the procedure **FormalPathAwithOrders**.

If $\alpha_1 = z_i - z_j$, we can replace — after taking the residue at $z_i = z_j$ — the variable z_i by the variable z_j in all the other roots. Thus we get rid of the variable z_i . The procedure **NewR** does produce the ordered path resulting from all these substitutions.

Recursively, we will have to compute the residue at $z_{i_0} - z_{j_0} = 0$ of an expression

$$(9) \quad f = \frac{A(z_i, i \in L)}{\prod_{i,j \in L; i < j} (z_i - z_j)^{m_{i,j}} \prod_{i \in L} z_i^{m_i}},$$

where L is a list of indices taken in $\{1, \dots, n-1\}$. Denote by maxi the order m_{i_0, j_0} of the root $z_{i_0} - z_{j_0}$ (the exponent maxi is recorded in the procedure **FormalPathAwithOrders**). Remark that computing the residue is exactly the same as computing the coefficient of z of degree $\text{maxi}-1$ of the expansion of $f \times (z_{i_0} - z_{j_0})^{\text{maxi}}$ at $z_{i_0} = z + z_{j_0}$. Let us describe in detail the procedure **ComputeRes**, performing this task.

Let $e_j = m_{i_0, j}$ and $e_j = m_{j, i_0}$ for $j \in L \setminus \{i_0, j_0\}$, and $e_0 = m_{i_0}$. Then $f \times (z_{i_0} - z_{j_0})^{\text{maxi}}$ can be written as $g \times h \times R$ where

$$(10) \quad g = \frac{A(z_i, i \in L)}{\prod_{i,j \in L; i, j \neq i_0; i < j} (z_i - z_j)^{m_{i,j}}} \times \frac{1}{\prod_{i \in L; i \neq i_0} z_i^{m_i}},$$

$$(11) \quad h = \frac{1}{\prod_{j \in L; j \neq i_0, j_0} (z_j - z_{i_0})^{e_j}} \times \frac{1}{z_{i_0}^{e_{i_0}}},$$

$$(12) \quad R = (-1)^{\sum_{j > i_0} e_j}.$$

Let $z_{i_0} = z + z_{j_0}$. So to get the desired residue, we need to calculate the expansion of g and h at $z = 0$. More precisely if $g = \sum_{i=0}^{maxi-1} g_i z^i$ and $h = \sum_{i=0}^{maxi-1} h_i z^i$, then the coefficient of degree $maxi-1$ of $f \times (z_{i_0} - z_{j_0})^{maxi}$ is simply $R \times \sum_{i=0}^{maxi-1} g_i h_{maxi-1-i}$. Let us describe how the procedure **ComputeRes** performs this task.

Rewrite the fraction h defined in Equation (11) as $B \times \tilde{h}$, where

$$\begin{aligned} B &= \frac{1}{\prod_{j \in L; j \neq i_0, j_0} (z_j - z_{j_0})^{e_j}} \times \frac{1}{z_{j_0}^{e_0}}, \\ \tilde{h} &= \frac{1}{\prod_{j \in L; j \neq i_0, j_0} \left(1 - \frac{z}{z_j - z_{j_0}}\right)^{e_j}} \times \frac{1}{\left(1 + \frac{z}{z_{j_0}}\right)^{e_0}}. \end{aligned}$$

Consequently to expand h we only need to expand \tilde{h} . This is done in the procedure **CoeffBin** using the binomial coefficients.

In the procedure **CoeffFun** we calculate the expansion at $z_{i_0} - z_{j_0} = 0$ of the fraction

$$\begin{aligned} (13) \quad & f \times (z_{i_0} - z_{j_0})^{maxi} \times z_{i_0}^{e_0} \times \prod_{j \in L; j \neq i_0, j_0} (z_{i_0} - z_j)^{e_j} \\ &= g \times R \times (-1)^{\sum_{j \neq i_0, j_0} e_j}. \end{aligned}$$

Finally the procedure **ComputeRes** performs the sum over i ranging from 0 to $maxi$ of

$$\begin{aligned} & (S := (-1)^{\sum_{j \neq i_0, j_0} e_j}) \times B \\ & \times (\text{the component of degree } i \text{ of } \mathbf{CoeffFun}) \\ & \times (\text{the component of degree } maxi - 1 - i \text{ of } \mathbf{CoeffBin}). \end{aligned}$$

Rewrite this as the sum over i of

$$\begin{aligned} & R \\ & \times (\text{the component of degree } i \text{ of } \mathbf{CoeffFun}) \times R \times S \\ & \times B \times (\text{the component of degree } maxi - 1 - i \text{ of } \mathbf{CoeffBin}), \end{aligned}$$

or equivalently as $\sum_{i=0}^{maxi-1} R \times g_i \times h_{maxi-1-i}$: this is exactly the desired coefficient.

Remark 7.4. For residues along roots of type z_{i_0} instead of $z_{i_0} - z_{j_0}$ the procedure **ComputeRes** also calls procedures **srCoeffFun** and **srCoeffBin**, similar to **CoeffFun** and **CoeffBin**.

7.4. The procedure MNS_KostantA. We finish the Section dedicated to A_{n-1} by giving the global outline of the procedure **MNS_KostantA(v)** computing the Kostant partition number of a vector v lying in the root lattice. We begin by slightly deforming v so that it lies on no admissible hyperplanes, with the command $v' := \mathbf{DefVector}(v, n)$. We compute all MPNSs for v' with the procedure **MNSs(v')**.

Given such a MPNS $M = \{S_k\}$, we extract highest roots of its irreducible components with the call $R := \text{ThetaMNS}(M)$. We obtain a set R where each element of R is a root represented as $[i, j]$ together with the cardinal of the set S_k it comes from. We then transform this set R into a path p keeping track of order of poles by setting $p := \text{FormalPathAwithOrders}(R)$.

Finally we compute the residue associated to this path with the command $\text{OneIteratedResidue}(p, v, n)$. Summing all these residues over the set of MNSs, we obtain thanks to Theorem 7.2 the desired partition number for v .

Let us describe in detail the procedure $\text{OneIteratedResidue}(p, v, n)$ computing the iterated residue along a path p for a vector v lying in the root lattice for A_{n-1} . We first compute Kostant fraction (second item of Definition 7.1, procedure KostantFunctionA). After, we replace in the path all roots $z_i - z_n$ by z_i (with Kpath). We also build an upper bound for the orders of the roots m_{ij} (with Multiplicity). Keep in mind that the exponent $maxi$ needed in the residue calculation is computed a priori, and our computation seems quite optimal.

Then we compute iteratively the residues, using the procedure ComputeRes (Section 7.3). Remark that at each step we have to update the list of orders (with MultRoots) and the list of remaining variables (with ListOfVariables).

7.5. Parametrized version of the algorithm. Our algorithm can work with formal parameters, only needing slight modifications of procedures. This has been implemented in a program named Para.KostantA.MNS . We are then able to compute directly the polynomial $h \mapsto N_{\mathcal{K}_n}(h)$ giving the number of integral points in the polytope $\Pi_{\mathcal{K}_n}(h)$, on the chamber determined by h . As a consequence we can easily get the Ehrhart polynomial $t \mapsto N_{\mathcal{K}_n}(th_1, \dots, th_n)$ of the polytope. See [7].

Now let us outline how works this modified program. Given an element $h = (h_1, \dots, h_n)$ of the root lattice for A_{n-1} , we want to compute the Kostant partition function for the vector (h_1, \dots, h_n) , when h varies in $\mathbb{Z}^n \cap \bar{\epsilon}$.

Recall that we have to perform the residue at $z_{i_0} - z_{j_0} = 0$ of the fraction defined in Equations (8) and (9).

Remark that the numerator of the fraction $K_a(h)$ contains terms of the form $(1 + z_i)^{h_i + n - i}$ that we must formally expand.

For any parameter b , we write at $z_{i_0} = z + z_{j_0}$,

$$(1 + z_{i_0})^b = (1 + z_{j_0})^b \left(1 + \sum_{j=1}^{maxi-1} \binom{b}{j} \left(\frac{z}{1 + z_{j_0}} \right)^j \right) + O(z^{maxi}).$$

Hence, for any parameter b , the procedure CoeffFun (see Equation (13)) now computes the expansion at $z_{i_0} - z_{j_0} = 0$ of the fraction

$$\begin{aligned}
f &\times \left(z_{i_0}^{e_0} \times \prod_{j \in L; j \neq i_0, j_0} (z_{i_0} - z_j)^{e_j} \right) \\
&\times \left(1 + \sum_{j=1}^{maxi-1} \binom{b}{j} \left(\frac{z_{i_0} - z_{j_0}}{1 + z_{j_0}} \right)^j \right) \times (1 + z_{j_0})^{maxi-1}.
\end{aligned}$$

For the residue along a root of type z_{i_0} instead of $z_{i_0} - z_{j_0}$, the procedure `srCoeffFun` has been modified in a similar way.

8. THE TYPE B_n

8.1. The formulae to be implemented. Consider a vector space V with basis e_1, e_2, \dots, e_n . We choose on V the standard Lebesgue measure dh .

Let

$$\mathcal{B}_n = \{e_i \mid 1 \leq i \leq n\} \cup \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

Then \mathcal{B}_n is a positive roots system of type B_n and generates V .

We denote by U the dual of V . The lattice $V_{\mathbb{Z}}$ generated by roots is equal to \mathbb{Z}^n .

The cone $\mathcal{C}(\mathcal{B}_n)$ is simplicial and spanned by the n simple roots $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n$. A vector $v = [v_1, v_2, \dots, v_n]$ is in $\mathcal{C}(\mathcal{B}_n)$ if and only if it satisfies the inequations $v_1 + \dots + v_i \geq 0$ for all $i = 1, \dots, n$.

Let v be in the cone $\mathcal{C}(\mathcal{B}_n)$. Consider the polytope

$$\Pi_{\mathcal{B}_n}(v) = \left\{ (x_{\alpha})_{\alpha} \geq 0 \mid \sum_{\alpha \in \mathcal{B}_n} x_{\alpha} \alpha = v \right\}.$$

If h is a point in V with integral coordinates, we are interested in computing the number $N_{\mathcal{B}_n}(h)$ of integral points in $\Pi_{\mathcal{B}_n}(h)$.

Let $U_{\mathbb{Z}}$ be the lattice dual to $V_{\mathbb{Z}}$. We identify the torus $T = U/U_{\mathbb{Z}} = \mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$ by

$$(u_1, u_2, \dots, u_n) \mapsto (e^{2\pi\sqrt{-1}u_1}, \dots, e^{2\pi\sqrt{-1}u_n}).$$

If G is a representative of $g = (g_1, g_2, \dots, g_n) \in T$, and $h = \sum_{i=1}^n h_i e_i$ in $V_{\mathbb{Z}}$, then $e^{\langle h, 2\pi\sqrt{-1}G \rangle}$ is equal to $\prod_{i=1}^n g_i^{h_i} = g^h$.

As the set \mathcal{B}_n is not unimodular, the sets $T(\sigma)$ are not reduced to 1.

Example 8.1. Let σ be the basic set $\{e_1 + e_2, e_1 - e_2\}$ for B_2 . Then $T(\sigma) = \{(1, 1), (-1, -1)\}$.

We now determine a set F containing all sets $T(\sigma)$.

Lemma 8.2. Let σ be a basic subset of \mathcal{B}_n . Assume $g \in T(\sigma)$. Then all the coordinates of g are equal to ± 1 . Furthermore, if g is not 1, there are at least two coordinates of g which are equal to -1 .

Proof. We prove this by induction on n . For \mathcal{B}_2 , we have seen this by direct computation.

Let σ be a basic subset of \mathcal{B}_n . Assume first that σ contains a root e_i . Up to renumbering, we may assume that this root is e_n . Then the basis σ produces a basis σ' of \mathcal{B}_{n-1} by putting $e_n = 0$. Let $g = (g_1, g_2, \dots, g_n)$ in $T(\sigma)$. We see that $g' = (g_1, g_2, \dots, g_{n-1})$ is in $T(\sigma')$. Thus, by induction the first $n-1$ coordinates of g' are equal to ± 1 . But since e_n is in σ we get $1 = g_n$. Remark that $g \neq 1$ if and only if $g' \neq 1$, hence by induction hypothesis g' has at least two coordinates not equal to 1.

Consider now the case where σ does not contain any root e_i . Up to renumbering, it contains a root $e_{n-1} - e_n$ or $e_{n-1} + e_n$.

Let us examine first the case where σ contains the root $\alpha = e_{n-1} - e_n$. Let $g = (g_1, g_2, \dots, g_{n-1}, g_n)$ in $T(\sigma)$. This implies $g_{n-1} = g_n$. Consider the map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} . Then q sends $\sigma \setminus \{e_{n-1} - e_n\}$ to a basis σ' of \mathcal{B}_{n-1} . The element $g' = (g_1, g_2, \dots, g_{n-1})$ is easily seen to belong to $T(\sigma')$. Indeed if α equals $e_i \pm e_j$ with $1 \leq i < j < n$, this is by definition. On the other hand $q(e_i \pm e_n) = e_i \pm e_{n-1}$ and $g_{n-1} = g_n$ imply that $g_i g_{n-1}^{\pm 1}$ coincides with the value of $g_i g_n^{\pm 1}$. By induction hypothesis, all coordinates of g' are equal to ± 1 . Moreover $g \neq 1$ if and only if $g' \neq 1$, so that g is of the wanted form.

Finally, the same argument works if σ contains $\alpha = e_{n-1} + e_n$ by considering the map q sending e_i to e_i if $i < n$ and e_n to $-e_{n-1}$. \square

Definition 8.3. If I is a subset of $\{1, 2, \dots, n\}$ with at least two elements, we consider the set $F(I) := \{(g_1, g_2, \dots, g_n) \mid g_i = -1, i \in I; g_j = 1, j \notin I\}$.

We define $F \subset T$ to be the finite subset of T union of such sets $F(I)$ together with the identity $(1, 1, \dots, 1)$.

Let $v = \sum_{i=1}^n v_i e_i \in V$ be a vector with real coordinates and $h = \sum_{i=1}^n h_i e_i \in V$ a vector with integral coordinates. We will compute the normalized volume of $\Pi_{\mathcal{B}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{B}_n}(h)$ using Theorem 3.3.

Thus we introduce the function $J_B(v)$ on U defined by:

$$J_B(v)(u) = \frac{e^{\sum_{i=1}^n u_i v_i}}{\prod_{i=1}^n u_i \prod_{1 \leq i < j \leq n} (u_i - u_j) \prod_{1 \leq i < j \leq n} (u_i + u_j)}.$$

For $g = (g_1, g_2, \dots, g_n) \in F$ and $h \in V_{\mathbb{Z}} \cap \mathcal{C}(\mathcal{B}_n)$ the Kostant fraction (3) is the function on U defined by:

$$\begin{aligned} q_B(g, h)(u) &= \frac{\prod_{i=1}^n g_i^{h_i} e^{\sum_{i=1}^n u_i h_i}}{\prod_{i=1}^n (1 - g_i^{-1} e^{-u_i}) \times \prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j e^{-(u_i - u_j)})} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j^{-1} e^{-(u_i + u_j)})}. \end{aligned}$$

We have then

Theorem 8.4. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{B}_n)$.*

- *For any $v \in \bar{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{B}_n}(v) = JK_{\mathfrak{c}}(J_B(v)).$$

- *For any $h \in V_{\mathbb{Z}} \cap \bar{\mathfrak{c}}$, the value of the partition function is given by:*

$$N_{\mathcal{B}_n}(h) = \sum_{g \in F} JK_{\mathfrak{c}}(q_B(g, h)).$$

As in the case of A_n , we will use the change of variable $1 + z_i = e^{u_i}$ to compute more easily $N_{\mathcal{B}_n}(h)$. However, let us note that this transformation does not leave the hyperplane $u_i + u_j = 0$ fixed. This hypersurface is transformed into the hypersurface $z_i + z_j + z_i z_j = 0$. So we use the expression of $JK_{\mathfrak{c}}$ as an integral over the cycle $H(\mathfrak{c})$ defined in Theorem 4.13. This cycle is stable by the transformation $e^{u_i} = 1 + z_i$ which is close to the identity.

Thus define the following function on U :

$$\begin{aligned} \mathcal{K}_B(g, h)(z) &= \frac{\prod_{i=1}^n (1 + z_i)^{h_i + 2n - i - 1} \times \prod_{i=1}^n g_i^{h_i}}{\prod_{i=1}^n (1 + z_i - g_i) \times \prod_{1 \leq i < j \leq n} (1 + z_i - g_i g_j (1 + z_j))} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 + z_i)(1 + z_j) - g_i g_j}. \end{aligned}$$

Performing the change of variables $e^{u_i} = 1 + z_i$ on the function $q_B(g, h)(u)$ and computing the Jacobian, the Theorem 3.3 becomes:

Theorem 8.5. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{B}_n)$.*

- *For any $v \in \bar{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{B}_n}(v) = JK_{\mathfrak{c}}(J_B(v)).$$

- *For any $h \in V_{\mathbb{Z}} \cap \bar{\mathfrak{c}}$, the value of the partition function is given by:*

$$N_{\mathcal{B}_n}(h) = \sum_{g \in F} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{K}_B(g, h)(z) dz.$$

As in the case of type A , in order to implement these formulae we first have to describe the set $\mathcal{P}(v, \mathcal{B}_n)$ (Section 8.2), then we will explain how the integral over the cycle $H(\mathfrak{c})$ is calculated similarly as to an iterated residue formula associated to these paths (Section 8.3), using an estimate of the order of poles. Finally we explain how these computations fit together to get a global procedure for Kostant partition function for B_n (Section 8.4).

8.2. The search for maximal proper nested sets. A height function is

$$\text{ht}(v) = \sum_{i=1}^n (n + 1 - i) v_i$$

which takes value 1 on all simple roots. We will deform ht later on in order to have a function taking different values on roots.

We now proceed to describe hyperplanes for \mathcal{B}_n . If $P = [P^+, P^-]$ are two disjoint subsets of $\{1, 2, \dots, n\}$, we denote by $\langle u_P, v \rangle$ the linear form $\sum_{i \in P^+} v_i - \sum_{j \in P^-} v_j$. Consider the hyperplane

$$H_P = \{v \in V, \langle u_P, v \rangle = 0\}$$

in V . It is equal to the hyperplane determined by the reverse list $[P^-, P^+]$. Thus to each set $P = \{P^+, P^-\}$ of two disjoint sets P^+, P^- such that at least one is non empty, we associate a hyperplane H_P .

We denote by Z the complement of $P^+ \cup P^-$ in $\{1, 2, \dots, n\}$ and by $\mathcal{B}(Z)$ the subset of \mathcal{B}_n defined by

$$\mathcal{B}(Z) = \{e_i \mid i \in Z\} \cup \{e_i \pm e_j \mid 1 \leq i < j \leq n; i, j \in Z\}.$$

This is the positive roots system of type $B_{|Z|}$, with the positivity induced by the lexicographic order.

Let $\mathcal{K}(P^+, P^-)$ be the subset of \mathcal{B}_n defined by

$$\begin{aligned} & \{e_i - e_j \mid 1 \leq i < j \leq n; i, j \in P^+\} \\ \cup & \{e_i + e_k \mid i \in P^+, k \in P^-\} \\ \cup & \{e_k - e_\ell \mid 1 \leq k < \ell \leq n; k, \ell \in P^-\}. \end{aligned}$$

Remark that by defining $f_i = e_i$ if $i \in P^+$ and $f_k = -e_{|P^-| - k + 1}$ if $k \in P^-$, the set $\mathcal{K}(P^+, P^-)$ coincides with

$$\begin{aligned} & \{f_i - f_j \mid 1 \leq i < j \leq n; i, j \in P^+\} \\ \cup & \{f_i - f_k \mid i \in P^+, k \in P^-\} \\ \cup & \{f_k - f_\ell \mid 1 \leq k < \ell \leq n; k, \ell \in P^-\}. \end{aligned}$$

Thus the set $\mathcal{K}(P^+, P^-)$ is a positive roots system of type $A_{|P^+| + |P^-| - 1}$. However the positivity is induced by the lexicographic order on P^+ and the reverse lexicographic order on P^- .

Observe also that H_P is the vector space spanned by $\mathcal{K}(P^+, P^-) \cup \mathcal{B}(Z)$.

Lemma 8.6. • *The hyperplane H_P is a \mathcal{B}_n -admissible hyperplane.*

- *The set $\mathcal{B}_n \cap H_P$ is the union of $\mathcal{B}(Z)$ and $\mathcal{K}(P^+, P^-)$.*
- *Every \mathcal{B}_n -admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. We prove the third assumption by induction on n , the case $n = 2$ being trivial. Let H be a \mathcal{B}_n -admissible hyperplane and let α be a root in H . There are 3 possibilities for α : up to renumbering roots, we can consider the cases $\alpha = e_n$, $\alpha = e_{n-1} - e_n$ and $\alpha = e_{n-1} + e_n$.

In the first case, the map q sending e_i to e_i if $i < n$ and e_n to 0 maps the set $\mathcal{B}_n \setminus \{\alpha\}$ to \mathcal{B}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{B}_{n-1} -admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$, where P'^+ and P'^-

are two disjoint sets contained in $\{1, 2, \dots, n-1\}$. Then the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$.

In the second case, the map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} sends the set $\mathcal{B}_n \setminus \{\alpha\}$ to \mathcal{B}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{B}_{n-1} -admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$. If neither P'^+ nor P'^- contain $n-1$, the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Otherwise assume that for example P'^+ contains $n-1$. Then the hyperplane H is equal to the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+ \cup \{n\}$ and $P^- = P'^-$.

In the third case, the map q sending e_i to e_i if $i < n$ and e_n to $-e_{n-1}$ sends the set $\mathcal{B}_n \setminus \{\alpha\}$ to \mathcal{B}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{B}_{n-1} -admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$. If neither P'^+ nor P'^- contains $n-1$, the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Assume that P'^+ contains $n-1$. Then the hyperplane H is equal to the the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+$ and $P^- = P'^- \cup \{n\}$. \square

We now give a detailed description of our algorithm computing maximal nested sets.

We describe a vector as an array $v = [v_1, v_2, \dots, v_n]$. To check if v is in the cone $\mathcal{C}(\mathcal{B}_n)$, we need to verify if $\sum_{j=1}^i v_j \geq 0$ for $1 \leq i \leq n$. This is done by the procedure **CheckBvector**, which returns the answer true or false.

For the system \mathcal{B}_n the highest root $\theta^B(n)$ is equal to

$$\theta^B(n) = [1, 1, 0, 0, 0, \dots, 0].$$

We recall here that P is divided in two sets $P^+ \cup P^-$, one of them being non empty. The first task is to list the hyperplanes in $\text{Hyp}(v, \mathcal{B}_n)$. This set of hyperplanes is obtained by the command line **AllPossibleBwalls**(v). The input of this procedure is the vector v . The output is a set of elements $P = \{P^+, P^-\}$, where $P^+ = [i_1, i_2, \dots, i_p]$ and $P^- = [j_1, j_2, \dots, j_q]$ are two ordered disjoint lists made from indices taken in $\{1, \dots, n\}$, with at least one of P^+ or P^- being non empty. Let $\langle u_P, v \rangle = \sum_{i \in P^+} v_i - \sum_{j \in P^-} v_j$ be the normal vector to H_P . Then as stated in Lemma 5.3 we need to test if $\langle u_P, \theta^B(n) \rangle$ is not zero and if $\langle u_P, v \rangle \times \langle u_P, \theta^B(n) \rangle$ is non-negative.

We therefore construct the vector

$$\text{proj}_H(v) = v - \frac{\langle u_P, v \rangle}{\langle u_P, \theta^B(n) \rangle} \theta^B(n).$$

This vector is represented as $\{[v_1, P^+], [v_2, P^-], [w, Z]\}$. The sum of coordinates of v_1 is equal to the sum of the coordinates of v_2 . Now Z is the ordered list $[k_1, k_2, \dots, k_\ell]$ of complementary indices to $P^+ \cup P^-$ and

$$w = [\text{proj}_H(v)[k_1], \dots, \text{proj}_H(v)[k_\ell]].$$

Remark that equations of the cone $\mathcal{C}(\mathcal{K}(P^+, P^-))$ can be given on the convenient form $v_1 \oplus v_2 \in \mathcal{C}(\mathcal{K}(P^+, P^-))$ if and only if **CheckBvector**(v_1)

and `CheckBvector`(\mathbf{v}_2) are true. Equations of the cone $\mathcal{C}(\mathcal{B}(Z))$ are given in the form $w \in \mathcal{C}(\mathcal{B}(Z))$ if and only if `CheckBvector`(\mathbf{w}) is true.

Thus the condition that H is in $Hyp(v, \mathcal{B}_n)$ is equivalent to the series of conditions:

$$\begin{aligned} \langle u_P, \theta^B(n) \rangle &\neq 0, \\ \langle u_P, v \rangle \times \langle u_P, \theta^B(n) \rangle &\geq 0, \\ \text{CheckBvector}(\mathbf{v}_1) &= \text{true}, \\ \text{CheckBvector}(\mathbf{v}_2) &= \text{true}, \\ \text{CheckBvector}(\mathbf{w}) &= \text{true}. \end{aligned}$$

Those five conditions are checked by the command line `CheckBwall(v,H)`, that gives an answer true or false.

Remark 8.7. *Of course, we can first construct all disjoint subsets P^+ , P^- of $\{1, 2, \dots, n\}$ and test these 5 conditions successively on all of them. However it is highly desirable to throw away a priori a great number of these partitions by noticing the following restrictive conditions on the possible lists to be considered.*

Let $\{P^+, P^-\} = \{[i_1, i_2, \dots, i_p], [j_1, j_2, \dots, j_q]\}$ be a set of two disjoint subset of $\{1, 2, \dots, n\}$ represented as lists with strictly increasing indices. Let $Z = [k_1, k_2, \dots, k_\ell]$ be the list of complementary indices to $P^+ \cup P^-$ in $\{1, \dots, n\}$. The following linear forms are positive on the cone $\mathcal{C}(\mathcal{K}(P^+, P^-))$ generated by $\mathcal{K}(P^+)$ and $\mathcal{K}(P^-)$:

$$\begin{aligned} v_{i_1} + v_{i_2} + \dots + v_{i_s} &\geq 0 && \text{for all } 1 \leq s \leq p, \\ v_{j_1} + v_{j_2} + \dots + v_{j_t} &\geq 0 && \text{for all } 1 \leq t \leq q, \end{aligned}$$

as well as

$$v_{k_1} + v_{k_2} + \dots + v_{k_s} \geq 0 \quad \text{for all } 1 \leq s \leq \ell.$$

Remark that all the above linear forms take positive values on $\theta^B(n)$. We employ Lemma 8.6. Thus if $v[i_1] < 0$, the index i_1 cannot start the list P^+ of an element $\{P^+, P^-\}$ in `AllPossibleBwalls`(\mathbf{v}) and we reject all such $\{P^+, P^-\}$.

Similarly assume that we have constructed a list of indices $[i_1, i_2]$ satisfying conditions $v[i_1] \geq 0$ and $v[i_1] + v[i_2] \geq 0$. Then if $v[i_1] + v[i_2] + v[i_3] < 0$, a list starting with $[i_1, i_2, i_3]$ cannot be the first three indices of the component P^+ of an element $\{P^+, P^-\}$ in the set `AllPossibleBwalls`(\mathbf{v}) and we skip it right away.

This achieves the description of the procedure **AllPossibleBwalls**. We now have to perform the next step of our algorithm.

As for type A we build MNSs iteratively. At each step we get a set of partial MNSs, to which we will apply recursively our algorithm.

Remark that after Lemma 8.6 the intersection of a \mathcal{B}_n -admissible hyperplane H_P with \mathcal{B}_n is the union of a system of type A and a system of type B .

The part of the MNS coming from the subsystem of type A is computed with the procedure **AddAnests**. It performs a reordering of the result of a call to the procedure **MNSs** described in Section 7.2.

The part of the MNS coming from the subsystem of type B is computed with the procedure **Bsplits**, calling the previously described procedure **AllPossibleBwalls**.

Procedures **AddAnests** and **Bsplits** are enclosed in **MoreNSs**, thus giving a new iteration of the process. After regularization of the result we hence get a procedure named **AllNewNSs**, performing a new step in the building of MNSs.

Finally the procedure **B_MNSs**, computing MNSs for a given vector v for type B , is the following. First, we use a procedure named **B_NS1** to calculate the first seed of all MNSs. After, repeated calls to the procedure **AllNewNSs** build the desired MNSs.

8.3. Residues associated to maximal proper nested sets. A proper maximal nested set M gives rise to an ordered basis α_i , and a cycle $H(M)$. We need to compute

$$\int_{H(M)} \mathcal{K}_B(g, h)(z) dz$$

where

$$H(M) := \{z, |\langle \alpha_i, z \rangle| = \epsilon_i\}.$$

The function $z \mapsto \mathcal{K}_B(g, h)(z)$ is deduced from the function $q_B(g, h)(u)$ in the space \widehat{R}_A by the change of variable $e^{u_i} = 1 + z_i$. Thus its denominator is a product of factors, either of the form z_i corresponding to the root u_i , either of the form $z_i - z_j$ corresponding to the root $u_i - u_j$ or $z_i + z_j + z_i z_j$ corresponding to the root $u_i + u_j$. We denote by $u(z)$ the point with coordinates u_i satisfying $e^{u_i} = 1 + z_i$.

We start integrating our function $\mathcal{K}_B(g, h)(z)$ over the smaller circle $|\langle \alpha_1, z \rangle| = \epsilon_1$ keeping the other variables fixed. By our condition on the cycle, the function we integrate has poles on the domain $|\langle \alpha_1, z \rangle| \leq \epsilon_1$ only when $\alpha_1(u(z)) = 0$. If $\alpha_1(u(z)) = u_i - u_j$ or $\alpha_1(u(z)) = u_i$, the poles are obtained for $z_i = z_j$ or $z_i = 0$. If $\alpha_1(u(z)) = u_i + u_j$, the pole on the domain $|\langle \alpha_1, z \rangle| \leq \epsilon_1$ is obtained for $z_i = -z_j/(1 + z_j)$. Thus we compute the integral over the circle by the residue theorem in one variable, and proceed. From the general theory, the poles of the function we obtain, replacing z_i

by one of the value above are again of the same form with respect to the remaining variables, as easily checked.

As in case A_{n-1} , for a root $\alpha = u_i$ (resp. $\alpha = u_i \pm u_j$) we can replace after taking the residue at $\alpha = 0$ the variable z_i by 0 (resp. by $\mp z_j$) in all other roots. Thus we get rid of the variable z_i . The procedure **FormalPathB** does produce the ordered path resulting from all these substitutions.

In the case of B we compute the residue, by directly checking the order of the pole at $\alpha = 0$, and then using differentiation. In the near future, we will complete the program in a version with parameters. The function will be locally polynomial with polynomial coefficients depending of the parity of the integers h_i .

8.4. The procedure MNS_KostantB. We finish the Section dedicated to B_n by giving the global outline of the procedure **MNS_KostantB(v)** computing the Kostant partition number of a vector v lying in the root lattice of B_n . We begin by slightly deforming v so that it lies on no wall, by setting $v' := \text{DefVectorB}(v, n)$. We then compute all MNSs for v' with the call **B_MNSs(v')** (Section 8.2). For every MNS M , we extract the list R of highest roots of its irreducible components by setting $R := \text{BthetaMNS}(M)$. We sort these roots by their height with the command line $R' := \text{BorderThetas}(R, n)$.

Then for every g in F we do the following. We transform the list of roots R' into a path p by setting $p := \text{FormalPathB}(R')$.

Remark that our procedures are designed to take residues along positive roots, using the fact that $\text{res}_{-\alpha} = -\text{res}_\alpha$ for any root α . The sign that appears (more precisely -1 to the power the number of negative roots in the path p) is computed with the procedure **PathSign(p, n)**.

The iterated residue along the path p is then obtained by the command line **OneIteratedBresidue(p, g, v, n)**. Let us briefly describe its implementation. We first compute Kostant fraction (second item of Definition 7.1, procedure **KostantFunctionB**). Then for every root of the path we apply the procedure **ComputeOneResidue** (Section 8.3) and update the order of the pole with a procedure named **OrderPoleB**.

Finally summing all products $\text{PathSign}(p, n) \times \text{OneIteratedBresidue}(p, g, v, n)$ over the sets of g 's and of M 's, we get the desired result. ■

Remark 8.8. Let us fix a list $R' = [\alpha_1, \dots, \alpha_n]$ of ordered roots coming from a MNS, and an element g . We say that R' and g are compatible if the following condition is satisfied. If indices of monomial(s) of α_k have not yet occurred among indices of roots α_ℓ with $\ell < k$, then g must satisfy $g^{\alpha_k} = 1$ (that is $g_i g_j^{\pm 1} = 1$ if $\alpha_k = e_i \pm e_j$ and $g_i = 1$ if $\alpha_k = e_i$). Remark that the iterated residue for g and for the path p associated to R' is zero if g and R' are not compatible. Hence summing only over g 's that are compatible with a given list R' saves useless computations. The check of compatibility is performed by the procedure **ListAndGAreCompatible(R', g, n)**.

9. THE TYPE C_n

Consider a vector space V with basis e_1, e_2, \dots, e_n . We choose on V the standard Lebesgue measure dh .

Let

$$\mathcal{C}_n = \{2e_i \mid 1 \leq i \leq n\} \cup \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

Then \mathcal{C}_n is a positive roots system of type C_n , and generates V . Remark that elements of \mathcal{C}_n and \mathcal{B}_n are proportional, so they determine the same hyperplane arrangement and same chambers.

Let L be the lattice defined by $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \oplus \mathbb{Z}e_n$. We remark that the lattice $V_{\mathbb{Z}}$ generated by \mathcal{C}_n is the sublattice of index 2 in L consisting of all elements $v = [v_1, v_2, \dots, v_n]$ with integral coordinates and such that the sum $\sum_{i=1}^n v_i$ is an even integer. A \mathbb{Z} -basis of $V_{\mathbb{Z}}$ is for example

$$\mathbb{Z}(e_1 - e_n) \oplus \mathbb{Z}(e_2 - e_n) \oplus \dots \oplus \mathbb{Z}(e_{n-1} - e_n) \oplus \mathbb{Z}(2e_n).$$

The dual lattice $U_{\mathbb{Z}}$ is the lattice of vectors $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ such that γ_i are half integers and such that $\gamma_i + \gamma_j$ is an integer for all i, j . The set $U_{\mathbb{Z}}/\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$ is of cardinal 2 with representative elements $(0, 0, \dots, 0, 0)$ and $(1/2, \dots, 1/2)$.

As before, we identify the torus $\tilde{T} = U/(\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n) = \mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$ by

$$(u_1, u_2, \dots, u_n) \mapsto (e^{2\pi\sqrt{-1}u_1}, \dots, e^{2\pi\sqrt{-1}u_n}).$$

Then

$$T = \tilde{T}/\{\pm 1\} = U/U_{\mathbb{Z}}.$$

Let G be a representative of $g = (g_1, g_2, \dots, g_n) \in \tilde{T}$ and $h = \sum_{i=1}^n h_i e_i$ in $V_{\mathbb{Z}}$. Then $e^{\langle h, 2\pi\sqrt{-1}G \rangle}$ is equal to $\prod_{i=1}^n g_i^{h_i} = g^h$. This function is well defined on $T = \tilde{T}/\{\pm 1\}$ since $\sum_{i=1}^n h_i$ is even.

For σ a basic subset of \mathcal{C}_n , define

$$\tilde{T}(\sigma) = \left\{ g \in \tilde{T} \mid e^{\langle \alpha, 2\pi\sqrt{-1}G \rangle} = 1 \text{ for all } \alpha \in \sigma \right\}.$$

As the set \mathcal{C}_n is not unimodular, sets $\tilde{T}(\sigma)$ are not reduced to 1.

Lemma 9.1. *Let σ be a basic subset of \mathcal{C}_n . Then $\tilde{T}(\sigma) \subset \{\pm 1\}^n$.*

Proof. We prove by induction on n that if σ is basic then the condition $g = (g_1, \dots, g_n) \in \tilde{T}(\sigma)$ forces $g_i^2 = 1$ ($1 \leq i \leq n$). In other words $g^\alpha = 1$ for all long roots α . If so then $g_i = \pm 1$ for all i . The base of the induction, that is \mathcal{C}_2 , is straightforward and we omit it. We thus proceed considering various possibilities for our σ .

If there exists a long root in σ we may assume that this long root is $2e_n$. We embed the system \mathcal{C}_{n-1} in \mathcal{C}_n via the first $(n-1)$ coordinates. Then the basis σ of \mathcal{C}_n produces a basis σ' of \mathcal{C}_{n-1} consisting of roots $\{e_i \pm e_j \in \sigma \mid 1 \leq i < j \leq n-1\}$, of roots $\{2e_i \in \sigma \mid 1 \leq i \leq n-1\}$ and of roots

$\{2e_i \mid e_i \pm e_n \in \sigma; i \neq n\}$. It is easy to see that elements $(g_1, g_2, \dots, g_{n-1})$ are in $\tilde{T}(\sigma')$. Indeed $g_i^2 = 1$ if $e_i \pm e_n \in \sigma$ as $g_i g_n^{\pm 1} = 1$ and $g_n^2 = 1$; and similarly $g_i^2 = 1$ if $2e_i \in \sigma$. Thus by induction we obtain $g_i^2 = 1$ for every i .

Now assume that there is no long root in σ . We may assume that there is a root of the form $e_{n-1} - e_n$ or $e_{n-1} + e_n$.

In the first case, consider the basis σ' of \mathcal{C}_{n-1} consisting of roots $\{e_i \pm e_j \in \sigma \mid 1 \leq i < j \leq n-1\}$ and of roots $\{e_i \pm e_{n-1} \mid e_i \pm e_n \in \sigma\}$. It is easy to see that elements $(g_1, g_2, \dots, g_{n-1})$ are in $\tilde{T}(\sigma')$. Indeed for example $g_i g_{n-1}^{\pm 1} = 1$ if $e_i \pm e_{n-1} \in \sigma'$, as $g_i g_n^{\pm 1} = 1$ and $g_n = g_{n-1}$. Thus by induction hypothesis we obtain $g_i^2 = 1$ for all $i \neq n$. Since $g_n = g_{n-1}$, we also obtain $g_n^2 = 1$.

The second case is similar. \square

Let $v = \sum_{i=1}^n v_i e_i \in V$ be a vector with real coordinates and $h = \sum_{i=1}^n h_i e_i \in V$ a vector with integral coordinates and such that $\sum_{i=1}^n h_i$ is even. We will compute the normalized volume of $\Pi_{\mathcal{C}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{C}_n}(h)$ using Theorem 3.3. We will use the JK residue with respect to the measure dh associated to the basis e_1, e_2, \dots, e_n . However, the normalized volume $\text{vol}_{\mathbb{Z}, \mathcal{C}_n}(h)$ is computed for the measure determined by the lattice spanned by \mathcal{C}_n which is of index 2 on $\oplus_{i=1}^n \mathbb{Z} e_i$.

We introduce the function $J_C(v)$ on U defined by:

$$J_C(v)(u) = \frac{e^{\sum_{i=1}^n u_i v_i}}{\prod_{i=1}^n 2u_i \prod_{1 \leq i < j \leq n} (u_i - u_j) \prod_{1 \leq i < j \leq n} (u_i + u_j)}.$$

For $g = (g_1, g_2, \dots, g_n) \in \{\pm 1\}^n$ the Kostant fraction (3) is the function on U defined by:

$$\begin{aligned} q_C(g, h)(u) &= \frac{\prod_{i=1}^n g_i^{h_i} e^{\sum_{i=1}^n u_i h_i}}{\prod_{i=1}^n (1 - e^{-2u_i}) \times \prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j e^{-(u_i - u_j)})} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j^{-1} e^{-(u_i + u_j)})} \end{aligned}$$

We have then

Theorem 9.2. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{C}_n)$.*

- *For any $v \in \bar{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{C}_n}(v) = 2 JK_{\mathfrak{c}}(J_C(v)).$$

- *For any $h \in V_{\mathbb{Z}} \cap \bar{\mathfrak{c}}$ be a vector with integral coordinates and such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N_{\mathcal{C}_n}(h) = \sum_{g \in \{\pm 1\}^n} JK_{\mathfrak{c}}(q_B(g, h)).$$

In the second formula, there should be a multiplication by a factor 2 as the volume of the fundamental domain of the lattice spanned by \mathcal{C}_n is 2. However, we should sum only on $T = \tilde{T}/\{\pm 1\}$. Thus the two factors of 2

compensate. In fact, we will indeed sum over T represented as $\{\pm 1\}^{n-1} \times \{1\}$ and multiply the result by the constant 2.

As in the case of B_n , we will use the change of variable $1 + z_i = e^{u_i}$ to compute more easily the formula for $N_{C_n}(h)$. As explained in the case of B_n we need to use the integral formulation of the Jeffrey-Kirwan residue.

Thus define

$$\begin{aligned} \mathcal{K}_C(g, h)(z) &= \frac{\prod_{i=1}^n (1 + z_i)^{h_i + 2n - i} \times \prod_{i=1}^n g_i^{h_i}}{\prod_{i=1}^n ((1 + z_i)^2 - 1) \times \prod_{1 \leq i < j \leq n} (1 + z_i - g_i g_j (1 + z_j))} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 + z_i)(1 + z_j) - g_i g_j}. \end{aligned}$$

Performing the change of variables $e^{u_i} = 1 + z_i$ on the function $q_C(g, h)(u)$ and computing the Jacobian, the Theorem 3.3 becomes:

Theorem 9.3. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{C}_n)$.*

- *For any $v \in \overline{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, C_n}(v) = 2 JK_{\mathfrak{c}}(J_C(v)).$$

- *For any $h \in V_{\mathbb{Z}} \cap \overline{\mathfrak{c}}$ a vector with integral coordinates h_i with $\sum_{i=1}^n h_i$ even, the value of the partition function is given by:*

$$N_{C_n}(h) = \sum_{g \in \{\pm 1\}^n} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{K}_C(g, h)(z) dz.$$

Similarly we will sum over T represented as $\{\pm 1\}^{n-1} \times \{1\}$ and multiply the result by the constant 2.

The cycle $H(\mathfrak{c})$ associated to a chamber \mathfrak{c} containing a regular element $v = \sum_{i=1}^n v_i e_i$ is the same cycle that we computed in the preceding section for B_n . Hence we can reuse most of procedures from the type B_n . Paths are the same, and the residue calculations are the same. More precisely, the only two changes are in the computation of the set G (procedure **GC(n)**) and in the computation of the Kostant function (procedure **UCKostant**). This terminates the case of C_n .

10. THE TYPE D_n

10.1. The formulae to be implemented. Consider a vector space V with basis e_1, e_2, \dots, e_n . We choose the standard Lebesgue measure dh .

Let

$$\mathcal{D}_n = \{e_i - e_j \mid 1 \leq i < j \leq n\} \cup \{e_i + e_j \mid 1 \leq i < j \leq n\}.$$

Then \mathcal{D}_n is a positive roots system of type D_n , and generates V .

We remark that the lattice $V_{\mathbb{Z}}$ generated by roots of \mathcal{D}_n is the same lattice than the lattice generated by the roots of \mathcal{C}_n . It is of index 2 in $L := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \oplus \mathbb{Z}e_n$ and consists of elements $v = [v_1, v_2, \dots, v_n]$ with

integral coordinates and such that the sum $\sum_{i=1}^n v_i$ is an even integer. The group $T = U/U_{\mathbb{Z}}$ is thus the quotient of $\tilde{T} = U/\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, obtained by identifying g and $-g$, that is $T = \tilde{T}/\{\pm 1\}$. As in section 9, we identify the torus $\tilde{T} = U/(\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n) = \mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$ by

$$(u_1, u_2, \dots, u_n) \mapsto (e^{2\pi\sqrt{-1}u_1}, \dots, e^{2\pi\sqrt{-1}u_n}).$$

Consider the set $F = \{\pm 1\}^n \subset (S^1)^n$. For σ a basic subset of \mathcal{D}_n , define

$$\tilde{T}(\sigma) = \left\{ g \in \tilde{T} \mid e^{\langle \alpha, 2\pi\sqrt{-1}G \rangle} = 1 \text{ for all } \alpha \in \sigma \right\}.$$

Lemma 10.1. *Let σ be a basic subset of \mathcal{D}_n . Then $\tilde{T}(\sigma)$ is contained in F .*

Proof. Basic subsets of \mathcal{D}_n are basic subsets of \mathcal{C}_n so that we can choose the same set $F = \{\pm 1\}^n$. \square

Let $v = \sum_{i=1}^n v_i e_i \in V$ be a vector with real coordinates and $h = \sum_{i=1}^n h_i e_i \in V$ a vector with integral coordinates and such that $\sum_{i=1}^n h_i$ is even. We will compute the normalized volume of $\Pi_{\mathcal{D}_n}(v)$ and the number of integral points in $\Pi_{\mathcal{D}_n}(h)$ using Theorem 3.3.

Thus we introduce the function $J_D(v)$ on U defined by:

$$J_D(v)(u) = \frac{e^{\sum_{i=1}^n u_i v_i}}{\prod_{1 \leq i < j \leq n} (u_i - u_j) \prod_{1 \leq i < j \leq n} (u_i + u_j)}.$$

For $g = (g_1, g_2, \dots, g_n) \in \{\pm 1\}^n$ the Kostant fraction (3) is the function on U defined by:

$$\begin{aligned} q_D(g, h)(u) &= \frac{\prod_{i=1}^n g_i^{h_i} \times e^{\sum_{i=1}^n u_i h_i}}{\prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j e^{-(u_i - u_j)})} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 - g_i^{-1} g_j^{-1} e^{-(u_i + u_j)})}. \end{aligned}$$

We have then

Theorem 10.2. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{D}_n)$.*

- *For any $v \in \bar{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{D}_n}(v) = 2 JK_{\mathfrak{c}}(J_D(v)).$$

- *For any $h \in V_{\mathbb{Z}} \cap \bar{\mathfrak{c}}$ be a vector with integral coordinates and such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N_{\mathcal{D}_n}(h) = \sum_{g \in \{\pm 1\}^n} JK_{\mathfrak{c}}(q_D(g, h)).$$

We use the change of variable $1 + z_i = e^{u_i}$ to compute more easily the formula for $N_{\mathcal{D}_n}(h)$ and thus introduce integration over a cycle.

Thus define

$$\begin{aligned} \mathcal{K}_D(g, h)(z) &= \frac{\prod_{i=1}^n (1 + z_i)^{h_i + 2n - i - 2} \times \prod_{i=1}^n g_i^{h_i}}{\prod_{1 \leq i < j \leq n} (1 + z_i - g_i g_j (1 + z_j))} \\ &\quad \times \frac{1}{\prod_{1 \leq i < j \leq n} (1 + z_i)(1 + z_j) - g_i g_j}. \end{aligned}$$

After performing the change of variables $e^{u_i} = 1 + z_i$ on the function $q_D(g, h)(u)$ and after computing the Jacobian, the Theorem 3.3 becomes:

Theorem 10.3. *Let \mathfrak{c} be a chamber of $\mathcal{C}(\mathcal{D}_n)$.*

- *For any $v \in \bar{\mathfrak{c}}$, we have*

$$\text{vol}_{\mathbb{Z}, \mathcal{D}_n}(v) = 2 JK_{\mathfrak{c}}(J_D(v)).$$

- *For any h in $V_{\mathbb{Z}} \cap \bar{\mathfrak{c}}$ a vector with integral coordinates h_i and such that $\sum_{i=1}^n h_i$ is even, the value of the partition function is given by:*

$$N(\mathcal{D}_n, h) = \sum_{g \in \{\pm 1\}^n} \frac{1}{(2\pi\sqrt{-1})^n} \int_{H(\mathfrak{c})} \mathcal{K}_D(g, h)(z) dz.$$

As for types A and B , in order to implement these formulae we first have to describe the set $\mathcal{P}(v, \mathcal{D}_n)$ (Section 10.2). We finish to explain the implementation of case D in Section 10.3, using the fact that types B and D are similar.

10.2. The search for maximal proper nested sets. A height function is

$$\text{ht}(v) = \sum_{i=1}^n (n - i) v_i$$

which takes value 1 on all simple roots. We will deform it later on in order to have a function taking different values on roots.

We now proceed to describe hyperplanes for \mathcal{D}_n . If $P = [P^+, P^-]$ are two disjoint subsets of $\{1, 2, \dots, n\}$, we denote by $\langle u_P, v \rangle$ the linear form $\sum_{i \in P^+} v_i - \sum_{j \in P^-} v_j$. Consider the hyperplane in V defined by

$$H_P = \{v \in V, \langle u_P, v \rangle = 0\}$$

and remark that it is equal to the hyperplane determined by the reverse list $[P^-, P^+]$. Thus to each set $P = \{P^+, P^-\}$ of two disjoint sets P^+, P^- such that at least one is non empty, is associated a hyperplane H_P .

We denote by Z the complement of $P^+ \cup P^-$ in $\{1, 2, \dots, n\}$ and by $\mathcal{D}(Z)$ the subset of \mathcal{D}_n defined by

$$\mathcal{D}(Z) = \{e_i \pm e_j \mid 1 \leq i < j \leq n; i, j \in Z\}.$$

This is the positive roots system of type $D_{|Z|}$, with the positivity induced by the lexicographic order.

Let $\mathcal{K}(P^+, P^-)$ be the subset of \mathcal{D}_n defined by

$$\begin{aligned}
& \{e_i - e_j \mid 1 \leq i < j \leq n; i, j \in P^+\} \\
& \cup \{e_i + e_k \mid i \in P^+, k \in P^-\} \\
& \cup \{e_k - e_\ell \mid 1 \leq k < \ell \leq n; k, \ell \in P^-\}.
\end{aligned}$$

As we observed in Section 8.2 for B_n , by defining $f_i = e_i$ if $i \in P^+$ and $f_k = -e_{|P^+|-k+1}$ if $k \in P^-$, the set $\mathcal{K}(P^+, P^-)$ is a positive roots system of type $A_{|P^+|+|P^-|-1}$. Here the positivity is induced by the lexicographic order on P^+ and the reverse lexicographic order on P^- .

Observe also that H_P is the vector space spanned by $\mathcal{K}(P^+, P^-) \cup \mathcal{D}(Z)$.

Lemma 10.4. • *The hyperplane H_P is a \mathcal{D}_n -admissible hyperplane.*
• *The set $\mathcal{D}_n \cap H_P$ is the union of $\mathcal{D}(Z)$ and $\mathcal{K}(P^+, P^-)$.*
• *Every \mathcal{D}_n -admissible hyperplane is of this form.*

Proof. The first two assumptions are easy to see. We prove the third assumption by induction on n , the case $n = 2$ being trivial. Let H be a \mathcal{D}_n -admissible hyperplane. For any root $\alpha = e_i \pm e_j$ in \mathcal{D}_n , denote by $\hat{\alpha}$ the root $\hat{\alpha} = e_i \mp e_j$. We have two possibilities: either whenever a root α is in H then also $\hat{\alpha}$ is in H ; or there exists a root α in H such that $\hat{\alpha}$ is not in H .

In the first case, since H has dimension $n - 1$, there exists precisely one index $i_0 \in \{1, \dots, n\}$ such that the set of roots $\{e_i \pm e_{i_0} \mid i < i_0\} \cup \{e_{i_0} \pm e_i \mid i > i_0\}$ is not a subset of H . Hence H is determined by $[P^+, P^-]$ where $P^+ = \{i_0\}$ and $P^- = \emptyset$.

In the second case, up to renumbering roots we can assume that $\alpha = e_{n-1} - e_n$ or $\alpha = e_{n-1} + e_n$. Let us first examine the case when $\alpha = e_{n-1} - e_n$. The map q sending e_i to e_i if $i < n$ and e_n to e_{n-1} maps the set $\mathcal{D}_n \setminus \{\alpha, \hat{\alpha}\}$ to \mathcal{D}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{D}_{n-1} -admissible hyperplane. It is thus determined by $P' = [P'^+, P'^-]$, where P'^+ and P'^- are two disjoint sets contained in $\{1, 2, \dots, n-1\}$. If neither P'^+ nor P'^- contain $n-1$, the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Otherwise assume that for example P'^+ contains $n-1$. Then the hyperplane H is equal to the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+ \cup \{n\}$ and $P^- = P'^-$.

Now if $\alpha = e_{n-1} + e_n$ then the map q sending e_i to e_i if $i < n$ and e_n to $-e_{n-1}$ sends the set $\mathcal{D}_n \setminus \{\alpha, \hat{\alpha}\}$ to \mathcal{D}_{n-1} . The space $H/\mathbb{R}\alpha$ becomes a \mathcal{D}_{n-1} -admissible hyperplane, thus determined by $P' = [P'^+, P'^-]$. If neither P'^+ nor P'^- contain $n-1$, then the hyperplane H is equal to the hyperplane determined by $[P'^+, P'^-]$. Assume now that for example P'^+ contains $n-1$. Then the hyperplane H is equal to the the hyperplane determined by $[P^+, P^-]$, where $P^+ = P'^+$ and $P^- = P'^- \cup \{n\}$. \square

10.3. The procedure MNS_KostantD. Most of procedures from type B_n are kept unchanged. More precisely iterated residue calculation, estimate of the order of poles and global procedures coordinating computations are exactly the same as for type B_n .

The only serious adaptations to the case of D_n appears in the procedure **CheckDvector**(\mathbf{n}, \mathbf{v}). In fact now we check that

$$\begin{aligned} v_1 + \cdots + v_i &\geq 0 \quad \text{for } 1 \leq i \leq n-1, \\ v_1 + \cdots + v_{n-1} + v_n &\geq 0 \quad \text{and is even,} \\ v_1 + \cdots + v_{n-1} - v_n &\geq 0 \quad \text{and is even.} \end{aligned}$$

Other modifications are in procedures that are parent of **CheckDvector**. For example the procedure **CheckDwall** works exactly as **CheckBwall**, but now calls **CheckDvector** instead of **CheckBvector**. See Section 8.2.

11. PERFORMANCE OF THE PROGRAMS

In this Section, we describe several tests of our programs implementing the above algorithms for types A_n, B_n, C_n, D_n . We also compare with five previous algorithms: raw force using nested loops, raw force using extraction of coefficients of polynomials, LATTE's implementation of Barvinok's algorithm [11], its refinement **Barvinok** by Beyls-Bruynooghe-Loechner-Seguir-Verdoolaege's [3], Baldoni-DeLoera-Vergne's algorithm using *special permutations* [2].

Let us begin by describing the two raw methods.

As partition function counts the number of integral points in a bounded area of the space, we can simply enumerate all possible cases. Let us describe what happens in the case of the number of non-negative integral solutions $x \in \mathbb{R}^6$ of the equation $\sum_{i=1}^6 x_i \Phi_i = v$, where Φ_i are the columns of the matrix

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

of positive roots for A_3 , and $v = \sum_{i=1}^4 v_i e_i \in \mathbb{R}^4$ is in $V_{\mathbb{Z}} \cap \mathcal{C}(\mathcal{A}_3)$. Remark that this number of solutions is the same as the number of non-negative integral solutions $y \in \mathbb{R}^6$ of the equation $\sum_{i=1}^6 y_i \Phi'_i = v'$, where

$$\Phi' = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad v' = \begin{pmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \end{pmatrix}.$$

Thus Kostant partition for the vector v can be computed with the nested loops described in Figure 9.

Although several refinements (by examining more carefully bounds for loops) can save time, this raw method is untractable if the size of coordinates of the vector v' augments (say, two digits). Note that there are as many loops as the number of positive roots in the algebra, and as many linear conditions to check as the rank of the algebra.

```

for  $y_1$  from 1 to  $v_1$  do
  for  $y_2$  from 1 to  $v_1 + v_2$  do
    for  $y_3$  from 1 to  $v_1 + v_2 + v_3$  do
      for  $y_4$  from 1 to  $v_1$  do
        for  $y_5$  from 1 to  $v_1 + v_2$  do
          for  $y_6$  from 1 to  $v_1$  do
            if  $y_1 + y_4 + y_6 = v_1$ 
              and  $y_2 + y_4 + y_5 + y_6 = v_1 + v_2$ 
              and  $y_3 + y_5 + y_6 = v_1 + v_2 + v_3$ 
              then add 1 to the number of solutions
end of loops on  $y$ .

```

FIGURE 9. Nested loops for Kostant partition function for A_3

The second method to which we compared our algorithms comes from the second point in Proposition 2.8. More precisely Kostant partition function for the previously defined v for A_3 is the coefficient of the monomial $y_1^{v_1} y_2^{v_1+v_2} y_3^{v_1+v_2+v_3}$ in the polynomial

$$\prod_{i=1}^3 \left(\sum_{k=0}^u y_i^k \right) \times \prod_{i=1}^2 \left(\sum_{k=0}^u y_i^k y_{i+1}^k \right) \times \left(\sum_{k=0}^u y_1^k y_2^k y_3^k \right)$$

coming from the columns of the matrix Φ' , where $u = \max\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$. This method is faster than the previous one but still explodes when size of coordinates of v' augments.

All these various methods helped us to verify our algorithms on various examples.

PUT THE EXPLANATION FOR COMPUTATION ON CUBES.

NOW PUT TABLES WHERE WE COMPARE DIFFERENT METHODS.

A WORD ON COMPOSITE METHODS?

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