Vector partition function and representation theory

Charles Cochet

ABSTRACT. We apply some recent developments of Baldoni-Beck-Cochet-Vergne [**BBCV04**] on vector partition function, to Kostant's and Steinberg's formulae, for classical Lie algebras of types A_n, B_n, C_n. We therefore get efficient MAPLE programs that compute for these Lie algebras: the multiplicity of a weight in an irreducible finite-dimensional representation; the decomposition coefficients of the tensor product of two irreducible finite-dimensional representations. A similar forthcoming program will take care of D_n.

Nous appliquons des résultats récents de Baldoni-Beck-Cochet-Vergne [**BBCV04**] sur la fonction de partition vectorielle, aux formules de Kostant et de Steinberg, dans le cas des algèbres de Lie classiques A_n, B_n, C_n. Ceci donne lieu à des programmes MAPLE efficaces qui calculent pour ces algèbres de Lie : la multiplicité d'un poids dans une représentation irréductible de dimension finie ; les coefficients de décomposition du produit tensoriel de deux représentations irréductibles de dimension finie. Un programme similaire traitera prochainement D_n.

Extended abstract

In this note, we are interested in the two following computational problems for classical Lie algebras of types A_n, B_n, C_n, D_n:

- The multiplicity c_λ^μ of the weight μ in the representation $V(\lambda)$ of highest weight λ.
- Littlewood-Richardson coefficients, that is the multiplicity ${c_\lambda}^\nu{}_\mu$ of the representation $V(\nu)$ in the tensor product of representations of highest weights λ and μ.

Softwares LiE (from van Leeuwen *et al.* [**vL94**]) and GAP ([**GAP**]), and MAPLE packages `coxeter/weyl` (from Stembridge [**S95**]) use Freudenthal's and Klymik's formulae, and work for any semi-simple Lie algebra $\mathfrak{g}$ (not only for classical Lie algebras). Unfortunately, these formulae are really sensitive to the size of coefficients of weights.

Here the approach to these two problems is through *vector partition function*, that is the function computing the number of ways one can decompose a vector as a linear combination with non-negative integral coefficients of a fixed set of

2000 *Mathematics Subject Classification.* Primary 22E46, 52B55.

Key words and phrases. Representation theory, Kostka numbers, Littlewood-Richardson coefficients, convex polytopes.

vectors. For example the number $p(x)$ of ways of counting x euros with coins, that is

$$p(x) = \sharp\{n \in \mathbb{N}^8 \,;\, x = 200n_1 + 100n_2 + 50n_3 + 20n_4 + 10n_5 + 5n_6 + 2n_7 + n_8\},$$

can be seen as the partition of the 1-dimensional vector (x) with respects to the set $\{(200), (100), (50), (20), (10), (5), (2), (1)\}$ of 1-dimensional vectors. In the case of the decomposition with respects to the set of positive roots of a Lie algebra, we speak of *Kostant partition function*.

Recall that any r-dimensional rational convex polytope can be written as the set $P(\Phi, a)$ of non-negative solutions $x = (x_i) \in \mathbb{R}^N$ $(N \geq r)$ of an equation $\sum_{i=1}^{N} x_i \phi_i = a$ (for vectors ϕ_i and a in $\mathbb{R}^r$). It follows that evaluationg the vector partition is equivalent to computing the number of integral points in a lattice polytope.

The vector partition function arises in many areas of mathematics: representation theory, flows in networks, magic squares, statistics, crystal bases of quantum groups. Its complexity is polynomial in the size of input when the dimension of the polytope is fixed, and NP-hard if it can vary ([**B94**], [**B97**] and [**BP99**]).

There are several approaches to the vector partition problem. For example Barvinok's decomposition algorithm ([**B94**]), recently implemented by the LattE team ([**DHTY03**]), works for almost general sets of vectors. Beck-Pixton ([**BP03**]) also created an algorithm dedicated to vector sets arising from transportation polytopes (hence magic squares).

In this note, we use recent results of Baldoni-Beck-Cochet-Vergne ([**BBCV04**]) to obtain a fast algorithm for Kostant partition function *via* inverse Laplace formula. This involves DeConcini-Procesi's *maximal nested sets* – in short MNSs ([**DCP04**]), and iterated residues of rational functions computed by formal power series development.

We combine the resulting procedure with Kostant's and Steinberg's formulae giving c_λ^μ and $c_{\lambda\,\mu}^{\;\nu}$ in terms of vector partition function. We then obtain three Maple programs computing for Lie algebras A_n, B_n, C_n the multiplicity of a weight in an irreducible finite-dimensional representation, as well as the decomposition coefficients of the tensor product of two irreducible finite-dimensional representations (the case of D_n is in progress).

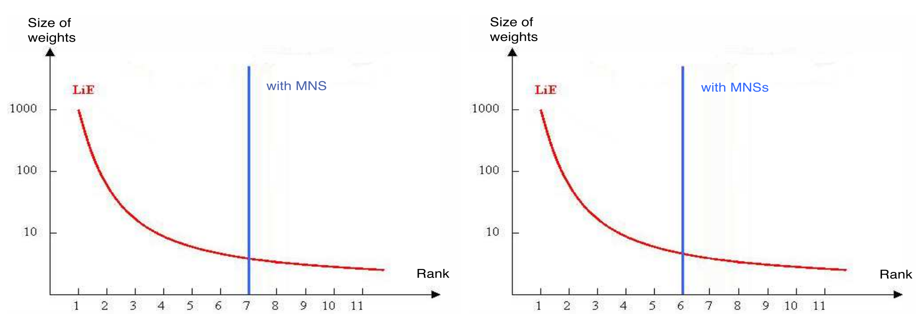

Figure 1. Multiplicities and tensor product coefficients for A_n

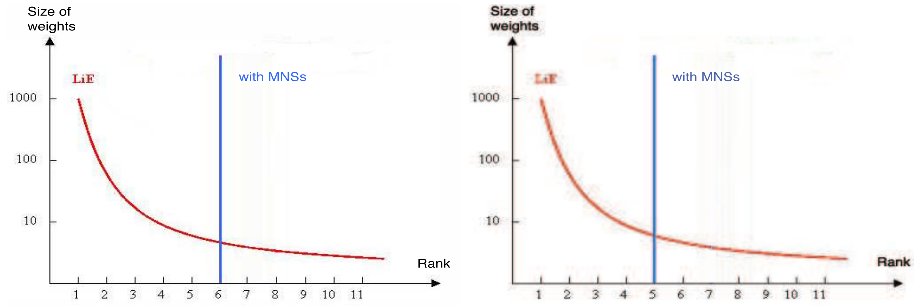

FIGURE 2. Multiplicities and tensor product coefficients for B_n (same figures for C_n)

These programs (freely available at [**C**]) are specially designed for large parameters of weights. Indeed although only written in MAPLE they can perform examples with weights with 5 digits coordinates, beyond the classical softwares written in C++. We also stress that our programs are absolutely clear, easy to use and require no installation (only a MAPLE worksheet to download). They are fully commented, so that a curious user can figure out their internal mechanisms.

However, other softwares and packages are not limited by the rank of the algebra like our programs. For example computation of non-trivial examples in Lie algebras of rank 10 is possible with the software LiE, whereas our programs are efficient up to rank 5–7. These facts make our programs complementary to traditional softwares. Figures 1 and 2 show comparative tests of the software LiE and our programs using MNSs; any area located to the left of a colored line represents the range where a program can compute examples in a reasonable time.

Remark that Kostant's and Steinberg's formulae have already been implemented once in the case of A_n ([**C03**]), using Kostant partition function. This previous program relies on results of Baldoni-Vergne ([**BV01**]) implemented by Baldoni-DeLoera-Vergne ([**BdLV03**]), computing Kostant partition function only in the case of A_n. Tools were *special permutations* and again iterated residues of rational fraction.

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Institut Mathématique de Jussieu, Université Paris 7 – Denis Diderot, 75251 Paris cedex 05, France

E-mail address: cochet@math.jussieu.fr