Equivariant cohomology and some applications.

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Contents

1 Introduction

Very lousy for the moment.

The aim of this article is to give my insights on how tools of localization in equivariant cohomology not only provide beautiful mathematical formulae, but also help in algorithmic computations. I mostly center around my favorite themes: quantification of symplectic manifolds and algorithms for polytopes, and I neglect many other applications. Thus I will discuss several mathematical objects related to Lie group actions on a manifold M which can be described by integral formulae of equivariant cohomology classes, as the equivariant volume of Hamiltonian manifolds, and the equivariant index of (transversally elliptic) operators. In cases of compact manifolds, these integrals can be described by the AB-BV fixed point formulae (also called abelian localization formula), similar to the Atiyah-Bott Lefschetz formulae for elliptic operators. In the case of equivariant volume of non compact manifolds, or the equivariant index of transversally elliptic operators, there are not always fixed point formulae available (the action of the group could be free). Thus, and for other reasons (EXPLAIN), it is also important to write "delocalized formulae".

More generally, we explain the use of equivariant integrals in the sense of generalized functions, and their localization (for good cases) in the sense of Witten non abelian localization. The use of equivariant cohomology classes with generalized coefficients is not only absolutely necessary when dealing with non compact problems, but also very powerful in "localizing" problems on a (compact or non compact) Hamiltonian manifold. So we have to use of localization and delocalization as the yin and yang principle...

new impulse to research on some open questions....

intersection number on toric manifolds, a proof (among others) of Guillemin-Sternberg conjecture, etc...

I will insist also that we can "compute" (with computer) effectively these quantities. Inspired by these new theorems, we implemented algorithms for various problems such as computing the value of the convolution of large number of Heaviside distributions at a point, Kostant partitions functions, local Euler-MacLaurin formula for a rational simplex, etc... These applications to polytopes have elementary proofs, but it was through interaction with Hamiltonian geometry that some of these tools were discovered.

For lack of space, I could include only central references to the topics discussed in this text. For more bibliographical comments, references and motivations, one might consult [**?**],[14], [26],[**?**] [**?**] and my home page (notably, the text called "Exégèse") at math.polytechnique.fr/cmat/vergne/

SAY THE LAST SECTION IS INDEPENDENT.

THANKS TO BERLINE, BRION, DUFLO, BALDONI, POPESCU-PAMPU

CITE SAwin ??

2 Baby Examples

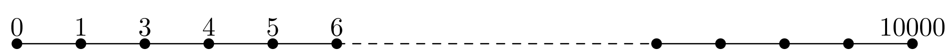
Some formulae in mathematics condense a very long information in very short expressions.

2.1 Geometric progressions

The most striking formula perhaps is the one that sums a very long geometric progression:

$$\sum_{i=0}^{10000} q^i = \frac{1}{1-q} + \frac{q^{10000}}{1-q^{-1}}.$$

For a straightforward calculation of the left hand side for a given value q, one needs to know the value of the function q^i at all the 10001 integral points of the interval $[0, 10000]$, while for the right hand side one needs only the value of this function at the end points $0, 10000$. We will say informally that this sum localizes at the end points.



The short formula (here A, B, i are integers)

$$\sum_{i=A}^{B} q^i = \frac{q^A}{1-q} + \frac{q^B}{1-q^{-1}} = -\frac{q^{A-1}}{1-q^{-1}} - \frac{q^{B+1}}{1-q} \tag{1}$$

is related to the following equalities of characteristic functions:

$$\begin{aligned}\chi([A,B]) &= \chi([A,\infty[) + \chi(]B,\infty[) - \chi(\mathbb{R}) \\ &= \chi(\mathbb{R}) - \chi(]-\infty,A[) - \chi(]B,\infty[).\end{aligned}$$

We draw the picture of the last equality.

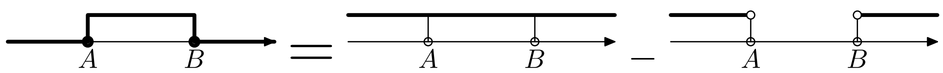


Figure 1: Decomposition of an interval

Then to sum q^i from A to B, we first sum q^i from $-\infty$ to ∞ and subtract the two sums over the integers strictly less than A and over the integers strictly greater than B. Thus, if

$$S_0 := \sum_{i=-\infty}^{\infty} q^i, \qquad S_A := \sum_{-\infty}^{A-1} q^i, \qquad S_B := \sum_{B+1}^{\infty} q^i,$$

we get formally, or, setting $q = e^{2i\pi y}$, in the sense of generalized functions on the unit circle,

$$S = S_0 - S_A - S_B. \tag{2}$$

For a value $q \neq 1$, the first sum S_0 is 0 as follows from $(1-q)S_0 = 0$, while S_A, S_B are just geometric progressions and we come back to the short formula (1).

Formula (2) illustrates a very simple example of Paradan's localization of elliptic operators, which we describe in Section 5. Indeed, Formula (2) is an example of the decomposition of the equivariant index of an elliptic operator on the Riemann sphere in a sum of indices of 3 transversally elliptic operators (see Example 9).

2.2 Integration over a interval

Let me write the following "continuous analogue" of summation of a geometric series: the integration of an exponential $e^{i\phi x}$ ($\phi \in \mathbb{R}$) on an interval:

$$I(\phi) := \int_A^B e^{i\phi x} dx = \frac{e^{i\phi A}}{-i\phi} + \frac{e^{i\phi B}}{i\phi}, \quad \text{here } A \leq B \text{ are any real numbers.} \tag{3}$$

This formula illustrates a very simple example of the "Atiyah-Bott-Berline-Vergne" (AB-BV) localization formula (also called abelian localization) that we will state in Section 4, Theorem 4. Indeed, the fundamental theorem of calculus:

$$\int_A^B f'(x)dx = f(B) - f(A)$$

is a very simple example of "AB-BV localization formula" !!.

If we break the integral (3) according to Figure 1, we get:

$$I(\phi) = I_0(\phi) - I_A(\phi) - I_B(\phi) \tag{4}$$

with

$$I_0(\phi) := \int_{-\infty}^{\infty} e^{i\phi x} dx = \delta_0(\phi), \text{ the Dirac function at } 0,$$

$$I_A(\phi) := \int_{-\infty}^{A} e^{i\phi x} dx, \qquad I_B(\phi) := \int_{B}^{\infty} e^{i\phi x} dx.$$

The three functions $I_0(\phi), I_A(\phi), I_B(\phi)$ are generalized functions of ϕ, with values for $\phi \neq 0$ respectively equal to $0, e^{i\phi A}\frac{1}{i\phi}, e^{i\phi B}\frac{1}{-i\phi}$.

The equality (4) above illustrates a very simple case of Witten's non abelian localization formula (see Section 4, Example 7).

2.3 Integration over polyhedra

Trapezoid.

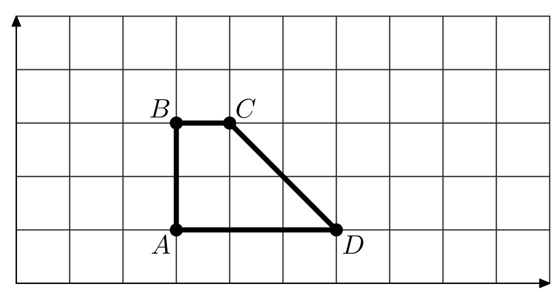


Figure 2: The Trapezoid H

Consider the trapezoid H in the plane $\mathbb{R}^2$, with vertices $A := [3,1], B := [3,3], C := [4,3], D := [6,1]$. Try to compute

$$I_H(\phi_1,\phi_2) := \int_H e^{i(\phi_1 x_1+\phi_2 x_2)} dx_1 dx_2.$$

To do it by hand, best to use a primitive:

$$e^{i(\phi_1 x_1+\phi_2 x_2)} dx_1 dx_2 = \frac{1}{2} d\left(e^{i(\phi_1 x_1+\phi_2 x_2)} \frac{\phi_2 dx_2 - \phi_1 dx_1}{i\phi_1\phi_2}\right).$$

Use Stokes formula and obtain an integral on the boundary of H (which consists of 4 edges) of an exponential, apply again Stokes formula in dimension 1, and we obtain the result, a priori as a sum of 8 expressions. They combine beautifully. The function I_H is a sum of 4 terms, each corresponding to a vertex:

$$(5) \qquad I_H(\phi_1,\phi_2) = \frac{e^{3i\phi_1+i\phi_2}}{-\phi_1\phi_2} + \frac{e^{3i\phi_1+3i\phi_2}}{\phi_2\phi_1} + \frac{e^{4i\phi_1+3i\phi_2}}{\phi_1(\phi_1-\phi_2)} + \frac{e^{6i\phi_1+i\phi_2}}{(\phi_2-\phi_1)\phi_1}.$$

This formula illustrates again an example of AB-BV localization formula. The tool of equivariant cohomology on a manifold (the Hirzebruch surface of complex dimension 2 described in Example 14 in Subsection 7.2), which projects on the polytope H by the moment map, allows us to do these iterated Stokes computations in just one step and to come right away at Formula (5). Remark that this final result involves only the value of the function $e^{i\langle\phi,x\rangle}$ at the four vertices of H and the tangent cone at these four vertices: the linear forms in the denominator in each of the four terms corresponding to the vertex v are the directions of the edges passing through v (with some sign).

Formula (5) is a special case of Brion's formulae for integrating or summing exponentials on polytopes (see [13] for elementary proofs, via cone decompositions).

A strip.

Consider now the strip S with edge $[A,B]$ and 2 infinite horizontal edges starting at A, B. The polyhedron S has just two vertices A, B. We may compute the integral $I_S(\phi_1,\phi_2) := \int_S e^{i(\phi_1 x_1+\phi_2 x_2)} dx_1 dx_2$.

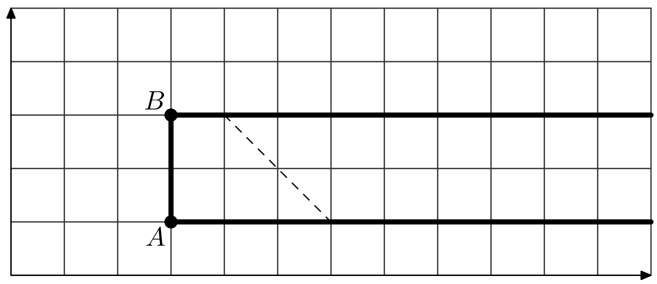


Figure 3: The Strip S

This is a generalized function, analytic when $\phi_1\phi_2 \neq 0$, given by Formula (5) provided we suppress in it the third and fourth terms corresponding to C, D. We get

$$I_S(\phi_1, \phi_2) = \frac{e^{3i\phi_1 + i\phi_2}}{-\phi_1\phi_2} + \frac{e^{3i\phi_1+3i\phi_2}}{\phi_2\phi_1}. \tag{6}$$

However, I_S is not an analytic function and is not determined by its value on $\{\phi_1\phi_2 \neq 0\}$. Denote

$$Y_+(\phi) := \int_0^\infty e^{i\phi x}dx, \qquad Y_-(\phi) := \int_{-\infty}^0 e^{i\phi x}dx.$$

Writing the characteristic function of interval $[A, B]$ as a difference, as in Figure 1, we get the more precise formula

$$I_S(\phi_1, \phi_2) = -e^{3i\phi_1+i\phi_2}Y_+(\phi_1)Y_-(\phi_2) - e^{3i\phi_1+3i\phi_2}Y_+(\phi_1)Y_+(\phi_2) + e^{3i\phi_1}Y_+(\phi_1)\delta_0(\phi_2). \tag{7}$$

Remark that we have replaced $\frac{-1}{i\phi_1}$ by $Y_+(\phi_1)$ and $\frac{\pm 1}{i\phi_2}$ by $Y_\mp(\phi_2)$ in Formula (6), and we added another term equal to 0 when $\phi_2 \neq 0$.

Formula (7) illustrates a simple example of the technique of non abelian localization formula in non compact spaces. This technique is necessary to obtain a proof of Guillemin-Sternberg conjecture for discrete series. Indeed this example arises when describing the restriction to $SO(4)$ of a discrete series of the non compact group $SO(4, 1)$, the Kirwan polytope being this strip S. We will discuss this example in Section 6.

2.4 Inverse problem

The inverse problem is: given a short expression for a sum, compute an individual term of the sum.

Here is an example. Consider the following product of geometric progressions $G := (\sum_{i=0}^{\infty} q_1^i)^3(\sum_{j=0}^{\infty} q_2^j)^3(\sum_{k=0}^{\infty} q_1^k q_2^k)^3$ given by the short expression:

$$S(q_1, q_2) := \frac{1}{(1-q_1)^3}\frac{1}{(1-q_2)^3}\frac{1}{(1-q_1q_2)^3}.$$

We might want to compute the coefficient $c(a,b)$ of $q_1^a q_2^b$ in G. If $a \geq b$, an iterated application of the residue theorem in one variable leads to

$$c(a,b) = \operatorname{res}_{x_2=0}\left(\operatorname{res}_{x_1=0}\frac{e^{ax_1}e^{bx_2}}{(1-e^{-x_1})^3(1-e^{-x_2})^3(1-e^{-(x_1+x_2)})^3}\right).$$

This is easily computed. Let

$$g(a,b) = \frac{(b+1)(b+2)(b+3)(b+4)(b+5)(7a^2-7ab+2b^2+21a-9b+14)}{14\cdot 5!}$$

Then we obtain

$$\text{If } a \geq b, \text{ then } c(a,b) = g(a,b). \tag{8}$$

$$\text{If } a \leq b, \text{ then } c(a,b) = g(b,a). \tag{9}$$

This inverse problem occurs when solving linear diophantine inequations (see Subsection7.3). Guillemin-Sternberg conjecture (see Section 6) is an example where a similar inverse problem has an answer in geometric terms. We will discuss in Section 7 a residue theorem (Theorem 13) in several variables in order to solve this problem efficiently.

2.5 Stationary phase

Let M be a compact manifold of dimension n, f a smooth function on M and dm a smooth density. Consider the function

$$F(t) := \int_M e^{itf(m)}dm.$$

The major contribution to the value of its integral when t tends to ∞ arise from the neighborhood of the set C of critical points of f. We indicate a proof, as we will sketch a very similar proof for Witten's non abelian localization formula, see Section 4. Consider the image of M by the map $x = f(m)$ and the push-forward of the density dm. Then $F(t) = \int_{\mathbb{R}} e^{itx} f_*(dm)$. Choose a function χ equal to 1 in the neighborhood of the set C and supported near C. Then $F(t) = F_C(t) + R(t)$ where

$$F_C(t) := \int_{\mathbb{R}} e^{itx} f_*(\chi dm), \qquad R(t) = \int_{\mathbb{R}} e^{itx} f_*((1-\chi)dm).$$

$R(t)$ is the Fourier transform of a smooth compactly supported function, and thus decreases rapidly at ∞. It is not hard to show that, if f has a finite number of non degenerate critical points,

$$F(t) \sim F_C(t) \sim \sum_{p\in C} e^{itf(p)} \sum_{i\geq 0}^{\infty} a_{p,i} t^{-\frac{n}{2}+i}$$

where the constants $a_{p,i}$ can be computed in function of f, dm near $p \in C$. Asymptotically, the integral "localizes" at a finite number of points p.

When $f(m)$ is the Hamiltonian of an action of the circle group $S^1 := \{e^{2i\pi\phi}\}$ on a compact symplectic manifold M of dimension 2ℓ and dm the Liouville measure, then Duistermaat-Heckman [27] showed that $f_*(dm)$ is locally polynomial on $f(M)$ and that

$$F(t) = \sum_{p\in C} e^{itf(p)} a_{p,0} t^{-\ell}. \tag{10}$$

This is referred to as the exact stationary phase formula (or D-H formula).

Example 1 *Let us give a simple example. Take the (dilated) sphere* $M := \{x^2 + y^2 + z^2 = A^2\}$ *with Liouville measure* $dm := \frac{dy\wedge dz}{2\pi x}$. *The function* $f = x$ *is the Hamiltonian of the rotation around the axe* x. *The critical points of* f *are the points* $[\pm A, 0, 0]$. *We can see immediately that* $f_*(dm)$ *is the characteristic function of the interval* $[-A, A]$. *We obtain again the formula:* $F(t) = \int_{-A}^{A} e^{itx} dx = \frac{e^{-iAt}}{it} + \frac{e^{iAt}}{it}$.

3 Equivariant differential forms

Let M be a manifold with an action of the circle group S^1. The Atiyah-Bott fixed point formula for the equivariant index of an elliptic operator on

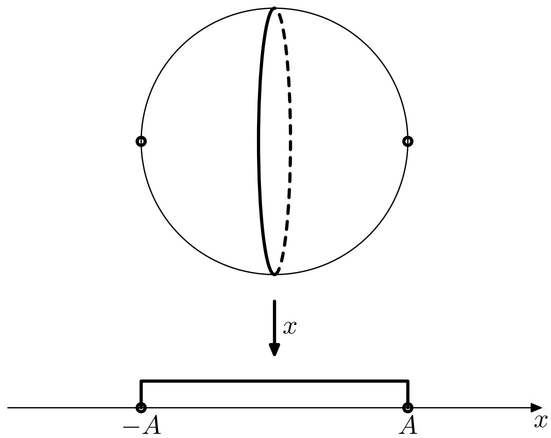


Figure 4: Projecting the sphere $x^2 + y^2 + z^2 = A^2$

M was obtained via localization for kernels of operators on M near fixed points of the action. In particular, Atiyah-Bott formula explains the Weyl formula for characters of a compact Lie group K. Our original motivation with Nicole Berline was to explain its continuous analogue, that is the very similar Harish-Chandra formula for the Fourier transform of a coadjoint orbit of K (the symplectic manifold attached to the Weyl character via Kirillov orbit method) and more generally Rossmann formula [44] (for the non compact coadjoint orbit attached to a discrete series character). We understood Rossmann fixed point formula via a Stokes formula for integrals of equivariant forms ([15]). That is was possible to condense certain integrals on M in short formulae localized near "fixed points" was already transparent from Bott residue formulae ([20]), and D-H stationary phase formula. The corresponding cohomological tool (a deformation of de Rham complex with use of vector fields) which lies beyond was formalized independently by Berline-Vergne ([15]) and Witten ([53]) with different motivations. SAY SOMETHING FOR ATIYAH-BOTT. We discovered later that this complex was already known by Henri Cartan in an algebraic context. However, this revival of "de Rham" theory of equivariant cohomology in analytic terms was very fruitful, in particular in order to work on non compact spaces or allowing stationary phase type of arguments. Duistermaat-Heckman formula on loop spaces, as formally suggested by Atiyah, gave a new impulse on index theorems. In this text, we will emphasize the point of view of equivariant forms with generalized coefficients.

3.1 Equivariant forms

We first take a C^∞-point of view. Let G be a Lie group acting on a manifold N. I do not assume for the moment that either N or G are compact. It is difficult for the moment to push things very far when G is not compact. A puzzling example is Kashiwara's fixed point formulae [31] for a discrete series attached to a coadjoint orbit N of a real semi-simple Lie group G (the fixed points belonging not to N, but to the closure of N in a specific compact space). Here the work of Libine [37] shows that equivariant cohomology together with new ideas on deformation of Lagrangian cycles in cotangent spaces help explain these formulae.

I keep the notation N for non necessarily compact manifolds, and M for compact manifolds. Similarly a compact group will be denoted by the letter K while G will be an arbitrary real Lie group. The letters T, H will be reserved for a torus. Here a torus is a compact connected abelian Lie group, thus just a product of circles groups $\{e^{2i\pi\phi_a}\}$. In this case, I take as basis of the Lie algebra $\mathfrak{t}$, elements J_a such that $\exp(\phi_a J_a) := e^{2i\pi\phi_a}$ ($\phi_a \in \mathbb{R}$). The gothic german letters $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}, \mathfrak{h}$ etc.. denote the corresponding Lie algebras, $\mathfrak{g}^*, \mathfrak{k}^*, \mathfrak{t}^*, \mathfrak{h}^*$ the dual vector spaces, J^a the dual basis to a basis J_a, etc.. The letter ϕ denotes an element of $\mathfrak{g}$.

For $\phi \in \mathfrak{g}$, we denote by $V\phi$ the vector field on N generated by the infinitesimal action of $-\phi$. At a point x of N, $V_x\phi := \frac{d}{d\epsilon}\exp(-\epsilon\phi)\cdot x|_{\epsilon=0}$. Let $\mathcal{A}(N)$ be the algebra of differential forms on N (with complex coefficients), d the exterior derivative. If V is a vector field, let $\iota(V)$ be the contraction by V. If $\nu := \sum_{i=0}^{\dim N}\nu_{[i]}$ is a differential form on an oriented manifold N, then the integral of ν over N is by definition the integral of the top term of ν: $\int_N \nu = \int_N \nu_{[\dim N]}$ (provided the integral is convergent).

A smooth map $\alpha : \mathfrak{g} \to \mathcal{A}(N)$ is called an equivariant form, if α commutes with the action of G on both sides. The equivariant de Rham operator D ([53],[15]) is viewed as a deformation of the de Rham operator d with the help of the vector field $V\phi$. It is defined on equivariant forms by:

$$(D(\alpha))(\phi) := d(\alpha(\phi)) - u\iota(V\phi)\alpha(\phi),$$

u being a parameter in $\mathbb{C}$. Then $D^2 = 0$. When $u = 0$, we recover the usual differential d. Here we fix $u = 1$. An equivariant form α is equivariantly closed if $D\alpha = 0$. The cohomology space, denoted by $\mathcal{H}^\infty(\mathfrak{g}, N)$, is as usual the kernel of D modulo its image. This is only $\mathbb{Z}/2\mathbb{Z}$ graded in even and odd forms.

The integral of an equivariant differential form is very naturally defined as a generalized function. Take $F(\phi)$ a test function on $\mathfrak{g}$, then $\int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi$ is a differential form on N. If it is integrable on N, then $\int_N \alpha$ is defined by

$$\langle \int_N \alpha, F d\phi\rangle = \int_N \int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi.$$

3.2 Hamiltonian spaces

Examples of equivariantly closed forms arise immediately in Hamiltonian geometry.

Let N be a symplectic manifold with symplectic form Ω. By definition, the action of G on N is Hamiltonian with moment map $\mu : N \to \mathfrak{g}^*$ if, for every $\phi \in \mathfrak{g}$, $d(\langle \phi, \mu\rangle) = \iota(V\phi)\cdot\Omega$. Thus the zeroes of the vector field $V\phi$ (that is the **fixed points** of the one parameter group generated by ϕ) are the critical points of $\langle \phi, \mu\rangle$.

The equivariant symplectic form $\Omega(\phi) := \langle \phi, \mu\rangle + \Omega$ is a closed equivariant form. A particularly important closed form for us is $e^{i\Omega(\phi)}$. If $\dim N = 2\ell$, then

$$e^{i\Omega(\phi)} = e^{i\langle \phi,\mu\rangle}\left(1 + i\Omega + \frac{(i\Omega)^2}{2!} + \cdots + \frac{(i\Omega)^\ell}{\ell!}\right).$$

Let M be a K-Hamiltonian manifold of dimension 2ℓ. By definition, the equivariant symplectic volume of M is the function of $\phi \in \mathfrak{k}$ given by

$$\mathrm{vol}_M(\phi) := \frac{1}{(2i\pi)^\ell}\int_M e^{i\Omega(\phi)} = \int_M e^{i\langle\phi,\mu(m)\rangle}\frac{\Omega^\ell}{\ell!(2\pi)^\ell},$$

($\mathrm{vol}_M(0)$ is the symplectic volume of M). The last integral, according to Duistermaat-Heckman exact stationary phase formula [27], localizes as a sum of integrals on the connected components of the zeroes of $V\phi$. If this set of zeros is finite, we obtain the D-H formula:

$$\mathrm{vol}_M(\phi) = \sum_{p\in \text{zeros of } \mathrm{V}\phi} \frac{e^{i\langle\phi,\mu(p)\rangle}}{i^\ell\sqrt{\det_{T_pM} L_p(\phi)}}, \tag{11}$$

where $L_p(\phi)$ is the endomorphism of T_pM determined by the infinitesimal action of ϕ at p. As the action comes from a compact group, there is a well-defined polynomial square root of the function $\phi \mapsto \det_{T_pM} L_p(\phi)$, the sign being determined by the orientation.

Equivariant volumes of non compact Hamiltonian spaces.

Let us point out some examples of non compact manifolds N where the equivariant symplectic volume exists in the sense of generalized functions.

- **T^*S^1.**

  The simplest example is the manifold T^*S^1. If $[e^{i\theta}, t]$ is a point of T^*S^1 with $t \in \mathbb{R}$, the Liouville form is $\omega := td\theta$ and $\Omega = d\omega$. S^1 acts by rotations and we take $\mathfrak{g} := \mathbb{R}J$ with $VJ := -2\pi\partial_\theta$. Thus $e^{i\Omega(\phi)} = e^{2i\pi\phi t}(1 + idt \wedge d\theta)$, and

  $$\mathrm{vol}_N(\phi J) = \int_{\mathbb{R}} e^{2i\pi t\phi} dt = \delta_0(2\pi\phi).$$

  This is consistent with the fact that the action of S^1 on T^*S^1 is free, so that the set of zeros of $V(\phi J)$ is empty when $\phi \neq 0$.

- **Coadjoint orbits.** Let G be a real Lie group. Recall (Kostant [34]) that when $N := G\lambda$ is the orbit of an element $\lambda \in \mathfrak{g}^*$ by the coadjoint representation, then N has a unique structure of G-Hamiltonian space, such that the moment map is just the inclusion map $N \to \mathfrak{g}^*$. Then the equivariant volume $\mathrm{vol}_N(\phi)$ is (usually) defined as a generalized function on $\mathfrak{g}$. This is just the Fourier transform of the G-invariant Liouville measure supported on $G\lambda \subset \mathfrak{g}^*$.

  When M is a coadjoint orbit of a compact Lie group K, Harish-Chandra gave a fixed point formula for $\mathrm{vol}_M(\phi)$, now seen as a special case of D-H formula (11). Rossmann [44] and Libine [37] extended Harish-Chandra's formula to the case of closed coadjoint orbits of reductive (non compact) Lie groups, involving delicate constants at fixed points at "infinity" defined combinatorially by Harish-Chandra and Hirai and topologically by Kashiwara.

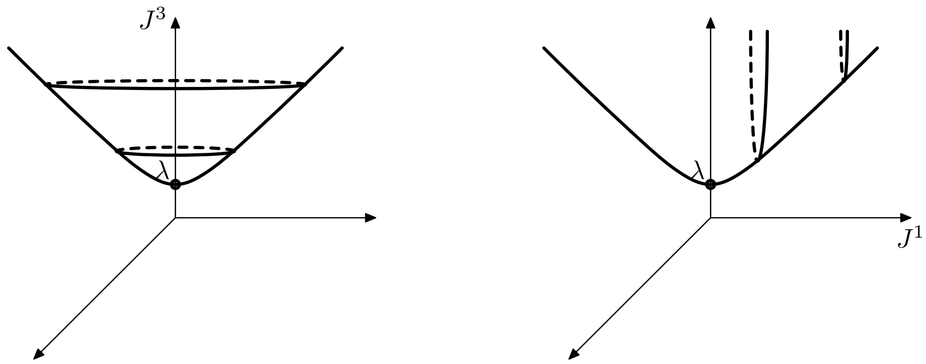
  Here is an example. Consider the group $SL(2, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ with basis

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The one-parameter group generated by J_3 is compact, while those generated by J_1 and J_2 are non compact. The manifold

$$N := \{\xi_1 J^1 + \xi_2 J^2 + \xi_3 J^3; \xi_3^2 - \xi_1^2 - \xi_2^2 = \lambda^2, \xi_3 > 0\}$$

is a coadjoint orbit. Then the generalized function $\mathrm{vol}_N(\phi_1 J_1 + \phi_2 J_2 + \phi_3 J_3)$ is given by an invariant locally L_1-function, analytic outside $\phi_1^2 + \phi_2^2 - \phi_3^3 = 0$.



$$\mathrm{vol}_N(\phi_3 J_3) = \frac{-e^{i\lambda\phi_3}}{i\phi_3}, \qquad \mathrm{vol}_N(\phi_1 J_1) = \frac{e^{-|\lambda\phi_1|}}{|\phi_1|}.$$

CHECK

The formula for the generator J_3 of a compact group action is in agreement with "fixed point formula philosophy" and coincides with what would be D-H formula: there is just one fixed point λJ^3 for the action.

The formula for J_1 is not so easy to explain within a general framework. Indeed the non compact group $\exp(\phi_1 J_1)$ acts freely on N, however the value of the generalized function $\mathrm{vol}_N(\phi_1 J_1)$ is non zero although there are no fixed points on N. In [36], N is embedded in the Riemann sphere $M = P_1(\mathbb{C})$ as an open hemisphere, and a subtle argument of deformations to fixed points of J_1 in M "explain" the formula for $\mathrm{vol}_N(\phi_1 J_1)$. formula.

- **Symplectic vector spaces.** Let (V, B) be a symplectic vector space of dimension 2ℓ and consider the linear action of the Lie group $G := \mathrm{Sp}(V, B)$ on V. Then

$$\mathrm{vol}_V(\phi) = \frac{1}{(2\pi)^\ell}\int_V e^{i\langle\phi v, v\rangle} dv$$

exists as a generalized function. For a point $\phi \in \mathfrak{g}$, such that the eigenvalues $\{a_i\}$ of the action of ϕ on $V_\mathbb{C}$ are all non zero, roughly $\mathrm{vol}_V(\phi) = \frac{1}{\sqrt{\prod_{i=1}^{2\ell} a_i}}$. But the choice of the square root of $\det_V \phi = \prod_{i=1}^{2\ell} a_i$ involves delicate constants depending if the $\{a_i\}$ are real, imaginary or complex. REFERENCE HORMANDER CHECK PAGE

A remark. For all examples of non compact symplectic spaces given above, the form Ω is exact. Indeed, if $N := T^*M$ is a cotangent bundle to a G-manifold M, then $\Omega = d\omega$, with ω the Liouville form pdq. Similarly if $N := (V, B)$ is a symplectic vector space, then $\Omega = d\omega$ with $\omega = B(v, dv)$. In de Rham cohomology, the closed form e^Ω is congruent to 1. Similarly the form $e^{i\Omega(\phi)}$ is congruent to 1 in $\mathcal{H}^\infty(\mathfrak{g}, N)$. The following obvious formula will be important in Section 4.

Proposition 2 *Assume that $\Omega = d\omega$, where ω is a G-invariant 1-form. Then $\Omega(\phi) = D\omega(\phi)$ and*

$$e^{i\Omega(\phi)} = 1 + D(B_N)(\phi), \text{ with } B_N(\phi) := i\omega \wedge (\sum_{k=1}^{\infty} \frac{(i\Omega(\phi))^{k-1}}{k!}).$$

3.3 Equivariant cohomology groups

The result of the integration of an equivariant form $\alpha(\phi)$ with C^∞ coefficients on a non compact space is often defined in the generalized sense, as we just saw. It is thus natural to consider more generally the $C^{-\infty}$-point of view where $\alpha(\phi)$ is a generalized function of ϕ ([26]). This allows us to consider push-forward of equivariant forms from vector bundles to the base, etc .... Also the $C^{-\infty}$ point of view encompasses the rational point of view arising right away in any localization theorem and that we already saw appearing in our baby examples. For example the rational function $\frac{\pm 1}{i\phi}$ may be extended at 0 as the generalized function $\int_{\mathbb{R}^{\mp}} e^{i\phi x}dx$ (the side $\mp$ to be determined by the local problem). Also generalized functions supported at 0 like the Dirac function $\delta_0(\phi)$ occur when studying free actions.

Thus we may define several types of equivariant cohomology groups, depending of our aims. I have already introduced $H^\infty(\mathfrak{g}, N)$. Now I introduce two other equivariant cohomology groups.

Cartan's complex. The first one is equivalent to the topological equivariant cohomology, defined via classifying spaces. Here we consider, for a K-manifold N, the space $\mathcal{A}^{pol}(\mathfrak{k}, N)$ of equivariant forms $\alpha(\phi)$ depending polynomially of ϕ. The cohomology group is $\mathcal{H}^{pol}(\mathfrak{k}, N)$. This is a $\mathbb{Z}$-graded group, where element of $\mathfrak{k}^*$ have degree two, and forms their exterior degree. A basic theorem of H. Cartan is the following: If K acts on a compact manifold M with finite stabilizers, then $\mathcal{H}^{pol}(\mathfrak{k}, M) = H^*(M/K)$.
Details on Cartan's theory and further developments can be found in the stern monograph (which contains treasures) of Duflo-Kumar-Vergne [26], or in the attractive book of Guillemin-Sternberg [28].

This de Rham point of view for topological equivariant cohomology seems to be adapted only to smooth spaces. However, the use of equivariant Poincaré dual allows us to work also on algebraic varieties, where Joseph polynomials and Rossmann localization formula (see [45]) are important tools. For lack of space, I will not pursue this topic. Let me also mention the theory of equivariant Chow groups, initially due to Totaro and developed by Edidin-Graham and Brion (see [23]), for algebraic actions on algebraic varieties defined over any field.

Generalized coefficients. An equivariant form $\alpha(\phi)$ with $C^{-\infty}$ coefficients is a generalized function on $\mathfrak{g}$ with values in $\mathcal{A}(N)$. Thus for any smooth function F with compact support, the integral $\int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi$ is a differential form on N. If $N := \bullet$, an equivariant form with $C^{-\infty}$-coefficients is just an element of $(C^{-\infty}(\mathfrak{g}))^G$, that is an invariant generalized function on $\mathfrak{g}$. The operator D is well defined. For example if $S^1 := \{|z| = 1\}$ is the circle acted on by rotation, then $Q(\phi J) := \delta_0(\phi)\frac{dz}{z}$ is an equivariant form with

$C^{-\infty}$ coefficients such that $DQ = 0$. We denote the corresponding cohomology group by $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$. Some basic theorems on $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$ are proved in [26].

If N is non compact, it is important to consider the case where the form $\int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi$ is rapidly decreasing at ∞ (our non compact manifold N will be most of the time a vector bundle over a compact base, and this notion is well defined). We denote by $\mathcal{H}^{-\infty,dec}(\mathfrak{g}, N)$ the corresponding cohomology group.

Integration is well defined on these cohomology groups. The first integration $\int_M \alpha(\phi)$ is well defined on $\mathcal{H}^{pol}(\mathfrak{k}, M)$ if M is compact oriented and gives us an invariant polynomial on $\mathfrak{k}$. The second integration $\int_N \alpha(\phi)$ is well defined on $\mathcal{H}^{-\infty,dec}(\mathfrak{g}, N)$ and gives us an invariant generalized function on $\mathfrak{g}$.

3.4 Reduction of symplectic spaces

Let N be a Hamiltonian K-manifold. Assume that $\xi \in \mathfrak{k}^*$ is a regular value of the moment map μ and let K_ξ be the stabilizer of ξ. Then K_ξ acts with finite stabilizers in $\mu^{-1}(\xi)$ so that $\mu^{-1}(\xi)/K_\xi$ is a symplectic orbifold, called the reduced space at ξ and denoted by $N_{\xi,red}$. If $\xi = 0$, we simply denote it by $N//K := \mu^{-1}(0)/K$. When N is a projective manifold, then Kirwan [33] shows that $N//K$ is the quotient in the sense of Mumford's Geometric invariant theory (GIT). By considering the symplectic manifold $Q = N \times (K \cdot (-\xi))$ (the shifting trick), we may always consider reduction at 0.

If 0 is a regular value, Kirwan associates to an equivariant closed form $\alpha(\phi)$ on N a cohomology class α_{red} on $N//K$: $\alpha(\phi)|_{\mu^{-1}(0)}$ is equivalent to the pull-back of α_{red}. Kirwan's map $\chi : H_T^*(N) \to H^*(N//K)$ is surjective [33], at least when N is compact.

4 Witten's non abelian localization

Let M be a K-manifold. Let $\kappa \in \mathcal{A}^1(M)$ be a K-invariant 1-form. Let

(12) $$C := \{x \in M; \langle \kappa_x, V_x\phi\rangle = 0\,, \text{for all}\, \phi \in \mathfrak{k}\}.$$

Witten considers the exact equivariant form $D\kappa(\phi) := -\langle \kappa, V\phi\rangle + d\kappa$.

Then for any test function $F(\phi)$ on $\mathfrak{k}$, the integral

$$\int_{M\times\mathfrak{k}} \alpha(\phi)F(\phi)d\phi = \int_M \int_{\mathfrak{k}} e^{iaD\kappa(\phi)}\alpha(\phi)F(\phi)d\phi \tag{13}$$

localizes on C when a tends to ∞.

Paradan systematized this localization, by a partition of unity argument. Indeed, the following theorem is immediate to prove.

Theorem 3 *(Paradan) Let χ be a K-invariant function on M supported on a small neighborhood of C and such that $\chi = 1$ on a smaller neighborhood of C. Then*

$$\mathrm{Par}(\phi) := \chi - id\chi \int_0^\infty e^{iaD\kappa}\kappa da$$

is a closed equivariant form in $\mathcal{A}^{-\infty}(\mathfrak{g}, M)$ supported near C. Furthermore, we have the equation in $\mathcal{A}^{-\infty}(\mathfrak{g}, M)$:

$$\mathrm{Par} = 1 + DB \qquad \text{with } B := (1-\chi)i\int_{a=0}^\infty e^{iaD\kappa}\kappa da.$$

The equation is immediate to verify. The fact that forms $\int_{a=0}^\infty e^{iaD\lambda}\lambda da$ converges against a test function $Test(\phi)$ at each point $x \in M$ not in C follows from standard estimates on Fourier transforms.

In particular, the integral $I(\phi) := \int_M \alpha(\phi)$ of a closed equivariant form with C^∞-coefficients can be replaced by $\int_M \alpha(\phi)\mathrm{Par}(\phi)$ which is localized in an integral supported near C. As application, we recover the exact stationary phase, or the AB-BV localization formula, with the following tool. For a S^1-action with generator J, we choose $\kappa := \langle VJ, ?\rangle$, where $\langle,\rangle$ is an S^1-invariant Riemannian metric on M and we can use Paradan's partition of unity argument in a very elementary way as described in [50]. We state AB-BV in the case of isolated fixed points (this formula is also called abelian localization, as it concerns essentially an action of S^1 on M).

Theorem 4 *([53],[15],[4]) Let S^1 acting on M with isolated fixed points. Let $\alpha(\phi)$ be a closed equivariant form with C^∞ coefficients. Then*

$$\frac{1}{(2\pi)^{\frac{\dim M}{2}}}\int_M \alpha(\phi) = \sum_{p\in\{\text{fixed points}\}} \frac{i_p^*\alpha(\phi)}{\sqrt{\det_{T_pM} L_p(\phi)}}.$$

CHECK

The most important application of Witten deformation method (13) takes place in the following context. Assume that M is a compact Hamiltonian manifold with moment map $\mu : M \to \mathfrak{k}^*$. We choose a K-invariant identification $\mathfrak{k}^* \to \mathfrak{k}$ and a K-invariant Riemannian metric on M. Then the following vector field $V\mu$ with $V_m\mu := \exp(-\epsilon\mu(m)) \cdot m$ is a K-invariant vector field and $\kappa := \langle V\mu, ?\rangle$ is a K-invariant form. In this case the set C (Formula (12)) is the set of critical points of the invariant function $\|\mu\|^2$ on M. One connected component of C is the set $\mu^{-1}(0)$ of zeros of the moment map (if not empty). Recall that $\mu^{-1}(0)/K$ is the reduced space $M//K$.

The following theorem follows from Witten's deformation argument. It is referred to as "non abelian localization formula" as the group K is not assumed to be a torus. Let me point out that Witten non abelian localization formula is powerful also for torus actions.

Theorem 5 *(Witten) Let M be a compact Hamiltonian K-manifold and $\alpha(\phi)$ a equivariantly closed form with polynomial coefficients. Assume that* 0 *is a regular value of the moment map. Then*

$$\int_{\mathfrak{k}}(\int_M e^{i\Omega(\phi)}\alpha(\phi))d\phi = (2i\pi)^{\dim \mathfrak{k}}\operatorname{vol}(K)\int_{M//K} e^{i\Omega_{red}}\alpha_{red}.$$

Let me explain the meaning of the first integral. Let $I_M(\phi) := \int_M e^{i\Omega(\phi)}\alpha(\phi)$. This is an analytic function on $\mathfrak{k}$ with at most polynomial growth. We compute $\int_{\mathfrak{k}} e^{i\langle\xi,\phi\rangle}I_M(\phi)d\phi$ in the sense of Fourier transform. This Fourier transform is a polynomial near 0 (this is part of the theorem). The meaning of $\int_{\mathfrak{k}} I_M(\phi)d\phi$ is the value of this polynomial at $\xi = 0$.

The theorem above is used to compute integrals on reduced spaces. Indeed the right member of the equality in the theorem above is an integral of a cohomology class over the reduced space $M//K$ of M, which is difficult to compute. Instead, we first compute an equivariant integral on the original space M (easy to do thanks to the usual reduction to the maximal torus T and the AB-BV localization formula, and we obtain $I_M(\phi)$ as a sum of polynomials functions on $\mathfrak{t}$ multiplied by imaginary exponential functions and divided by product of linear forms. Then we have to compute the value of the Fourier transform of $I_M(\phi)$ at the point 0.

We stated the theorem for the reduction at 0, so this is the reason why only the component $\mu^{-1}(0)$ of C occurs in this formula. By the shifting trick, we can also compute by a similar method integrals on reduced spaces at any point of reduction ξ.

In Subsection 7.1, we will explain how the value of the Fourier transform of the inverse of product of linear functions can be computed efficiently at any point $\xi \in \mathfrak{t}^*$.

Many important symplectic manifolds arise as reduction of symplectic manifolds. Atiyah-Bott using Yang-Mills theory constructed the moduli space M_g of flat bundles over curves of genus g, as a symplectic reduction of an infinite dimensional manifold. Later Jeffrey-Kirwan [30] and Alekseev-Malkin-Meinrenken [1] gave finite dimensional reduction models. Applications on intersection numbers on M_g have been obtained using Witten's theorem and diverse refinements ([30], [1]). Ourselves, we give results on toric manifolds and computation on polytopes in Section 7, inspired by this theorem.

Remark 6 *We may also consider the case where N is non compact with proper moment map (and conditions to assure convergence). There is also a doubly-equivariant version of this theorem which allows to compute equivariant integrals on reduced manifolds (when the original manifold N is acted on by two commuting groups K_1, K_2).*

Witten's theorem has been reproved and sharpened by many authors ([29],[49], [40],[**?**]). Let me comment on Paradan's method which gives precise information on contributions of all connected components of C, the set of critical points of $\|\mu\|^2$. This refinement is necessary for applications such as asymptotic behavior of Witten's integral [39], Guillemin-Sternberg conjecture for discrete series (see Section 6), jumps [43] of symplectic volumes by flips, etc....

We write $C = \cup C_F$ where C_F are the connected components of C. The set $\mu^{-1}(0)$ is one connected component (if non empty). If C_{fix} is a connected component of the set of fixed points of the action, then C_{fix} is also a component of C. In general, the structure of C_F is a mixture of these cases and inducing. Consider Paradan's form Par (in Theorem 3) supported near C. Write $\mathrm{Par} := \sum_F \mathrm{Par}_F$ where Par_F is supported on a small neighborhood of a connected component C_F of C. The formula for Par may look awesome. However it is possible to determine it "concretely" near each connected component. Assume (for simplicity) that C_F is smooth with normal bundle N_F. Imbedding a neighborhood of C_F in N_F, we may consider Par_F

as a compactly supported form on the non compact space N_F. It is then almost uniquely determined by the fact that it is congruent to 1. Replace the form $\kappa := \langle V\mu, ?\rangle$ (vanishing on C_F) by its smallest homogeneous component κ' in the normal direction. Then consider the form $P' := e^{iD\kappa'}$ (which is with C^∞ coefficients, but over the non compact space N_F) so that $P'\mathrm{Par}_F$ makes sense. As $\mathrm{Par}_F = 1 + DB_F$ (in N_F), $P' = 1 + DB'$, we obtain $\mathrm{Par}_F = P' + D(B_F P' + B'\mathrm{Par}_F)$. It remains to see that $B_F P'$ is in the space $\mathcal{A}^{-\infty,dec}(\mathfrak{k}, N_F)$. The two extreme cases are when K acts (infinitesimally) freely on $C_0 := \mu^{-1}(0)$, or when $C_p := \{p\}$ is a fixed point for the action of K, with moment image $\xi \neq 0$. In both cases, we easily identify Par_F as equivalent to $e^{iD\kappa'}$, a C^∞ form congruent to 1 on the normal bundle, similar to the ones described by Formula (2) in Subsection 3.2. The general case is a mixture of these cases and inducing.

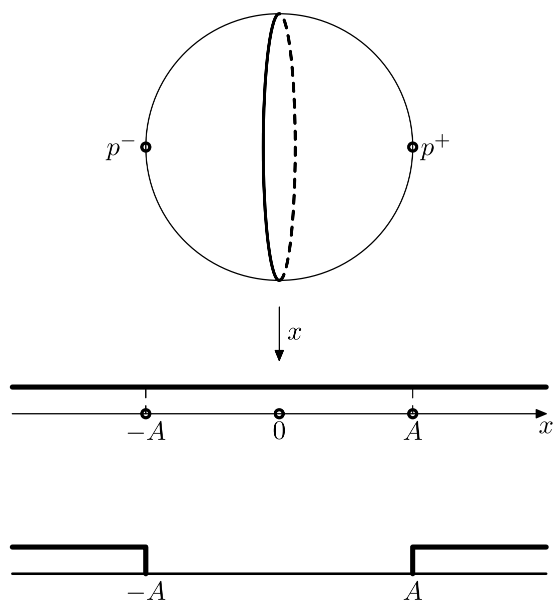


Figure 5: Decomposition of equivariant volumes

Example 7 *Return to Example 4 of the sphere $M = \{x^2 + y^2 + z^2 = A^2\}$, with moment map $\mu(x, y, z) = x$. The critical values of x^2 are $0, A, -A$. The set of critical points has three connected components. The circle C_0 drawn in black in Figure 5 and $\{p^+\}$,$\{p^-\}$. The normal bundle to C_0 is identified* *to T^*S^1 and the normal bundle to p^+, p^- to $\mathbb{R}^2$. The forms* Par_F *are easily identified to forms given by Formulae (2) for T^*S^1 or the symplectic space $\mathbb{R}^2$. Let*

$$v_M(\phi) := \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)}.$$

We obtain $v_M(\phi) = v_0(\phi) + v_{p^-}(\phi) + v_{p^+}(\phi)$ with

$$v_0(\phi) := \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)}\mathrm{Par}_{C_0}(\phi) = \int_{-\infty}^{\infty} e^{i\phi x}dx,$$

$$v_{p^-}(\phi) := \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)}\mathrm{Par}_{p^-}(\phi) = -\int_{-\infty}^{-A} e^{i\phi x}dx,$$

$$v_{p^+}(\phi) := \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)}\mathrm{Par}_{p^+}(\phi) = \int_{A}^{\infty} e^{-i\phi x}dx.$$

Thus we reobtain our decomposition of $I(\phi) = \int_{-A}^{A} e^{i\phi x}dx$ as the sum (4) of 3 generalized functions given in Section 2.

5 Index of transversally elliptic operators

Consider a compact even dimensional oriented manifold M. For simplicity we assume M provided with an almost complex structure. We choose an Hermitian metric $\|\xi\|^2$ on T^*M. For $[x,\xi] \in T^*M$, the symbol of the Dolbeaut-Dirac operator $\overline{\partial} + \overline{\partial}^*$ is the Clifford multiplication $c(\xi)$ on $\Lambda T^*_x M$. It is invertible for $\xi \neq 0$, since $c(\xi)^2 = -\|\xi\|^2$. Let $\mathcal{E}$ be an auxiliary vector bundle over M, then $c_{\mathcal{E}}([x,\xi]) := c(\xi)\otimes \mathrm{Id}_{\mathcal{E}_x}$ defines an element of the $\mathbf{K}$-group of T^*M. Assume that a compact group K acts on M and $\mathcal{E}$. Now the topological index $\mathrm{Index}(c_{\mathcal{E}})$ of $c_{\mathcal{E}} \in \mathbf{K}_K(T^*M)$ is an invariant function on K (which computes the equivariant index of the K-invariant operator $\overline{\partial}_{\mathcal{E}} + \overline{\partial}^*_{\mathcal{E}}$). Atiyah-Bott-Segal-Singer expresses $\mathrm{Index}(c_{\mathcal{E}})(k)$ $(k \in K)$ in function of the fixed points of k on M. We constructed (see [14]) the equivariant Chern character $\mathrm{ch}(\phi,\mathcal{E})$ of the bundle $\mathcal{E}$ and the equivariant Todd class $\mathrm{Todd}(\phi, M)$ such that (for ϕ small)

$$\mathrm{Index}(c_{\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim M)/2}\int_M \mathrm{ch}(\phi,\mathcal{E})\mathrm{Todd}(\phi,M). \tag{14}$$

For $\phi = 0$, this is Atiyah-Singer formula (in the conventions of [14], $\mathrm{ch}(0,\mathcal{E}), \mathrm{Todd}(0,M)$ are not exactly $\mathrm{ch}(\mathcal{E}), \mathrm{Todd}(M)$ so this is why unfortunately factors $(2i\pi)$ arrive.) Formula (17) is a "delocalization" of the A-B-B-S formula. The delocalized index formula can be adapted to new cases. Indeed in the following two contexts, the index exists in the sense of generalized functions but cannot be always computed in terms of fixed point formulae.

- Index of transversally elliptic operators.
- L^2-index of some elliptic operators on some non compact manifolds (as in Narasimhan-Okamoto, Parthasarathy, Atiyah-Schmid, Connes-Moscovici, etc...).

Recall Atiyah's definition of transversally elliptic operators [3]. Let N be a K-manifold and $T^*_K N$ be the conormal bundle to K-orbits. The symbol of a transversally elliptic pseudo-differential operator S defines an element $\sigma(S)$ of $\mathbf{K}_K(T^*_K N)$ (we say briefly that an element of $\mathbf{K}_K(T^*_K N)$ is a transversally elliptic bundle map). The index of the operator S (the virtual vector space of K-finite solutions of S) is a generalized function on K. In [17], we gave a cohomological formula for the index of S in function of $\sigma(S) \in \mathbf{K}_K(T^*_K N)$, as an equivariant integral on T^*N in the spirit of the delocalized formula (17). This formula was strongly inspired by Bismut's ideas on delocalizations [19] and Quillen superconnection formalism.

The following example shows that, contrary to the melancholic remark of Atiyah about his work on transversally elliptic operators (page 6, vol 4, [2]), there are many transversally elliptic bundle maps of great interest.

Consider a K-Hamiltonian manifold N with moment map μ. Consider the form $\kappa := \langle V\mu, ?\rangle$. The analogue of Witten's deformation is the bundle map

$$c_{\mu,\mathcal{E}}(x,\xi) := c(x, \xi + \kappa_x) \otimes \mathrm{Id}_{\mathcal{E}_x}, \tag{15}$$

QUESTION PARADAN :IL ME SEMBLE QUE JE DOIS METTRE + car ma definition de l'application moment n'est pas la meme !! (COMMENT DECIDER ??) defined by Paradan [**?**]. Remark that $c_{\mu,\mathcal{E}}(x,\xi)$ is invertible except if $\xi = -\kappa_x$. If furthermore $[x,\xi] \in T^*_K N$, this implies $\xi = 0, \kappa_x = 0$. Indeed, in our identification of T^*M with TM, κ_x is tangent to Kx while ξ is normal.

A related operator is defined by Braverman [22]. When N is compact, $c_{\mu,\mathcal{E}}$ is transversally elliptic and equal in $\mathbf{K}$-theory to the elliptic symbol $c_{\mathcal{E}}$, via the deformation $c(x, \xi + a\kappa_x) \otimes 1_{\mathcal{E}}$, for $a \in [0,1]$.

Paradan's construction may also define a transversally elliptic element for some non compact manifolds.

Proposition 8 *[41] Consider a K-Hamiltonian manifold N with proper moment map μ. Assume that the set C of critical points of the function $\|\mu\|^2$ is compact. Then $c_{\mu,\mathcal{E}}$ is transversally elliptic on T^*N with support the zero section $[C, 0]$.*

Conjecture: The index of $c_{\mu,\mathcal{E}}$ is given (for ϕ small) by the formula

$$\text{Index}(c_{\mu,\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim N)/2}\int_N \text{ch}(\phi,\mathcal{E})\text{Todd}(\phi,N)\text{Par}(\phi) \tag{16}$$

and similar formulae for $\text{Index}(c_{\mu,\mathcal{E}})(s\exp\phi)$ with $s\cdot\phi = \phi$ at any point s of K.

When M is compact, Formula (16) is true, as it reduces to Formula (17) since $\text{Par}(\phi)$ is equal to 1 in cohomology. But even in this case, Formula (16) has strong implications,as the symbol $c_{\mathcal{E}}$ is broken to several parts according to the connected components of C.

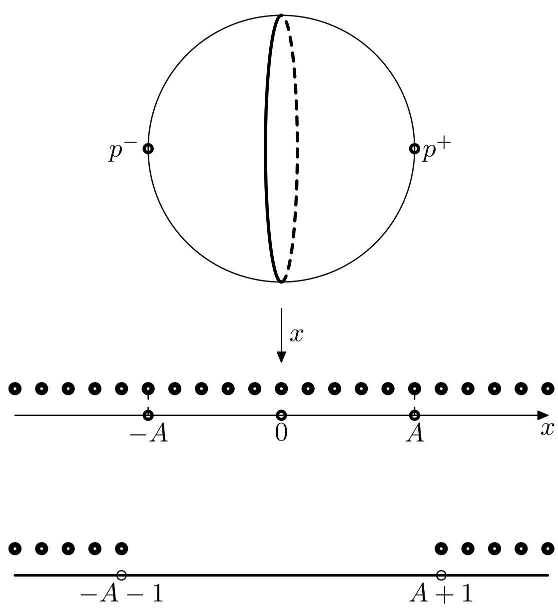


Figure 6: Decomposition of equivariant indices

Example 9 *Return to Example 7. Let A be a positive integer. We identify $P_1(\mathbb{C})$ to $M_A := \{x^2+y^2+z^2 = A^2\}$ by the map*

$$[z_1, z_2] \mapsto (A\frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}, 2A\frac{\Re(z_1\overline{z_2})}{|z_1|^2 + |z_2|^2}, 2A\frac{\Im(z_1\overline{z_2})}{|z_1|^2 + |z_2|^2}),$$

the action $(e^{2i\pi\phi}z_1, z_2)$ becoming the rotation around the axe x. We consider the Dolbeaut-Dirac operator D_{2A} on $P_1(\mathbb{C})$ with solution space $\oplus_{j+k=2A}\mathbb{C}z_1^j z_2^k$. Twisting the action by $e^{2i\pi\phi A}$, its equivariant index is $\sum_{k=-A}^{A} q^k$ with $q = e^{2i\pi\phi}$. We may deform $D_{2A} = D_0 + D_{p^+} + D_{p^-}$ in the sum of 3 transversally elliptic operators with support $[C_0, 0]$, $[p^+, 0]$, $[p^-, 0]$, that is each operator is supported on a connected component of the critical set of $\|\mu\|^2$. Near C_0, to compute the index of D_0, we are led to compute the set of solutions of the Dolbeaut operator on the complex manifold $\mathbb{C}/\mathbb{Z} = S^1 \times \mathbb{R}$ (the action of $S^1 = \mathbb{R}/\mathbb{Z}$ being by translations) and we get all function $e^{2i\pi kz}$ for any $k \in \mathbb{Z}$. Thus

$$\mathrm{IndexD_0} = \sum_{k=-\infty}^{\infty} q^k.$$

Near the fixed points p^+, p^-, we get the index of the operators $[\overline{\partial}^{\pm}]$ (see [3]) on $\mathbb{C}$ (shifted). We obtain

$$\mathrm{IndexD_{p^+}} = -\sum_{k=(A+1)}^{\infty} q^k,$$

$$\mathrm{IndexD_{p^-}} = -\sum_{k=-\infty}^{-A-1} q^k.$$

The equality

$$\mathrm{IndexD_{2A}} = \mathrm{IndexD_0} + \mathrm{IndexD_{p^+}} + \mathrm{IndexD_{p^-}}$$

is Formula (2) in Section 2.

It might happen that the integral $\int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N)$ over a non compact manifold N is already convergent (in the generalized sense), and as $P = 1$ in cohomology, it might happen (after checking convergence of the boundary term) that the following equality holds

$$\mathrm{Index}(c_{\mu,\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim N)/2}\int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N).$$

This is indeed the case for discrete series. To be precise, we have to rephrase our theorem in the spin context. If N is an even dimensional oriented spin manifold, and $\mathcal{E}$ a twisting bundle, we denote by $\sigma(\xi)$ the Clifford action of $\xi \in T^*M_x$ on spinors, and by $\sigma_{\mathcal{E}}$ the symbol of the twisted Dirac operator $D_{\mathcal{E}}$. If M is a compact K-manifold, the equivariant index of $D_{\mathcal{E}}$ is given by a formula similar to (17)

$$\text{Index}(\sigma_{\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim M)/2} \int_M \text{ch}(\phi, \mathcal{E})\hat{\text{A}}(\phi, M), \tag{17}$$

where the $\hat{A}$ equivariant class replaces the equivariant Todd class.

If N is a K-Hamiltonian manifold with proper moment map μ, under the same hypothesis as in Theorem 8, the bundle map $\sigma_{\mu,\mathcal{E}}(x,\xi) = \sigma(\xi + \kappa_x) \otimes \text{I}_{\mathcal{E}_x}$ is transversally elliptic and its equivariant index is a generalized function on K.

Let G be a real reductive Lie group with maximal compact subgroup K. We assume that the maximal torus T of K is a maximal torus in G. Let $N := G\lambda$ be the orbit of a regular admissible elliptic element $\lambda \in \mathfrak{k}^*$. Harish-Chandra associates to λ a representation of G, realized as a L^2-index of the twisted Dirac operator D_λ. The moment map μ for the K-action on N is the projection $G\lambda \to \mathfrak{k}^*$ and is proper. Furthermore the set C is very easy to compute in this case, it is connected and consists of the compact orbit $K \cdot \lambda$.

Theorem 10 *(Paradan [42]) The character of discrete series $\Theta^G(\lambda)$ restricted to K is the index of the transversally elliptic element $\sigma_{\mu,\mathcal{L}_\lambda}$ on N.*

Here $\mathcal{L}_\lambda$ is the Kostant line bundle $G \times_{G(\lambda)} \mathbb{C}_\lambda$ on $N = G/G(\lambda)$. A calculation of the index of $\sigma_{\mu,\mathcal{L}_\lambda}$ (which is supported near $K\lambda$) leads immediately to Blattner's formula for $\Theta^G(\lambda)|_K$.

6 Quantization and symplectic quotients

Let N be a G-manifold (N, G non necessarily compacts), and $\mathcal{E}$ a G-equivariant bundle on N with G-invariant connection ∇. We can then construct the closed equivariant form $\text{ch}(\phi, \mathcal{E})$ ([16],[21]). For simplicity, I assume the existence of a G-invariant complex structure (see exact formulations in [48]). Then I conjectured (under additional conditions that i do not know how to control)

Conjecture: There exists a representation $Q(N,\mathcal{E})$ of G such that the character $\mathrm{Tr}_{Q(N,\mathcal{E})}(g)$ is given by the formula

$$\mathrm{Tr}_{Q(N,\mathcal{E})}(\exp\phi) = (2i\pi)^{-(\dim N)/2}\int_N \mathrm{ch}(\phi,\mathcal{E})\mathrm{Todd}(\phi,N) \tag{18}$$

for ϕ small (and similar formulae near any elliptic point s of G).

Thus, via integration of equivariant cohomology forms, it should be possible to define a push-forward map from a generalized **K**-theory of vector bundles with connections on G-manifolds N (with some conditions such that a "quotient $N//G$" is compact) to $R^{-\infty}(G)$.

Remark 11 *When N is a coadjoint admissible regular orbit of any real algebraic Lie group G and $\mathcal{L}$ the Kostant line bundle (more precisely only $\mathcal{L}^2$ is a line bundle), Formula (18) (with the $\hat{A}$ class instead of the Todd class) becomes Kirillov's universal formula [32] for characters (proved by Kirillov for compact and nilpotent groups, by Duflo, Rossmann, Khalgui, Vergne, etc, for any real algebraic group). If N,G are compact, Formula* **??** *(with $\hat{A}$) is the equivariant index formula for the Dirac operator twisted by $\mathcal{E}$. Thus Formula 18 (modified as in [48]) is a fusion of the universal character formula, and of the formulae of Atiyah-Segal-Singer for indices of twisted Dirac operators.*

Let now (M,Ω) be a symplectic manifold with Hamiltonian action of a compact group K. We assume the existence of a K-equivariant line bundle $\mathcal{L}$ on M with connection ∇ of curvature equal to $-i\Omega$ (in other words, M is prequantizable in the sense of [34]). We take an almost complex structure compatible with Ω, as in Guillemin-Sternberg (see [38]). Then we denote $Q(M,\mathcal{L})$ simply by $Q(M)$. This is a canonical finite dimensional (virtual) representation $Q(M)$ of K, the quantification of the symplectic manifold M. Levels of energy of the elements $\phi\in\mathfrak{k}$ in $Q(M)$ should be the "quantum" version of the level of energy of the Hamiltonian function $\langle\mu,\phi\rangle$ on M (see [51] for survey). Guillemin-Sternberg [**?**] conjectured in 1982 that the multiplicity of the irreducible representation V_ξ of K (of highest weight $\xi\in\mathfrak{t}^*_+\subset\mathfrak{k}^*$) in the representation $Q(M)$ is equal to $Q(M_{\xi,red})$ and proved it for the case of Kaehler manifolds. This is summarized by the slogan: "Quantification commutes with Reduction". In other words, when $\xi=0$, we should have the equality

$$\int_K \mathrm{Tr}_{Q(M)}(k)dk = \int_{M//K}\mathrm{ch}(\mathcal{L}//K)\mathrm{Todd}(M//K).$$

Although thanks to Atiyah-Bott Lefschetz formula, a fixed point formula exists for $\mathrm{Tr}_{Q(M)}(k)$, it is difficult to extract Guillemin-Sternberg conjecture directly from this formula. Thus this conjecture (fundamental for the credo of quantum mechanics) remained unproved for years. Witten's inversion formula [54]

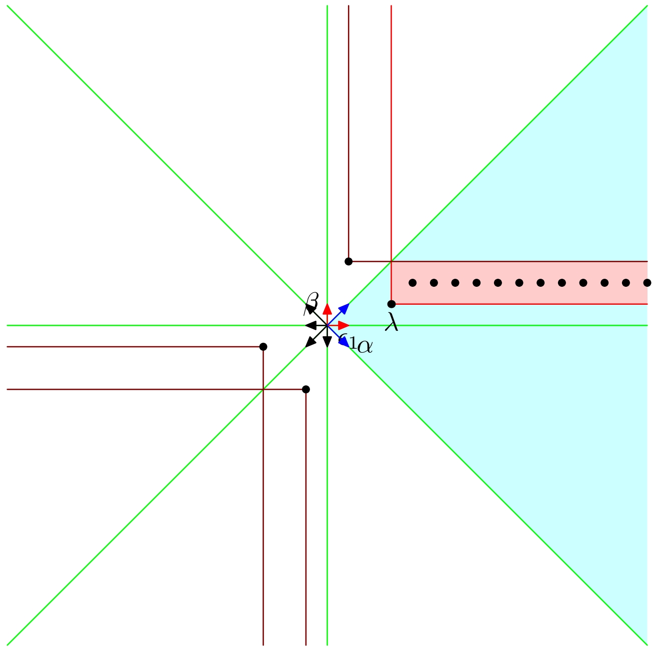
$$\int_k (\int_M e^{i\Omega(\phi)}\alpha(\phi))d\phi = (2i\pi)^{\dim \mathfrak{k}} \mathrm{vol}(K) \int_{M//K} e^{i\Omega_{red}}\alpha_{red}$$

is in strong analogy with this conjecture. In particular, apart from factors of $2i\pi$, the form $e^{i\Omega_{red}}$ is just $\mathrm{ch}(\mathcal{L}//K)$. So it gave a new impulse to research on this question. Shortly after Witten's article, Meinrenken and I, we produced independently proofs in line with Witten's argument for the special (and relatively easy) case of the torus. Later, Meinrenken and Sjamaar [38] used in a very subtle way Atiyah-Bott Lefschetz formula, and symplectic cutting, to produce a proof for any compact K-Hamiltonian manifold.

Clearly the conjecture make sense even for a non compact Hamiltonian K-manifold N, if the moment map is proper, whenever a Hilbert space $Q(N)$ can be constructed, let's say via L^2-cohomology.

With the help of his deformation via the transversally elliptic operator $\sigma_{\mu,\mathcal{L}_\lambda}$, Paradan [42] can prove this conjecture (in the spin context) when $N = G\lambda$ is an admissible coadjoint orbit of a reductive real Lie group G and K a subgroup of G such that the moment map $\mu : N \to \mathfrak{k}^*$ is proper. This is the case when K is the maximal compact subgroup of G. In particular, irreducible representations Θ^K_ξ (of highest weight $\xi - \rho_{\mathfrak{k}}$) of K occurring in the discrete series $\Theta^G(\lambda)|_K$ are such that ξ lies in the interior of the Kirwan polytope $\mu(N) \cap \mathfrak{t}^*_+$. This is a strong constraint on representations appearing in $\Theta^G(\lambda)|_K$.

Here is the drawing for the restriction of the representation $\Theta^G(\lambda)$ of $SO(4,1)$ to $SO(4)$. The black dots are the ξ such that $\Theta^K(\xi)$ occurs in $\Theta^G(\lambda)$ (they all occur with multiplicity 1).



7 Applications

7.1 Convolution of Heaviside distributions and cycles in the complement of a set of hyperplanes

To numerically compute integrals on reduced spaces, we need to compute the value at a point of $\mathfrak{t}^*$ of the convolution of Heaviside distributions supported on the lines $\mathbb{R}^+\beta$ (with $\beta \in \mathfrak{t}^*$). This becomes algorithmically hard if there is a large number of convolutions. Some ideas coming from Jeffrey-Kirwan [29], Brion-Vergne [**?**], Szenes-Vergne [46],[47], de Concini-Procesi [25] have led to progress on this topic.

Let us consider a set $\mathcal{B} := \{\beta_1, \ldots, \beta_n\}$ of linear forms β_a on a vector space V (of dimension r) all in a open half-space of V^*. We assume that the set $\mathcal{B}$ span V^*. By definition, an element $\xi \in V^*$ is regular if it does not lie in a cone spanned by $(r-1)$ elements of $\mathcal{B}$. The convolution H of Heaviside distributions of all the half lines $\mathbb{R}^+\beta_a$ is a locally polynomial function on V^* and is continuous on the cone $\mathrm{Cone}(\mathcal{B})$ spanned by $\mathcal{B}$. Our problem is to compute $H(\xi)$ at a particular point $\xi \in V^*$. In principle, $H(\xi)$ is given by the following limits of integrals (on the non compact "cycle" V of dimension r, and in the sense of Fourier transforms):

$$H(\xi) = \lim_{\epsilon \to 0} (2i\pi)^{-r} \int_V e^{-i\langle \xi, v\rangle} \frac{1}{\prod_{i=1}^n \langle \beta_i, v + i\epsilon\rangle} dv$$

where ϵ is in the dual cone to Cone$(\mathcal{B})$.

Consider the complement of the hyperplanes defined by $\mathcal{B}$ in the complexified space $V_{\mathbb{C}}$:

$$V(\mathcal{B}) := \{v \in V_{\mathbb{C}}; \langle v, \beta\rangle \neq 0 \, \text{for all} \, \beta \in \mathcal{B}\}.$$

Jeffrey-Kirwan introduced a residue calculus on the space of functions defined on $V(\mathcal{B})$. A rational function on $V(\mathcal{B})$ is of the form $R(v) = \frac{Q(v)}{\prod_{i=1}^n \langle \beta_i, v\rangle^{n_i}}$ where $Q(v)$ is a polynomial. The following theorem is a result of Jeffrey-Kirwan ideas, refined by Brion-Vergne and Szenes-Vergne. We still denote by dv the holomorphic r form $dv_1 \wedge \cdots \wedge dv_r$ on $V_{\mathbb{C}}$.

Theorem 12 *Let $\xi \in V^*$ be regular. There exists a compact oriented cycle $Z(\xi, \mathcal{B})$ of dimension r contained in $V(\mathcal{B})$ such that for any rational function R on $V(\mathcal{B})$*

$$\lim_{\epsilon \to 0} \int_V e^{-i\langle \xi, v\rangle} R(v + i\epsilon) dv = \int_{Z(\xi, \mathcal{B})} e^{-i\langle \xi, v\rangle} R(v) dv.$$

The homology class of $Z(\xi, \mathcal{B})$ depends only of the connected component $\mathfrak{c}(\xi)$ of the set of regular points of V^* containing ξ. If R is rational and homogeneous of degree $-r$, then the Fourier transform of R is a constant on each connected component $\mathfrak{c}$ of the set of regular points of V^*. On $\mathfrak{c}(\xi)$, this constant is $(2i\pi)^{-r} \int_{Z(\xi, \mathcal{B})} R(v) dv$.

Szenes-Vergne [47] gave a formula for the cycle $Z(\xi, \mathcal{B})$.

Theorem 13 *(Szenes-Vergne). Let $\xi \in \mathfrak{t}^*$ be regular. Write $\xi = \sum_{i=1}^n m_i \beta_i$ where all m_i are strictly positive. Then*

$$Z(\xi, \mathcal{B}) = \{v \in V(\mathcal{B}); \sum_{i=1}^n \log |\langle m_i \beta_i, v\rangle| m_i \beta_i = -\xi\}.$$

This description of $Z(\xi, \mathcal{B})$ is strongly related to mirror symmetry and quantum cohomology, but this is still highly mysterious for us.

We gave a simple algorithm (further simplified by De Concini-Procesi [25]) to compute the homology class of $Z(\xi, \mathcal{B})$ as a disjoint union of tori, so that integration is simply the algebraic operation of taking ordinary iterated residues. Indeed if $T(\boldsymbol{\epsilon}) \subset V(\mathcal{B})$ is a compact torus of the form (in some coordinates $(v_1, v_2, \ldots, v_r) \in \mathbb{C}^r := V_{\mathbb{C}}$)

$$T(\boldsymbol{\epsilon}) := \{v \in V(\mathcal{B}); |v_a| = \epsilon_a, \text{ for } a = 1, \ldots, r\},$$

with $\boldsymbol{\epsilon} := [\epsilon_1 << \epsilon_2 << \cdots << \epsilon_r]$ a sequence of increasing real numbers (here $\epsilon_1 << \epsilon_2$ meaning that ϵ_2 is significantly greater than ϵ_1, see [47] for precise definitions), then the integration on $T(\boldsymbol{\epsilon})$ of a function $F(v_1, v_2, \ldots, v_r)$ with poles on the hyperplanes $\mathcal{B}$ is

$$\frac{1}{(2i\pi)^r}\int_{T(\boldsymbol{\epsilon})} F(v_1, v_2, \ldots, v_r)dv = \mathrm{res}_{v_r=0}\,\mathrm{res}_{v_{r-1}=0}\cdots\mathrm{res}_{v_1=0}\,F(v_1, v_2, \ldots, v_r)d,$$

where each residue is taken assuming that the variables with higher indices have a fixed, nonzero value.

Let me explain why this algorithm is efficient for computing the covolution $H(\xi)$ of a large number of Heaviside distributions in a vector space of small dimension. The usual way to compute $H(\xi)$ would be by induction on the cardinal of Σ (the divide and conquer method ??). Here we fix ξ and we compute the cycle $Z(\xi, \mathcal{B})$ (depending of ξ) by induction on the dimension of V. It can be done quite quickly using the maximal nested sets of De Concini-Procesi, at least for classical root systems [6].

7.2 Intersection numbers on Toric manifolds

Let T be a torus of dimension r acting diagonally on $N := \mathbb{C}^n$ with weights $\mathcal{B} := [\beta_1, \beta_2, \ldots, \beta_n]$. We assume that the cone $\mathrm{Cone}(\mathcal{B})$ spanned by the vectors β_a is an acute cone in $\mathfrak{t}^*$ with non empty interior. The moment map $\mu : \mathbb{C}^n \to \mathfrak{t}^*$ for the action of T is $\mu(z_1, \ldots, z_n) = \sum_{a=1}^n |z_a|^2\beta_a$. The reduced space at a point $\xi \in \mathrm{Cone}(\mathcal{B})$ is $N_\xi = \mu^{-1}(\xi)/T$. It is an orbifold if ξ is regular. The space N_ξ is still provided with a Hamiltonian action of the full diagonal group $H := (S^1)^n$ with Lie algebra $\mathfrak{h} := \{\sum_{a=1}^n \nu_a J_a\}$. The image of N_ξ under the moment map for H is the convex polytope

$$P(\xi) := \{\sum_{a=1}^n x_a J^a \in \mathfrak{h}^*; x_a \geq 0; \sum_{a=1}^n x_a\beta_a = \xi\}.$$

Computing the volume of the polytope $P(\xi)$ is the same as computing the symplectic volume of N_ξ.

Example 14 :Hirzebruch surface. *Let* $\mathbb{C}^4 := \{[z_1, z_2, z_3, z_4]\}$. *We consider the action of* $T := S^1 \times S^1$ *on* $\mathbb{C}^4$ *with weights* $\phi_1, \phi_2, \phi_1 + \phi_2, \phi_1$. *Let* $\kappa := 3J_1^* + 2J_2^*$. *Then* $N_\kappa := \{[z_1, z_2, z_3, z_4]; |z_1|^2 + |z_2|^2 + |z_3|^2 = 3; |z_3|^2 + |z_4|^2 = 2\}/T$ *and*

$$P(\kappa) := \{[x_1, x_2, x_3, x_4] \geq 0; x_1 J^1 + x_2 J^2 + x_3(J^1 + J^2) + x_4 J^1 = 3J^1 + 2J^2\}.$$

The polytope $H \subset \mathbb{R}^2$*, considered in Subsection 2.3, Figure 2, is isomorphic to* $P(\kappa)$ *by* $I(x, y) := [x - 3, 3 - y, y - 1, 7 - (x + y)]$. *(By GIT,* $N_\kappa = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.*)*

The T-equivariant cohomology of N is $S(\mathfrak{t}^*)$. If ξ is regular, Kirwan map gives a surjective map $\chi(Q) := Q_{red}$ from $S(\mathfrak{t}^*)$ to $H^*(N_\xi)$. The following theorem allows us to compute integral on reduced spaces.

Theorem 15 *[47] Let* $\xi \in \mathfrak{t}^*$ *be regular and* $Q \in S(\mathfrak{t}^*)$*, then*

$$\int_{N_\xi} \chi(Q) = (2i\pi)^{-r} \int_{Z(\xi,\mathcal{B})} \frac{Q(\phi)}{\prod_{i=1}^n \langle \beta_i, \phi \rangle} d\phi$$

Remark 16 *This theorem can be proved directly. Let us see why it could also be seen as a special case of Witten's formula. It is easy to see that both members are* 0 *when* Q *is homogeneous of degree different from* $n - r$ *(the complex dimension of the orbifold* N_ξ*). Thus, we choose* Q *homogeneous of degree* $(n - r)$. *Let us compute on* $N := \mathbb{C}^n$ *the equivariant integral* $\int_N Q(\phi) e^{i\Omega(\phi)}$. *Computing the integral* $\int_{N_\xi} e^{i\Omega_{red}} Q_{red}$ *over the reduced space is the same (up to multiplicative constants) as computing the Fourier transform of the equivariant integral on* N *at the point of reduction* ξ. *Now, as* Q_{red} *is of top degree,* $\int_{N_\xi} e^{i\Omega_{red}} Q_{red} = \int_{N_\xi} Q_{red}$. *On the other hand,*

$$\int_N Q(\phi) e^{i\Omega(\phi)} = Q(\phi) \int_N e^{i \sum_{a=1}^n \langle \beta_a, \phi \rangle |z_a|^2} \prod_{a=1}^n dx_a dy_a$$

is (up to multiplicative constants) just $\frac{Q(\phi)}{\prod_{a=1}^n \langle \beta_a, \phi \rangle}$. *By Theorem 12, the value at* ξ *of the Fourier transform of the rational function* $\frac{Q}{\prod_{a=1}^n \beta_a}$ *can be calculated as an integral over the cycle* $Z(\xi, \mathcal{B})$.

If $Q(\phi) := \langle \phi, \xi \rangle^{n-r}$, the cohomology class Q_{red} is just the symplectic form of $N_{\xi,red}$. This way we obtain the formula:

Corollary 17 *Let $\xi \in \mathfrak{t}^*$ be regular, then*

$$\mathrm{vol}(N_\xi) = \frac{1}{(2\pi)^r} \int_{Z(\xi,\mathcal{B})} \frac{e^{-i\langle \xi, \phi \rangle}}{\prod_{i=1}^n \langle \beta_i, \phi \rangle} d\phi.$$

We recall that the homology class of the cycle $Z(\xi, \mathcal{B})$ is computed recursively so that the preceding integral is easily calculated by iterated residues.

7.3 Polytopes and computations

Of course, all theorems on toric varieties as integration of characteristic classes, equivariant Kawasaki-Riemann-Roch formulae, etc..., have a translation in the world of polytopes. With Brion, Szenes, Baldoni, Berline, we carefully gave elementary proofs of the corresponding theorems on polytopes, even if our inspiration came from equivariant cohomology on Hamiltonian manifolds. A pleasant (??) survey of some topics around counting integral points in polytopes is in [52].

The most convenient setting is that of partition polytopes: Let $\mathcal{B} := [\beta_1, \ldots, \beta_n]$ be a sequence of linear forms on a vector space V of dimension r strictly contained in a half-space of V^*. If $\xi \in V^*$, the partition polytope is

$$P_{\mathcal{B}}(\xi) := \{\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n; x_i \geq 0; \sum_{i=1}^n x_i \beta_i = \xi\}.$$

Any polytope can be realized as a partition polytope.

Example 18 **The transportation polytope.** *We consider two sequences $[r_1, r_2, \ldots, r_k], [c_1, c_2, \ldots, c_\ell]$ of positive numbers with $\sum_i r_i = \sum_j c_j$. Then* $\mathrm{Transport}(k, \ell, r, c)$ *is the polytope consisting of all real matrices with k rows and n columns, with non negative entries, and with sums of entries in row i equal to r_i and in column j equal to c_j. This is a special case of a network polytope (see [7],[8]) .*

The volume of $P_{\mathcal{B}}(\xi)$ is easily seen to be equal to the value at ξ of the convolution of the Heaviside distributions supported on $\mathbb{R}^+\beta_i$. The volume of $\mathrm{Transport}(k, \ell, r, c)$ necessitates the convolution of $k\ell$ Heaviside distributions in a space of dimension $k+\ell-1$. For example Beck-Pixton could compute the volume of Transport(k,ℓ,r,c) for $k=10, \ell=10$, for special values $r_i=c_j=1$ (thus convoluting 100 linear forms in a 19 dimensional space) in 17 years (scaled on 1 Ghz processor) ([12]) .

Theorem 19

$$\mathrm{vol}(P_{\mathcal{B}}(\xi)) = (2i\pi)^{-r}\frac{1}{(n-r)!}\int_{Z(\xi,\mathcal{B})}\frac{\langle \xi, v\rangle^{n-r}}{\prod_{i=1}^{n}\langle \beta_i, v\rangle}dv.$$

By De Concini-Procesi recursive determination of $Z(\xi,\mathcal{B})$, this formula is expressed as a sum of iterated residues.

Assume the β_i span a lattice Λ in V^*, and that ξ is in Λ. The discrete analogue of the volume of $P_{\mathcal{B}}(\xi)$ is the number $N_{\mathcal{B}}(\xi)$ of integral points in the rational polytope $P_{\mathcal{B}}(\xi)$. A fundamental result of Barvinok [9] asserts that $N_{\mathcal{B}}(\xi)$ can be computed in polynomial time, when n is fixed.

The function $N_{\mathcal{B}}(\xi)$ associates to the vector ξ the number of ways to represents the vector ξ as a sum of a certain number of vectors β_i is also called the (vector)-partition function of $\mathcal{B}$. There is also an integral formula ([46]) on the cycle $Z(\xi,\mathcal{B})$ for this number of points. It has many interesting applications, for example information on jumps of the partition function from chamber to chamber (see also [**?**]). For example, the appearance of the 5 linear factors in $g(a,b)$ (Formula 8 of Subsection 2.4) follows from our calculations. However, except for relatively good systems $\mathcal{B}$, this formula does not allow polynomial time computations. A general program for computing number of points of general rational polytopes following Barvinok's algorithm is done in Latte [35]. For unimodular systems $\mathcal{B}$ (as networks) or systems not too far from unimodularity (as root systems), our programs based on iterated residues are more efficient, in particular for the transportation polytopes [7] or for $\mathcal{B}$ any classical root system of semi-simple Lie algebras. It leads to the fastest computation of Kostant partition function, weight multiplicities c_μ^λ, tensor product multiplicities $c_{\lambda,\mu}^\nu$ of classical Lie algebras of small rank, but the bit size of the weights λ,μ,ν can be very large (cite [6], [24].)

Finally, let me describe the local Euler-MacLaurin formula which was conjectured by Barvinok-Pommersheim [11]. It was after observing the analogy of this conjecture with the local property of the non-abelian localization theorem that I fully realized the beauty of this conjecture. Nicole Berline and I, we proved it by elementary means.

Let P be a convex polytope in $\mathbb{R}^n$. For simplicity we assume that P has integral vertices. The detailed statement for any rational convex polytope and what we really mean by "depending only on" is in [18]. Let $\mathcal{F}$ be the set of faces of P (from dimensions 0 (vertices) to n (the polytope P itself)). For each face F of P, we denote by $N(P,F)$ the normal cone to P at F (it is a cone with vertex 0 of dimension equal to the codimension of F).

Theorem 20 *(Local Euler-MacLaurin formula.). There exist constant coefficients differential operators D_F (of infinite order) depending only of the normal cone $N(P,F)$ such that, for any polynomial function Φ on $\mathbb{R}^n$, then*

$$\sum_{\xi\in P\cap\mathbb{Z}^n}\Phi(\xi)=\sum_{F\in\mathcal{F}}\int_F D_F(\Phi).$$

The operators D_F have rational coefficients and can be computed in polynomial time when n and the order of the expansion are fixed (with the help of Barvinok signed decomposition of cones and LLL short vector algorithm.)

The local properties of D_F means that if two polytopes P and P' look the same in the neighborhood of a generic point of F, then the operators D_F for P or P' coincide. This is very similar to Paradan's localization formula near any C_F.

Let $E(P)(t) :=$ number of points in $(tP\cap\mathbb{Z}^n)$ (for t non negative integer) be the Ehrhart polynomial of P Then $E(P)(t)=\sum_{i=0}^n e_i t^{n-i}$, with $e_0=\mathrm{vol}(P)$. Using our local theorem, we hope to implement for rational simplices a poly-time algorithm (when k is fixed, but n **not fixed**) to compute all the coefficients e_i for $i\leq k$. The fact than e_i with $i\leq k$ could be computed in polynomial time was obtained recently by Barvinok [10].

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