Lecture 2
Residue formulae for volumes and number of integral points in convex rational polytopes

Introduction

As second topic, I shall present here some recent results on the number of points with integral coordinates in convex rational polytopes. My own interest in this topic comes from my efforts to understand the relations between symplectic manifolds and group representations, the existence of such relations being the *Credo* of quantum mechanics. Quantum mechanics enables us to associate discrete quantities to some geometric objects. For example, the volume of a compact symplectic manifold (M,ω), with an integral closed non-degenerate 2-form ω, has a discrete analogue which is the dimension of the vector space given by the quantum model $Q(M,\omega)$ for M. It is important to understand the relation between both quantities. The dimension $q(k)$ of the vector space $Q(M,k\omega)$ is a polynomial in k, and the volume of the manifold M is the limit of $k^{-\dim M/2}q(k)$ when k tends to ∞. The full expression for the dimension of $Q(M,k\omega)$ is the content of the Riemann-Roch theorem, which expresses this integer $q(k)$ as an integral over M. A similar comparison problem is the following: if $P \subset \mathbb{R}^n$ is a convex polytope with integral (or rational) vertices, can we compare the number $|P \cap \mathbb{Z}^n|$ of points in P with integral coordinates and the volume of P ? It is clear that the volume of P is obtained as the limit when k tends to ∞ of $k^{-n}|kP \cap \mathbb{Z}^n|$. Can we give more precise relations ?

There is a dictionary between polytopes and some classes of compact symplectic manifolds (toric manifolds). This dictionary inspired the formulation of several results, starting with the fascinating formula of Khovanskii-Pukhlikov (see Theorem 16). The point of view of these lectures will be algebraic and based on generating functions combined with an algebraic recipe due to Jeffrey-Kirwan ([21]) for the inversion of Laplace transforms. We shall only give some references on the correspondence between polytopes and symplectic geometry in the last section.

I hope to show, in this lecture, that calculating the volume of a convex rational polytope or calculating the number of points with integral coordinates inside this polytope are similar problems (both difficult). Convex polytopes arise in many fields of mathematics: symplectic geometry, representation theory, algebraic geometry. Computing volumes is important. It is also important to study integral points in rational polytopes: they are the integral solutions of systems of linear inequations with integral coefficients. For example, solving the equation $5x + 10y + 20z = 105$ with x, y, z positive integral numbers is an equation that we see everyday, when buying an item of value 105 cents, with coins of 5 cents, 10 cents and 20 cents (the number of solutions is 36). Similarly regulating flows in networks is reduced to linear inequalities. Notice that already the problem of finding if there exists an integral point in a rational polytope is non-trivial (a polytope is called rational if its vertices have rational coordinates). H. Lenstra ([23]) showed in 1983 that there is, for a fixed dimension n, a polynomial time algorithm checking if a rational polytope P in $\mathbb{R}^n$ contains an integral point: $|P \cap \mathbb{Z}^n| \neq \emptyset$, and A. Barvinok ([4]) showed in 1994 that there is, for a fixed dimension n, a polynomial time algorithm giving the number $|P \cap \mathbb{Z}^n|$ of integral points in P.

I shall explain here explicit and very similar formulae for both the volume and the number of integral points in rational convex polytopes. The formula for the volume is deduced in a straightforward way from the inversion formula of Jeffrey-Kirwan ([21]) for the Laplace transform. The formula for the number of points given in ([32]) follows from a multidimensional residue theorem, deduced from the one dimensional residue theorem with the help of a result due to A. Szenes ([31]) on separation of variables. From these formulae, the relations obtained by Khovanskii-Pukhlikov ([22]) and more generally by Brion-Vergne ([7]), Cappell-Shaneson ([10]), Guillemin ([20]) between volumes and the number of integral points will become clear. The residue formulae for volume and number of points have a theoretical interest: for example, the periodic-polynomial dependence of the formula in terms of the inequalities defining the polytope is clear from the given formulae. They can also be used for the computation of these quantities, at least for network polytopes. This is work in progress with Velleda Baldoni and Jesus de Loera.

1 Definition of the Ehrhart polynomial

By definition, a convex polytope in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$, or equivalently a compact set of $\mathbb{R}^n$ defined by linear inequations. A third way of representing a convex polytopes is as a set of positive solutions to linear equations: i.e. the first positive quadrant intersected with a linear space.

Example. The Tetrahedron. The convex hull in $\mathbb{R}^3$ of the points $0 = (0,0,0)$, $A = (a,0,0)$, $B = (0,b,0)$, $C = (0,0,c)$ is also described by the inequations

$$x \geq 0, y \geq 0, z \geq 0 \qquad \text{and} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$$

or is also clearly isomorphic to the convex polytope in $\mathbb{R}^4$ described by

$$x \geq 0, y \geq 0, z \geq 0, h \geq 0 \qquad \text{and} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + h = 1.$$

A convex polytope is called integral if its vertices have integral coordinates.

A convex polytope is called rational if its vertices have rational coordinates.

Our topic of discussion here is the calculation of the volume of convex polytopes, together with the calculation of the number of points with integral coordinates in an integral convex polytope P.

Let us start with the example of the standard simplex Δ in $\mathbb{R}^n$, with vertices $0, e_1, e_2, \ldots, e_n$. The polytope $k\Delta$ is defined by the inequations

$$k\Delta = \{x_1 \geq 0, \ldots, x_n \geq 0 | \ x_1 + x_2 + \cdots + x_n \leq k\}.$$

It is not difficult to show that the volume of $k\Delta$ is

$$\mathrm{vol}(k\Delta) = \frac{k^n}{n!}.$$

The number of integral points in $k\Delta$ is given (as we shall see shortly) by

$$p_n(k) = \frac{(k+1)(k+2)\cdots(k+n)}{n!}.$$

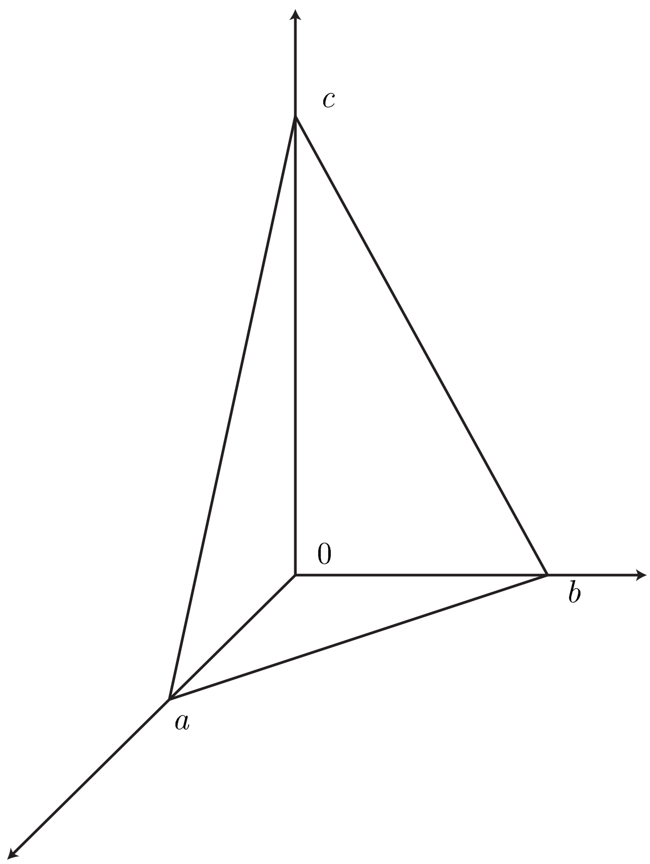
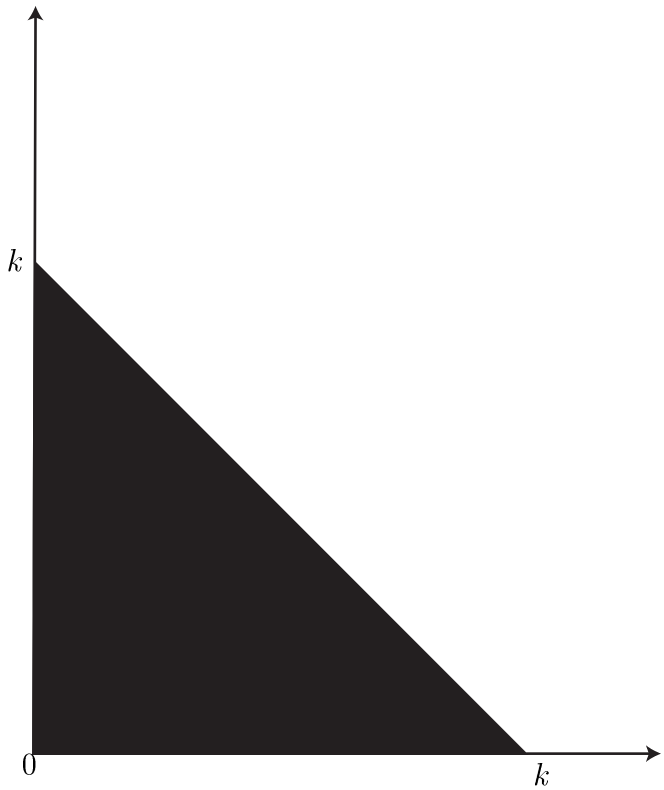
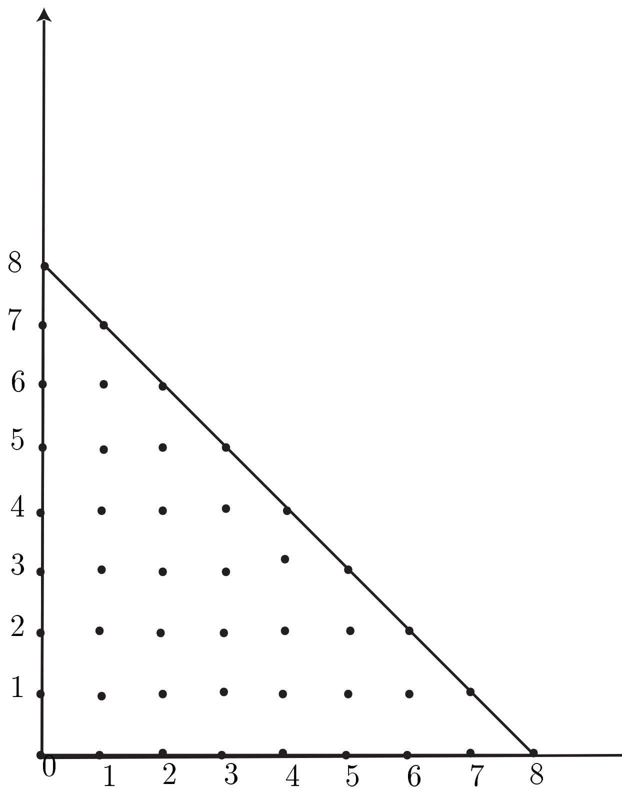


Figure 1: Tetrahedron



$\text{Volume}(k\Delta) = k^2/2$



$|(k\Delta \cap \mathbb{Z}^2)| = \frac{(k+1)(k+2)}{2}$

We refer to Section 12 for the relation between points in $k\Delta$ and a basis of vector spaces $V(k)$ attached to P.

Thus we see that the discrete analogue to the monomial $\frac{x^n}{n!}$ giving the volume of $k\Delta$ for $x = k$ is the polynomial $p_n(x) = \frac{(x+1)(x+2)\cdots(x+n)}{n!}$ which gives the number of integral points in $k\Delta$ for $x = k$. This polynomial $p_n(x)$ has same leading term $\frac{x^n}{n!}$, but it has the advantage that it takes integral values on all integers, and any polynomial which takes integral values on all integers is a linear combination with integral coefficients of such polynomials $p_n(x)$.

More generally, to any integral convex polytope P is associated a polynomial function: the Ehrhart polynomial ([15],[16],[17]).

Theorem 1 (The Ehrhart theorem). *Let $P \subset \mathbb{R}^n$ be an integral convex polytope with non-empty interior. Then the function on $\mathbb{N}$ defined by*

$$i_P(k) = \mathrm{cardinal}(kP \cap \mathbb{Z}^n), \quad k = 0, 1, 2, \dots$$

is given by a polynomial expression in k called the Ehrhart polynomial. The first two leading terms of this polynomial are $\mathrm{vol}(P)k^n + \frac{1}{2}\,\mathrm{vol}_{\mathbb{Z}}(\delta P)k^{n-1} + \cdots$. *The constant term of the Ehrhart polynomial $i_P(k)$ is equal to* 1.

Here $vol_{\mathbb{Z}}(\delta P)$ is the sum of the volumes of the faces of the boundary δP of the polytope P. The volume of a face F of the boundary δP is computed using a normalized measure built from the Lebesgue measure on the affine space A_F spanned by the face. The normalization is done in such a way that the fundamental domain of the lattice $\mathbb{Z}^n \cap A_F$ has volume 1.

The Ehrhart function $i_P(k)$ is our second example of a function of k, the polynomial feature of which does not follow evidently from its definition, the first example (given in Lecture 1) being the Verlinde sums $k \mapsto \mathrm{Ver}(q, k)$. After some further comments, we shall give here the proof of Ehrhart's theorem, following Ehrhart.

The following theorem directly generalizes Bernoulli Theorem on sums of the m-th powers of the first $k + 1$ numbers (see Lecture 1, Section 1).

Theorem 2 *Let $P \subset \mathbb{R}^n$ be an integral convex polytope with non-empty interior. Let f be a polynomial function on $\mathbb{R}^n$ homogeneous of degree m. Then*

$$k \mapsto \sum_{\xi \in kP \cap \mathbb{Z}^n} f(\xi)$$

is a polynomial function of k. The leading term of this polynomial is $k^{n+m} \int_P f(x)dx$.

We also note here the important reciprocity law for the Ehrhart polynomial.

Theorem 3 (Reciprocity law). *Let $P \subset \mathbb{R}^n$ be an integral convex polytope with non-empty interior. Let P^0 be the interior of P. Define*

$$i_{P^0}(k) = \mathrm{cardinal}(kP^0 \cap \mathbb{Z}^n)$$

to be the number of integral elements in the interior of kP. Then

$$i_{P^0}(k) = (-1)^n i_P(-k).$$

The obvious example of the reciprocity law is when $n = 1$ and $P = [0,1]$ is the unit interval. Then $i_P(k)$, the number of integral points in $[0,k]$, is $(k+1)$ and the number $i_{P^0}(k)$ is the number of integers strictly greater than 0 and strictly less than k. We have

$$i_{P^0} = (k-1) = -i_P(-k).$$

Ehrhart's theorem stating that the number of integral points in the dilated polytope kP is **a polynomial** in k, with leading term $k^n \operatorname{vol}(P)$ is far from being obvious.

Let us give an example involving dilatations in just one direction, where we shall see that even the asymptotics between the volume and the number of integral points is only true under dilations in all directions.

Example: The hanging pyramid.

Let $m \in \mathbb{R}^+$. The hanging pyramid P_m is the convex hull in $\mathbb{R}^3$ of the vertices $s_0 = (0,0,0), s_1 = (1,0,0)$, $s_2 = (0,1,0)$, and $s_3 = (1,1,m)$.

Its volume depends on m:

$$\operatorname{vol}(P_m) = m/6.$$

Assume that m is an integer, then P_m is an integral convex polytope, but there are no integral points in P_m other than its 4 vertices. Whatever the value of m might be:

$$i(P_m) = 4$$

where we have set $i(P) := i_P(1)$ taking $k = 1$. Indeed, as the full pyramid projects on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, any integral point in P_m

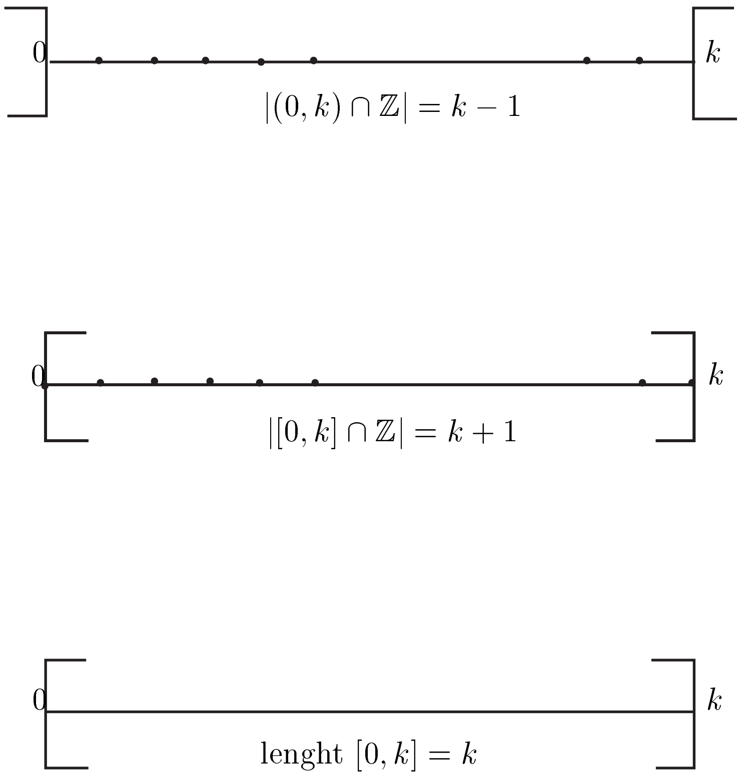
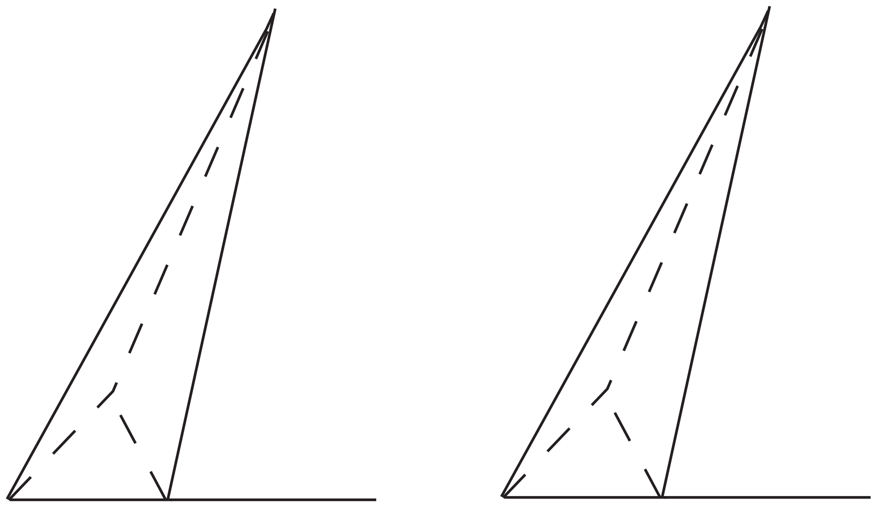


Figure 2: Reciprocity law for the interval [0,k]



$\mathrm{Vol}(P_m) = m/6$ $\qquad$ $|P_m \cap Z^3| = 4$

Figure 3: Hanging pyramid

is above one of the 4 points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, thus is one of the 4 vertices. The asymptotics in m of the volume and the number of integral points are therefore very different in this example.

Before going to the proof of Ehrhart's theorem, let us have a look at the formulae relating the number of points in an integral convex polytope and its volume in dimension 1 and 2:

In dimension 1: Let us consider the polytope given by the interval $P = [0,1]$. Then

$$i_P(k) = k + 1$$

which relates to the length by

$$i_P(k) = k\,\mathrm{length}(P) + 1.$$

In dimension 2: Pick's theorem tells us that the number of points in a convex polytope with integral vertices dilated by k is given by:

$$i_P(k) = k^2\,\mathrm{vol}(P) + \frac{k}{2}(\text{number of integral points on the boundary of } P) + 1.$$

2 Number of integral points in Simplices

To prove Ehrhart's theorem on the polynomial behavior of the function $i_P(k)$, it is sufficient to prove it for simplices, by decomposing an integral convex polytope in unions and differences of integral simplices. Thus we consider $(n+1)$ points in $\mathbb{Z}^n$ and the convex polytope given by the convex hull of these points. Ehrhart gave a formula for the number of integral points contained in this set.

Let us start with the standard simplex Δ in $\mathbb{R}^n$, with vertices $0, e_1, e_2, \ldots, e_n$. The polytope $k\Delta$ is defined by the inequations

$$x_1 \geq 0, \ldots, x_n \geq 0, x_1 + x_2 + \cdots + x_n \leq k.$$

We need to find the number of integer solutions of the inequations $x_1 \geq 0, \ldots, x_n \geq 0, x_1 + x_2 + \cdots + x_n \leq k$. In other words, we have to find the number $p(n,k)$ of solutions in non-negative integers $(x_0, x_1, \ldots, x_n)$ of the equation $x_0 + x_1 + x_2 + \cdots + x_n = k$.

We first prove:

Theorem 4 *For any integer $k \geq -n$, the number $p(n,k)$ of solutions in non-negative integers $(x_0, x_1, \ldots, x_n)$ of the equation*

$$x_0 + x_1 + x_2 + \cdots + x_n = k$$

is

$$p(n,k) = \frac{(k+1)(k+2)\cdots(k+n)}{n!}.$$

Proof. The result is obvious for $k = -1, -2, \ldots, -n$, both sides of the formula being equal to 0. Let us set $k \geq 0$ and consider the generating function:

$$Z_k(t) = \sum_{k=0}^{\infty} p(n,k) e^{-kt}, \quad t > 0.$$

We shall recover $p(n,k)$ as the coefficient of e^{-kt} in $Z_k(t)$.

We have

$$
\begin{aligned}
Z_k(t) &= \sum_{(x_0,\dots,x_n)\in(\mathbb{Z}^+)^{n+1}} e^{-t(x_0+x_1+\cdots+x_n)} \\
&= \prod_{i=0}^{n} \sum_{x_i=0}^{\infty} e^{-tx_i} \\
&= \frac{1}{(1-e^{-t})^{n+1}}.
\end{aligned}
$$

Differentiating $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, we get $\frac{1}{(1-z)^{n+1}} = \sum_{k=0}^{\infty} \binom{k+n}{n} z^k$ and hence

$$
\frac{1}{(1-e^{-t})^{n+1}} = \sum_{k=0}^{\infty} \binom{k+n}{n} e^{-kt}
$$

with $\binom{k+n}{n} = \frac{(k+1)(k+2)\cdots(k+n)}{n!}$.

Corollary 5

$$
i_\Delta(k) = \text{cardinal}(k\Delta \cap \mathbb{Z}^n) = \frac{(k+1)(k+2)\cdots(k+n)}{n!}.
$$

Notice that $i_\Delta(k)$ is indeed a polynomial in k with constant term 1 and leading terms $\frac{1}{n!}k^n + \frac{1}{2}\frac{(n+1)}{(n-1)!}k^{n-1} + \cdots$ as announced in Ehrhart's theorem since $\text{vol}(\Delta) = \frac{1}{n!}$ and the boundary of Δ is the union of $(n+1)$ standard simplices, each of volume $\frac{1}{(n-1)!}$.

Let us now generalize this to the case of a general simplex P in $\mathbb{R}^n$. We give a formula due to Ehrhart for the number of points in kP.

Theorem 6 *Let P be a simplex in $\mathbb{R}^n$ with integral vertices $\alpha_0, \alpha_1, \dots, \alpha_n$. The number of integer points in kP is*

$$
i_P(k) = \sum_{r=0}^{n} \text{cardinal}(\Box(P,r)) \binom{k-r+n}{n}
$$

where for an integer r between 0 and n

$$
\Box(P,r) = \left\{ u \in [0,1[^{n+1} \mid \sum_{i=0}^{n} u_i = r, \sum_{i=0}^{n} u_i\alpha_i \in \mathbb{Z}^n \right\}
$$

which is a finite set. Thus $i_P(k)$ is a polynomial in k.

Proof. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be the vertices of P which we assume to have integer coordinates. An element of kP is an element of the convex hull of the points $k\alpha_0, k\alpha_1, \ldots, k\alpha_n$. It can be uniquely written as

$$w = x_0\alpha_0 + x_1\alpha_1 + \cdots + x_n\alpha_n$$

with $x_0 + x_1 + \cdots + x_n = k$. By taking the integral part k_i of x_i, the point w can be written in an unique way as:

$$w = (u_0\alpha_0 + u_1\alpha_1 + \cdots + u_n\alpha_n) + (k_0\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n)$$

with k_i non-negative integers, $0 \leq u_i < 1$, and $\sum_{i=0}^n k_i + \sum_{i=0}^n u_i = k$. As the elements α_i are in $\mathbb{Z}^n$, the point w is in $\mathbb{Z}^n$ if and only if

$$u_0\alpha_0 + u_1\alpha_1 + \cdots + u_n\alpha_n \in \mathbb{Z}^n.$$

Notice that the number $\sum_{i=0}^n u_i = k - \sum_{i=0}^n k_i$ is a non-negative integer. This integer r is less or equal to n, as the value $(n+1)$ for r would compel all u_i to be equal to 1, while we have $u_i < 1$.

Conversely, let $0 \leq r \leq n$ be an integer. Now, if the non-negative integers k_i satisfy

$$k_0 + k_1 + \cdots + k_n = k - r$$

and $u \in \Box(P, r)$, the point

$$(u_0\alpha_0 + u_1\alpha_1 + \cdots + u_n\alpha_n) + (k_0\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n)$$

lies in kP. We enumerate this way all the points in kP. Since $r \leq n$ and $k \geq 0$, the integer $k - r$ is greater or equal to $-n$ and it follows from what we have proven before that the number of solutions of the equation

$$k_0 + k_1 + \cdots + k_n = k - r$$

is $\binom{k-r+n}{n}$, for any $k \geq 0$.

We therefore obtain Ehrhart's formula:

$$i_P(k) = \sum_{r=0}^{n} \mathrm{cardinal}(\Box(P, r)) \binom{k - r + n}{n}.$$

This expression is polynomial in k.

The sets $\Box(P,r)$ are not very easy to determine. Furthermore, the decomposition of P in integral simplices is also not immediate. So the formula above has mainly a theoretical interest. We shall give later on more specific result on the Ehrhart polynomial.

Let P be a **rational** convex polytope: the vertices of P have rational coordinates, instead of integral coordinates. Ehrhart analyzed more generally the behavior in k of the function $k \mapsto i_P(k) = \mathrm{cardinal}(kP \cap \mathbb{Z}^n)$ and proved that this function is given by a"periodic-polynomial", i.e. is a polynomial function in k with coefficients periodic functions of k. In other words, there exists an integer d, such that the function $k \mapsto i_P(dk)$ $(k \geq 0)$ is polynomial, as well as all functions $i_P(dk+j)$ where j is an integer such that $0 \leq j < d$. For example, if $P := [0, \frac{1}{2}]$, then, with $d = 2$, $i_P(2k) = k+1$, while $i_P(2k+1) = k+1$. These two formulae can be assembled together to give the periodic-polynomial function $i_P(k) = \frac{k}{2} + 1 - \frac{1}{4}(1-(-1)^k)$.

For a given rational polytope P, it is not easy to determine the smallest value p of the integer d with the property that functions $k \mapsto i_P(dk+j)$ are polynomials in k. Of course, when P has integral vertices, we just proved that $p = 1$. Similarly, if q is an integer such that, for all vertices s of P, $qs \in \mathbb{Z}^n$, then p divides q. It may happen that p is strictly smaller than q.

Example. Stanley's pyramid. Consider the following example given by Stanley ([27]): let P be the convex polytope in $\mathbb{R}^3$ with vertices $O = (0,0,0)$, $A = (1,0,0)$,$B = (0,1,0)$, $D = (1,1,0)$ and $C = (\frac{1}{2}, 0, \frac{1}{2})$. Then P is not integral, however

$$i_P(k) = \frac{(k+1)(k+2)(k+3)}{3!}$$

is a polynomial function.

3 Some examples of Ehrhart polynomials

I first give some easy examples, where the brutal formula above can be calculated.

Example 1. Let P be the standard simplex with $\alpha_0 = 0$, $\alpha_1 = e_1, \ldots,$ $\alpha_n = e_n$.

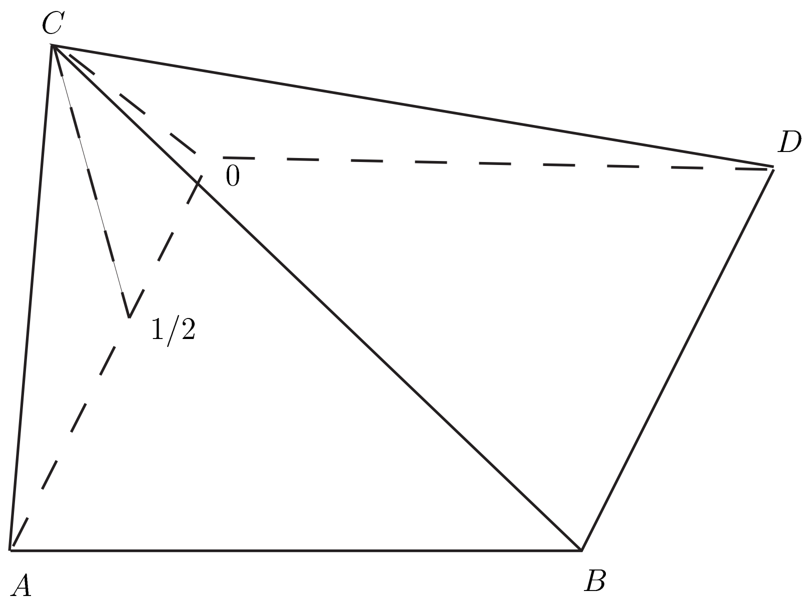


Figure 4: Stanley's pyramid

Then $\square(P,0) = \{0\}$, the other sets $\square(P,r)$ are empty. We have the formula that we used:

$$i_p(k) = \binom{k+n}{n}.$$

Example 2. Let P be the triangle in $\mathbb{R}^2$ with vertices $\alpha_0 = 0$, $\alpha_1 = e_1$, $\alpha_2 = 2e_2$.

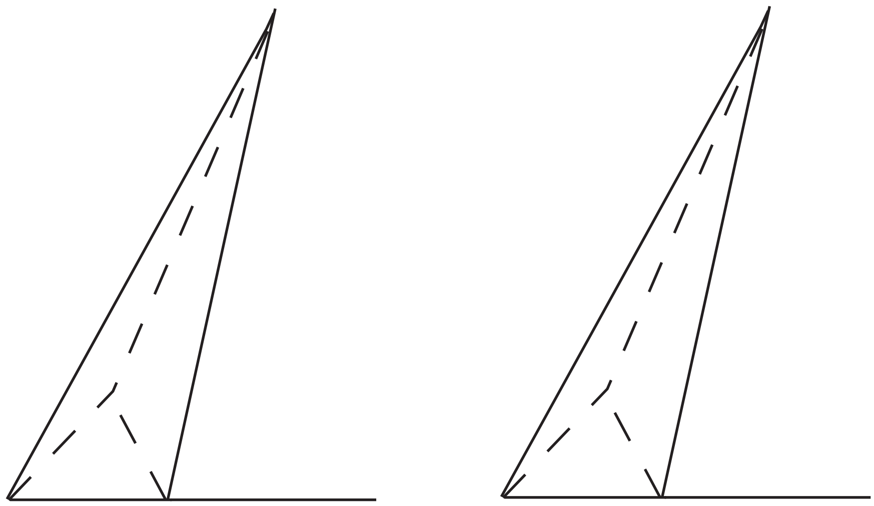
Then $\square(P,0) = \{0\}$, $\square(P,1)$ has 1 element and the others are empty. We have the formula:

$$i_P(k) = \binom{k+2}{2} + \binom{k+1}{2} = (k+1)^2.$$

(The volume of P is 1.)

Example 3. Let P be the simplex in $\mathbb{R}^3$ with vertices $\alpha_0 = 0$, $\alpha_1 = e_1$, $\alpha_2 = 2e_2$, $\alpha_3 = 3e_3$.

Then $\square(P,0) = \{0\}$, $\square(P,1)$ has 4 elements and $\square(P,2)$ has 1 element



$\mathrm{Vol}(kP_m) = \frac{mk^3}{6}$ $\qquad$ $|k(P_m) \cap Z^3| =$

Figure 5: Hanging pyramid

so that we have the formula:

$$\begin{aligned} i_P(k) &= \binom{k+3}{3} + 4\binom{k+2}{3} + \binom{k+1}{3} \\ &= (k+1)^3. \end{aligned}$$

(The volume of P is 1.)

Example 4. Let P the simplex in $\mathbb{R}^4$ with vertices $\alpha_0 = 0$, $\alpha_1 = e_1$, $\alpha_2 = 2e_2$, $\alpha_3 = 3e_3$, $\alpha_4 = 4e_4$.

Then $\square(P, 0) = \{0\}$ has 1 element, $\square(P, 1)$ has 12 elements, $\square(P, 2)$ has 11 elements. We obtain

$$i_P(k) = (k+1)(k+2)(k^2 + \frac{4}{3}k + \frac{1}{2}).$$

(The volume of P is 1.)

Example 5: The hanging pyramid P_m. Let P_m be the simplex with vertices $\alpha_0 = (0, 0, 0)$, $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (1, 1, m)$.

The set $\square(P, 0)$ has 1 element, $\square(P, 1)$ has 0 elements and $\square(P, 2)$ has $(m-1)$ elements. We obtain

$$i_{P_m}(k) = \frac{1}{6}(k+1)(k+2)(k+3) + (m-1)\frac{1}{6}(k-1)(k)(k+1)$$

$$= \frac{m}{6}k^3 + k^2 + (12-m)\frac{k}{6} + 1.$$

Notice on this last formula that we indeed have $i_{P_m}(1) = \frac{m}{6}+1+\frac{12-m}{6}+1 = 4$. But we see here that the coefficient of k in the Ehrhart polynomial is a negative number, when m is large enough. **In particular, the coefficients of Ehrhart polynomials are not necessarily positive.**

Finally, we include here a formula due to Mordell ([26]) for the number of points in the polytope $P(a,b,c)$ defined by the vertices 0, $A = ae_1$, $B = be_2$, $C = ce_3$ where a, b, c are relatively prime integers. The volume of $P(a,b,c)$ is $abc/6$.

The number of integral points $i_{P(a,b,c)}(1)$ with integral coordinates in $P(a,b,c)$ is not a rational function of a, b, c. Here is the Ehrhart polynomial for the number of points with integral coordinates contained in $kP(a,b,c)$:

$$\begin{aligned} i_{P(a,b,c)}(k) = \frac{1}{6}abck^3 + (\frac{1}{4}(bc+ca+ab+1))k^2 + \\ (\frac{1}{4}(a+b+c+3) + \frac{1}{12}(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}+\frac{1}{abc}))k \\ -(s(bc,a)+s(ca,b)+s(ab,c))k+1. \end{aligned}$$

Here $s(p,q)$ is the Dedekind sum, defined by

$$s(p,q) = \sum_{i=1}^{q}((\frac{i}{q}))((\frac{pi}{q}))$$

with $((x)) = 0$ if x is an integer and $((x)) = x - [x] - \frac{1}{2}$ otherwise.

4 The magic square polytope $\mathrm{Magic}(n)$ and the Chan-Robbins-Yuen polytope

Let us now consider a special polytope given by the convex set $\mathrm{Magic}(n)$ formed by the doubly stochastic $(n \times n)$-matrices. These are matrices (x_{ij}) with non-negative entries and such that, on each line and each column, the coefficients add up to 1. Thus $\mathrm{Magic}(n)$ is defined as the intersection of the positive quadrant in $\mathbb{R}^{n^2}$, cut by the $2n$ linear equations $\sum_i x_{ij} = 1$ for all $1 \leq j \leq n$ and $\sum_j x_{ij} = 1$ for all $1 \leq i \leq n$. Clearly the sum of coefficients in all lines is equal to the sum of coefficients on all columns and its value is n. Thus this polytope is of dimension $n^2-(2n-1)=(n-1)^2$. It is not difficult ([18]) to see that $\mathrm{Magic}(n)$ is also the convex hull of the $n!$ permutation matrices so that its vertices are the $n!$ permutation matrices. In particular $\mathrm{Magic}(n)$ is an integral convex polytope.

Example. The set Magic(5) is the set of matrices

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}$$

with $x_{ij}\geq 0$ and

$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}=1,$$

$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25}=1,$$

$$\dots$$

$$x_{11}+x_{21}+x_{31}+x_{41}+x_{51}=1,$$

$$x_{12}+x_{22}+x_{32}+x_{42}+x_{52}=1,$$

$$\dots$$

As follows from the general theory of the Ehrhart polynomial, the number of integral elements in $k\mathrm{Magic}(n)$ is a polynomial of the form

$$m_n k^{(n-1)^2}+\cdots+1,$$

where m_n is the volume of $\mathrm{Magic}(n)$. Finding $i_{\mathrm{Magic}(n)}(k)$ boils down to counting magic squares: square $(n\times n)$-matrices filled up with non-negative integers and such that the lines and the columns add up to k.

Here is an example of an element of $8\,\mathrm{Magic}(3)$.

$$\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix}.$$

We have

$$i_{\mathrm{Magic}(1)}(k)=1,$$

$$i_{\mathrm{Magic}(2)}(k) = k+1,$$
$$i_{\mathrm{Magic}(3)}(k) = \binom{k+4}{4} + \binom{k+3}{4} + \binom{k+2}{4},$$

Notice that:

$$i_{\mathrm{Magic}(n)}(0) = 1,$$
$$i_{\mathrm{Magic}(n)}(1) = n!$$

(number of permutation matrices.)

The Ehrhart polynomial $i_{\mathrm{Magic}(n)}(k)$ is known only when $n \leq 9$. A computation due to Chan-Robbins (co/9806076) gives a formula for $i_{\mathrm{Magic}(n)}(k)$ when $n \leq 8$. The recent calculation by Beck and Pixton (co/0202267) of the Ehrhart polynomial for $n = 9$ requires 325 days of computer time on a 1GHz PC running under Linux. The leading term of the Ehrhart polynomial (the volume) requires 15 seconds for $n = 7$, 54 minutes for $n = 8$, 317 hours for $n = 9$.

We now consider the Chan-Robbins-Yuen polytope CRY_n, namely the subset of the set of $(n \times n)$ doubly stochastic matrices consisting of those with just one non-zero line above the diagonal permitted. For example, CRY_5 is the set of matrices

$$\begin{pmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & x_{34} & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}$$

with $x_{ij} \geq 0$ and

$$\sum_i x_{ij} = 1, \text{ for all } 1 \leq j \leq 5,$$

and

$$\sum_j x_{ij} = 1, \text{ for all } 1 \leq i \leq 5.$$

The dimension of this convex polytope is $n(n-1)/2$. The corresponding Ehrhart polynomial is not known but the leading term is known (Chan-Robbins-Yuen ([13]) and Zeilberger ([33])). It is given by

$$(\prod_{i=1}^{n-2} \frac{(2i)!}{(i+1)!i!}) \frac{k^{n(n-1)/2}}{(n(n-1)/2)!}.$$

We shall see why this calculation is possible in Section 10.

5 Brion's formulae

The volume of a convex polytope and the Ehrhart polynomial are both difficult to compute. There are very few cases for which they are explicitly known. One can however hope to find some relations between them.

A relation together with a way to compute the volume and the number of integral convex polytopes follows from Brion's formulae ([5]), see also ([2],[3],[6]).) This formula has been used by Barvinok ([4]) to give an algorithm to compute in polynomial time (when the dimension is fixed) the number of integral points in a rational convex polytope.

I shall give Brion's formulae here since they are very beautiful. However, we shall indicate later a method of generating function and separation of variables, which leads to a direct comparison between volumes and number of points via residue formulae. Furthermore, the residue formulae seem to yield a more efficient computational tool than Brion's formulae, at least for transportation polytopes.

Michel Brion provided a formula for integral of exponential functions over convex polytopes or for sums of exponentials over integral points. They generalize the following formulae:

- For any $a, b \in \mathbb{R}$
$$\int_a^b e^{xy} dx = -\frac{e^{ay}}{y} + \frac{e^{by}}{y}$$
which follows by integration.

- For a, b integers
$$\sum_{u=a, u\in\mathbb{Z}}^{b} e^{uy} = \frac{e^{ay}}{1-e^{y}} + \frac{e^{by}}{1-e^{-y}}.$$

This last result follows from writing integers between a and b as differences of integers strictly greater than b and integers greater or equal than a and summing arithmetic progressions.

The first integral formula generalizes from the interval polytope $[a, b]$ to a general convex polytope $P \subset \mathbb{R}^n$. Let P be a convex polytope in $\mathbb{R}^n$, and let $\mathcal{V}(P)$ be the finite set of its vertices. We assume (this is the generic case) that at each vertex $s \in \mathcal{V}(P)$ start n edges $a_1^s, a_2^s, \ldots, a_n^s$ of the polytope P: vectors a_i^s are elements of $\mathbb{R}^n$ such that near s the convex polytope P is the

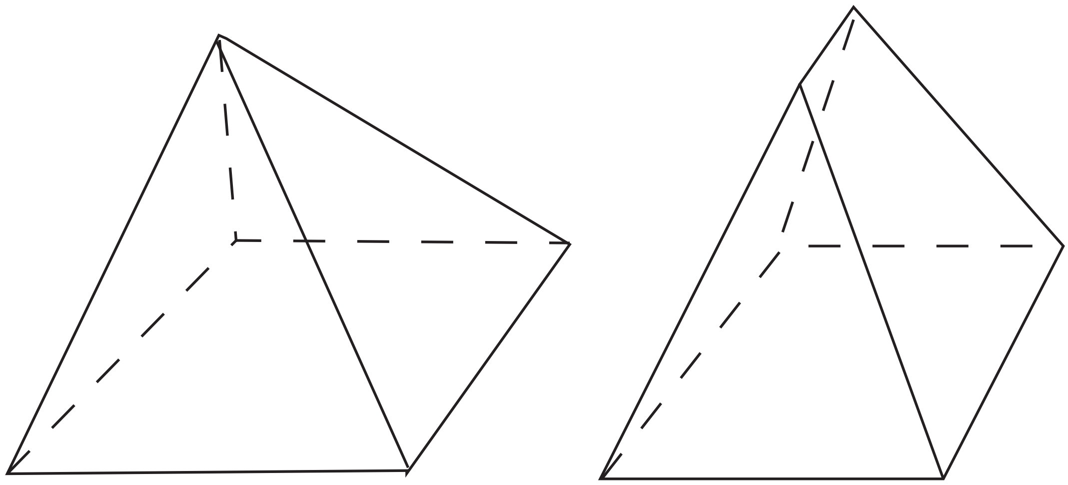


Figure 6: Non-generic pyramid versus generic pyramids

set of points of the forms $s+\sum_{i=1}^{n} t_i a_i^s$, with $t_i \geq 0$ (t_i small). In other words, the tangent cone at s to P is the affine cone $s+\sum_{i=1}^{n} \mathbb{R}^+ a_i^s$.

We identify the exterior product $\Lambda^n\mathbb{R}^n$ with $\mathbb{R}$, and $|a_1^s \wedge a_2^s \wedge \cdots \wedge a_n^s|$ is, by definition, the absolute value of the determinant of the $(n\times n)$-matrix with a_i^s as column vectors. Elements a_i^s are defined here only up to proportionality.

Then, we have:

$$(-1)^n \int_P e^{\langle x,y\rangle} dx = \sum_{s\in\mathcal{V}(P)} \frac{|a_1^s \wedge \cdots \wedge a_n^s| e^{\langle s,y\rangle}}{\langle a_1^s, y\rangle \cdots \langle a_n^s, y\rangle}.$$

Notice that this formula does not depend of the choice of the length of the edges a_i^s passing through the vertex s.

With $n=1$, setting $P=[a,b]$, then there are 2 vertices a and b. We can choose $a_1^a = b-a$, $a_1^b = a-b$, and the above formula yields back

$$\int_a^b e^{xy} = -\left(\frac{|(b-a)|e^{ay}}{(b-a)y} + \frac{|(a-b)|e^{ay}}{(a-b)y}\right) = -\frac{e^{ay}}{y} + \frac{e^{by}}{y}.$$

In particular, Brion's formula gives a formula for $\mathrm{vol}(P)$ knowing the vertices s and the edges through s.

$$\mathrm{vol}(P) = (-1)^n \sum_{s\in\mathcal{V}(P)} \frac{|a_1^s \wedge \cdots \wedge a_n^s| \langle s,y\rangle^n}{\langle a_1^s, y\rangle \cdots \langle a_n^s, y\rangle}$$

for any generic y.

Brion established similar formulae for sums of exponentials. To simplify the statement of Brion's formula, we assume that the convex polytope P is an integral convex polytope. Furthermore, we assume that at each vertex s, we can choose integral vectors $a_i^s \in \mathbb{Z}^n$ such that $|a_1^s \wedge \cdots \wedge a_n^s| = 1$, and such that, as before, the tangent cone at s to P is the affine cone $s + \sum_{i=1}^n \mathbb{R}^+ a_i^s$. We call such a convex polytope a Delzant polytope.

An integral convex polytope is rarely a Delzant polytope:

Example. Let $T(r)$ be the triangle in $\mathbb{R}^2$ with vertices $A = (0,0)$, $B = (1,r)$, $C = (1,0)$. It is a Delzant polytope, if and only if $|r| = 1$.

Let us however state Brion's formula in this case:

Theorem 7 *Let P be a Delzant polytope in $\mathbb{R}^n$. Then we have*

$$\sum_{\xi \in P \cap \mathbb{Z}^n} e^{\langle \xi, y \rangle} = \sum_{s \in \mathcal{V}(P)} \frac{e^{\langle s, y \rangle}}{(1 - e^{\langle a_1^s, y \rangle}) \cdots (1 - e^{\langle a_n^s, y \rangle})}.$$

Example. Brion's formula to integrate an exponential over the standard simplex Δ dilated by k in $\mathbb{R}^2$ is

$$\int_{k\Delta} e^{y_1 x_1 + y_2 x_2} dx_1 dx_2 = \frac{1}{y_1 y_2} + \frac{e^{k y_1}}{(-y_1)(y_2 - y_1)} + \frac{e^{k y_2}}{(-y_2)(y_1 - y_2)},$$

while the formula to sum up an exponential on all the integral points (p_1, p_2) in $k\Delta$ is

$$\sum_{(p_1, p_2) \in k\Delta} e^{y_1 p_1 + y_2 p_2} = \frac{1}{(1 - e^{y_1})(1 - e^{y_2})} + \frac{e^{k y_1}}{(1 - e^{-y_1})(1 - e^{y_2 - y_1})} + \frac{e^{k y_2}}{(1 - e^{-y_2})(1 - e^{y_1 - y_2})}.$$

6 Partition polytopes

We now introduce more general families of convex polytopes than just the dilated polytopes kP.

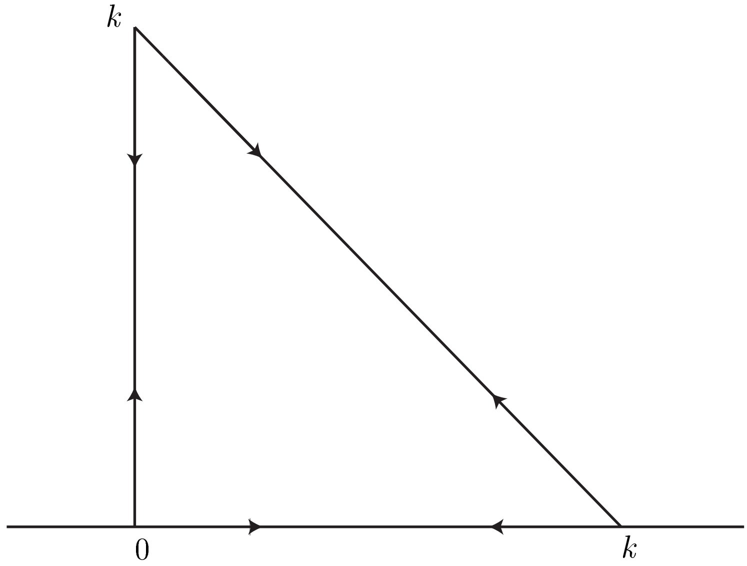


Figure 7: Simplex and edges through vertices

Let P and Q be two convex polytopes in $\mathbb{R}^n$. The Minkowski sum is defined as

$$P + Q = \{x + y |\ x \in P, y \in Q\}.$$

The following theorem, proved in ([25]), is a generalization of Ehrhart's theorem.

Theorem 8 • *Let $P_1, P_2, \ldots, P_r$ be convex polytopes in $\mathbb{R}^n$. Then the function*

$$v(t_1, .., t_r) = \text{volume}(t_1 P_1 + t_2 P_2 + \cdots + t_r P_r)$$

is a polynomial function of $(t_1, t_2, \ldots, t_r) \in (\mathbb{R}_+)^r$.

- *Assume $P_1, P_2, \ldots, P_r$ are integral convex polytopes. Then the function on $(\mathbb{Z}^+)^r$ given by*

$$i(k_1, k_2, \ldots, k_r) = \text{cardinal}((k_1 P_1 + \cdots + k_r P_r) \cap \mathbb{Z}^n)$$

is a polynomial function of $(k_1, k_2, \ldots, k_r)$.

We shall study more general families of convex polytopes.

Let

$$\Phi = [\alpha_1, \alpha_2, \ldots, \alpha_N]$$

be a sequence of N vectors in $\mathbb{R}^n$ (elements α_i may not be all distinct). We assume that all vectors α_i lie strictly on the same side of a hyperplane, so that Φ generates an acute cone $C(\Phi) = \{\sum_{i=1}^N t_i\alpha_i |\ t_i \geq 0\}$ in $\mathbb{R}^n$ (an acute cone is a cone which does not contain any straight line).

Let $\{w_1, w_2, \ldots, w_N\}$ be the canonical basis of $\mathbb{R}^N$. We define a linear map from $\mathbb{R}^N$ to $\mathbb{R}^n$ by:

$$A_\Phi(x_1, x_2, \ldots, x_N) = \sum_{i=1}^N x_i\alpha_i.$$

We may write the linear map $A := A_\Phi$ as a $(n \times N)$-matrix A with column vectors the vectors α_i.

For $a \in C(\Phi)$ define

$$P_\Phi(a) = A_\Phi^{-1}(a) \cap \mathbb{R}_+^N,$$

which is the intersection of an affine space with the standard quadrant.

In other words $P_\Phi(a)$ consists of all solutions of the equation $Ax = a$, where x is a N-vector with non-negative coordinates. The set $P_\Phi(a)$ is a convex polytope called partition polytope.

Example. Transportation polytopes

Let $\mathbb{R}^{2n}$ with canonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ and let Φ be the set of n^2 vectors

$$\Phi = [(e_i + f_j)|\ 1 \leq i, j \leq n].$$

Then

$$A(x_{ij}) = \sum x_{ij}(e_i + f_j).$$

The convex polytope $P_\Phi(a_1e_1 + a_2e_2 + \cdots + a_ne_n + b_1f_1 + b_2f_2 + \cdots + b_nf_n)$ can be identified to the set of $(n \times n)$-matrices (x_{ij}) with non-negative coefficients and with given sums of coefficients in each row and given sums of coefficients of each column. Indeed the vector equation $\sum x_{ij}(e_i + f_j) = (a_1e_1 + a_2e_2 + \cdots + a_ne_n + b_1f_1 + b_2f_2 + \cdots + b_nf_n)$ is equivalent to the series of $2n$ equations:

$$\sum_{j=1}^n x_{ij} = a_i,$$

$$\sum_{i=1}^{n} x_{ij} = b_j.$$

Of course, to get a solution, we need that $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$.

When $a_1 = a_2 = \cdots = a_n = b_1 = b_2 = \cdots = b_n = 1$, we recover the polytope Magic(n). For general (a_i, b_j) (such that $\sum a_i = \sum b_j$), this polytope is called a transportation polytope, and is very important in studying flow in networks.

If Φ is a sequence of vectors in $\mathbb{Z}^n$, then $P_\Phi(a)$ is a rational convex polytope.

Example: Let

$$A = (6, 10, 15),$$

then

$$A(x_1, x_2, x_3) = 6x_1 + 10x_2 + 15x_3.$$

The polytope $P_\Phi(k)$ is the convex hull of its 3 vertices

$$(\frac{k}{6}, 0, 0), \quad (0, \frac{k}{10}, 0), \quad (0, 0, \frac{k}{15}).$$

If A is surjective, then the polytopes $P_\Phi(a)$ are of dimension $N - n$, for a in the interior of $C(\Phi)$. The polytopes $P_\Phi(a)$ are contained in affine spaces parallel to $E := \operatorname{Ker} A$. If we denote by $f_1, f_2, ..., f_N$ the restrictions of the linear forms $x_1, x_2, \ldots, x_N$ on $\mathbb{R}^N$ to the subspace E, then for $u \in \mathbb{R}^N$, the polytope $P_\Phi(Au)$ is isomorphic to the polytope in $E = \operatorname{Ker} A$ described by the inequations

$$Q(u) := \{y \in E | \ \langle f_i, y\rangle + u_i \geq 0\}.$$

Indeed, if $y \in Q(u)$, the point $u + y$ lies in $P(Au)$. Some of the inequalities above might be irrelevant. If $Au = Av$, polytopes $Q(u)$ and $Q(v)$ are just translations of each other in the space $\operatorname{Ker} A$ by the vector $u - v$.

Consider the polytope $P_\Phi(A(u + hw_k)) = P_\Phi(Au + h\alpha_k)$. Then the polytope $Q(u + hw_k)$ is isomorphic to the polytope $P_\Phi(Au + h\alpha_k)$ and is defined by the same inequations as the polytope $Q(u)$ except that one of the inequalities $\langle f_i, y\rangle + u_i \geq 0$ has been replaced for $i = k$ by the inequality $\langle f_k, y\rangle + (u_k + h) \geq 0$. In particular all polytopes $P_\Phi(a + h)$, when a is generic and h varies in a small neighborhood of 0, have parallel faces.

Example

We draw the pictures of the polytopes $Q(Au)$ and their small deformations for the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The space $\operatorname{Ker} A$ is isomorphic to $\mathbb{R}^2$ with basis $b_1 = (w_3 - w_2), b_2 = (w_4 - w_1 - w_2)$. We write $y \in \operatorname{Ker} A$ as $y = y_1 b_1 + y_2 b_2$. To describe the family $Q(u)$, we may vary u in a supplementary subspace to $\operatorname{Ker} A$. We choose $u = u_1 w_1 + u_2 w_2$.

The 4 equations describing $Q(u)$ are

$$-y_2 + u_1 \geq 0,$$

$$-(y_1 + y_2) + u_2 \geq 0,$$
$$y_1 \geq 0,$$
$$y_2 \geq 0.$$

There are 6 cases leading to different polytopes, depending on the stratification of the cone $C(\Phi)$ in chambers (a topic that we shall discuss after this example).

- $u_1 = u_2 = 0$.

  Then $Q(0, 0) = \{0\}$

- $u_1 = 0$, $u_2 > 0$

  Then $Q(0, u_2)$ is the interval $[0, u_2]b_1$.

- $u_1 > 0$, $u_2 = 0$

  Then $Q(u_1, 0)$ is the interval $[0, u_1]b_2$.

- $u_2 > u_1$.

  Then $Q(u_1, u_2)$ has 4 vertices

  0, $A = u_1 b_2$, $B = u_2 b_1$, $C = (u_2 - u_1)b_1 + u_1 b_2$ and is a trapeze.

- $u_1 = u_2 = u$

  Then $Q(u_1, u_2)$ is a triangle with vertices 0, $A = ub_2$, $B = ub_1$.

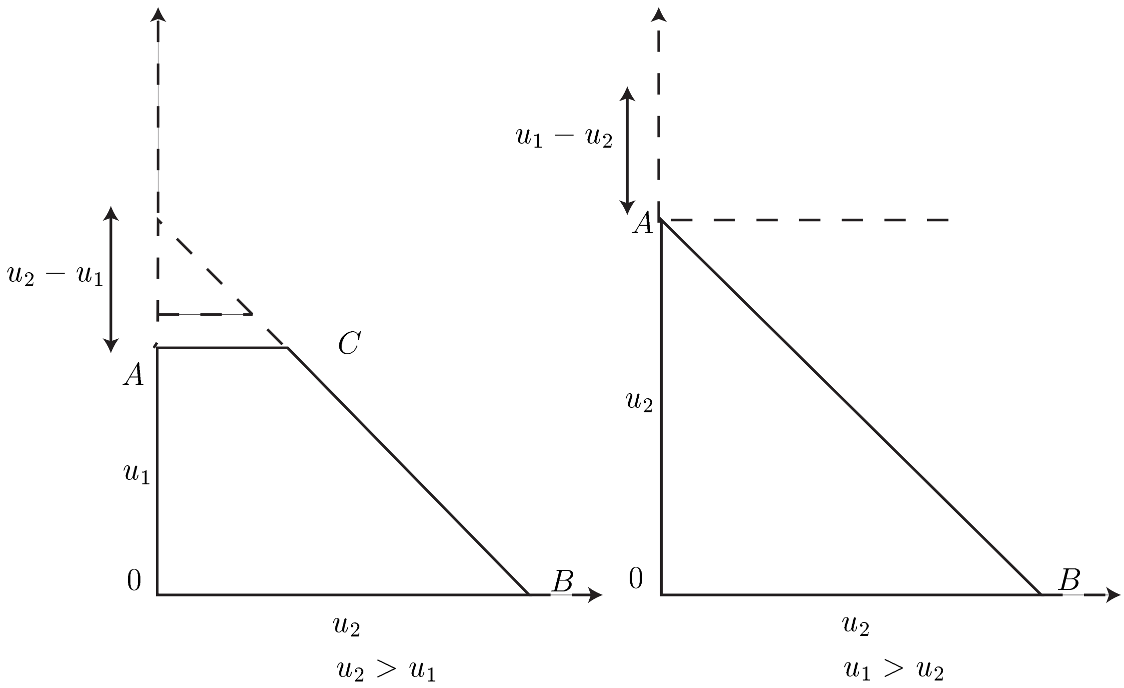


Figure 8: Variation of the partition polytopes: Q(u)

- $u_2 < u_1$
  Then $Q(u_1, u_2)$ is a triangle with vertices 0, $A = u_2 b_2$, $B = u_2 b_1$.

Notice that in the last two cases, the equation $-y_2 + u_1 \geq 0$ is irrelevant and does not produce a face of $Q(u)$.

Conversely, any convex polytope described by inequations can be realized canonically as a member of a family of partition polytopes, which contains also its small deformations obtained by moving faces parallel to themselves.

Let us, following Gelfand-Kapranov-Zelevinski ([19]), decompose the cone $C(\Phi)$ as a union of "chambers". By definition, a chamber is a connected component of the open subset of $C(\Phi)$ obtained by removing the boundaries of all the cones $C(\sigma)$ spanned by subsets σ of Φ forming a basis of V^*. In the case of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

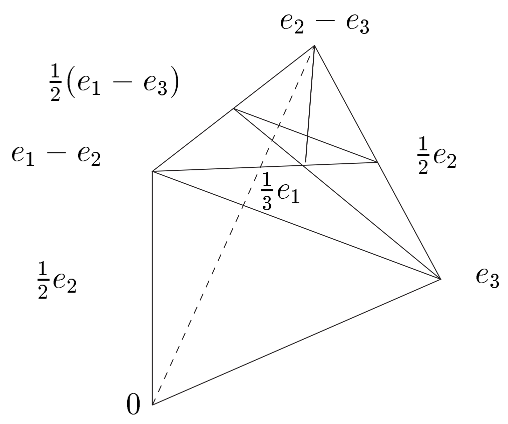


Figure 9: Chambers for A_3

then $\alpha_1 = e_1 = \alpha_3$, $\alpha_2 = e_2$ and $\alpha_4 = e_1 + e_2$. The cone $C(\Phi)$ is the first quadrant, but to define chambers, we have to remove the half-lines $\mathbb{R}^+ e_1, \mathbb{R}^+ e_2, \mathbb{R}^+(e_1 + e_2)$. This leads to the two chambers $\{x_1 > 0, x_2 > 0, x_1 > x_2\}$ or $\{x_1 > 0, x_2 > 0, x_1 < x_2\}$.

When a varies inside a chamber, the combinatorial nature of the polytope $P_\Phi(a)$ remains the same. But, as we have already seen in the last example, the combinatorial nature of the polytope $P_\Phi(a)$ varies when a crosses the wall of a chamber. If $\{P_1, P_2, \ldots, P_r\}$ is a set of convex polytopes in $\mathbb{R}^n$, then the family of polytopes obtained by taking their Minkowski sums

$$\{t_1 P_1 + \cdots + t_r P_r\}$$

with $t_i \geq 0$ can be embedded as a subset of a family of partition polytopes $P_\Phi(a)$ where a **varies in the closure of a chamber** $\mathcal{C}$ of $C(\Phi)$.

Figure 9 is the drawing of chambers for the matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

For a general matrix A, it is difficult to describe chambers of the cone $C(\Phi)$. The program PUNTOS available on the homepage of Jesus de Loera (www.math.ucdavis.edu/deloera) gives an algorithm to compute them, based on Gelfand-Kapranov-Zelevinski theory.

7 Generating functions

The method we used to compute the volume as well as number of points of a rational convex polytopes is based on generating functions.

We are in the setting of partition polytopes, with a surjective map $A : \mathbb{R}^N \to \mathbb{R}^n$. We denote by $(w_1, w_2, \ldots, w_N)$ the canonical basis of $\mathbb{R}^N$, by $\alpha_k = A(w_k)$ and by $\Phi := [\alpha_1, \alpha_2, \ldots, \alpha_N]$. We assume that all vectors α_k lies strictly on an open half-space delimited by an hyperplane. Then the dual cone to the cone $C(\Phi)$ generated by Φ is an open cone: it consists of all vectors $z \in \mathbb{R}^n$ such that $\langle z, \alpha_k \rangle > 0$, for all $1 \leq k \leq N$.

For $a \in C(\Phi)$, let

$$i(a) = \text{cardinal}(P_\Phi(a) \cap \mathbb{Z}^N)$$

and

$$v(a) = \text{vol}(P_\Phi(a)).$$

To compute $i(a)$ and $v(a)$ we shall use generating functions.

Proposition 9 *Let z in the dual cone to $C(\Phi)$, then*

$$\begin{aligned} \int_{C(\Phi)} v(a) e^{-\langle a,z \rangle} da &= \frac{1}{\prod_{\alpha \in \Phi} \langle \alpha, z \rangle}, \\ \sum_{a \in C(\Phi) \cap \mathbb{Z}^n} i(a) e^{-\langle a,z \rangle} &= \frac{1}{\prod_{\alpha \in \Phi} (1 - e^{-\langle \alpha, z \rangle})}. \end{aligned}$$

Proof. Let z in the dual cone to $C(\Phi)$ and let us compute

$$F(z) := \int_{\mathbb{R}_+^N} e^{-\langle A(x), z \rangle} dx.$$

We first write $A(x) = \sum_{i=1}^N x_i \alpha_i$. The integral reads

$$
\begin{aligned}
F(z) &= \int_{\mathbb{R}_+^N} e^{-\langle \sum x_i\alpha_i, z\rangle} dx_1 dx_2 \cdots dx_N \\
&= \prod_{i=1}^N \int_{\mathbb{R}^+} e^{-x_i\langle \alpha_i, z\rangle} dx_i \\
&= \prod_{i=1}^N \frac{1}{\langle \alpha_i, z\rangle}.
\end{aligned}
$$

On the other hand, we can use Fubini formula. We first integrate over the x such that $A(x) = a$, then we integrate on a. Thus

$$
F(z) = \int_{a\in C(\Phi)} \left(\int_{\{x\in\mathbb{R}_+^N \mid A(x)=a\}} e^{-\langle A(x), z\rangle} dx \right) da.
$$

The set $\{x \in \mathbb{R}_+^N \mid A(x) = a\}$ is our partition polytope $P_\Phi(a)$ and the integral over this set of $e^{-\langle A(x),z\rangle}$ is $e^{-\langle a,z\rangle}\operatorname{vol}(P_\Phi(a))$. We thus obtain the first formula of the proposition.

The second formula arises in the same way by calculating in two different ways the sum $\sum_{x\in\mathbb{Z}_+^N} e^{-\langle A(x),z\rangle}$.

The problem of computing $v(a)$ and $i(a)$ now boils down to computing the inverse on the right hand side of the two equations of Proposition 9. A similar problem was considered by Jeffrey and Kirwan ([21]), who, in the context of arrangements of hyperplanes, found an efficient calculus for the inversion of Laplace transforms.

8 Inversion of Laplace transforms and residue formulae

We explain our method ([32]) in the very simple case where $n = 1$.

Let $n = 1$, and let G be the space of functions of the form

$$
f(z) := \frac{P(z)}{z^M},
$$

where P is a polynomial such that $f(z)$ tends to 0 when z tends to ∞. Then there exists a polynomial function $v(h)$ such that

$$f(z) = \int_{h\geq 0} v(h)e^{-hz}dh$$

for $z > 0$.

I claim that v is given by the residue formula

$$v(h) = \text{residue}_{x=0}(f(x)e^{hx}),$$

which is obvious to check. Indeed, $f(z)$ is of the form $\sum_{p=1}^{M} \frac{u_p}{z^p}$, and we may directly check the formula for $\frac{1}{z^p} = \int_{h\geq 0} \frac{h^{p-1}}{(p-1)!}dh$.

The dependance of $v(h)$ in h is via the Taylor series of e^{hx} at $x = 0$. The function $f(x)$ has a pole at $x = 0$ of order M, thus we need to take the Taylor development of e^{hx} only up to order M. In particular, it is clear that $v(h)$ is a polynomial in h of degree less or equal to $M - 1$.

Let us turn to the calculation of $i(a)$.

Let $\mathcal{M}$ be the space of functions of the form

$$F(z) := \frac{P(z)}{(1-z)^M},$$

where $P(z)$ is a polynomial in z. We assume that $F(z)$ tends to 0 when z tends to ∞. Expand the function $F(z)$ as a Taylor series at the origin. Then there exists a polynomial function $i(k)$ such that

$$F(z) = \sum_{k=0}^{\infty} i(k)z^k$$

for $|z| < 1$.

I claim that i is given by the residue formula

$$i(k) = -\text{residue}_{z=1}\left(F(z)z^{-k}\frac{dz}{z}\right).$$

Indeed, integrating on a small circle near $z = 0$ the Taylor expansion of F, we obtain

$$i(k) = \text{residue}_{z=0}\left(F(z)z^{-k}\frac{dz}{z}\right).$$

From our assumption, it follows that the 1-form $F(z)z^{-k}\frac{dz}{z}$ has no residue at ∞. Thus we obtain, from the residue theorem on $P_1(\mathbb{C})$, that for any $k \geq 0$,

$$i(k) = -\text{residue}_{z=1}\left(F(z)z^{-k}\frac{dz}{z}\right).$$

Writing $z = e^{-x}$, we also obtain

$$i(k) = \text{residue}_{x=0}(F(e^{-x})e^{kx}),$$

which is strikingly similar to the formula

$$v(h) = \text{residue}_{x=0}(f(x)e^{hx}).$$

The function $F(e^{-x})$ has a pole of order M at $x = 0$. So it is clear that $i(k)$ is a polynomial in k of degree less or equal to $M - 1$.

A similar **multidimensional** residue formula is used in the final formulae of Theorems 15 and 17. The same idea of moving a residue from a contour near $z = 0$ to a contour near $z = 1$ is involved in the proof of this formula. The cohomology of the complement of an union of hyperplanes is the crucial tool we need (or better an algebraic version of this cohomology).

In a very simple example, consider functions $F(z_1, z_2)$ of the forms $F(z_1, z_2) = \frac{1}{(1-z_1)^M(1-z_2)^N(1-z_1z_2)^Q}$. Here M, N, Q are positive integers. Expand

$$F(z_1, z_2) = \sum_{k_1\geq 0, k_2\geq 0} v(k_1, k_2)z_1^{k_1}z_2^{k_2}$$

in Taylor series. Then there exists two polynomial functions $v_1(a_1, a_2)$ and $v_2(a_1, a_2)$ such that $v_1(a, a) = v_2(a, a)$ and such that

$$v(k_1, k_2) = v_1(k_1, k_2)$$

if $k_1 \geq k_2$, while

$$v(k_1, k_2) = v_2(k_1, k_2)$$

if $k_1 \leq k_2$.

Indeed, we have, from the Cauchy theorem,

$$v(k_1, k_2) = (\frac{1}{2i\pi})^2 \int_{|z_1|=\epsilon_1, |z_2|=\epsilon_2} \frac{1}{(1-z_1)^M(1-z_2)^N(1-z_1z_2)^Q} z_1^{-k_1}z_2^{-k_2}\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

whatever small non-zero real numbers ϵ_1, ϵ_2 we choose. It is tempting, as in the one-dimensional case, to use the other pole $z_1 = z_2 = 1$ to compute this integral. When choosing exponential coordinates e^{-x_1}, e^{-x_2} near this pole, we are led to consider the 2-form

$$\frac{1}{(1-e^{-x_1})^M(1-e^{-x_2})^N(1-e^{-(x_1+x_2)})^Q} e^{k_1x_1}e^{k_2x_2}dx_1dx_2.$$

In a neighborhood of $(0,0)$, the function

$$J(x_1,x_2) = \frac{1}{(1-e^{-x_1})^M(1-e^{-x_2})^N(1-e^{-(x_1+x_2)})^Q}$$

has now poles on the hyperplanes $\{x_1 = 0\}$, $\{x_2 = 0\}$ **and** $\{x_1 + x_2 = 0\}$. Thus its restriction to a cycle $C(\epsilon_1, \epsilon_2) = \{(x_1, x_2)|\ |x_1| = \epsilon_1, |x_2| = \epsilon_2\}$ is holomorphic provided $\epsilon_1 \neq \epsilon_2$. Now the cycles $C(\epsilon_1, \epsilon_2)$ for $\epsilon_1 > \epsilon_2$ or $\epsilon_2 > \epsilon_1$ cannot be deformed to each other without coming across a pole of $J(x_1, x_2)$.

It is not difficult to show that we have

$$v_1(a_1,a_2) = (\frac{1}{2i\pi})^2 \int_{C(\epsilon_1,\epsilon_2)} J(x_1,x_2)e^{a_1x_1+a_2x_2}dx_1dx_2$$

with $\epsilon_2 > \epsilon_1 > 0$, while

$$v_2(a_1,a_2) = (\frac{1}{2i\pi})^2 \int_{C(\epsilon_1,\epsilon_2)} J(x_1,x_2)e^{a_1x_1+a_2x_2}dx_1dx_2,$$

with $\epsilon_1 > \epsilon_2 > 0$.

In other words, the analogue of the residue formula for

$$i(k) = \text{residue}_{x=0}(F(e^{-x})e^{kx}),$$

for the Taylor series of F in one variable is replaced by the two formulae (both obviously polynomial in k_1, k_2.)

$$v_1(k_1,k_2) = \text{residue}_{x_2=0}\text{residue}_{x_1=0}(F(e^{-x_1},e^{-x_2})e^{k_1x_1+k_2x_2}),$$

$$v_2(k_1,k_2) = \text{residue}_{x_1=0}\text{residue}_{x_2=0}(F(e^{-x_1},e^{-x_2})e^{k_1x_1+k_2x_2}).$$

9 Arrangements of hyperplanes and the total residue

Let V^* be a real vector space of dimension n with a set Δ of non-zeros vectors. We assume Δ symmetric, that is $\Delta = -\Delta$. To each $\alpha \in \Delta$ we associate a hyperplane $\{\alpha = 0\}$ in V.

Let R_Δ be the ring of rational functions on $V_{\mathbb{C}}$ with poles on the hyperplanes $\{\alpha = 0\}$. If $F \in R_\Delta$, we have, for $z \in V_{\mathbb{C}}$

$$F(z) = \frac{P(z)}{\prod_{\alpha\in\Delta}\langle\alpha, z\rangle^{n_\alpha}}$$

for some polynomial P and non-negative integers n_α.

In dimension 1, with linear form $\alpha(z) = z$, our space R_Δ is just the space of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. The power z^k is the derivative of the function $\frac{1}{k+1}z^{k+1}$ except when $k = -1$. Thus there is just a particular function $\frac{1}{z}$ which has no primitive. We can write:

$$\mathbb{C}[z, z^{-1}] = \mathbb{C}\frac{1}{z} \oplus \partial_z\mathbb{C}[z, z^{-1}].$$

Let us come back to a general hyperplane arrangement and let us give a description of R_Δ which generalizes this decomposition of $\mathbb{C}[z, z^{-1}]$.

Let σ be a subset of Δ, such that elements of σ form a basis of V^*. We will say that σ is a basic subset of Δ. Setting

$$\sigma = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\},$$

we define the "simple fraction"

$$f_\sigma(z) = \frac{1}{\langle\alpha_{i_1}, z\rangle\langle\alpha_{i_2}, z\rangle\cdots\langle\alpha_{i_n}, z\rangle}$$

and the space S_Δ generated by simple fractions:

$$S_\Delta = \sum_\sigma \mathbb{C}f_\sigma,$$

where σ runs over all basic subsets of Δ.

Choose a basis $\{e^1, e^2, \ldots, e^n\}$ of V. The partial derivative ∂_i acts on R_Δ. We denote

$$\partial R_\Delta = \sum_{i=1}^{n} \partial_i R_\Delta.$$

The following theorem is proved in ([9]).

Theorem 10 *We have:*

$$R_\Delta = S_\Delta \oplus \partial R_\Delta.$$

The space S_Δ will be very important in our formulae. It is spanned by the functions f_σ, however the following examples show that the elements f_σ are not generally linearly independent.

Example. We consider V of dimension 2 with basis e^1, e^2. Consider the set

$$\Delta^+ = \{z_1, z_2, (z_1 + z_2)\}$$

of linear forms on $V := \{z = z_1 e^1 + z_2 e^2\}$. Let $\Delta = \Delta^+ \cup \Delta^-$. There are 3 basis of V^* formed with elements of Δ^+, namely $\sigma_1 = \{z_1, z_2\}$, $\sigma_2 = \{z_1, (z_1 + z_2)\}$, and $\sigma_3 = \{z_2, (z_1 + z_2)\}$. There is a linear relation between the 3 corresponding simple fractions:

$$\frac{1}{z_1 z_2} = \frac{1}{z_1(z_1 + z_2)} + \frac{1}{z_2(z_1 + z_2)}.$$

Example. We consider the vector space $\mathbb{R}^n$ and the set of $n(n+1)$ linear forms:

$$A_n = \{\pm(z_i - z_j)_{i<j}, \pm z_1, \pm z_2, \ldots, \pm z_n\}.$$

The following proposition can be proved by induction on n.

Proposition 11 *A basis $\{f_w\}$ of the space S_{A_n} is obtained as follows: set*

$$f_w(z) = \frac{1}{(z_{w(1)} - z_{w(2)})(z_{w(2)} - z_{w(3)}) \cdots (z_{w(n-1)} - z_{w(n)}) z_{w(n)}}$$

where w is a permutation on $\{1, \ldots, n\}$.

Then S_{A_n} has a basis indexed by elements w of the permutation group Σ_n and hence dim $S_{A_n} = n!$.

Let us come back to the general case. As

$$R_\Delta = S_\Delta \oplus \partial R_\Delta,$$

there is a projection:

$$\mathrm{Tres} : R_\Delta \mapsto S_\Delta$$

called the total residue. Notice that the total residue of F vanishes whenever F is a sum of derivatives. For instance, the residue of the following function

$$\frac{z_1^3}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)z_1 z_2 z_3} - \frac{1}{(z_1 - z_2)(z_2 - z_3)z_3} + \frac{1}{(z_1 - z_3)(z_3 - z_2)z_2}$$

vanishes. Indeed, we can verify that this function is equal to:

$$-\partial_2 \frac{(z_1 - 2z_3)}{z_3(z_1 - z_3)(z_2 - z_3)} + \partial_3 \frac{(z_1 - 2z_2)}{z_2(z_1 - z_2)(z_3 - z_2)}.$$

The total residue vanishes on homogeneous functions of R_Δ which are of degree m, whenever $m \neq -n$. Thus we can extend the total residue to functions $F(z) = \frac{P(z)}{\prod_{\alpha \in \Delta} \langle \alpha, z \rangle^{n_\alpha}}$, where P is a holomorphic function defined near 0.

The most important tool in computations is the following linear form on the space S_Δ.

For each chamber $\mathcal{C}$ there exists a linear functional denoted by

$$\phi \mapsto \langle\langle \mathcal{C}, \phi \rangle\rangle$$

on S_Δ, defined by

$$\langle\langle \mathcal{C}, |\det \sigma| f_\sigma \rangle\rangle = 1$$

if $\mathcal{C} \subset C(\sigma)$

$$\langle\langle \mathcal{C}, |\det \sigma| f_\sigma \rangle\rangle = 0$$

otherwise.

Via projections of R_Δ on S_Δ, we identify linear forms on S_Δ to linear forms on R_Δ vanishing on derivatives.

Determining the chambers $\mathcal{C}$ and the linear form $\langle\langle \mathcal{C}, \phi \rangle\rangle$ is difficult in general. The linear form $\langle\langle \mathcal{C}, \phi \rangle\rangle$ can be realized as an integration on a cycle in the space $V_{\mathbb{C}} \setminus \cup_{\alpha \in \Delta} \{\alpha = 0\}$ depending of the chamber $\mathcal{C}$.

It can be easy to describe the linear form $\langle\langle \mathcal{C}, \phi\rangle\rangle$ for some very special cases such as

$$\Delta := A_n := \{\pm(e_i - e_j)|\ 1 \le i < j \le n\} \cup \{\pm e_i|\ 1 \le i \le n\}$$

and

$$\Phi := A_n^+ := \{(e_i - e_j)|\ 1 \le i < j \le n\} \cup \{e_i|\ 1 \le i \le n\}.$$

The space R_Δ consists of rational functions of $(z_1, z_2, \dots, z_n)$ with poles on the hyperplanes $\{z_i = z_j\}$ or $\{z_i = 0\}$.

The cone $C(\Phi)$ is described as follows:

$$C(\Phi) = \{a_1e_1 + a_2e_2 + \cdots + a_ne_n|\ a_1 \ge 0, a_1 + a_2 \ge 0, \dots, a_1 + a_2 + \cdots + a_n \ge 0\}.$$

The number of chambers of the cone $C(\Phi)$ is not known (see de Loera-Sturmfels ([14]) for the description of chambers for small n). But in any dimension, one of the chambers of the cone $C(\Phi)$ above is

$$\mathcal{C}_{nice} = \{a_1e^1 + a_2e^2 + \cdots + a_ne_n|\ a_1 > 0, a_2 > 0, \dots, a_n > 0\}.$$

Furthermore, in this case the linear form attached to this chamber can be written in terms of iterated residues:

$$\langle\langle \mathcal{C}_{\text{nice}}, \phi\rangle\rangle = \text{residue}_{z_1=0} \cdots \text{residue}_{z_n=0}(\phi(z_1, z_2, \dots, z_n)).$$

The linear forms attached to any chamber $\mathcal{C}$ are not too difficult to compute, as we know an explicit basis of the space S_{A_n}.

Example. Let us consider the matrix:

$$A := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

The system Φ spanned by this matrix is the system A_3^+. There are 7 chambers.

Our space R_Δ consists of functions $f(z_1, z_2, z_3)$ with poles on $\{z_i = 0\}$ or on $\{z_i = z_j\}$. Thus when taking a contour

$$C(\epsilon_1, \epsilon_2, \epsilon_3) := \{|z_1| = \epsilon_1, |z_2| = \epsilon_2, |z_3| = \epsilon_3\},$$

provided that all ϵ_i are different, the function $f(z_1, z_2, z_3)$ is well defined on $C(\epsilon_1, \epsilon_2, \epsilon_3)$. However, the function $f(z_1, z_2, z_3)$ with poles on $\{z_i = z_j\}$, we see that the relative order on $\epsilon_1, \epsilon_2, \epsilon_3$ will lead to different cycles.

Let us give two examples for two different chambers.

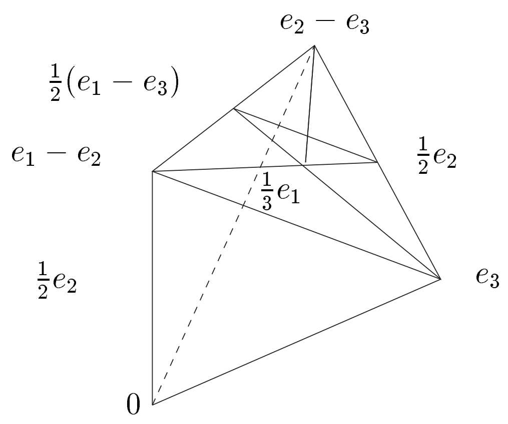


Figure 10: Chambers for A_3

- Let $\mathcal{C}_1$ be the the chamber spanned by e_1, e_2, e_3. (This is what we called the nice chamber). Then

$$\langle\langle \mathcal{C}_1, \phi \rangle\rangle = (\frac{1}{2i\pi})^3 \int_{C(\epsilon_1,\epsilon_2,\epsilon_3)} \phi(z_1, z_2, z_3) dz_1 dz_2 dz_3$$

  where $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$.

- Let $\mathcal{C}_2$ be the the chamber spanned by $e_1 - e_2, e_1, e_3$.

  Then

$$\langle\langle \mathcal{C}_2, \phi \rangle\rangle = (\frac{1}{2i\pi})^3 \int_{C(\epsilon_1,\epsilon_2,\epsilon_3)} \phi(z_1, z_2, z_3) dz_1 dz_2 dz_3$$
$$-(\frac{1}{2i\pi})^3 \int_{C(\epsilon'_1,\epsilon'_2,\epsilon'_3)} \phi(z_1, z_2, z_3) dz_1 dz_2 dz_3$$

  where $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$ and $\epsilon'_2 > \epsilon'_1 > \epsilon'_3 > 0$.

10 Jeffrey-Kirwan formula for the volume

Let $\Phi \subset V^*$ be a sequence of vectors belonging to Δ and all lying on the same side of some given hyperplane. We assume that Φ spans V^*.

Let us give a formula for the volume of the polytope $P_\Phi(a)$. A quick proof is given in ([1]).

Theorem 12 (Jeffrey-Kirwan formula)
Let $\mathcal{C}$ be a chamber of $C(\Phi)$. Then for $a \in \overline{\mathcal{C}}$

$$\begin{aligned}\mathrm{vol}(P_\Phi(a)) &= \langle\langle \mathcal{C}, \mathrm{Tres}\frac{e^{\langle a,z\rangle}}{\langle \alpha_1, z\rangle \cdots \langle \alpha_N, z\rangle}\rangle\rangle \\ &= \frac{1}{(N-n)!}\langle\langle \mathcal{C}, \mathrm{Tres}\frac{\langle a,z\rangle^{N-n}}{\langle \alpha_1, z\rangle \cdots \langle \alpha_N, z\rangle}\rangle\rangle\end{aligned}$$

which is a polynomial in a on the chamber $\mathcal{C}$.

From this formula, following Aomoto's proof of the calculation of Selberg-like integrals and the indication of Zeilberger ([33]), we show in ([1]) that it is possible to recover the conjectured formula for the volume of the Chan-Robbins-Yuen polytope.

Theorem 13 (Zeilberger).
Let $N := n(n-1)/2$. Then

$$\begin{aligned}\mathrm{vol}(CRY_n) &= \frac{1}{N!}\mathrm{residue}_{z_1=0} \cdots \mathrm{residue}_{z_n=0}\frac{z_1^N}{z_1 z_2 \cdots z_n \prod_{i<j}(z_i - z_j)} \\ &= \frac{1}{N!}\prod_{i=1}^{n-2} C_i\end{aligned}$$

where the $C_i = \frac{2i!}{(i+1)!i!}$ are the Catalan numbers.

In the general case of a convex polytope realized as $P_\Phi(a)$, the algorithm to compute its volume is the following. We choose Δ^+ any subset of Δ containing Φ, and contained in a half-space. We compute (or better, we know as in the case of the system A_n) a basis f_σ of the space S_Δ, indexed by a finite set F of subsets of Δ^+. Then, given a generic point $a \in V^*$, we compute if a belongs to the cone $C(\sigma)$, **only for those σ belonging to F**. If $\mathcal{C}$ is the chamber containing a, this determines entirely the form $\langle\langle \mathcal{C}, \cdot\rangle\rangle$, as $\langle\langle \mathcal{C}, |\det\sigma| f_\sigma\rangle\rangle = 0$ or 1 according to the fact that $a \in C(\sigma)$ or not. Then we can realize the form $\langle\langle \mathcal{C}, \cdot\rangle\rangle$ as an iterated residue with respect to special orders, entirely determined by our chamber $\mathcal{C}$.

11 Residue formula for the number of integral points in rational polytopes

Let Φ be a sequence of $\mathbb{Z}^n$ all lying on the same side of a given hyperplane and spanning $\mathbb{R}^n$. Then for $a \in \mathbb{Z}^n \cap C(\Phi)$, the polytope $P_\Phi(a)$ is a rational polytope. Let me now state the relation between the partition function and the number of points in the rational polytope $P_\Phi(a)$ and indicate some properties of the function $i(a)$. Let us introduce a notation:

$$\square(\Phi) = \sum_{\alpha \in \Phi} [0,1]\alpha$$

Notice that the box $\square(\Phi)$ grows larger as the set Φ gets larger. Furthermore, as Φ spans V^*, then for any chamber $\mathcal{C}$ of $C(\Phi)$, the open set $\mathcal{C} - \square(\Phi)$ contains the closure of the open set $\mathcal{C}$.

The following qualitative theorem generalizes Theorem 8.

Theorem 14 *Let Φ be a sequence of vectors in $\mathbb{Z}^n$ all in the same side of an hyperplane and spanning $\mathbb{R}^n$. For z in the dual cone to $C(\Phi)$, the following equality holds:*

$$\sum_{a \in C(\Phi) \cap \mathbb{Z}^n} i(a) e^{-\langle a, z \rangle} = \frac{1}{\prod_{\alpha \in \Phi} (1 - e^{-\langle \alpha, z \rangle})}$$

where $i(a)$ is the cardinal of the set of solutions in non-negative integers n_k of the equation

$$a = n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_N \alpha_N.$$

Then, for each chamber $\mathcal{C}$ of the cone $C(\Phi)$, there exists a periodic-polynomial function $i_{\mathcal{C}}$ on $\mathbb{R}^n$ such that $i(a) = i_{\mathcal{C}}(a)$ for any $a \in (\mathcal{C} - \square(\Phi)) \cap \mathbb{Z}^n$.

The closure $\overline{\mathcal{C}}$ of the chamber $\mathcal{C}$ is a subset of $\mathcal{C} - \square(\Phi)$. The periodic-polynomial behavior of the function $i(a)$ on the set $\overline{\mathcal{C}}$ is due to Sturmfels ([29]).

In fact, we (i.e. Szenes and myself) proved in ([32]) an "explicit formula" for the function $i(a)$. This formula is proven in a rather straightforward way by a separation of variable argument due to A. Szenes ([31]) and the residue theorem in one variable. We state it first in the unimodular case.

Let now Δ be a set of vectors in $\mathbb{Z}^n$ and $\Phi = [\alpha_1, \ldots, \alpha_N]$ a sequence of elements of the set Δ. We assume that Φ spans $\mathbb{Z}^n$. We consider the unimodular case where $|\det \sigma| = 1$ for any subset σ of Φ consisting of n linearly independent vectors. In this case the convex polytopes $P_\Phi(a)$ have integral vertices, whenever a is in $\mathbb{Z}^n$. We get almost the same formulae for the volume or for the number $i_\Phi(a)$ of integral points in $P_\Phi(a)$.

Theorem 15 *Let $\mathcal{C}$ be a chamber. For $a \in (\mathcal{C} - \square(\Phi)) \cap \mathbb{Z}^n$:*

$$i_\Phi(a) = \langle\langle \mathcal{C}, \mathrm{Tres} \frac{e^{\langle a,z\rangle}}{(1-e^{-\langle \alpha_1,z\rangle})\cdots(1-e^{-\langle \alpha_N,z\rangle})} \rangle\rangle.$$

Both formulae for the volume $v(a)$ (Theorem 12) and the number of points $i(a)$ (Theorem 15) in terms of the linear form $\langle\langle \mathcal{C}, \cdot \rangle\rangle$ attached to $\mathcal{C}$ can be proven in a completely parallel way by reducing to the 1-dimensional case (treated in Section 8). The basic formula to reduce to the 1-dimensional case is an analogue of the two formulae below:

$$\frac{1}{z_1 z_2(z_1+z_2)} = \frac{1}{z_1^2 z_2} - \frac{1}{z_1^2(z_1+z_2)},$$

$$\frac{1}{(1-z_1)(1-z_2)(1-z_1z_2)} = \frac{1}{(1-z_1)^2(1-z_2)} + \frac{1}{(1-z_1)(1-z_1^{-1})(1-z_1z_2)}.$$

Using the formula of Theorem 15 (together with a change of variable in residues ([1])), with V. Baldoni, we have implemented a simple Maple program for calculations of number of integral points in transportation polytopes or more generally for networks. It computes number of points in (5×5)-transportation polytopes in approximately 4 hours. Previous algorithm were based on Brion's formula ([14]).

As an easy consequence of the two parallel formulae (Theorem 12 and Theorem 15), we recover the Riemann-Roch formula of Khovanskii-Pukhlikov ([22]). Let us explain how.

Let $\alpha \in V^*$. We denote by $\partial(\alpha)$ the differential operator acting on functions on functions on V^* by

$$\partial(\alpha)P(h) = \frac{d}{d\epsilon}P(h+\epsilon\alpha)\Big|_{\epsilon=0}.$$

Consider the function $\mathrm{Todd}_N(z_1, z_2, \ldots, z_N) := \prod_{i=1}^{N} \frac{z_i}{1-e^{-z_i}}$ and its Taylor series at the origin:

$$\mathrm{Todd}_N(z) := 1 + \frac{1}{2}\sum_{i=1}^{N} z_i + \cdots .$$

Let $\Phi = [\alpha_1, \alpha_2, \ldots, \alpha_N]$ our unimodular set of vectors. We now substitute $z_i = \partial(\alpha_i)$ in this series. We obtain a series of constant coefficients differential operators that we denote by

$$\mathrm{Todd}(\Phi, \partial) := \prod_{\alpha\in\Phi} \frac{\partial(\alpha)}{1-e^{-\partial(\alpha)}}.$$

We can apply this series of differential operators to a polynomial function on V^*.

Theorem 16 (Khovanskii-Pukhlikov)

Let Φ be an unimodular system of vectors in $\mathbb{Z}^n$. Let $\mathcal{C}$ be a chamber of $C(\Phi)$. Then, for $a \in \overline{\mathcal{C}} \cap \mathbb{Z}^n$, we have the equality

$$i(a) = \mathrm{Todd}(\Phi, \partial) v(h)|_{h=a}.$$

Proof. We argue in exactly the same way as in Lecture 1. To apply the series of differential operators $\mathrm{Todd}(\Phi, \partial)$ to the function $v(h)$ given by the residue formula

$$v(h) := \langle\langle \mathcal{C}, \mathrm{Tres} \frac{e^{\langle h,z\rangle}}{\langle\alpha_1, z\rangle \cdots \langle\alpha_N, z\rangle}\rangle\rangle$$

we can commute the residue and the series (the residue operates automatically by truncation of series). Thus we obtain

$$\begin{aligned}\mathrm{Todd}(\Phi, \partial) v(h) &= \langle\langle \mathcal{C}, \mathrm{Tres} \prod_{i=1}^{N} (\frac{\langle\alpha_i, z\rangle}{1-e^{-\langle\alpha_i,z\rangle}}) \frac{e^{\langle h,z\rangle}}{\langle\alpha_1, z\rangle \cdots \langle\alpha_N, z\rangle}\rangle\rangle \\ &= \langle\langle \mathcal{C}, \mathrm{Tres} \frac{e^{\langle h,z\rangle}}{(1-e^{-\langle\alpha_1,z\rangle}) \cdots (1-e^{-\langle\alpha_N,z\rangle})}\rangle\rangle.\end{aligned}$$

We recognize here the residue formula for $i(a)$.

In the general case of a system Φ spanning $\mathbb{Z}^n$, but not necessarily unimodular, the formula for $i(a)$ is periodic-polynomial over the sectors $\mathcal{C} - \Box(\Phi)$. Here is the formula:

Theorem 17 *Let $\mathcal{C}$ be a chamber of $C(\Phi)$. For $a \in (\mathcal{C} - \Box(\Phi)) \cap \mathbb{Z}^n$:*

$$i_\Phi(a) = \sum_{\gamma \in \mathbb{R}^n/2\pi\mathbb{Z}^n} \langle\langle \mathcal{C}, \mathrm{Tres}[\frac{e^{\langle a, z+i\gamma\rangle}}{(1-e^{-\langle \alpha_1, z+i\gamma\rangle})\cdots(1-e^{-\langle \alpha_N, z+i\gamma\rangle})}]\rangle\rangle.$$

The infinite sum $\sum_{\gamma \in \mathbb{R}^n/2\pi\mathbb{Z}^n}$ means the sum over a finite set F of representatives of γ, for which the function

$$z \mapsto \frac{e^{\langle a, z+i\gamma\rangle}}{(1-e^{-\langle \alpha_1, z+i\gamma\rangle})\cdots(1-e^{-\langle \alpha_N, z+i\gamma\rangle})}$$

has a non-zero total residue. (For γ generic, it is holomorphic at zero, thus its total residue is zero. It is easy to prove that indeed the set F described above is finite.)

This result implies the periodic-polynomial behavior of $i(a)$ on the sector $\mathcal{C} - \Box(\Phi)$, which contains $\overline{\mathcal{C}}$. In particular, this formula along rays yields back Ehrhart's theorem (Section 1).

As in the case of Khovanskii-Pukhlikhov, this residue formula implies the formula of Cappell-Shaneson ([11]) and Brion-Vergne ([7]) for the partition function as derivatives of the volumes of nearby polytopes.

12 Polytopes and symplectic geometry: very few references

Let P be a convex integral polytope in $\mathbb{R}^n$. Let T be the torus $S^1 \times S^1 \times \cdots \times S^1$, i.e. the product of n groups of circular rotations $\{e^{i\theta}\}$. Under some conditions (the polytope has to be a Delzant polytope, see Section 5), there exists a compact symplectic manifold M_P of dimension $2n$, an action of the group T on M_P and a map from M_P to P, such that the preimage of the point $p \in P$ is an orbit of T. In other words, the polytope P is exactly the parameter space for the orbits of the action of commuting rotations on M_P. The map f is the moment map. We can thus think of the manifold M_P as a sort of inflation of the polytope P. We refer to the lecture of Michèle Audin in the proceedings of the 5th EWM meeting held in Luminy for more details.

The inflated symplectic manifold corresponding to the interval $[-1, 1]$ is the 2-sphere $S \subset \mathbb{R}^3$ with radius 1. We project S on $\mathbb{R}$ via the height z. The image of S is the interval $[-1, 1]$. The rotation group $T = S^1$ acts by rotation around the axis Oz.

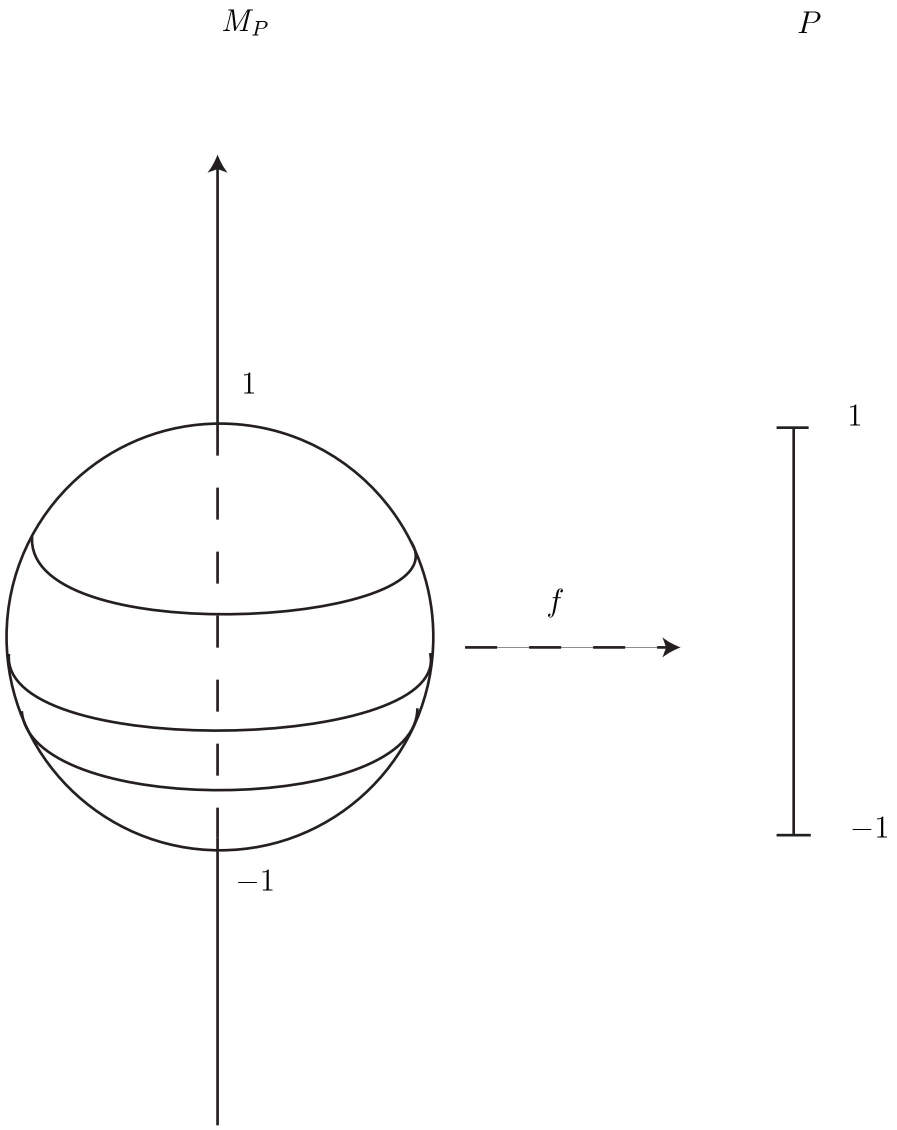


Figure 11: Moment map

The inflated symplectic manifold corresponding to the standard simplex $\Delta \subset \mathbb{R}^n$ is the projective space $P_n(\mathbb{C})$. We realize $P_n(\mathbb{C})$ as the space

$$\{(z_1, z_2, \ldots, z_{n+1}) | \ |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2 = 1\}/\{z \mapsto e^{i\theta} z\}$$

with identification of all proportional points z and $e^{i\theta}z$ in the sphere S^{2n+1} in $\mathbb{C}^{n+1}$, so that $P_n(\mathbb{C})$ is of dimension $2n$. Rotations are given by

$$(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \cdot (z_1, z_2, \ldots, z_n, z_{n+1}) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n, z_{n+1}).$$

The moment map $F : P_n(\mathbb{C}) \to \mathbb{R}^n$ is given by

$$F(z_1, z_2, \ldots, z_n, z_{n+1}) = (|z_1|^2, |z_2|^2, \ldots, |z_n|^2).$$

It is clear that the image of points in $P_n(\mathbb{C})$ are in Δ, indeed $x_1 = |z_1|^2 \geq 0, \ldots, x_n = |z_n|^2 \geq 0$, and $x_1 + x_2 + \cdots + x_n = 1 - |z_{n+1}|^2 \geq 0$. Furthermore two points having the same image are conjugated by $(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$.

The set of integral points in $k\Delta_n$ provides a basis for the vector space $V(k)$ of homogeneous polynomials of degree k in $(n+1)$ variables (in other terms, $V(k)$ is the space $H^0(P_n(\mathbb{C}), \mathcal{O}(k))$), the point $(p_1, p_2, \cdots, p_n)$ with $p_i \geq 0$ and $p_1 + p_2 + \cdots + p_n \leq k$ indexing the monomial $z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} z_{n+1}^{(k - \sum_{i=1}^n p_i)}$.

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