Residue formulae for Verlinde sums, and for number of integral points in convex rational polytopes

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Introduction

These two lectures are dedicated to two topics and their contents are technically independent. In the first lecture on Bernoulli series and Verlinde sums, I present some algebraic formulae concerning a (fascinating) subject on which I am rather ignorant. I believe I am more knowledgeable on the second topic : volume and number of integral points of rational polytopes. In both topics, we shall encounter polynomial functions $w(k)$ of k, where k is a non negative integer, and the fact that these functions are polynomials is not obvious at all from the definition of $w(k)$. There is a common geometric concept underlying this fact for both topics: compact symplectic manifolds. If (M, ω) is a compact symplectic manifold and if the form ω is integral, then we can associate to M a quantized vector space $Q(M, \omega)$. The Riemann-Roch theorem asserts in particular that the dimension of $Q(M, k\omega)$ is a polynomial in k with leading term $k^{\dim M/2} \operatorname{vol}(M)$, where $\operatorname{vol}(M)$ is the symplectic volume of M.

The underlying manifold M to the first topic is the moduli space of flat connections on a Riemann surface with holonomy t around one hole. Verlinde sums are the dimension of $Q(M, k\omega)$ while Bernoulli series compute the volume of such manifolds.

Underlying manifolds to the second topic are toric manifolds. An integral polytope determines a toric manifold M of dimension $2 \dim P$ together with a symplectic form ω on M. The volume of the polytope P is the volume of M and the number of integral points in kP is the dimension of $Q(M, k\omega)$. It is a polynomial in k with leading term $k^{\dim P} \operatorname{vol}(P)$.

In both cases, inspired by the Riemann-Roch theorem, it is possible to develop a purely algebraic Riemann-Roch calculus, and to prove directly a beautiful relation between Bernoulli series and Verlinde sums, as well as between volumes of polytopes and number of integral points contained in the interior. The main idea of this purely algebraic relation in the first case is an idea of A. Szenes while in the second case it goes back to G.Khovanskii and A.V.Pukhlikov.

We have a sort of Riemann-Roch relation between two functions, one depending on a continuous parameter t, the volume, and the other on a discrete parameter (the integer k), the dimension of a vector space. What about showing directly the relation between these two quantities by "computing" them explicitly? Thus my aim in both lectures is to give a hint of "explicit residue formulae" for all these quantities (Bernoulli series, Verlinde sums, volume of polytopes, number of integral points in polytopes), formulae which

allows both **concrete calculations**, and quick proofs of the Riemann-Roch relations.

Many thanks to Sylvie Paycha, for writing and expanding these notes and to Velleda Baldoni-Silva for illustrating.

Lecture 1

Residue formulae for Bernoulli polynomials and Verlinde sums.

In this lecture, I will present some simple sum formulae, which relate Bernoulli series to Verlinde sums. I hope to show in particular that residue formulae for series or sums are very efficient tools to relate these sums **and** calculate them. The one dimensional case is presented here in detail and is already very amusing (and amazing).

These lectures are mainly based on the following articles by A. Szenes (all available on Arkiv) :

–[Sz 1]: The combinatorics of the Verlinde formulas. *Vector bundles in algebraic geometry (Durham 1993)*, 241-253, London Math. Soc. Lecture Note Ser., 208, *Cambridge Univ. Press, Cambridge*, 1995.

–[Sz 2]: Iterated residues and multiple Bernoulli polynomials. *Internat. Math. Res. Notices* **1998**, 18, pp 937–956.

–[Sz 3]: A residue formula for rational trigonometric sums and Verlinde's formula (math CO/0109038)

I do not explain here the underlying geometry. These lectures may (also) serve as an introduction to the algebraic aspects (multi-dimensional residue formulae) of articles of Jeffrey-Kirwan and Bismut-Labourie on the Verlinde formulae. References for the geometry underlying this calculation are given at the end of this lecture.

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1 Bernoulli's theorem

We have all tried to work out formulae for sums of the m-th powers of the first $k + 1$ numbers, and to compare them to the corresponding integral $\int_0^k x^m dx = \frac{k^{m+1}}{m+1}$.

For $m = 0$, we have:

$$0^0 + 1^0 + 2^0 + \cdots + k^0 = k + 1.$$

For $m = 1$, we have:

$$0 + 1 + 2 + 3 + \cdots + k = \frac{k^2 + k}{2}.$$

For $m = 2$, we have:

$$0^2 + 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}.$$

For $m = 3$, we have:

$$0^3 + 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^4}{4} + \frac{k^3}{2} + \frac{k^2}{4}.$$

$$\cdots$$

This polynomial behavior is a general feature of such sums as the following theorem shows:

Theorem 1 *(Jakob BERNOULLI -(1654-1705))*
For any positive integer m, the sum $S_m(k) = \sum_{a=0}^{k} a^m$ of the mth-powers of the first $k + 1$ integers is given by a polynomial formula in k.

We will prove this theorem and relate the sum $S_m(k)$ to the Bernoulli polynomials.

Given a real number t, the Bernoulli polynomials $B(m, t)$ are defined by the following relation:

$$z\frac{\exp(tz)}{\exp(z) - 1} = \sum_{m=0}^{\infty} B(m, t)\frac{z^m}{m!}.$$

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This means, we expand $z\frac{\exp(tz)}{\exp(z)-1}$ in a Taylor series at $z = 0$. The coefficient of $\frac{z^m}{m!}$ in this Taylor series depends polynomially on t, and is defined to be the Bernoulli polynomial $B(m,t)$. The first ones are

$$B(0,t) = 1,$$

$$B(1,t) = t - \frac{1}{2},$$

$$B(2,t) = t^2 - t + \frac{1}{6},$$

$$B(3,t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,$$

$$B(4,t) = t^4 - 2t^3 + t^2 - \frac{1}{30},$$

$$\cdots$$

It follows immediately from the definition of the Bernoulli polynomials that

$$\frac{d}{dt}B(p,t) = pB(p-1,t),$$

$$\int_0^1 B(p,t)dt = 0 \;\; \text{if } p > 1.$$

(Notice also that $(-1)^p B_{2p} = (-1)^p B(2p,0)$ is positive.)

The Bernoulli number B_m is defined to be $B(m,0)$. They satisfy the following relation:

$$\frac{z}{\exp(z)-1} = \sum_{m=0}^{\infty} B_m \frac{z^m}{m!}.$$

Since

$$\frac{1}{1-e^{-z}} = \frac{1}{e^z-1} + 1,$$

we have $\sum_{m=0}^{\infty} B_m \frac{z^m}{m!} + z = \sum_{m=0}^{\infty} B_m \frac{(-z)^m}{m!}$ so that the Bernoulli numbers are equal to 0 for m odd, except when $m = 1$ in which case we have $B_1 = -\frac{1}{2}$.

Now remark that

$$\frac{1}{(m+1)!}(B(m+1,t+1) - B(m+1,t))$$

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is the coefficient of z^{m+1} in the Taylor series of

$$z\left(\frac{e^{(t+1)z}}{e^z - 1} - \frac{e^{tz}}{e^z - 1}\right) = z\frac{e^{tz}(e^z - 1)}{e^z - 1} = ze^{tz}.$$

Thus we obtain:

$$B(m+1, t+1) - B(m+1, t) = (m+1)t^m.$$

We write this equality for $t = 0, 1, \ldots, k$:

$$B(m+1, 1) - B(m+1, 0) = (m+1)0^m,$$

$$B(m+1, 2) - B(m+1, 1) = (m+1)1^m,$$

$$\ldots$$

$$B(m+1, k) - B(m+1, k-1) = (m+1)(k-1)^m,$$

$$B(m+1, k+1) - B(m+1, k) = (m+1)k^m.$$

Adding up this equalities, we obtain

$$B(m+1, k+1) - B(m+1, 0) = (m+1)S_m(k).$$

As

$$z\frac{e^{(t+1)z}}{e^z - 1} = (-z)\frac{e^{((-t)(-z))}}{e^{(-z)} - 1},$$

it follows that $B_m(t+1) = (-1)^m B_m(-t)$.

Thus we obtain Bernoulli Theorem:

Proposition 2 *The function $k \mapsto S_m(k)$ is given by the polynomial formula in k:*

$$S_m(k) = \frac{1}{m+1}((-1)^{m+1}B(m+1, -k) - B_{m+1}).$$

2 Bernoulli series and residues.

In order to describe the Bernoulli polynomials in terms of series, it is useful to consider rational functions of the type $\phi(z) = \frac{E(z)}{z^p}$ where $E(z)$ is a polynomial. The sum over all non zero integers $n \in \mathbb{Z}, n \neq 0$:

$$B(\phi)(t) = \sum_{n \neq 0} \phi(2i\pi n) e^{2i\pi nt}$$

converges absolutely at each real number t if p is a sufficiently large integer since $\sum_{n \neq 0} \frac{1}{n^\alpha}$ converges for α sufficiently large. It defines a periodic function of t. It always converges as a generalized function of t, which is smooth when $t \notin \mathbb{Z}$. For $0 < t < 1$ and p sufficiently large, the residue formula in the plane yields:

$$B(\phi)(t) = \text{residue}_{x=0}(\phi(x)\frac{e^{xt}}{1 - e^x}).$$

(For $-1 < t < 0$, the formula is $B(\phi)(t) = -\text{residue}_{x=0}(\phi(x)\frac{e^{xt}}{1-e^{-x}}).$)

In particular, $t \mapsto B(\phi)(t)$ is given by a polynomial formula whenever $0 < t < 1$.

From the definition of $B(p, t)$ it follows that

$$B(p, t) = -p!\,\text{residue}_{x=0}(x^{-p}\frac{e^{xt}}{1 - e^x})$$

so that setting $\phi(z) = z^{-p}$, we obtain, for $0 < t < 1$, the following formula for the Bernoulli polynomial $B(p, t)$:

Proposition 3 *Let $0 < t < 1$. Then, we have:*

$$\frac{B(p, t)}{p!} = -\text{residue}_{x=0}(x^{-p}\frac{e^{xt}}{1 - e^x}) = -\sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^p}.$$

It is important to notice that the expression on the right hand side is periodic with respect to t, while the left hand side is polynomial. We will call the right hand side the Bernoulli series. The Bernoulli polynomial and the Bernoulli series coincide ONLY on the interval $0 < t < 1$. If $p \geq 2$, the sum in the right hand side is absolutely convergent and the formula is valid for all $0 \leq t \leq 1$.

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In particular, for $p = 2g$ and $t = 0$, we get:

$$B_{2g} = -(2g)! \sum_{n \neq 0} \frac{1}{(2i\pi n)^{2g}}$$

$$= 2(-1)^{g+1}(2g)!(2\pi)^{-2g}\zeta(2g)$$

where $\zeta(k) = \sum_{n=0}^{\infty} n^{-k}$ is the zeta function at point k.
For $g = 1$, as $B_2 = \frac{1}{6}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3   Trigonometric sums

We now consider a closely related sum, which will be a special case of Verlinde sum. Let p be an integer and let

$$F(z) = \frac{1}{(1 - z)^p}.$$

Given a positive **integer** t, let us consider the expression:

$$W(p, t)(k) = \sum_{\omega^k = 1, \omega \neq 1} \omega^t F(\omega) = \sum_{1 \leq n \leq (k-1)} \frac{e^{2i\pi nt/k}}{(1 - e^{2i\pi n/k})^p}.$$

It is very similar to the formula:

$$k^p \frac{B(p, t/k)}{p!} = -\sum_{n \neq 0} \frac{e^{2i\pi nt/k}}{(2i\pi n/k)^p}.$$

In fact, when k tends to ∞, it is not difficult to see that $k^{-p}W(p, t)(k)$ tends to 0 if p is odd, while if p is even, $k^{-p}W(p, t)(k)$ tends to $-\frac{B_p}{p!}$.

It is not at all obvious from its definition that the function $k \mapsto W(p, t)(k)$ is polynomial in k. We will prove it now.

Theorem 4 *Assume $0 \leq t \leq p$, and $k \geq 1$, then the function $k \mapsto W(p, t)(k)$ is given by a polynomial formula in k. This polynomial is of degree less or equal to $p - 1$ if p is odd, and is of degree p if p is even. When p is even,*

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the highest degree term of this polynomial is $-k^p \frac{B_p}{p!}$. *We have the residue formula:*

$$W(p,t)(k) = -k\left(\text{residue}_{x=0}\left(\frac{e^{tx}}{(1-e^x)^p}\frac{1}{(1-e^{-kx})}dx\right)\right).$$

Proof. Consider the 1-form $\frac{z^t}{(1-z)^p}\frac{z^k}{(1-z^k)}\frac{dz}{z}$. From the conditions $0 \le t \le p$, and $k \ge 1$, this form has no residues at 0 and ∞. Poles of the factor $\frac{1}{(1-z^k)}$ are obtained at $z = e^{2i\pi n/k}$ ($0 \le n \le (k-1)$). They are simple, when $n \ne 0$, and their residues add up to the sum $W(p,t)(k)$. As the sum of residues of this 1-form is equal to 0, we obtain from the residue theorem

$$W(p,t)(k) = k\left(\text{residue}_{z=1}\left(\frac{z^t}{(1-z)^p}\frac{z^k}{(1-z^k)}\frac{dz}{z}\right)\right).$$

A change of variable $z = e^x$ in the residue yields:

$$W(p,t)(k) = -k\left(\text{residue}_{x=0}\left(\frac{e^{tx}}{(1-e^x)^p}\frac{1}{(1-e^{-kx})}dx\right)\right).$$

This last expression depends on k via the Laurent expression of $\frac{k}{1-e^{-kx}}$. Recall from the definition of the Bernoulli numbers that

$$\frac{kx}{e^{kx}-1} = \sum_{r=0}^{\infty} B_r \frac{(kx)^r}{r!}.$$

Hence

$$W(p,t)(k) = -\sum_{r=0}^{\infty} B_r R_r(p,t)\frac{k^r}{r!}$$

where

$$R_r(p,t) = (-1)^r \text{residue}_{x=0}\left(\frac{e^{tx}}{(1-e^x)^p}x^{r-1}\right)$$

so that $R_r(p,t)$ vanishes for large r. In fact, as $\frac{1}{(1-e^x)^p}$ has a pole at 0 of order p, we see that $R_r(p,t)$ vanishes for $r > p$, so that $W(p,t)(k)$ is a polynomial of degree less or equal than p. More precisely, it is of degree p for even p and of degree less or equal to $p-1$ for odd p. For p even, the highest degree term is $-B_p\frac{k^p}{p!}$, while if p is odd, the term of degree p in k is equal to 0 as $B_p = 0$, while the term of degree $(p-1)$ is $-(p/2 - t)B_{p-1}\frac{k^{p-1}}{(p-1)!}$.

4   A relation between Bernoulli series and trigonometric sums.

Define:

$$V(q,k) = \sum_{n=1}^{k-1} \frac{1}{4^q(\sin(\pi n/k))^{2q}}.$$

As we will see later on, $V(q,k)$ is a special case of a Verlinde sum. We have $V(q,k) = (-1)^q W(2q,q)(k)$. Indeed

$$(-1)^q W(2q,q)(k) = (-1)^q \sum_{1 \le n \le (k-1)} \frac{e^{2i\pi nq/k}}{\left(1 - e^{2i\pi n/k}\right)^{2q}}$$

$$= \sum_{n=1}^{k-1} \frac{1}{(1 - e^{2i\pi n/k})^q (1 - e^{-2i\pi n/k})^q}$$

$$= \sum_{n=1}^{k-1} \frac{1}{4^q(\sin(\pi n/k))^{2q}}.$$

Thus, from Theorem 4, we obtain that $k \mapsto V(q,k)$ is a polynomial in k, of degree $2q$. Notice that $V(q,k)$ is a sum of positive real numbers so that $V(q,k)$ is positive.

The polynomial $k \mapsto V(q,k)$ is of degree $2q$ and its highest degree term is

$$(-1)^{q+1} \frac{B_{2q}}{(2q)!} k^{2q}.$$

(In our conventions for Bernoulli numbers, $(-1)^{q+1} B_{2q}$ is positive.)

There is a more precise relation between the polynomial function $t \mapsto B(2q,t)$ and the polynomial function $k \mapsto V(q,k)$. Consider the Taylor series at the origin of the function of x

$$\hat{A}(q,x) := \frac{(x/2)^{2q}}{(\sinh(x/2))^{2q}}$$

$$= 1 - \frac{q}{12}x^2 + (\frac{q}{1440} + \frac{q^2}{288})x^4 + \cdots.$$

Substitute $x = \frac{\partial}{\partial t}$ in $\hat{A}(q, x)$, and consider $\hat{A}(q, \partial)$ as a series of differential operators in powers of $\partial := \frac{\partial}{\partial t}$. The action of $\hat{A}(q, \partial)$ on a polynomial function of t is well defined.

Theorem 5

$$V(q, k) = (-1)^{(q+1)} \frac{k^{2q}}{2q!} (\hat{A}(q, \partial/k) \cdot B(2q, t))|_{t=0}.$$

On this expression, we see again that the highest degree term of the polynomial function $k \mapsto V(q, k)$ is $(-1)^{q+1} B_{2q} \frac{k^{2q}}{2q!}$.

(Remark. In the context of the Verlinde formula, this theorem is closely related to the Riemann-Roch theorem on the manifold $M_g := M(SU(2), g)$ of flat connections on vector bundles of rank 2 on a Riemann surface of genus g. This manifold is provided with a line bundle $\mathcal{L}$. The above expression arises when calculating the integral $\int_{M_g} ch(\mathcal{L}^{k-2}) \hat{A}(M_g)$, (where ch is the Chern character and $\hat{A}(M_g)$ is the $\hat{A}$ genus). This evaluates the dimension of so-called conformal blocks: the dimension of the space of holomorphic sections of the holomorphic line bundle $\mathcal{L}^{k-2}$ over M_g. We shall come back to this analogy later.)

Proof. How to prove this theorem: Consider the residue formula for the Bernoulli polynomial

$$B(2q, t) = -(2q!) \text{residue}_{x=0}(x^{-2q} \frac{e^{xt}}{(1 - e^x)}).$$

We can apply the series $\hat{A}(q, \partial/k)$ to this expression. Under the residue, any analytic function is automatically replaced by its Taylor series. Thus we obtain

$$(-1)^{q+1} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t) =$$

$$(-1)^q k^{2q} \text{residue}_{x=0} \left(\hat{A}(q, x/k) x^{-2q} \frac{e^{xt}}{(1 - e^x)} \right)$$

$$= \text{residue}_{x=0} \left(\frac{1}{(1 - e^{x/k})^q (1 - e^{-x/k})^q} \frac{e^{xt}}{(1 - e^x)} \right).$$

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Thus at $t = 0$, we obtain

$$(-1)^{q+1} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t)|_{t=0}$$

$$= \text{residue}_{x=0} \left(\frac{1}{(1 - e^{x/k})^q (1 - e^{-x/k})^q} \frac{1}{(1 - e^x)} \right).$$

The change of variables $x \mapsto -kx$ leads to

$$(-1)^{q+1} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t)|_{t=0}$$

$$= -k (\text{residue}_{x=0} \left(\frac{1}{(1 - e^x)^q (1 - e^{-x})^q} \frac{1}{(1 - e^{-kx})} \right))$$

$$= -k(-1)^q (\text{residue}_{x=0} \left(\frac{e^{qx}}{(1 - e^x)^{2q}} \frac{1}{(1 - e^{-kx})} \right))$$

We recognize here the residue expression given in Theorem 4 for $(-1)^q W(q, 2q)(k) = V(q, k)$. Thus we obtain

$$(-1)^{q+1} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t)$$

$$= V(q, k) = \sum_{n=1}^{k-1} \frac{1}{(1 - e^{2i\pi n/k})^q (1 - e^{-2i\pi n/k})^q}.$$

It is amusing to give a false proof of this theorem, by interverting differentiation and summations:

Indeed if we apply formally $\hat{A}(q, \partial/k)$ to the sum $- \sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^{2g}}$ expressing the Bernoulli polynomial $\frac{1}{2q!} B(2q, t)$, we would obtain:

$$(-1)^{(q+1)} \frac{k^{2q}}{2q!} \hat{A}(q, \partial/k) B(2q, t)|_{t=0}$$

$$= (-1)^q k^{2q} \sum_{n \neq 0} \hat{A}(q, 2i\pi n/k) \frac{1}{(2i\pi n)^{2q}}$$

$$= \sum_{n \neq 0} \frac{1}{(1 - e^{2i\pi n/k})^q (1 - e^{-2i\pi n/k})^q}.$$

This last expression is highly divergent, for at least two reasons: first, for $n \neq 0$ multiple of k, the term to add to the sum is equal to ∞, second, all the terms in the arithmetic progression $n + kj$ give the same summand. However, it gives a hint of why this operator $\hat{A}(q, \partial/k)$ occurs in the comparison. The "renormalized" sum consists in restricting the sum to $0 < n < k$, a set of representatives of the non zero elements of $\mathbb{Z}/k\mathbb{Z}$.

The function $V(q, k)$ has some remarkable integral property (which follows from the representation theory of $SL(2, \mathbb{C})$). Indeed, the function

$$\text{Ver}(q, k) := (2(k+2))^q V(q, k+2) = 2^{-q}(k+2)^q \sum_{n=0}^{k} \frac{1}{\sin((n+1)\pi/k)^{2q}}$$

takes positive integral values on integers k. Notice that for $k = 0$, we have $\text{Ver}(q, 0) = 1$.

We have

$$\text{Ver}(1, k) := \frac{1}{6}(k+1)(k+2)(k+3),$$

$$\text{Ver}(2, k) = \frac{1}{180}(k+2)^2(k+3)(k+1)(k^2+4k+15),$$

$$\text{Ver}(3, k) = \frac{1}{7560}(k+2)^3(k+1)(k+3)(2k^4+16k^3+71k^2+156k+315),$$

$$\cdots$$

You can indeed verify on a few numbers k the amazing fact that these functions take integral values on integers.

5 Preliminaries on semi-simple Lie algebras

What we did in the previous sections correspond to the Lie group $SL(2, \mathbb{C})$ with Lie algebra $sl(2)$ and compact form $SU(2)$. Before introducing more general Verlinde sums, we need to recall some basic facts on the representation theory of semi-simple Lie algebras.

A Lie algebra $\mathfrak{g}$ over a field $\mathbb{C}$ is called semi-simple if its Cartan-Killing form $\langle \cdot, \cdot \rangle$:

$$\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

$$\langle X, Y \rangle = \text{tr}(ad(X)ad(Y))$$

is non degenerate. Any semi-simple Lie algebra is the direct product of simple Lie algebras (a Lie algebra $\mathfrak{g}$ is called simple if it is semi-simple and has no proper ideals). An example of simple Lie algebra is the algebra $sl(n)$ of all $n \times n$ matrices with trace equal to 0. Then, up to normalization, the Killing form is $\langle X, Y \rangle = \operatorname{tr}(XY)$ where tr is the ordinary trace.

In particular $sl(2) := \{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{C} \}$ is a simple Lie algebra.

In order to describe the finite dimensional representations of $\mathfrak{g}$, we need to introduce a few preliminary definitions.

A Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that:

i) $\mathfrak{h}$ is abelian and every element $X \in \mathfrak{h}$ is such that the transformation $ad(X)$ is diagonalizable,

ii) $\mathfrak{h}$ is its own normalizer in $\mathfrak{g}$ i.e. $\{X \in \mathfrak{g}, [X, \mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}$.

Such an algebra $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and is unique up to conjugacy. The dimension r of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$. In the case of $sl(n)$, the algebra $\mathfrak{h}$ is the set of diagonal matrices (with zero trace). The rank of $sl(n)$ is $n - 1$.

For $sl(2)$, we write an element t of $\mathfrak{h}$ as

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & -t_1 \end{pmatrix}.$$

The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate. We can then identify $\mathfrak{h}$ and $\mathfrak{h}^*$.

A root is an element $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$, such that the corresponding root space

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}, [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{h}\}$$

is non zero.

Let $\Delta := \{\alpha \in \mathfrak{h}^*, \alpha \text{ is a root}\}$.

A positive system of roots is a set $\Delta^+ \subset \Delta$ such that $\Delta^+ \cap (-\Delta^+) = \phi$, $\Delta^+ \cup (-\Delta^+) = \Delta$, and such that for any $\alpha, \beta \in \Delta^+$, $\alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta^+$.

A simple system of roots is a set $S \subset \Delta^+$, $S = \{\alpha_1, \ldots, \alpha_r\}$ such that given $\alpha \in \Delta^+$, there are uniquely defined non negative integers $m_i, i = 1, \ldots, r$ such that $\alpha = m_1 \alpha_1 + \cdots + m_r \alpha_r$.

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Let $S = \{\alpha_1, \ldots, \alpha_r\}$ be a simple system of roots; we set H_i to be the unique element of $\mathfrak{h}$ such that $\lambda(H_i) = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \langle \lambda, \alpha_i \rangle$, for any $\lambda \in \mathfrak{h}^*$. This element is well defined, since $\langle \cdot, \cdot \rangle_{\mathfrak{h} \times \mathfrak{h}}$ is non degenerate. Thus we have $\alpha_i(H_i) = 2$.

We denote by $\mathfrak{h}_{\mathbb{R}}$ the real vector space of $\mathfrak{h}$ spanned by the elements H_i of the complex Cartan subalgebra $\mathfrak{h}$. An element $\lambda \in \mathfrak{h}^*$ is called an integral weight if $\lambda(H_i)$ is an integer for all $1 \leq i \leq r$. It is called dominant if $\lambda(H_i)$ is real and non negative for all $1 \leq i \leq r$.

We denote by $P \subset \mathfrak{h}_{\mathbb{R}}^*$ the lattice of integral weights. We denote by Q its dual lattice in $\mathfrak{h}_{\mathbb{R}}$: the lattice Q is exactly the set of elements $t \in \mathfrak{h}_{\mathbb{R}}$, where all integral weights take integral values. A weight λ is called regular, if $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$. We denote by $P_{reg} \subset \mathfrak{h}_{\mathbb{R}}^*$ the set of regular integral weights.

The set

$$\Gamma := \{m_1 \alpha_1 + \cdots + m_r \alpha_r, m_i \text{ non negative integers}\}$$

induces a partial ordering on the weights: $\lambda \leq \lambda'$ whenever $\lambda' - \lambda \in \Gamma$.

Given a finite-dimensional representation U of $\mathfrak{g}$ in a complex vector space V, a weight λ of U is an element $\lambda \in \mathfrak{h}^*$ such that

$$V_\lambda := \{v \in V, U(H)v = \lambda(H)v \ \forall H \in \mathfrak{h}\}$$

is not reduced to zero. We have $V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$. All weights of a finite dimensional representation are integral weights.

If V is an irreducible finite-dimensional representation of $\mathfrak{g}$, then V has a unique highest weight λ (all other weights λ' of the representation V satisfy $\lambda' < \lambda$). This weight λ is an integral and dominant weight. Reciprocally, consider the dominant cone

$$\mathcal{D} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^*, \lambda(H_i) \text{ non negative integer for } 1 \leq i \leq r\}$$

of all dominant integral weights. Given $\lambda \in \mathcal{D}$, there is exactly one equivalence class of finite dimensional irreducible representations of $\mathfrak{g}$ admitting λ as its highest weight. Let U_λ be a representative of this class. In other words, the representations $U_\lambda, \lambda \in \mathcal{D}$, exhaust all the irreducible representations of finite dimensions of $\mathfrak{g}$ up to equivalence.

6 Witten series.

One interesting generalization of the Bernoulli series are the Witten series $B(p, \mathfrak{g})(t)$. Here t is an element of $\mathfrak{h}_{\mathbb{R}}$, and

$$B(p, \mathfrak{g})(t) = \sum_{\lambda \in P_{reg}} \frac{e^{(2i\pi\lambda, t)}}{(\prod_{\alpha \in \Delta^+} 2i\pi\langle\alpha, \lambda\rangle)^p}$$

This series converges for p sufficiently large. For any p, it is well defined as a generalized function of t. The function of $t \in \mathfrak{h}_{\mathbb{R}}$ defined by this series is periodic with respect to the lattice Q. On $\mathfrak{h}_{\mathbb{R}}/Q$ (represented by a domain in $\mathfrak{h}_{\mathbb{R}}$), the expression $B(p, \mathfrak{g})(t)$ above is polynomial in sectors delimited by hyperplanes. On each sector, these series can be in fact expressed in terms of the Bernoulli polynomials in one variable.

In the case of $sl(2)$ one recovers:

$$B(p, t_1) = -B(p, sl(2))(t) = -\sum_{n \neq 0} \frac{e^{2i\pi t_1 n}}{(2i\pi n)^p}.$$

Here

$$t \in \mathfrak{h}_{\mathbb{R}} = \begin{pmatrix} t_1 & 0 \\ 0 & -t_1 \end{pmatrix}$$

with $t_1 \in]0, 1[$.

Similarly to the case of $sl(2)$, a residue formula due to A. Szenes ([Sz1]) can be given for these infinite sums, a formula which allows to calculate them effectively and to prove their polynomial behavior in sectors.

Consider the example of $sl(3)$: we get

$$B(q, sl(3))(t_1, t_2) = \sum_{n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0} \frac{e^{2i\pi(t_1 n_1 + (t_1 + t_2)n_2)}}{(2i\pi n_1)^q (2i\pi n_2)^q (2i\pi(n_1 + n_2))^q}.$$

Here the point $(t_1, t_2) \in \mathbb{R}^2$ represents the diagonal matrice in $\mathfrak{h}_{\mathbb{R}}$

$$\begin{pmatrix} t_1 & 0 & 0 \\ & t_2 & 0 \\ 0 & 0 & -(t_1 + t_2) \end{pmatrix}.$$

Changing n_1 to $n + m$, and n_2 to $-m$ this is also equal to

$$(-1)^q \sum_{n \neq 0, m \neq 0, n + m \neq 0} \frac{e^{2i\pi(t_1 n - t_2 m)}}{(2i\pi n)^q (2i\pi m)^q (2i\pi(n + m))^q},$$

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The residue formula depends on the position of (t_1, t_2) in sectors (as in the one dimensional case, the residue formula for the similar sum $\sum_{n \neq 0} \frac{e^{2i\pi nt}}{(2i\pi n)^p}$ was only valid for $0 < t < 1$). It reads

$$(-1)^q B(q, su(3))(t_1, t_2) =$$

$$-\text{residue}_{x=0}[\text{residue}_{y=0} \frac{e^{\{t_1\}x - \{t_2\}y}}{x^q y^q (x+y)^q} \frac{1}{(1-e^x)(1-e^{-y})}]$$

$$+\text{residue}_{x=0}[\text{residue}_{y=0} \frac{e^{\{t_1+t_2\}x + \{t_2\}y}}{x^q y^q (x+y)^q} \frac{1}{(1-e^x)(1-e^y)}].$$

Here $\{t\}$ denote $t - [t]$ where $[t]$ is the integral part of t.

For example, we have $B(2, su(3))(0, 0) = -\frac{1}{30240}$.

Very naively, looking at the sum of residues at the points $x = 2i\pi n, y = 2i\pi m$, the first iterated residue $-\text{residue}_{x=0}[\text{residue}_{y=0}\cdot]$ should already lead to

$$\sum_{n \neq 0, m \neq 0, n+m \neq 0} \frac{e^{2i\pi(t_1 n - t_2 m)}}{(2i\pi n)^q (2i\pi m)^q (2i\pi)(n+m)^q}$$

and the second residue should lead to the sum

$$\sum_{n \neq 0, m \neq 0, n+m \neq 0} \frac{e^{2i\pi(t_1 + t_2)n + (t_2)(n+m)}}{(2i\pi n)^q (2i\pi m)^q (2i\pi)(n+m)^q}$$

which is equal, as seen from the first formula for $B(q, sl(3))(t_1, t_2)$, after changing n_1 and n_2.

But, in fact the two iterated residues are not equal, and Szenes formula shows that the correct answer is the sum of both iterated residues. Furthermore we must be careful with the order in which we take residues.

*(Remark: Let G be the compact simply connected group whose Lie algebra is a compact form of $\mathfrak{g}$. Up to some normalization, for $p = 2g - 1$, the Witten series compute the symplectic **volume** of the manifold $M(G, g, t)$ of moduli space of flat connections on G-vector bundles on a Riemann surface of genus g with one hole (the variable $t \in \mathfrak{h}_{\mathbb{R}}$ parametrize the holonomy of the flat connection around the hole). We describe this manifold in Section 8, and give some references for its geometric meaning.)*

7 Verlinde sums

One interesting generalization of the sum

$$V(q,k) = \sum_{n=1}^{k-1} \frac{1}{4^q(\sin(\pi n/k))^{2q}}$$

considered in Section 4 is the Verlinde sum, that we are going to describe.

Let $\mathfrak{g}$ be a simple Lie algebra of rank r. Let S be a simple system of roots, Δ^+ the corresponding positive system of roots and $\mathcal{D}$ the associated cone of dominant integral weights. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Let θ be the highest root. Let us also introduce the set (called an alcove):

$$A_k := \{\lambda \in \mathcal{D}, 2\frac{\langle\lambda,\theta\rangle}{\langle\theta,\theta\rangle} \le k\}$$

(the definition of A_k does not depend of the choice of the scalar product.)

Let h be the dual Coxeter number $h = 2\frac{\langle\rho,\theta\rangle}{\langle\theta,\theta\rangle} + 1$. When $\mathfrak{g} = sl(n)$, then $h = n$.

Consider the scalar product $< \cdot, \cdot >$ such that $<\theta,\theta> = 2$.

Let q be an integer. The Verlinde sum is

$$\text{Ver}(q,\mathfrak{g})(k) = c_G^q(k+h)^{rq}\sum_{u\in A_k}\frac{1}{\prod_{\alpha\in\Delta^+}(1-e^{\frac{2i\pi<\alpha,u+\rho>}{k+h}})^q(1-e^{-\frac{2i\pi<\alpha,u+\rho>}{k+h}})^q}$$

$$= c_G^q(k+h)^{rq}\sum_{u\in A_k}\frac{1}{\prod_{\alpha\in\Delta^+}(2\sin(\frac{\pi<\alpha,u+\rho>}{k+h}))}^{2q}$$

Here c_G is an explicit constant depending on G. It is such that $\text{Ver}(1,\mathfrak{g})(0) = 1$.

When $\mathfrak{g} = sl(2)$, $\text{Ver}(q,\mathfrak{g})(k)$ is the polynomial $\text{Ver}(q,k)$ we considered at the beginning of this lecture. Similarly to this case, there is a residue formula (due to Szenes) for the Verlinde sum, which allows to show its polynomial behavior in k and to compare it to the Witten series.

For $\mathfrak{g} = sl(3)$, the above sum is

$$\text{Ver}(q,sl(3))(k) = 3^q(k+3)^{2q}$$

$$\times \sum_{n_1 \geq 0, n_2 \geq 0, n_1 + n_2 \leq k} (8 \sin(\frac{\pi(n_1 + 1)}{k + 3}) \sin(\frac{\pi(n_2 + 1)}{k + 3}) \sin(\frac{\pi(n_1 + n_2 + 2)}{k + 3}))^{-2q}.$$

It is known (and amazing) that the function $k \mapsto \mathrm{Ver}(q, \mathfrak{g})(k)$ takes integral values on integers: it is the dimension of a vector space arising in conformal field theory, namely the space of conformal blocks (a vector space of holomorphic sections of the space of holomorphic sections of the holomorphic line bundle $\mathcal{L}$ over the manifold $M(G, g) = M(G, g, 0)$) of a Riemann surface of genus $q + 1$ with central charge k. This was conjectured by Verlinde and proven by Beauville-Laszlo, G.Faltings , S. Kumar, M.S. Narasimhan and A. Ramanathan, in various degrees of generality.

There is a relation between the Witten series and the Verlinde sum: The Verlinde sum is obtained from the Witten series by applying a series of differential operators $\hat{A}(q, \partial/(k+h))$ to the Witten series. The form of the wanted operator $\hat{A}(q, \partial/(k + h))$ can be guessed from the same intuitive argument of "differentiating" under the sum sign. The correct argument follows immediately from the explicit residue formula. It allows to compute the integral $\int_{M(G,g)} ch(\mathcal{L}^{k+h}) \hat{A}(M(G, g)))$ (ch is the Chern character and $\hat{A}(M(G, g))$ is the $\hat{A}$ genus), at least when k is sufficiently large.

It is known that there exists an integer d such that the function $k \mapsto \mathrm{Ver}(q, \mathfrak{g})(dk)$ is polynomial in k. What is not known is the value of the **smallest** possible d. It is known that $d = 1$ for $\mathfrak{g} = sl(n)$.

The following conjecture on d is natural in view of Kumar-Narasimhan work on the Picard group of $M(g, G, 0)$ (Math. Ann. 308 (1997) 155-173):

$$d = 1 \text{ for } C_r$$

$$d = 2 \text{ for } B_r(r \geq 3), D_r(r \geq 4), G_2$$

$$d = 6 \text{ for } F_4, E_6$$

$$d = 12 \text{ for } E_7$$

$$d = 60 \text{ for } E_8$$

where C_r, B_r, D_r are respectively the series of simple Lie algebras and F_4, E_6, E_7, E_8 are the exceptional Lie algebras. (Kumar-Narasimhan proved that d was smaller or equal to these values).

8 Geometry beyond. A very few references

The underlying geometric object to the theory of Witten series and Verlinde sums is the manifold $M(G, g, t)$. Here G is a compact (simply connected) Lie group with Lie algebra $\mathfrak{g}$, g is a positive integer and t is an element of the Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ of the complex semi-simple Lie algebra $\mathfrak{g}_\mathbb{C}$. If $u_1, u_2 \in G$, we denote by $[u_1, u_2] = u_1 u_2 u_1^{-1} u_2^{-1}$

The manifold $M(G, g, t)$ is defined to be:

$$M(G, g, t) = \{(u_1, v_1, \ldots, u_g, v_g) \mid u_i, v_i \in G \text{ such that } \prod_{i=1}^{g} [u_i, v_i] = e^{2i\pi t}\}/T.$$

Here T denotes the maximal compact torus of G with Lie algebra $i\mathfrak{h}_\mathbb{R}$. The notation $/T$ means that we identify $(2g)$-tuples $(u_1, v_1, \ldots, u_g, v_g)$ and $(u_1', v_1', \ldots, u_g', v_g')$ if there exists $h \in T$ such that $u_i' = t u_i t^{-1}$ and $v_i' = t v_i t^{-1}$.

- The main reference on the geometric properties of the manifold $M(G, g, t)$ is:

  M.Atiyah and R.Bott: The Yang-Mills equation on a Riemann surface. *Phil. Trans. R. Soc. A* **1982** 308.

- The computation of the volume of $M(G, g, t)$ when $G = SU(2)$ is due to M. Thaddeus, who related volumes and Bernoulli numbers.

  M. Thaddeus: Conformal field theory and the cohomology of the moduli space of stable bundles. *J. Differential Geom.* **1992**, pp 131-149.

  In the general case, the volume of $M(G, g, t)$ is determined in the form of Witten series in:

  E. Witten: On quantum gauge theories in two dimensions. *Comm. Math. Phys.* **1991**, 141, pp 153-209.

- A simple description of the manifold $M(G, g, t)$ together with the computation of its symplectic form is in

  –A. Alekseev, A. Malkin, E. Meinrenken: Lie group valued moment maps. *J. Differential geom* **1998**, 48, pp 445-495

  A quick computation of Witten formulas for its symplectic volume is in:

  –A. Alekseev, E. Meinrenken and C. Woodward: Duistermaat-Heckman distributions for group valued moment maps.

- The Verlinde formula was conjectured by:

  E. Verlinde : Fusion rules and modular transformations in 2D-conformal field theory. *Nuclear Physics B* **1988** 300, pp 360-376.

- It was proved using fusion rules, by

  – A. Beauville and Y. Laszlo (for SL(n)): Conformal blocks and generalized theta functions. *Comm. Math. Phys.* **1994** 164, pp 385–419.

  – G. Faltings: A proof of the Verlinde formula. *J. of Alg. Geometry* **1994**, 3, pp 347–374.

  – S. Kumar, M.S. Narasimhan and A. Ramanathan: Infinite Grassmannian and moduli spaces of G-bundles. *Math. Annal.* **1994**, 300, pp 41–75.

- Results on computation of intersection numbers (and thus the Riemann-Roch formula) are obtained via multi-dimensional residues in:

  –Lisa Jeffrey and Frances Kirwan (for SL(n)): Intersection theory on moduli spaces of moduli spaces of holomorphic vector bundles of arbitrary rank on a Riemann surface. *Ann. of Math.* **1998**, 148, pp 109–196.

- A proof of the Verlinde formula for general compact simply connected groups and any number of holes (but with restrictions on k) is obtained via the Riemann-Roch theorem and multi-dimensional residue calculus is obtained in:

  –Jean-Michel Bismut and François Labourie. Symplectic geometry and the Verlinde formulas. *Surveys in differential geometry: differential geometry inspired by string theory* pp 97-311, Surv. Differ. Geom., 5, *Int. Press, Boston*, MA , 1999.

- Another approach, which handles the general case, is due

  – A. Alekseev, E. Meinrenken and C. Woodward: Formulas of Verlinde type for non simply connected groups. (SG/0005047).