A note on unimodular matrices and the normality of certain Rees semigroups

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Abstract

Let A be an integral matrix. If A is unimodular, we present a self-contained direct proof of the fact that the semigroup generated by the columns of A is normal. One of the main results proves that if A is the incidence matrix of a bipartite graph, then adding a row of 1's to A preserves total unimodularity, and as a noteworthy consequence the Rees semigroup associated to A is normal. The corresponding applications to semigroup rings are explained.

1 Introduction

Let $A = (a_{ij})$ be an integral matrix of order $n \times q$ with nonzero distinct columns and let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of column vectors of A. The *integral closure* or *normalization* of the semigroup

$$\mathbb{N}\mathcal{A} = \mathbb{N}v_1 + \cdots + \mathbb{N}v_q \subset \mathbb{Z}^n,$$

associated to A, is defined as:

$$\overline{\mathbb{N}\mathcal{A}} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}.$$

Here $\mathbb{R}_+\mathcal{A}$ denotes the *rational polyhedral cone* in $\mathbb{R}^n$ given by

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} \lambda_i v_i \,\middle|\, \lambda_i \in \mathbb{R}_+ \quad \text{for all } i \right\},$$

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where $\mathbb{R}_+$ is the set of non negative real numbers, and $\mathbb{Z}\mathcal{A}$ is the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{A}$. By Gordan's Lemma [5] there are $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}^n$ such that

$$\overline{\mathbb{N}\mathcal{A}} = \mathbb{N}\mathcal{A} + \mathbb{N}\gamma_1 + \cdots + \mathbb{N}\gamma_r\,,$$

the problem of describing the γ_i in terms of combinatorial properties of A remains open, however there is an algorithm [4] to compute $\gamma_1, \ldots, \gamma_r$. If A is the incidence matrix of a simple graph G, then the γ_i's are described in [6, 10] in terms of special circuits of the graph.

The semigroup $\mathbb{N}\mathcal{A}$ is said to be *normal* or *integrally closed* if $\overline{\mathbb{N}\mathcal{A}} = \mathbb{N}\mathcal{A}$. Our aim is to show some interesting families of normal semigroups derived from unimodular and totally unimodular matrices. The tools we will be using come from graph theory, linear algebra and polyhedral geometry.

Recall that the matrix A is called *totally unimodular* if each $i \times i$ minor of A is 0 or ± 1 for all $i \geq 1$, in particular each entry of A must be 0 or ± 1. Note that the bipartite simple graphs are characterized as those graphs whose incidence matrix is totally unimodular [8, Chapter 19]. We say that the matrix A is *t-unimodular* if all the nonzero $r \times r$ minors of A have absolute value equal to t, where $r \neq 0$ is the rank of A, if $t = 1$ we say that A is *unimodular*.

One of our results is a self-contained simple proof of the fact that $\mathbb{N}\mathcal{A}$ is normal if A is a t-unimodular matrix, this is essentially shown in [11] because "A is t-unimodular" is equivalent to "all regular triangulations of the convex hull of $v_1, \ldots, v_q$ are unimodular", see also [3] for a deep discussion of covering properties. The other main result is that if A is the incidence matrix of a bipartite graph, then the matrix B obtained from A by adding a row of 1's is again totally unimodular. Combining those two results we obtain a "natural" new proof of an important result of [9] showing that if A is the incidence matrix of a bipartite graph, then the *Rees semigroup*

$$\mathbb{N}(v_1, 1) + \cdots + \mathbb{N}(v_q, 1) + \mathbb{N}e_1 + \cdots + \mathbb{N}e_n \subset \mathbb{N}^{n+1}$$

is normal, where e_i is the *ith* unit vector of $\mathbb{R}^{n+1}$. As a byproduct we derive new information about the toric ideal of the Rees algebra associated to a bipartite graph, see Remark 4.4.

Toward the end of this note we present some applications to commutative algebra by introducing the semigroup ring and the Rees algebra associated to the matrix A.

2 Semigroups of unimodular matrices

For simplicity we keep the notation introduced above throughout the rest of this note. If $(M, +)$ is an abelian group its torsion subgroup, denoted by $T(M)$, is the set of all x in M such that $px = 0$ for some $0 \neq p \in \mathbb{N}$.

Let $b \in \mathbb{Z}^n$ be a fixed column vector such that $r = \mathrm{rank}(A) = \mathrm{rank}([A\, b])$. By a classical result of Kronecker [8, p. 51] the system

$$Ax = b$$

has an integral solution if and only if $\Delta_r(A) = \Delta_r([A\, b])$, where $\Delta_i(B)$ denotes the greatest common divisor of all the nonzero $i \times i$ minors of B.

To give some variants of Kronecker's theorem note that there is a positive integer k such that

$$kb = \lambda_1 v_1 + \cdots + \lambda_q v_q \quad (\lambda_i \in \mathbb{Z}\, \forall i). \tag{$*$}$$

Therefore there is a canonical epimorphism of finite groups

$$\varphi \colon T(\mathbb{Z}^n/\mathbb{Z}\mathcal{A}) \longrightarrow T(\mathbb{Z}^n/\mathbb{Z}\mathcal{B}) \quad (\alpha + \mathbb{Z}\mathcal{A} \overset{\varphi}{\longmapsto} \alpha + \mathbb{Z}\mathcal{B}),$$

where $\mathcal{B} = \mathcal{A} \cup \{b\}$.

Lemma 2.1 (Kronecker) The following conditions are equivalent

(i) φ is injective.

(ii) $b \in \mathbb{Z}\mathcal{A}$.

(iii) The abelian groups $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$ and $\mathbb{Z}^n/\mathbb{Z}\mathcal{B}$ have the same invariant factors.

(iv) The matrices $[A\, 0]$ and $[A\, b]$ have the same Smith normal form.

(v) $\Delta_r(A) = \Delta_r([A\, b])$, where $r = \mathrm{rank}(A)$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward. There are invertible integral matrices P_i and Q_i such that

$$D_1 = Q_1[A\ 0]P_1 = \mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0),$$
$$D_2 = Q_2[A\ b]P_2 = \mathrm{diag}(d'_1, \ldots, d'_r, 0, \ldots, 0),$$

are the Smith normal forms of $[A\, 0]$ and $[A\, b]$ respectively, that is, d_i, d'_i are positive integers satisfying that d_i divides d_{i+1} and d'_i divides d'_{i+1} for all i. By the fundamental structure theorem of finitely generated abelian groups there are isomorphisms:

$$T(\mathbb{Z}^n/\mathbb{Z}\mathcal{A}) \simeq (\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_r\mathbb{Z}),$$
$$T(\mathbb{Z}^n/\mathbb{Z}\mathcal{B}) \simeq (\mathbb{Z}/d'_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d'_r\mathbb{Z}).$$

Thus (iii) $\Leftrightarrow$ (iv). Note that $\Delta_i(A) = d_1 \cdots d_i$ and $\Delta_i([A\, b]) = d'_1 \cdots d'_i$ for all i. Hence (iv) $\Rightarrow$ (v). To prove (v) $\Rightarrow$ (i) observe $|T(\mathbb{Z}^n/\mathbb{Z}\mathcal{A})| = |T(\mathbb{Z}^n/\mathbb{Z}\mathcal{B})|$ and consequently φ must be injective, as required. $\square$

Remark 2.2 Recall that $\mathbb{R}_+\mathcal{A}$ can be expressed as an intersection of closed half spaces through the origin, that is, there is a rational matrix C such that

$$\mathbb{R}_+\mathcal{A} = \{x \in \mathbb{R}^n \,|\, Cx \leq 0\}.$$

Thus this representation together with Lemma 2.1 yield a membership test to decide when a given α in $\mathbb{Z}^n$ belongs to $\overline{\mathbb{N}\mathcal{A}} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}$.

Proposition 2.3 *If A is a t-unimodular matrix and $v_{i_1}, \ldots, v_{i_r}$ is a $\mathbb{Q}$-basis for the column space of A, then*

$$\mathbb{Z}\mathcal{A} = \mathbb{Z}v_{i_1} \oplus \cdots \oplus \mathbb{Z}v_{i_r}$$

Proof. For each j one has $\Delta_r([v_{i_1} \cdots v_{i_r}]) = \Delta_r([v_{i_1} \cdots v_{i_r}\, v_j]) = t$, then the result is a consequence of Lemma 2.1. $\Box$

The next result follows from [11, Remark 8.10 and Proposition 13.15], note that the proof given there basically works in general, not only for homogeneous ideals. Our proof below, in contrast to that of [11], is direct and do not make any use of Gröbner bases techniques.

Theorem 2.4 *If A is a t-unimodular matrix, then $\mathbb{N}\mathcal{A}$ is a normal semigroup.*

Proof. Let $b \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}$. By Carathéodory's Theorem [5, Theorem 2.3] there are $v_{i_1}, \ldots, v_{i_r}$ linearly independent columns of A, where r is the rank of A, such that

$$b \in \mathbb{R}_+v_{i_1} + \cdots + \mathbb{R}_+v_{i_r}. \tag{1}$$

On the other hand by Proposition 2.3 one has

$$b \in \mathbb{Z}v_{i_1} + \cdots + \mathbb{Z}v_{i_r}. \tag{2}$$

Since $v_{i_1}, \ldots, v_{i_r}$ are linearly independent by comparing the coefficients of b with respect to the two representations given by (1) and (2) one derives $b \in \mathbb{N}\mathcal{A}$. $\Box$

The material in this section is related to covering properties. For a discussion on the existence of unimodular (Hilbert) covers of rational cones see [3] and the references there.

3 Rees semigroups of bipartite graphs

Here we show that a Rees semigroup obtained from an incidence matrix A of a bipartite graph is normal. Recall that such semigroup is defined by a matrix C obtained by first adding a row of 1's to the matrix A and then adding the canonical vector $e_1, \ldots, e_n$ as column vectors. Unfortunately, in general, the first operation does not preserve total unimodularity even if one assumes that A has $\{0, 1\}$ entries and that all the columns have exactly k entries equal to 1. The case $k = 2$, which is the interesting case here, is treated below.

Definition 3.1 A $\{0, \pm 1\}$ matrix is *Eulerian* if each column and each row of the matrix have an even number of nonzeros.

Theorem 3.2 *Let G be a simple bipartite graph with n vertices and q edges and let $A = (a_{ij})$ be its incidence matrix. If $e_1, \ldots, e_n$ are the first n unit vectors in $\mathbb{R}^{n+1}$ and C is the matrix*

$$C = \begin{pmatrix} a_{11} & \ldots & a_{1q} & e_1 & \cdots & e_n \\ \vdots & \vdots & \vdots & & & \\ a_{n1} & \ldots & a_{nq} & & & \\ 1 & \ldots & 1 & & & \end{pmatrix}$$

obtained from A by adjoining a row of 1's *and the column vectors $e_1, \ldots, e_n$, then C is totally unimodular.*

Proof. Since total unimodularity is preserved under the operation of adding columns of unit vectors, it suffices to prove that the matrix B obtained from A by adding a row of 1's is totally unimodular.

If B is not totally unimodular, it contains a minimal violation submatrix V that contains part of the last row. That is V is not totally unimodular, but this is so for every proper submatrix of V. Hence according to [12, Corollary 12.3.12(b)], V is Eulerian, is square, is nonsingular, and has a row with all 1's.

Since A has two 1's in each column, B has three 1's in each column. Hence, as V is Eulerian, V has two 1's in each column. Using that the columns of V are different because V is nonsingular, this implies that the first row of V has exactly one nonzero entry, a contradiction with the fact that V is Eulerian.

Second Proof. We consider interesting the following alternative proof of this theorem since it gives a different insight for possible extensions of the result.

Suppose that $\{1, \ldots, m\}$ and $\{m+1, \ldots, n\}$ is the bipartition of the graph G. Let C' be the matrix obtained by deleting the last $n-m$ columns from C. It suffices to show that C' is totally unimodular. First one successively subtracts the rows $1, 2, \ldots, m$ from the row $n+1$. Then one reverses the sign in the rows $m+1, \ldots, n$. These elementary row operations produce a new matrix C''. The matrix C'' is the incidence matrix of a directed graph, namely, consider G as a directed graph, and add one more vertex $n+1$, and add the edges $(i, n+1)$ for $i = 1, \ldots, m$. The matrix C'', being the incidence matrix of a directed graph, is totally unimodular [8, p. 274]. As the last m column vectors of C'' are

$$e_1 - e_{n+1}, \ldots, e_m - e_{n+1},$$

one can successively pivot on the first nonzero entry of $e_i - e_{n+1}$ for $i = 1, \ldots, m$ and reverse the sign in the rows $m+1, \ldots, n$ to obtain back the matrix C'. Here a pivot on the entry c'_{st} means transforming column t of C'' into the sth unit vector by elementary row operations. Since pivoting preserves total unimodularity [7, Lemma 2.2.20] one derive that C' is totally unimodular, and hence so is C. In the next specific example we display C' and C''. $\Box$

Example 3.3 To illustrate the constructions in the second proof above consider the complete bipartite graph $G = K_{2,3}$. In this case the two $\mathbb{Z}$-row equivalent matrices are

$$C' = \begin{bmatrix} 1&1&1&0&0&0&1&0\\ 0&0&0&1&1&1&0&1\\ 1&0&0&1&0&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&1&0&0&1&0&0\\ 1&1&1&1&1&1&0&0 \end{bmatrix}, \quad C'' = \begin{bmatrix} 1&1&1&0&0&0&1&0\\ 0&0&0&1&1&1&0&1\\ -1&0&0&-1&0&0&0&0\\ 0&-1&0&0&-1&0&0&0\\ 0&0&-1&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&-1 \end{bmatrix}.$$

Note that pivoting in the last two columns of C'' amounts to adding row 1 to row 6, and then adding row 2 to row 6.

Example 3.4 Consider the matrices

$$V = \begin{pmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&1\\ 1&1&1&1 \end{pmatrix}, \quad A = \begin{pmatrix} 0&0&1&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&1&0&0\\ 1&0&0&0\\ 1&0&0&0\\ 1&0&0&1\\ 0&1&0&1\\ 0&0&1&1 \end{pmatrix}, \quad B = \begin{pmatrix} A \\ 1\,1\,1\,1 \end{pmatrix}.$$

The matrix V has determinant -2 and is the unique minimal violation matrix of total unimodularity with $\{0,1\}$ entries and of order 4, see [12]. On the other hand A is totally unimodular but B is not because V is a submatrix of B. Thus Theorem 3.2 does not extend to incidence matrices of k-hypergraphs with $k > 2$.

There is a family of $\{0,1\}$ matrices called *complement totally unimodular* that preserve total unimodularity when adjoining a row or column consisting of 1's, see [12, Lemma 12.3.4].

Corollary 3.5 ([9]) *If A is the incidence matrix of a bipartite graph, then the Rees semigroup*

$$\mathbb{N}(v_1, 1) + \cdots + \mathbb{N}(v_q, 1) + \mathbb{N}e_1 + \cdots + \mathbb{N}e_n \subset \mathbb{N}^{n+1}$$

is normal, where e_i denotes the ith unit vector of $\mathbb{R}^{n+1}$.

Proof. It follows from Theorem 3.2 and Theorem 2.4. $\Box$

A generalization of Corollary 3.5 is shown in [2] using linear programming techniques, but this generalization does not give any special information about the matrix C that defines the Rees semigroup.

4 Semigroup rings

For the sake of applications to commutative algebra let us briefly explain the link between semigroups and semigroup rings.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. There is a one to one correspondence between $\mathbb{N}^n$ and the semigroup of monomials of R given by

$$a = (a_1, \ldots, a_n) \longleftrightarrow x^a = x_1^{a_1} \cdots x_n^{a_n},$$

under this correspondence the set $\mathcal{A}$ maps onto the set $F = \{x^{v_1}, \ldots, x^{v_q}\}$. If the matrix A has nonnegative entries we associate to A the *semigroup ring* or *monomial subring* given by

$$K[F] = K[\{x^a \mid a \in \mathbb{N}\mathcal{A}\}] \subset R,$$

that is, $K[F]$ is the subring of R generated by F over K. Thus $K[F]$ is generated as a K-vector space by the set of monomials of the form x^α, with $\alpha \in \mathbb{N}\mathcal{A}$.

The *integral closure* of $K[F]$, denoted by $\overline{K[F]}$, is the set of all z in the field of fractions of $K[F]$ satisfying an equation of the form

$$z^m + s_{m-1}z^{m-1} + \cdots + s_1 z + s_0 = 0 \quad (s_i \in K[F] \text{ and } m \geq 1).$$

If $K[F] = \overline{K[F]}$ the ring $K[F]$ is said to be *normal*. Using [13, Chapter 7] one has the following description:

$$\overline{K[F]} = K[\{x^a \mid a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}].$$

Thus $\mathbb{N}\mathcal{A}$ *is normal if and only if* $K[F]$ *is normal.*

Corollary 4.1 ([11]) *If A is a t-unimodular matrix with nonnegative entries and K is a field, then $K[F]$ is a normal domain.*

Proof. It follows from Theorem 2.4. □

Remark 4.2 There is a converse to Corollary 4.1 and Theorem 2.4 that holds for Lawrence ideals, that is, defining ideals of toric subvarieties of $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, see [1, Theorem 1.2].

Corollary 4.3 ([9]) *If G is a bipartite graph with vertices $x_1, \ldots, x_n$ and K is a field, then the Rees algebra*

$$\mathcal{R} = K[\{x_i x_j t \mid x_i \text{ is adjacent to } x_j\} \cup \{x_1, \ldots, x_n\}] \subset R[t]$$

is a normal domain. Here t denotes a new variable.

Proof. It follows from Corollary 3.5. □

Remark 4.4 Corollary 4.3 was shown in [9] using that all the powers of the *edge ideal*

$$I(G) = (\{x_i x_j \,|\, x_i \text{ is adjacent to } \; x_j\}) \subset R$$

of a bipartite graph G have the same set of associated primes. The proof presented here is quite straightforward compared to that of [9]. We focus in the matrix of exponent vectors of the monomials that generate the Rees algebra rather than on the powers of the ideal $I(G)$. Our approach has the advantage of giving new information on the ideal of relations of the Rees algebra. Indeed using [11, Proposition 8.11] one obtains that the toric ideal of the Rees algebra has a universal Gröbner basis consisting of square-free binomials.

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