On the normality of Rees algebras associated to totally unimodular matrices

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Abstract

Let A be an integral matrix with non negative entries and let K be a field. We give a sufficient condition for the normality of the monomial subring determined by the columns of A over the field K. One of the main results proves that if A is totally unimodular, then the Rees algebra of the monomial ideal over K defined by the columns of A is a normal domain.

1 Introduction

Let $A = (a_{ij})$ be an integral matrix of order $n \times q$ with entries in $\mathbb{N}$ and with non zero distinct columns. If $v_k = (a_{1k}, \ldots, a_{nk})$ is the kth column of A, it is customary to set

$$f_k = \mathbf{x}^{v_k} = x_1^{a_{1k}} \cdots x_n^{a_{nk}},$$

where $x_1, \ldots, x_n$ are indeterminates over a field K, that is, $f_1, \ldots, f_q$ denote monomials in a polynomial ring $R = K[x_1, \ldots, x_n]$.

We are interested in the normality of two types of monomial subrings. The first is the K-*subring* of R spanned by $F = \{f_1, \ldots, f_q\}$, which is usually denoted by $K[F]$. The second is the *Rees algebra* of the ideal $I = (f_1, \ldots, f_q)$, which is denoted by $\mathcal{R}(I)$. Recall that the Rees algebra is also a K-subring given by

$$\mathcal{R}(I) = K[f_1t, \ldots, f_qt, x_1, \ldots, x_n] \subset R[t],$$

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where t is a new variable.

The matrix A is called *totally unimodular* if each $i \times i$ minor of A is 0 or ± 1 for all $i \geq 1$, in particular each entry of A must be 0 or ± 1. Examples of totally unimodular matrices include incidence matrices of hypergraphs without odd cycles [1, Chapter 5], incidence matrices of directed graphs [6, p. 274] and network matrices [6, Chapter 19]. The bipartite simple graphs are characterized as those graphs whose incidence matrix is totally unimodular [6, Chapter 19].

Let us describe the two main results of this paper. First we give a sufficient condition for the normality of $K[F]$ in terms of the elementary integral vectors of the kernel of the matrix A (see Theorem 2.3). If A is a totally unimodular matrix, using linear programming techniques we are able to prove that $\mathcal{R}(I)$ is a normal domain (see Theorem 3.2).

If A is the incidence matrix of a bipartite graph, then an important result of [7] proves that $\mathcal{R}(I)$ is normal. As an application we obtain a quite different proof of this result (cf. [4]).

Another application is that $K[F]$ is normal if A is unimodular, this is a result of [8]. Note that here we do not assume $K[F]$ to be homogeneous, for another self-contained short proof of this result see [4].

2 A sufficient condition for normality

First let us recall the notion of elementary integral vector and introduce some more terminology and notation. This notion has played a role in convex analysis [5] and in the theory of toric ideals of graphs [8, 9].

If $\alpha \in \mathbb{R}^q$, its *support* is defined as

$$\mathrm{supp}(\alpha) = \{i \mid \alpha_i \neq 0\}.$$

Note that $\alpha = \alpha_+ - \alpha_-$, where α_+ and α_- are two non negative vectors with disjoint support called the *positive* and *negative* part of α respectively.

Let N be a linear subspace of $\mathbb{Q}^q$. An *elementary integral vector* or *circuit* of N is a nonzero vector α in N with integral entries such that: (i) the support of α is minimal with respect to inclusion, i.e. $\mathrm{supp}(\alpha)$ does not properly contain the support of any other nonzero vector in N, and (ii) the nonzero entries of α are relatively prime.

Since any two circuits with the same support are equal up to a sign [5], the number of circuits of N is finite. See [8] for an algorithm to compute the circuits of the kernel of a matrix.

The next lemma is well known and easy to prove, but we include it here to emphasize its usefulness in induction arguments.

Lemma 2.1 *Let A be an $n \times q$ integral matrix and let A' be the matrix obtained from A by removing its ith column. Then the inclusion map*

$$\begin{array}{ccc} \mathbb{Z}^{q-1} & \xrightarrow{j} & \mathbb{Z}^{q} \\ (a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_q) & \longrightarrow & (a_1,\ldots,a_{i-1},0,a_{i+1},\ldots,a_q) \end{array}$$

induces a bijection between the elementary integral vectors of $\ker(A')$ *and the elementary integral vectors of* $\ker(A)$ *whose ith entry is equal to zero.*

Proof. Since the inclusion map j takes $\ker(A')$ into $\ker(A)$, the result follows readily. $\Box$

Definition 2.2 Let S be an integral domain and let K_S be its field of fractions. The *integral closure* of S, denoted $\overline{S}$, is the set of all $a \in K_S$ satisfying an equation of the form $f(a) = 0$, where $0 \neq f(x)$ is a monic polynomial with coefficients in S. The ring S is called *normal* or *integrally closed* if $S = \overline{S}$.

If $S = K[F]$ is a subring generated by a finite set F of monomials over a field K, then $\overline{S}$ can be described as the set of all $a \in K_S$ satisfying a binomial equation:

$$a^n + s_0 = 0 \quad (s_0 \in S \text{ and } n \geq 1),$$

see [2, Chapter 6]. There is an implemented algorithm [3] that effectively compute the integral closure of $K[F]$.

Let $\mathcal{A} = \{v_1,\ldots,v_q\}$ be a set of points in $\mathbb{R}^n$, the *cone* generated by $\mathcal{A}$, denoted by $\mathbb{R}_+\mathcal{A}$, is given by

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i v_i \,\middle|\, a_i \in \mathbb{R}_+ \text{ for all } i \right\},$$

where $\mathbb{R}_+$ is the set of non negative real numbers. The additive subgroup of $\mathbb{Z}^n$ generated $\mathcal{A}$ is denoted by $\mathbb{Z}\mathcal{A}$.

Theorem 2.3 *Let $A = (a_{ij})$ be an $n \times q$ integral matrix with non negative entries and with distinct non zero column vectors $v_1,\ldots,v_q$. If each elementary integral vectors of* $\ker(A)$ *has a positive or negative part with entries consisting of* 0's *and* 1's, *then the monomial subring*

$$K[\{\mathbf{x}^{v_1},\ldots,\mathbf{x}^{v_q}\}] \subset R$$

is normal, where $R = K[x_1,\ldots,x_n]$ is a polynomial ring over a field K.

Proof. The proof is by induction on q, the number of generators of $K[F]$. The case $q = 1$ is clear. Assume $q \geq 2$ and that the result holds for monomial subrings with less than q generators. To simplify the notation set

$$\mathcal{A} = \{v_1, \ldots, v_q\} \text{ and } F = \{f_1, \ldots, f_q\},$$

where $f_i = \mathbf{x}^{v_i}$. Using [10, Theorem 7.2.28] one has:

$$\overline{K[F]} = K[\{\mathbf{x}^a \mid a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}].$$

Let $z = \mathbf{x}^a \in \overline{K[F]}$ be a minimal generator of the integral closure of $K[F]$. One can write

$$z = \mathbf{x}^a = f_1^{\lambda_1} \cdots f_q^{\lambda_q},$$

for some $\lambda_1, \ldots, \lambda_q \in \mathbb{Z}$. There is a positive integer m such that

$$\begin{aligned} z^m &= f_1^{m\lambda_1} \cdots f_q^{m\lambda_q} \\ &= f_1^{b_1} \cdots f_q^{b_q}, \end{aligned} \tag{1}$$

for some $b_1, \ldots, b_q \in \mathbb{N}$. First we assume $b_i = \lambda_i = 0$ for some i. Consider the matrix A' obtained form A by removing its ith column, then Lemma 2.1 and the induction hypothesis yield $z \in K[F \setminus \{f_i\}]$. Hence we may assume that for each i either $\lambda_i \neq 0$ or $b_i > 0$.

If $\ker(A) = (0)$, the columns of A are linearly independent and $K[F]$ is normal because it is a polynomial ring. Assume $\ker(A) \neq (0)$ and take an elementary integral vector $0 \neq \alpha \in \ker(A)$, one can write

$$\alpha = \underbrace{(e_{i_1} + \cdots + e_{i_k})}_{\alpha_+} - \underbrace{(\epsilon_1 e_{j_1} + \cdots + \epsilon_t e_{j_t})}_{\alpha_-},$$

where e_i is the ith unit vector, $\epsilon_i \in \mathbb{N}$ for all i, and $i_1, \ldots, i_k, j_1, \ldots, j_t$ distinct. Thus one has the equality

$$\sum_{\ell=1}^{k} v_{i_\ell} = \sum_{\ell=1}^{t} \epsilon_\ell v_{j_\ell}. \tag{2}$$

If $b_{i_r} = 0$ for some $1 \leq r \leq k$, then using Eq. (2) one may rewrite Eq. (1) as

$$\begin{aligned} z^m &= f_1^{m\mu_1} \cdots f_q^{m\mu_q} \\ &= f_1^{b_1} \cdots f_q^{b_q}, \end{aligned}$$

where $\mu_i \in \mathbb{Z}$ for all i and $\mu_{i_r} = b_{i_r} = 0$, which by induction yields $z \in K[F \setminus \{f_{i_r}\}]$. It remains to consider the case $b_{i_r} > 0$ for all $1 \leq r \leq k$. One may assume $b_{i_1} \leq \cdots \leq b_{i_k}$. Using Eq. (1) and Eq. (2) we obtain

$$ma \overset{(1)}{=} \sum_{i=1}^{q} b_i v_i = b_{i_1} \sum_{\ell=1}^{k} v_{i_\ell} + \sum_{i \neq i_1} c_i v_i \overset{(2)}{=} b_{i_1} \sum_{\ell=1}^{t} \epsilon_\ell v_{j_\ell} + \sum_{i \neq i_1} c_i v_i, \tag{3}$$

where the c_i's are non negative integers. Therefore using Eq. (2) and Eq. (3) we can rewrite Eq. (1) as

$$\begin{aligned} z^m &= f_1^{m\delta_1}\cdots f_q^{m\delta_q} \\ &= f_1^{d_1}\cdots f_q^{d_q}, \end{aligned}$$

where $\delta_i \in \mathbb{Z}$, $d_i \in \mathbb{N}$ for all i, and $d_{i_1} = \delta_{i_1} = 0$. Hence by the induction hypothesis one obtains $z \in K[F \setminus \{f_{i_1}\}]$. $\Box$

We say that an integral matrix A is *t-unimodular* if all the nonzero $r \times r$ minors of A have absolute value equal to t, where $r \neq 0$ is the rank of A, if $t = 1$ we say that A is *unimodular*.

Corollary 2.4 ([8]) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let $F = \{\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_q}\}$ be a set of monomial in R. If the matrix A whose columns are the vectors $v_1, \ldots, v_q$ is t-unimodular, then $K[F]$ is normal.*

Proof. It follows from the results of [8, Chapter 4] that each elementary integral vector α of $\ker(A)$ has the form

$$\alpha = \epsilon_1 e_1 + \cdots + \epsilon_n e_n,$$

where $\epsilon_i \in \{0, \pm 1\}$. Hence by Theorem 2.3 the subring $K[F]$ is normal. $\Box$

Example 2.5 Let $F = \{x_5x_6x_7t, x_3x_4x_8t, x_1x_2x_9t, x_7x_8x_9t, x_1, \ldots, x_9\}$ and let K be a field, here t is just another variable. In this case the matrix A is:

$$A = \begin{pmatrix} 0&0&1&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&0&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&1&1&0&0&0&0&0&0&0&0&1 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The circuits of $\ker(A)$ are up to a sign given by the rows of the matrix.

$$E = \begin{pmatrix} 1&-1&-1&1&1&1&1&1&-1&-1&-2&0&0 \\ -1&1&-1&1&1&1&-1&-1&1&1&0&-2&0 \\ -1&-1&1&1&-1&-1&1&1&1&1&0&0&-2 \\ 0&0&1&-1&-1&-1&0&0&0&0&1&1&0 \\ 0&1&0&-1&0&0&-1&-1&0&0&1&0&1 \\ 0&-1&1&0&-1&-1&1&1&0&0&0&1&-1 \\ -1&0&1&0&-1&-1&0&0&1&1&1&0&-1 \\ 1&0&0&-1&0&0&0&0&-1&-1&0&1&1 \\ -1&1&0&0&0&0&-1&-1&1&1&1&-1&0 \end{pmatrix}$$

Thus by Theorem 2.3 the monomial subring $K[F]$ is a normal domain. Note that removing tha last row of A gives a totally unimodular matrix and that A itself in not unimodular.

3 Normality of the Rees algebra

The main references for linear programming are [6, 11]. In the proof below we use standard notation and terminology as described in those references, in particular note that if $a = (a_i)$ and $b = (b_i)$ are two vectors in $\mathbb{R}^m$ then $a \geq b$ means that $a_i \geq b_i$ for all i.

Lemma 3.1 *If $\eta_1, \ldots, \eta_q, b$ are non-negative integers such that*

$$\eta_1 + \cdots + \eta_q \geq b,$$

then there are integers $\epsilon_1, \ldots, \epsilon_q$ such that

$$0 \leq \epsilon_i \leq \eta_i \ \forall i \ \ \text{and} \ \ \epsilon_1 + \cdots + \epsilon_q = b.$$

Proof. By induction on q. Set

$$\epsilon = b - \sum_{i=1}^{q-1} \eta_i.$$

If $\epsilon \geq 0$, it suffices to take

$$\epsilon_1 = \eta_1, \ldots, \epsilon_{q-1} = \eta_{q-1}, \epsilon_q = \epsilon.$$

If $\epsilon < 0$, by induction there are $\epsilon_1, \ldots, \epsilon_{q-1}$ such that

$$0 \leq \epsilon_i \leq \eta_i \ \text{ for } \ i = 1, \ldots, q-1 \ \text{ and } \ \epsilon_1 + \cdots + \epsilon_{q-1} = b,$$

thus setting $\epsilon_q = 0$ one obtains the required sequence. □

Our main result is

Theorem 3.2 *Let $A = (a_{ij})$ be a $\{0, 1\}$ totally unimodular matrix of order $n \times q$ with non zero column vectors $v_1, \ldots, v_q$. If $R = K[x_1, \ldots, x_n]$ is a polynomial ring over a field K and I is the ideal of R generated by the monomials $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_q}$, then the Rees algebra $\mathcal{R}(I)$ of I is a normal domain.*

Proof. We define $\mathcal{A}' = \{(v_1, 1), \ldots, (v_q, 1), (e_1, 0), \ldots, (e_n, 0)\}$, where e_i is the ith unit vector in $\mathbb{R}^n$. According to [10, Theorem 7.2.28] one has

$$\overline{\mathcal{R}(I)} = K[\{\mathbf{x}^\alpha t^b | \, (\alpha, b) \in \mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+\mathcal{A}'\}] \subset R[t].$$

Note $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{n+1}$. Let $\mathbf{x}^\alpha t^b \in \overline{\mathcal{R}(I)}$. One can write

$$(\alpha, b) = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1) + \mu_1(e_1, 0) + \cdots + \mu_n(e_n, 0)$$

where λ_i and μ_j are non negative rational numbers for all i, j. Hence one obtains the equalities

$$\begin{aligned} \alpha &= \lambda_1 v_1 + \cdots + \lambda_q v_q + \mu_1 e_1 + \cdots + \mu_n e_n, \\ b &= \lambda_1 + \cdots + \lambda_q. \end{aligned}$$

Let A' be the $n \times (q + n)$ matrix

$$A' = (A\, e_1 \cdots e_n)$$

obtained from A by adjoining the column vectors $e_1, \ldots, e_n$. As total unimodularity is preserved under the operation of adding columns of unit vectors, the matrix A' is totally unimodular. Consider the linear program

Maximize $x_1 + \cdots + x_q$ $\qquad (*)$

Subject to

$A'x = \alpha$ and $x \geq 0$,

where x is the column vector $x = (x_1, \ldots x_q, x_{q+1}, \ldots, x_{q+n})$. Since the column vector

$$c = (c_1, \ldots, c_{n+q}) = (\lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_n)$$

satisfies

$$A'c = \alpha, \quad c \geq 0 \;\text{ and }\; c_1 + \cdots + c_q = b,$$

one concludes that the linear program $(*)$ has an optimal value greater or equal than b, which is attained at a vertex x_0 of the rational polytope

$$P = \{x \in \mathbb{R}^{n+q} \mid A'x = \alpha \text{ and } x \geq 0\},$$

see [11, Theorem 4.1.6]). Observe that by [6, Theorem 19.2] all the vertices of P have integral entries because A' is totally unimodular. Thus x_0 is a vector with non negative integral entries:

$$x_0 = (\eta_1, \ldots, \eta_q, \delta_1, \ldots, \delta_n)$$

such that $\eta_1 + \cdots + \eta_q \geq b$. By Lemma 3.1 there are integers $\epsilon_1, \ldots, \epsilon_q$ such that

$$0 \leq \epsilon_i \leq \eta_i \;\; \forall i \;\text{ and }\; \sum_{i=1}^{q} \epsilon_i = b.$$

Therefore

$$\begin{aligned}\mathbf{x}^{\alpha}t^{b} &= (\mathbf{x}^{v_1})^{\eta_1}\cdots(\mathbf{x}^{v_q})^{\eta_q}x_1^{\delta_1}\cdots x_n^{\delta_n}t^{b}\\ &= [(t\mathbf{x}^{v_1})^{\epsilon_1}\cdots(t\mathbf{x}^{v_q})^{\epsilon_q}]\left[(\mathbf{x}^{v_1})^{\eta_1-\epsilon_1}\cdots(\mathbf{x}^{v_q})^{\eta_q-\epsilon_q}x_1^{\delta_1}\cdots x_n^{\delta_n}\right],\end{aligned}$$

and $\mathbf{x}^{\alpha}t^{b}\in\mathcal{R}(I)$, as required. $\Box$

Example 3.3 Let $I=(\mathbf{x}^{v_1},\ldots,\mathbf{x}^{v_4})$, where $v_1,\ldots,v_4$ denote the column vectors of the matrix:

$$A=\begin{pmatrix}2&1&2&1\\9&6&8&4\\0&0&1&1\\1&6&1&2\end{pmatrix}.$$

Using [3] one derives that $\mathcal{R}(I)$ is not normal. Indeed the monomial $x_1^2x_2^8x_4^3$ is in I and is not in the integral closure of I. Since $\det(A)=-1$, this proves that Theorem 3.2 does not extend to unimodular matrices.

Corollary 3.4 ([7]) *Let G be a bipartite graph with vertices $x_1,\ldots,x_n$ and let K be a field. If $I=I(G)$ is the edge ideal of R generated by the set of monomials x_ix_j such that x_i is adjacent to x_j, then $\mathcal{R}(I)$ is a normal domain.*

Proof. Let A be the incidence matrix of G. Since G is bipartite, the matrix A is totally unimodular. Therefore $\mathcal{R}(I)$ is normal by Theorem 3.2. $\Box$

Remark 3.5 Note that Corollary 3.4 was shown in [7] using that $I(G)$ is a normally torsion free ideal if G is bipartite; see [4] for an alternative proof of this fact that gives new information on the toric ideal of $\mathcal{R}(I)$.

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