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CONDITIONS FOR THE EXISTENCE OF SOLUTIONS OF THE THREE-DIMENSIONAL PLANAR TRANSPORTATION PROBLEM

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A survey of conditions for the nonemptiness of the three planar sums transportation polytope is given together with several open problems.

1. Introduction

Let m, n and p be natural numbers and let $A=[a_{jk}]$, $B=[b_{ik}]$ and $C=[c_{ij}]$ be real matrices of type $n\times p$, $m\times p$ and $m\times n$ respectively. The *three planar sums transportation polytope* for triple (A, B, C) is defined as the set $T(A, B, C)$ of all nonnegative real three-dimensional matrices $X=[x_{ijk}]$ satisfying the system of equations

$$\sum_{i=1}^{m} x_{ijk} = a_{jk} \quad (j=1,2,\ldots,n;\ k=1,2,\ldots,p), \tag{1.1}$$

$$\sum_{j=1}^{n} x_{ijk} = b_{ik} \quad (i=1,2,\ldots,m;\ k=1,2,\ldots,p), \tag{1.2}$$

$$\sum_{k=1}^{p} x_{ijk} = c_{ij} \quad (i=1,2,\ldots,m;\ j=1,2,\ldots,n). \tag{1.3}$$

Several practical problems can be formulated as optimization problems over $T(A, B, C)$ for appropriately chosen matrices A, B, C. For examples of such formulations, see Schell (1955), Haley (1963), Schmid (1966), Junginger (1972) and Raskin–Kiričenko (1982).

The main purpose of this paper is to provide a survey of necessary conditions on the matrices A, B, C for $T(A, B, C)$ to be nonempty and to indicate some open problems. It is hoped that it may serve to stimulate interest in this problem.

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2. Obvious necessary conditions

Let $M=\{1, 2, \ldots, m\}$, $N=\{1, 2, \ldots, n\}$ and $P=\{1, 2, \ldots, p\}$. It follows directly from the constraints (1.1)–(1.3) that the conditions

$$a_{jk}\ge 0,\ b_{ik}\ge 0,\ c_{ij}\ge 0 \quad \text{for all } i\in M, j\in N, k\in P; \tag{2.1}$$

$$\sum_{j\in N} a_{jk}=\sum_{i\in M} b_{ik} \quad \text{for all } k\in P; \tag{2.2}$$

$$\sum_{k\in P} b_{ik}=\sum_{j\in N} c_{ij} \quad \text{for all } i\in M; \tag{2.3}$$

$$\sum_{i\in M} c_{ij}=\sum_{k\in P} a_{jk} \quad \text{for all } j\in N \tag{2.4}$$

are necessary for the nonemptiness of $T(A, B, C)$. Henceforth we shall assume that these obvious necessary conditions are satisfied by A, B, C.

3. The Schell conditions

In 1955, Schell gave an example demonstrating that the obvious necessary conditions are not sufficient to ensure that $T(A, B, C)$ is nonempty. Indeed, although the matrices

$$A=[a_{jk}]=\begin{bmatrix}1 & 8\\ 7 & 2\end{bmatrix},$$

$$B=[b_{ik}]=\begin{bmatrix}6 & 1\\ 2 & 9\end{bmatrix},$$

$$C=[c_{ij}]=\begin{bmatrix}4 & 3\\ 5 & 6\end{bmatrix}$$

satisfy all of the obvious necessary conditions the corresponding polytope $T(A, B, C)$ is empty since any $X=[x_{ijk}]$ belonging to $T(A, B, C)$ would have to satisfy

$$x_{111}\le a_{11}=1,$$

$$x_{112}\le b_{12}=1,$$

$$x_{111}+x_{112}=c_{11}=4.$$

Based on this example, Schell introduced a new set of necessary conditions for the nonemptiness of $T(A, B, C)$.

It is easy to see that for each $X\in T(A, B, C)$ the inequality

$$x_{ijk}\le \min\{a_{jk}, b_{ik}, c_{ij}\} \tag{3.1}$$

must hold for each $(i, j, k)\in M\times N\times P$. We will denote the right hand side of (3.1) by m_{ijk}. By taking the sum of the inequalities (3.1) over $i \in M$ and using (1.1) we obtain that if $T(A, B, C)$ is nonempty, then

$$a_{jk} \leq \sum_{i \in M} m_{ijk}, \quad j \in N, k \in P.$$

Similarly, we have

$$b_{ik} \leq \sum_{j \in N} m_{ijk}, \quad i \in M, k \in P;$$

$$c_{ij} \leq \sum_{k \in P} m_{ijk}, \quad i \in M, j \in N$$

as necessary conditions for the nonemptiness of $T(A, B, C)$.

Schell also noticed that the upper bounds m_{ijk} induce nontrivial lower bounds on x_{ijk}. If we fix $i \in M, j \in N$ and $k \in P$, the constraint

$$\sum_{\alpha \in M} x_{\alpha jk} = a_{jk}$$

implies that

$$x_{ijk} = a_{jk} - \sum_{\alpha \in M \setminus \{i\}} x_{\alpha jk} \geq a_{jk} - \sum_{\alpha \in M \setminus \{i\}} m_{\alpha jk}.$$

Similarly, we have

$$x_{ijk} \geq b_{ik} - \sum_{\alpha \in N \setminus \{j\}} m_{i\alpha k},$$

$$x_{ijk} \geq c_{ij} - \sum_{\alpha \in P \setminus \{k\}} m_{ij\alpha}.$$

Denoting the number

$$\max\left\{0,\ a_{jk} - \sum_{\alpha \in M \setminus \{i\}} m_{\alpha jk},\ b_{ik} - \sum_{\alpha \in N \setminus \{j\}} m_{i\alpha k},\ c_{ij} - \sum_{\alpha \in P \setminus \{k\}} m_{ij\alpha}\right\}$$

by M_{ijk}, we have

$$x_{ijk} \geq M_{ijk}, \quad i \in M, j \in N, k \in P.$$

Again, by summing and using (1.1)–(1.3) we have, together with the previous conditions, that if $T(A, B, C)$ is nonempty, then the inequalities

$$\sum_{\alpha \in M} M_{\alpha jk} \leq a_{jk} \leq \sum_{\alpha \in M} m_{\alpha jk}, \quad j \in N, k \in P; \tag{3.2}$$

$$\sum_{\alpha \in N} M_{i\alpha k} \leq b_{ik} \leq \sum_{\alpha \in N} m_{i\alpha k}, \quad i \in M, k \in P; \tag{3.3}$$

$$\sum_{\alpha \in P} M_{ij\alpha} \leq c_{ij} \leq \sum_{\alpha \in P} m_{ij\alpha}, \quad i \in M, j \in N \tag{3.4}$$

must hold. We will call these conditions the *Schell conditions.*

An immediate question is whether the Schell conditions are sufficient to ensu[re] that $T(A, B, C)$ is nonempty. Morávek and Vlach (1967) gave an examp[le] demonstrating that they are not. Indeed, a little calculation shows that for A, B a[nd] C given by

$$A=\begin{pmatrix}1&4\\1&4\\1&4\\4&1\\4&1\\4&1\\4&1\\3&2\end{pmatrix},\qquad B=\begin{pmatrix}1&7\\2&6\\7&1\\6&2\\6&2\end{pmatrix},$$

$$C=\begin{pmatrix}1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1\end{pmatrix}$$

all lower bounds M_{ijk} are equal to 0 and all upper bounds m_{ijk} are equal to 1. The[n] it is easy to verify directly that all of the Schell conditions are satisfied. Neverthele[ss] $T(A, B, C)$ is empty since any $X\in T(A, B, C)$ would have to satisfy the relation[s]

$$x_{1j2}+x_{2j2}\le c_{1j}+c_{2j}=2 \quad \text{for all } j\in\{1, 2, 3, 8\},$$

$$x_{1j2}+x_{2j2}\le \quad a_{j2} \quad =1 \quad \text{for all } j\in\{4, 5, 6, 7\},$$

$$\sum_{j=1}^{8} x_{1j2}+\sum_{j=1}^{8} x_{2j2}=b_{12}+b_{22}=13,$$

which it clearly cannot.

4. The Haley conditions

The idea of Schell was further developed by Haley (1963) who noticed that the lower bounds M_{ijk} induce a new tighter set of upper bounds by a similar procedure to that by which the upper bounds m_{ijk} induced the lower bounds M_{ijk}. Moreover, these new upper bounds induce new lower bounds, and so on. Formally, we can describe the procedure as follows:

Let $M^0_{ijk}=0$, $m^0_{ijk}=\infty$ for all $(i, j, k)\in M\times N\times P$ and define by induction

$$m^{r+1}_{ijk}=\min\left\{m^r_{ijk},\ a_{jk}-\sum_{\alpha\in M\setminus\{i\}}\right.$$

$$M^{r+1}_{ijk}=\max\left\{M^r_{ijk},\ a_{jk}-\sum_{\alpha\in M\setminus\{i\}}\right.$$

It is easy to see that if $X\in T(A$

$$M^0_{ijk}\le M^1_{ijk}\le\ldots\le x_{ijk}\le$$

Therefore the limits

$$H_{ijk}:=\lim_{r\to\infty} M^r_{ijk}, \qquad h$$

exist and

$$H_{ijk}\le x_{ijk}\le h_{ijk} \quad \text{for al}$$

Hence, by summing over i (ove following necessary conditions – le emptiness of $T(A, B, C)$:

$$\sum_{i\in M} H_{ijk}\le a_{jk}\le \sum_{i\in M} h_{ijk},$$

$$\sum_{j\in N} H_{ijk}\le b_{ik}\le \sum_{j\in N} h_{ijk},$$

$$\sum_{k\in P} H_{ijk}\le c_{ij}\le \sum_{k\in P} h_{ijk},$$

The previous example shows on since in this case $H_{ijk}=0$ and h_{ijk} are satisfied.

Problem 1. Haley (1965) claims th are stationary, i.e. there exists an This is certainly true if all entries for real matrices?

5. The Morávek–Vlach condition[s]

The previous example suggests t which may be tighter than the bou dividual variables. For $I\subset M$, $J\subset$ $X(I, J, K)$ by

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$$m^{r+1}_{ijk} = \min\left\{ m^r_{ijk},\ a_{jk} - \sum_{\alpha\in M\setminus\{i\}} M^r_{\alpha jk},\ b_{ik} - \sum_{\alpha\in N\setminus\{j\}} M^r_{i\alpha k},\ c_{ij} - \sum_{\alpha\in P\setminus\{k\}} M^r_{ij\alpha} \right\},$$

$$M^{r+1}_{ijk} = \max\left\{ M^r_{ijk},\ a_{jk} - \sum_{\alpha\in M\setminus\{i\}} m^r_{\alpha jk},\ b_{ik} - \sum_{\alpha\in N\setminus\{j\}} m^r_{i\alpha k},\ c_{ij} - \sum_{\alpha\in P\setminus\{k\}} m^r_{ij\alpha} \right\},$$

It is easy to see that if $X\in T(A, B, C)$, then

$$M^0_{ijk} \le M^1_{ijk} \le \ldots \le x_{ijk} \le \ldots \le m^1_{ijk} \le m^0_{ijk}.$$

Therefore the limits

$$H_{ijk} := \lim_{r\to\infty} M^r_{ijk}, \qquad h_{ijk} := \lim_{r\to\infty} m^r_{ijk}$$

exist and

$$H_{ijk} \le x_{ijk} \le h_{ijk} \quad \text{for all } (i, j, k)\in M\times N\times P.$$

Hence, by summing over i (over j, over k) and using (1.1)–(1.3) we obtain the following necessary conditions – let us call them the *Haley conditions* – for the non-emptiness of $T(A, B, C)$:

$$\sum_{i\in M} H_{ijk} \le a_{jk} \le \sum_{i\in M} h_{ijk}, \quad j\in N,\ k\in P; \tag{4.1}$$

$$\sum_{j\in N} H_{ijk} \le b_{ik} \le \sum_{j\in N} h_{ijk}, \quad i\in M,\ k\in P; \tag{4.2}$$

$$\sum_{k\in P} H_{ijk} \le c_{ij} \le \sum_{k\in P} h_{ijk}, \quad i\in M,\ j\in N. \tag{4.3}$$

The previous example shows once again that these conditions are not sufficient, since in this case $H_{ijk}=0$ and $h_{ijk}=1$ for all i, j, k and hence conditions (4.1)–(4.3) are satisfied.

Problem 1. Haley (1965) claims that if $T(A, B, C)\neq\emptyset$, then the sequences m^r_{ijk}, M^r_{ijk} are stationary, i.e. there exists an s such that $m^r_{ijk}=m^s_{ijk}$ and $M^r_{ijk}=M^s_{ijk}$ for all $r>s$. This is certainly true if all entries of A, B and C are rational numbers. Is this true for real matrices?

5. The Morávek–Vlach conditions I

The previous example suggests that bounds on sums of variables can be obtained which may be tighter than the bounds obtained by summing the bounds of the individual variables. For $I\subset M$, $J\subset M$, $K\subset P$ define $A(J, K)$, $B(I, K)$, $C(I, J)$ and $X(I, J, K)$ by

$$A(J, K) = \sum_{j \in J} \sum_{k \in K} a_{jk},$$

$$B(I, K) = \sum_{i \in I} \sum_{k \in K} b_{ik},$$

$$C(I, J) = \sum_{i \in I} \sum_{j \in J} c_{ij},$$

$$X(I, J, K) = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} x_{ijk}.$$

It follows directly from (1.1)–(1.3) that

$$X(I, J, K) \le \min\{A(J, K),\ B(I, K),\ C(I, J)\}$$

for every $X \in T(A, B, C)$ and every I, J, K. Since

$$X(M, J, K) = A(J, K),\ X(I, N, K) = B(I, K),\ X(I, J, P) = C(I, J)$$

for each $X \in T(A, B, C)$, we obtain the following necessary conditions for the nonemptiness of $T(A, B, C)$:

$$A(J, K) \le \sum_{i \in M} \min\{A(J, K),\ B(\{i\}, K),\ C(\{i\}, J)\}, \tag{5.1}$$

$$B(I, K) \le \sum_{j \in N} \min\{A(\{j\}, K),\ B(I, K),\ C(I, \{j\})\}, \tag{5.2}$$

$$C(I, J) \le \sum_{k \in P} \min\{A(J, \{k\}),\ B(I, \{k\}),\ C(I, J)\} \tag{5.3}$$

for each $I \subset M$, $J \subset N$, $K \subset P$.

Note that matrices A, B and C from the previous example, which passed the Haley conditions, fail to pass conditions (5.1)–(5.3). On the other hand, the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

constructed by Smith (1975) pass all of conditions (5.1)–(5.3) and fail to pass the Haley conditions. Thus, conditions (5.1)–(5.3) are not sufficient to ensure that $T(A, B, C)$ is nonempty.

Nevertheless, conditions (5.1)–(5.3) are sufficient for the class of $T(A, B, C)$ with $\min(m, n, p) \le 2$. The case $\min(m, n, p) = 1$ is trivial – the obvious necessary conditions are also sufficient. Assume that A, B, C with $p = 2$, $m \ge 2$, $n \ge 2$ satisfy (5.1) and consider the system

$$\sum_{i=1}^{m} y_{ij} = a_{j1} \quad (j = 1, 2, \ldots, n),$$

$$\sum_{j=1}^{n} y_{ij} = b_{i1} \quad (i = 1, 2, \ldots, m), \tag{5.4}$$

$0 \le y_{ij} \le c_{ij} \quad (i = 1, 2,$

For $K = \{1\}$ the condition (5.1)

$\sum_{j \in J} a_{j1} \le \sum_{i=1}^{m} \min\{b_{i1},$

Since

$\sum_{j=1}^{n} a_{j1} = \sum_{i=1}^{m} b_{i1},$

condition (5.5) is sufficient fo
Yemelichev–Kovalev–Kravtsov (

$x_{ij1} = y_{ij},\ x_{ij2} = c_{ij} - y_{ij}$

we obtain an $X \in T(A, B, C)$.
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6. The Morávek–Vlach conditio

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(5.1)–(5.3) do not imply conditi
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Let us set

$M^0(S) = m^0(S) = \begin{cases} A \\ B \\ C \end{cases}$

$M^0(S) = 0$ and $m^0(S) = \infty$ for ot

$m^{r+1}(S) = \min\{\min_{U, V} [$

$M^{r+1}(S) = \max\{\max_{U, V}$

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We shall prove by induction t

$M^0(S) \le M^1(S) \le \ldots \le$

for all $S \subset M \times N \times P$. Here

$$0 \le y_{ij} \le c_{ij} \quad (i = 1, 2, \dots, m;\ j = 1, 2, \dots, n).$$

For $K = \{1\}$ the condition (5.1) gives

$$\sum_{j \in J} a_{j1} \le \sum_{i=1}^{m} \min\left\{b_{i1}, \sum_{j \in J} c_{ij}\right\} \quad \textit{for all } J \subset N. \tag{5.5}$$

Since

$$\sum_{j=1}^{n} a_{j1} = \sum_{i=1}^{m} b_{i1},$$

condition (5.5) is sufficient for the existence of a solution $[y_{ij}]$ to (5.4) – see Yemelichev–Kovalev–Kravtsov (1981) or Gale (1957). Setting

$$x_{ij1} = y_{ij}, \quad x_{ij2} = c_{ij} - y_{ij}$$

we obtain an $X \in T(A, B, C)$.

Motzkin (1952) observed that all the extreme points of $T(A, B, C)$ lie in the ring generated by elements of A, B, C if and only if $\min(m, n, p) \le 2$. Therefore, conditions (5.1)–(5.3) are, for this case, also necessary and sufficient for the existence of a nonnegative integer solution to (1.1)–(1.3) with integer right hand sides.

6. The Morávek–Vlach conditions II

Since conditions (4.1)–(4.3) do not imply conditions (5.1)–(5.3) and conditions (5.1)–(5.3) do not imply conditions (4.1)–(4.3) it is natural to combine both ideas and develop an analogue of the Haley iterative procedure for sums of variables. Let us set

$$M^0(S) = m^0(S) = \begin{cases} A(J, K), & \text{whenever } S = M \times J \times K, \\ B(I, K), & \text{whenever } S = I \times N \times K, \\ C(I, J), & \text{whenever } S = I \times J \times P. \end{cases}$$

$M^0(S) = 0$ and $m^0(S) = \infty$ for other $S \subset M \times N \times P$ and define by induction

$$m^{r+1}(S) = \min\left\{\min_{U, V}[m^r(U) + m^r(V)], \min_{W}[m^r(S \cup W) - M^r(W)]\right\},$$

$$M^{r+1}(S) = \max\left\{\max_{U, V}[M^r(U) + M^r(V)], \max_{W}[M^r(S \cup W) - m^r(W)]\right\}$$

where the minimum and maximum over U, V are meant to be the minimum and maximum over all disjoint subsets U, V of S satisfying $U \cup V = S$, and the minimum and maximum over W are meant to be the minimum and maximum over all subsets W of the complement of S in $M \times N \times P$.

We shall prove by induction that if $X \in T(A, B, C)$, then

$$M^0(S) \le M^1(S) \le \dots \le X(S) \le \dots \le m^1(S) \le m^0(S) \tag{6.1}$$

for all $S \subset M \times N \times P$. Here $X(S)$ denotes the sum $\sum_{(i,j,k) \in S} x_{ijk}$. Obviously

$M^0(S) \le X(S) \le m^0(S)$. Assume that $M^r(Q) \le X(Q) \le m^r(Q)$ for all $Q \subset M \times N \times P$ and consider an arbitrary $S \subset M \times N \times P$. It is clear from the definition of $m^{r+1}(S)$ that either

$$m^{r+1}(S) = m^r(U_0) + m^r(V_0) \tag{6.2}$$

for some U_0, $V_0 \subset S$ satisfying $U_0 \cap V_0 = \emptyset$, $U_0 \cup V_0 = S$, or

$$m^{r+1}(S) = m^r(S \cup W_0) - M^r(W_0) \tag{6.3}$$

for some $W_0 \subset (M \times N \times P) \setminus S$. From our assumption it follows that

$$X(U_0) \le m^r(U_0), \qquad X(V_0) \le m^r(V_0),$$

$$X(S \cup W_0) \le m^r(S \cup W_0), \qquad X(W_0) \ge M^r(W_0).$$

Since $X(\cdot)$ is additive we have

$$X(S) = X(U_0) + X(V_0) \le m^r(U_0) + m^r(V_0) = m^{r+1}(S)$$

in case (6.2) and

$$X(S) = X(S \cup W_0) - X(W_0) \le m^{(r+1)}(S)$$

in case (6.3). The monotonicity of the sequence $\{m^r(S)\}$ is obvious directly from the definition. It can be verified analogously that

$$M^0(S) \le M^1(S) \le \ldots \le X(S).$$

Now it is clear that if $T(A, B, C)$ is nonempty, then for each $S \subset M \times N \times P$ the limits

$$\alpha(S) := \lim_{n \to \infty} M^r(S), \qquad \beta(S) := \lim_{n \to \infty} m^r(S)$$

exist and

$$\alpha(S) \le X(S) \le \beta(S) \tag{6.4}$$

for each $X \in T(A, B, C)$. Therefore, both the existence of limits $\alpha(S)$ and $\beta(S)$ and the validity of the inequality

$$\alpha(S) \le \beta(S) \tag{6.5}$$

for each $S \subset M \times N \times P$ are necessary for the nonemptiness of $T(A, B, C)$.

It follows directly from the definition of m^r and M^r that for each one-element set $\{(i, j, k)\}$ and each r there is an s such that $s > r$ and

$$m^s(\{(i, j, k)\}) \le m^r_{ijk}, \qquad M^s(\{(i, j, k)\}) \ge M^r_{ijk}. \tag{6.6}$$

Therefore, every A, B, and C which satisfy (6.5) also satisfy the Haley conditions. It turns out that the Morávek–Vlach conditions I are also implied by conditions (6.5). Indeed, the Morávek–Vlach conditions I can be restated in the following form – see Haley (1967):

$A(J, K) \le B(I, K) + C$

Here $\bar{I}$ stands for the complemen
and $\bar{K}$ for the complements of J
for each $S \subset M \times N \times P$, then

$A(J, K) = M^0(M \times J \times$

$\le m^2(M \times J \times$

$\le m^0(I \times N \times$

$\le m^0(I \times N \times$

Problem 2. Is the existence of the
(6.5) sufficient for the nonemptir

Remark. The procedure described
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where X is a nonempty (not nece

$-\infty \le f(S) < \infty, f(\emptyset) =$

$-\infty < g(S) \le \infty, g(\emptyset) =$

and consider the problem of dete
function x on T satisfying the ine

$f(S) \le x(S) \le g(S)$

for each $S \in$ T. Define

$M^0(S) = f(S), m^0(S) =$

$m^{r+1}(S) = \min\{\inf_{U, V} [m^r$

$M^{r+1}(S) = \max\{\sup_{U, V} [M$

where the infima and suprema over
to the minima and maxima describ
– see Morávek and Vlach (1968) – th
(6.8), then there exist limits

$\alpha(S) := \lim_{r \to \infty} M^r(S),$

and $\alpha(S) \le \beta(S)$ for each $S \in$ T.

$$A(J, K) \leq B(I, K) + C(\bar{I}, J) \quad \textit{for all } I, J, K. \tag{6.7}$$

Here $\bar{I}$ stands for the complement of I in M. Similarly we shall use the notation $\bar{J}$ and $\bar{K}$ for the complements of J in N and K in P respectively. Now, if $\alpha(S) \leq \beta(S)$ for each $S \subset M \times N \times P$, then

$$\begin{aligned} A(J, K) &= M^0(M \times J \times K) \leq \alpha(M \times J \times K) \leq \beta(M \times J \times K) \\ &\leq m^2(M \times J \times K) \leq m^1(I \times J \times K) + m^1(\bar{I} \times J \times K) \\ &\leq m^0(I \times N \times K) - M^0(I \times \bar{J} \times K) + m^0(\bar{I} \times J \times P) - M^0(\bar{I} \times J \times \bar{K}) \\ &\leq m^0(I \times N \times K) + m^0(\bar{I} \times J \times P) = B(I, K) + C(\bar{I}, J). \end{aligned}$$

Problem 2. Is the existence of the limits $\alpha(S)$ and $\beta(S)$ together with the validity of (6.5) sufficient for the nonemptiness of $T(A, B, C)$?

Remark. The procedure described in this section can be applied to more general problems. Let f and g be extended real-valued functions defined on an algebra $\mathrm{T} \subset 2^X$ where X is a nonempty (not necessarily finite) set. Assume that

$$-\infty \leq f(S) < \infty, \; f(\emptyset) = 0,$$

$$-\infty < g(S) \leq \infty, \; g(\emptyset) = 0$$

and consider the problem of determining whether or not there exists an additive function x on T satisfying the inequalities

$$f(S) \leq x(S) \leq g(S) \tag{6.8}$$

for each $S \in \mathrm{T}$. Define

$$M^0(S) = f(S), \; m^0(S) = g(S),$$

$$m^{r+1}(S) = \min\left\{\inf_{U, V}[m^r(U) + m^r(V)], \inf_{W}[m^r(S \cup W) - M^r(W)]\right\}$$

$$M^{r+1}(S) = \max\left\{\sup_{U, V}[M^r(U) + M^r(V)], \sup_{W}[M^r(S \cup W) - m^r(W)]\right\}$$

where the infima and suprema over U, V and W are defined in a manner analogous to the minima and maxima described in the beginning of this section. It turns out – see Morávek and Vlach (1968) – that if there exists an additive function x satisfying (6.8), then there exist limits

$$\alpha(S) := \lim_{r \to \infty} M^r(S), \qquad \beta(S) := \lim_{r \to \infty} m^r(S)$$

and $\alpha(S) \leq \beta(S)$ for each $S \in \mathrm{T}$.

7. The Smith conditions I

If follows directly from (1.1)–(1.3) that

$$A(J,K)=X(I,J,K)+X(\bar{I},J,K),$$

$$B(I,K)=X(I,J,K)+X(I,\bar{J},K),$$

$$C(\bar{I},J)=X(\bar{I},J,K)+X(\bar{I},J,\bar{K}).$$

Hence, for $\Delta(I,J,K)$ defined by

$$\Delta(I,J,K)=B(I,K)+C(\bar{I},J)-A(J,K)$$

we have

$$\Delta(I,J,K)=X(I,\bar{J},K)+X(\bar{I},J,\bar{K})$$

and every bound on $X(I,\bar{J},K)$ and $X(\bar{I},J,\bar{K})$ gives a bound on $\Delta(I,J,K)$.

Smith (1973) used the Haley lower and upper bounds on the invidual variables to obtain the following necessary conditions for the nonemptiness of $T(A,B,C)$:

$$H(I,\bar{J},K)+H(\bar{I},J,\bar{K})\leq\Delta(I,J,K)\leq h(I,\bar{J},K)+h(\bar{I},J,\bar{K}) \qquad (7.1)$$

for all I, J, K. Here the notation is used in a way analogous to that of Section 5.

Obviously, the conditions (7.1) imply the Morávek–Vlach conditions I because – see (6.7) – the latter are equivalent to $\Delta(I,J,K)\geq 0$. It is also not difficult – see Smith (1973) – to verify that the Haley conditions are a subset of the conditions (7.1). Smith (1973) gave an example demonstrating that the conditions (7.1) are in fact tighter than both the Haley conditions and the Morávek–Vlach conditions I. Indeed, the matrices

$$A=\begin{pmatrix}1&1&1&1&1&1&1\\1&0&0&0&1&1&1\\1&0&0&0&0&0&1\\1&0&0&0&0&1&0\\1&1&1&1&0&0&0\\1&1&1&1&1&1&1\\1&1&1&1&1&1&1\end{pmatrix},$$

$$B=\begin{pmatrix}1&1&1&1&1&1&1\\1&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&0&0&0&0&1&1\\1&1&1&1&1&1&1\\1&1&1&1&1&1&1\\1&1&1&1&1&1&1\end{pmatrix},$$

$$C=\begin{pmatrix}1&0&2&0&1\\1&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\4&1&0&0&0\\0&2&0&1&2\\0&1&0&1&1\end{pmatrix}$$

pass all of the Haley conditions an
conditions (7.1) for $I=\{1,5\}$, $J=$

Instead of the Haley bounds w

$$\alpha_{ijk}:=\alpha(\{(i,j,k)\}),$$

on the individual variables or th
$\bar{I}\times J\times\bar{K}$. The former gives the co

$$\alpha(I,\bar{J},K)+\alpha(\bar{I},J,\bar{K}):$$

for all I, J, K; the latter gives the

$$\alpha(I\times\bar{J}\times K)+\alpha(\bar{I}\times J\times$$

for all I, J, K.

Because of the subadditivity of
are at least as tight as conditions
least as tight as conditions (7.1).

Problem 3. Are conditions (7.2) c

Problem 4. Are conditions (7.1) c

It is shown in the next section t
Morávek–Vlach conditions II.

8. The Smith conditions II

Stating the Morávek–Vlach con

$$A(J,K)\leq B(I,K)+C($$

or, by symmetry, in the form

$$B(I,K)\leq A(J,K)+C($$

$$C=\begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 4 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 4 & 0 \end{pmatrix},$$

pass all of the Haley conditions and the Morávek–Vlach Conditions I but fail to pass conditions (7.1) for $I=\{1,5\}$, $J=\{1,7\}$, $K=\{6,7\}$.

Instead of the Haley bounds we can use in the same manner the bounds

$$\alpha_{ijk}:=\alpha(\{(i,j,k)\}), \qquad \beta_{ijk}:=\beta(\{(i,j,k)\})$$

on the individual variables or the bounds $\alpha(\cdot)$, $\beta(\cdot)$ for the sets $I\times\bar{J}\times K$ and $\bar{I}\times J\times\bar{K}$. The former gives the conditions

$$\alpha(I,\bar{J},K)+\alpha(\bar{I},J,\bar{K})\leq\Delta(I,J,K)\leq\beta(I,\bar{J},K)+\beta(\bar{I},J,\bar{K}) \tag{7.2}$$

for all I, J, K; the latter gives the conditions

$$\alpha(I\times\bar{J}\times K)+\alpha(\bar{I}\times J\times\bar{K})\leq\Delta(I,J,K)\leq\beta(I\times\bar{J}\times K)+\beta(\bar{I}\times J\times\bar{K}) \tag{7.3}$$

for all I, J, K.

Because of the subadditivity of β and superadditivity of α, the conditions (7.3) are at least as tight as conditions (7.2) and because of (6.6) conditions (7.2) are at least as tight as conditions (7.1).

Problem 3. Are conditions (7.2) or (7.3) tighter than conditions (7.1)?

Problem 4. Are conditions (7.1) or (7.2) or (7.3) sufficient?

It is shown in the next section that all of conditions (7.1)–(7.3) follow from the Morávek–Vlach conditions II.

8. The Smith conditions II

Stating the Morávek–Vlach conditions I in the form

$$A(J,K)\leq B(I,K)+C(\bar{I},J)$$

or, by symmetry, in the form

$$B(I,K)\leq A(J,K)+C(I,\bar{J}) \quad \text{or} \quad C(I,J)\leq A(J,K)+B(I,\bar{K})$$

we see that the sums of the right hand sides of (1.1)–(1.3) are always over sets in the form of cartesian products. Smith (1973) extended these conditions to sums over more general sets. To restate these conditions let us take $U\subset N\times P$, $V\subset M\times P$, $W\subset M\times N$ and define $Q_1(U)$, $Q_2(V)$ and $Q_3(W)$ by

$$Q_1(U)=\{(i,j,k)\mid (j,k)\in U\},$$

$$Q_2(V)=\{(i,j,k)\mid (i,k)\in V\},$$

$$Q_3(W)=\{(i,j,k)\mid (i,j)\in W\}.$$

Now the second set of the Smith conditions can be restated in the following form: For $T(A,B,C)$ to be nonempty it is necessary that

$$A(U)\le B(V)+C(W)\quad \textit{whenever } Q_1(U)\subset Q_2(V)\cup Q_3(W), \tag{8.1}$$

$$B(V)\le A(U)+C(W)\quad \textit{whenever } Q_2(V)\subset Q_1(U)\cup Q_3(W), \tag{8.2}$$

$$C(W)\le A(U)+B(V)\quad \textit{whenever } Q_3(W)\subset Q_1(U)\cup Q_2(V). \tag{8.3}$$

To see the necessity, observe that $A(U)=X(Q_1(U))$, $B(V)=X(Q_2(V))$, and $C(W)=X(Q_3(W))$.

It is straightforward that these conditions include the Morávek–Vlach conditions I – it suffices to take $U=J\times K$, $V=I\times K$ and $W=\tilde{I}\times J$ and noted that $Q_1(U)=M\times J\times K$, $Q_2(V)=I\times N\times K$ and $Q_3(W)=\tilde{I}\times J\times P$. The example in the previous section shows that the Morável–Vlach conditions I are a proper subset of the conditions (8.1)–(8.3) since it fails to satisfy (8.3) for

$$U=\{(1,1),\ (1,5),\ (1,6),\ (1,7),\ (3,5),\ (3,6),\ (3,7),\ (5,5),\ (5,6),\ (5,7),\ (7,1),\ (7,5),\ (7,6),\ (7,7)\},$$

$$V=\{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (2,2),\ (2,3),\ (2,4),\ (4,2),\ (4,3),\ (4,4),\ (5,2),\ (5,3),\ (5,4)\},$$

$$W=\{(1,1),\ (1,3),\ (1,5),\ (1,7),\ (2,1),\ (2,7),\ (4,1),\ (4,7),\ (5,1),\ (5,7)\}.$$

Moreover, the conditions (8.1)–(8.3) do not follow from the Haley conditions since this example passes all of the Haley conditions.

We shall prove now that the conditions (8.1)–(8.3) follow from the Morávek–Vlach conditions II. In fact, we shall prove it for the conditions (8.3) only, since (8.1) and (8.2) follow in a similar way.

Let us assume that the Morávek–Vlach condition II are satisfied for given A, B and C. If $Q_3(W)\subset Q_1(U)\cup Q_2(V)$, then for each $(i,j)\in W$ there is a set $K_{ij}\subset P$ such that

$$k\in K_{ij}\Rightarrow (j,k)\in U,$$

$$k\in \tilde{K}_{ij}\Rightarrow (i,k)\in V.$$

Let us set

$$S_1:=\bigcup_{(i,j)\in W}\{i\}\times\{j$$

$$S_2:=\bigcup_{(i,j)\in W}\{i\}\times\{j$$

Obviously

$$Q_3(W)=S_1\cup S_2,$$

Hence, for sufficiently large s

$$C(W)=\sum_{(i,j)\in W}M^0(\{$$

$$\le\alpha\Big(\bigcup_{(i,j)}\{i\}\times$$

$$\le m^{r+2}(S_1\cup S$$

$$\le m^r(Q_1(U))$$

$$\le m^r(Q_1(U))$$

$$=m^r\Big(\bigcup_{(j,k)\in U}$$

$$\le\sum_{(j,k)\in U}m^0(\Lambda$$

$$=A(U)+B(V$$

G. Rote observed recently tha tions (7.3), (7.2) and (7.1) follo

Indeed, it follows from const

$$m^{r+1}(I\times J\times K)\le m^r$$

$$m^{r+1}(\tilde{I}\times J\times K)\le m^r$$

$$m^{r+2}(M\times J\times K)\le n$$

Therefore, the sum

$$M^r(I\times\tilde{J}\times K)+M^r(\tilde{I}$$

does not exceed the value

$$m^r(I\times N\times K)+m^r(\tilde{I}$$

At the same time

-(1.3) are always over sets in
these conditions to sums over
ıs take $U \subset N \times P$, $V \subset M \times P$,

stated in the following form:

$\subset Q_2(V) \cup Q_3(W)$, (8.1)

$\subset Q_1(U) \cup Q_3(W)$, (8.2)

$\subset Q_1(U) \cup Q_2(V)$. (8.3)

$(U))$, $B(V) = X(Q_2(V))$, and

ne Morávek-Vlach conditions
$\times J$ and noted that $Q_1(U) =$
The example in the previous
a proper subset of the condi-

(3,7),

(7,7)},

(2,4),

,

, (4,1), (4,7), (5,1), (5,7)}.

v from the Haley conditions

.1)-(8.3) follow from the
t for the conditions (8.3) only,

I are satisfied for given A, B
$,j) \in W$ there is a set $K_{ij} \subset P$

Let us set

$$S_1 := \bigcup_{(i,j) \in W} \{i\} \times \{j\} \times K_{ij},$$

$$S_2 := \bigcup_{(i,j) \in W} \{i\} \times \{j\} \times \bar{K}_{i,j}.$$

Obviously

$$Q_3(W) = S_1 \cup S_2, \quad S_1 \cap S_2 = \emptyset, \quad S_1 \subset Q_1(U), \quad S_2 \subset Q_2(V).$$

Hence, for sufficiently large s and r we have

$$\begin{aligned}
C(W) &= \sum_{(i,j) \in W} M^0(\{i\} \times \{j\} \times P) \leq M^s\Big(\bigcup_{(i,j) \in W} \{i\} \times \{j\} \times P\Big) \\
&\leq \alpha\Big(\bigcup_{(i,j)} \{i\} \times \{j\} \times P\Big) = \alpha(S_1 \cup S_2) \leq \beta(S_1 \cup S_2) \\
&\leq m^{r+2}(S_1 \cup S_2) \leq m^{r+1}(S_1) + m^{r+1}(S_2) \\
&\leq m^r(Q_1(U)) - M^r(Q_1(U) \setminus S_1) + m^r(Q_2(V)) - M^r(Q_2(V) \setminus S_2) \\
&\leq m^r(Q_1(U)) + m^r(Q_2(V)) \\
&= m^r\Big(\bigcup_{(j,k) \in U} M \times \{j\} \times \{k\}\Big) + m^r\Big(\bigcup_{(i,k) \in V} \{i\} \times N \times \{k\}\Big) \\
&\leq \sum_{(j,k) \in U} m^0(M \times \{j\} \times \{k\}) + \sum_{(i,k) \in V} m^0(\{i\} \times N \times \{k\}) \\
&= A(U) + B(V).
\end{aligned}$$

G. Rote observed recently that this method can also be used to prove that conditions (7.3), (7.2) and (7.1) follow from the Morávek-Vlach conditions II.

Indeed, it follows from construction of m^r and M^r that

$$m^{r+1}(I \times J \times K) \leq m^r(I \times N \times K) - M^r(I \times \bar{J} \times K),$$

$$m^{r+1}(\bar{I} \times J \times K) \leq m^r(\bar{I} \times J \times P) - M^r(\bar{I} \times J \times \bar{K}),$$

$$m^{r+2}(M \times J \times K) \leq m^{r+1}(\bar{I} \times J \times K) + m^{r+1}(I \times J \times K).$$

Therefore, the sum

$$M^r(I \times \bar{J} \times K) + M^r(\bar{I} \times J \times \bar{K})$$

does not exceed the value

$$m^r(I \times N \times K) + m^r(\bar{I} \times J \times P) - m^{r+2}(M \times J \times K).$$

At the same time

$$m^r(I\times N\times K)\le m^0(I\times N\times K)=B(I,K),$$
$$m^r(\tilde{I}\times J\times P)\le m^0(\tilde{I}\times J\times P)=C(\tilde{I},J).$$

Moreover,

$$m^{r+2}(M\times J\times K)\ge M^0(M\times J\times K)=A(J,K),$$

provided the Morávek–Vlach conditions II hold.

Consequently,

$$M^r(I\times\tilde{J}\times K)+M^r(\tilde{I}\times J\times\tilde{K})\le\Delta(I,J,K)$$

for all r and hence

$$\alpha(I\times\tilde{J}\times K)+\alpha(\tilde{I}\times J\times\tilde{K})\le\Delta(I,J,K).$$

The remaining part of (7.3), i.e. the inequality

$$\Delta(I,J,K)\le\beta(I\times\tilde{J}\times K)+\beta(\tilde{I}\times J\times\tilde{K})$$

can be obtained in a similar way.

Conditions (7.2) now follow from superadditivity of α and subadditivity of β. Finally, since

$$H_{ijk}\le\alpha(\{i\}\times\{j\}\times\{k\}),$$
$$h_{ijk}\ge\beta(\{i\}\times\{j\}\times\{k\}),$$

we obtain (again by superadditivity of α and subadditivity of β) the Smith conditions I.

Problem 5. Do the Haley conditions follow from conditions (8.1)–(8.3)?

Problem 6. What are the relations between conditions (8.1)–(8.3) and conditions (7.1), (7.2) and (7.3)?

Problem 7. Are conditions (8.1)–(8.3) sufficient?

9. The Smith conditions III

The Smith conditions II can be enhanced by bounds on the individual variables in the same manner as the Morávek–Vlach conditions I have been enhanced in obtaining the conditions (7.1) and (7.2). For example, using the Haley bounds as suggested in Smith (1973) we obtain from (8.3) the following conditions: If $T(A,B,C)$ is nonempty, then for each U, V and W satisfying $Q_3(W)\subset Q_1(U)\cup Q_2(V)$ we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}H_{ijk}(u_{jk}+v_{i}$$

where

$$\bar{\Delta}=\sum_{j=1}^{n}\sum_{k=1}^{p}a_{jk}u_{jk}+\sum_{i=1}^{m}$$

$$u_{jk}=\begin{cases}1, & \text{whenever (}\\ 0, & \text{whenever (}\end{cases}$$

$$v_{ik}=\begin{cases}1, & \text{whenever (}\\ 0, & \text{whenever (}\end{cases}$$

$$w_{ij}=\begin{cases}1, & \text{whenever (}\\ 0, & \text{whenever (}\end{cases}$$

To see the necessity, observe t

$$\bar{\Delta}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{p}x_{ijk}(u_{jk}\cdot$$

and that $Q_3(W)\subset W_1(U)\cup Q_1(V$

The resultant conditions imply Morávek–Vlach conditions I and

Problem 8. What are the relati Morávek–Vlach conditions II?

Problem 9. Are the Smith condit

Problem 10. The problems anal resulting from the bounds α_{ijk} an

10. Some other problems

It is easy to verify that the pol

$$A=\begin{pmatrix}m-1 & m & \dots\\ m & 1 & \dots\\ \vdots & \vdots & \\ m & 1 & \dots\end{pmatrix}$$

$$C=\begin{pmatrix}p & 1 & \dots & 1\\ \vdots & \vdots & & \vdots\\ p & 1 & \dots & 1\\ p-1 & p & \dots & \end{pmatrix}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} H_{ijk}(u_{jk}+v_{ik}-w_{ij}) \leq \bar{\Delta} \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} h_{ijk}(u_{jk}+v_{ik}-w_{ij}) \qquad (9.1)$$

where

$$\bar{\Delta} = \sum_{j=1}^{n}\sum_{k=1}^{p} a_{jk}u_{jk} + \sum_{i=1}^{m}\sum_{k=1}^{p} b_{ik}v_{ik} - \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij},$$

$$u_{jk} = \begin{cases} 1, & \text{whenever } (j,k) \in U, \\ 0, & \text{whenever } (j,k) \notin U, \end{cases}$$

$$v_{ik} = \begin{cases} 1, & \text{whenever } (i,k) \in V, \\ 0, & \text{whenever } (i,k) \notin V, \end{cases}$$

$$w_{ij} = \begin{cases} 1, & \text{whenever } (i,j) \in W, \\ 0, & \text{whenever } (i,j) \notin W, \end{cases}$$

To see the necessity, observe that

$$\bar{\Delta} = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{p} x_{ijk}(u_{jk}+v_{ik}-w_{ij})$$

and that $Q_3(W) \subset W_1(U) \cup Q_1(V)$ ensures nonnegativity of $u_{jk}+v_{ik}-w_{ij}$.

The resultant conditions imply all of the Smith conditions I and II, all of the Morávek-Vlach conditions I and all of the Haley conditions.

Problem 8. What are the relations between the Smith conditions III and the Morávek-Vlach conditions II?

Problem 9. Are the Smith conditions III sufficient?

Problem 10. The problems analogous to the previous two for the conditions resulting from the bounds α_{ijk} and β_{ijk} instead of H_{ijk} and h_{ijk}.

10. Some other problems

It is easy to verify that the polytope $T(A, B, C)$ defined by the matrices

$$A = \begin{pmatrix} m-1 & m & \dots & m \\ m & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ m & 1 & \dots & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} n & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ n & 1 & \dots & 1 \\ n-1 & n & \dots & n \end{pmatrix},$$

$$C = \begin{pmatrix} p & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ p & 1 & \dots & 1 \\ p-1 & p & \dots & p \end{pmatrix},$$

degenerate to a point, namely to the point $X=[x_{ijk}]$ with

$$x_{ijk}=\begin{cases}0, & \text{if } i=m,\ j=1,\ k=1,\\ 0, & \text{if } (i,j,k)\in M'\times N'\times P',\\ 1, & \text{otherwise}\end{cases}$$

where $M'=M\setminus\{m\}$, $N'=N\setminus\{1\}$ and $P'=P\setminus\{1\}$.

If $T(A, B, C)$ is nonempty and if the Haley lower bounds are equal to the corresponding Haley upper bounds, then $T(A, B, C)$ has one point only. The example demonstrates that the converse is not true. Morávek and Vlach (1970) applied the Haley device of lower and upper bounds to certain flow problems in networks and gave necessary and sufficient conditions for the existence of a unique feasible flow.

Problem 11. Characterize those A, B and C for which $T(A, B, C)$ degenerates into a point.

It is not difficult to verify that $m+n+p-1$ equations of system (1.1)–(1.3) follow from the remaining equations and that the rank of the subsystem

$$\sum_{i\in M} x_{ijk}=a_{jk},\quad j\in N,\ k\in P;$$

$$\sum_{j\in N} x_{ijk}=b_{ik},\quad i\in M',\ k\in P;$$

$$\sum_{k\in P} x_{ijk}=c_{ij},\quad i\in M,\ j\in N$$

is equal to

$$mp+np+mn-m-n-p+1.$$

Therefore, the dimension of $T(A, B, C)$ cannot exceed $(m-1)(n-1)(p-1)$. At the same time, $T(A, B, C)$ with

$$a_{jk}=m,\qquad b_{ik}=n,\qquad c_{ij}=p$$

for all i, j, k contains a point X all components of which are positive (e.g. $x_{ijk}=1$). Consequently, the dimension of this polytope is equal to $(m-1)(n-1)(p-1)$.

Problem 12. Do there exist matrices A, B and C such that the dimension of $T(A, B, C)$ is q for each integer q between 0 and $(m-1)(n-1)(p-1)$?

Since the matrix of the system (1.1)–(1.3) is not totally unimodular whenever $\min(m, n, p)>2$, it is possible to construct an example of a nonempty $T(A, B, C)$ with integer A, B and C which has no integer point. For example, $T(A, B, C)$ with

$$A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

consists of one noninteger pc

$$[x_{ij1}]=\begin{pmatrix}0&\frac12&0\\ \frac12&0&\frac12\\ 0&0&0\\ 0&0&0\\ \frac12&\frac12&0\\ 0&0&\frac12\end{pmatrix}$$

$$[x_{ij3}]=\begin{pmatrix}0&\frac12&0\\ \frac12&0&\frac12\\ \frac12&0&0\\ 0&\frac12&\frac12\\ 0&0&0\\ 0&0&0\end{pmatrix}$$

D. de Werra (1978) showed

$$a_{j1}=a_{j2}=\ldots=a_{jp},$$

$$b_{i1}=b_{i2}=\ldots=b_{ip},$$

then $T(A, B, C)$ with nonnega only if

$$\sum_{i\in M} c_{ij}\le pa_{j1},\quad j\in$$

$$\sum_{j\in N} c_{ij}\le pb_{i1},\quad i\in$$

Problem 13. Characterize thos contains an integer point.

Problem 14. The nonemptine algorithm in time polynomial i Is there an algorithm to decide v of arithmetic operations and co p? In other words, is there a

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

consists of one noninteger point only, namely of the point $X = [x_{ijk}]$ with

$$[x_{ij1}] = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad [x_{ij2}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$[x_{ij3}] = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

D. de Werra (1978) showed that if

$$a_{j1} = a_{j2} = \ldots = a_{jp},$$

$$b_{i1} = b_{i2} = \ldots = b_{ip},$$

then $T(A, B, C)$ with nonnegative integer A, B and C has an integer point if and only if

$$\sum_{i \in M} c_{ij} \le p a_{j1}, \quad j \in N,$$

$$\sum_{j \in N} c_{ij} \le p b_{i1}, \quad i \in M.$$

Problem 13. Characterize those integer matrices A, B and C for which $T(A, B, C)$ contains an integer point.

Problem 14. The nonemptiness of $T(A, B, C)$ can be decided by the ellipsoid algorithm in time polynomial in the length of the binary encoding of A, B and C. Is there an algorithm to decide whether $T(A, B, C)$ is nonempty in which the number of arithmetic operations and comparisons is bounded by a polynomial in m, n, and p? In other words, is there a genuinely polynomial algorithm to decide whether

$T(A, B, C)$ is nonempty. See Megiddo (1982) for the concept of genuinel polynomial algorithm.

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NOTE

ON THE MEMBERSH GRAMMARS

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There are (at least) three motiva *graph grammars* (*regular* DNLC subclasses of DNLC grammars, (it provides a useful framework for The complexity of (the memb investigated.

Introduction

The theory of graph grammars theory and it has become a well-e 1983). The potential applicabilit science provides substantial mot

One of such fields is the theory has become quite popular as an a e.g., Mazurkiewicz, 1984, Mazurk and Rozenberg, 1985a). In Aalbe was related to the theory of gra controlled graph grammars (abbr the directed node-label controllec berg, 1981). It also turns out that into the hierarchy of various su natural subclass of the (directed berg and Welzl, 1984). Moreover very nicely the notion of a righ grammars.